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Configuration space properties of the S-matrix and time delay in potential scattering (*)

by

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ABSTRACT. — We prove estimates on time decay for products of Schrödinger evolution groups between weighted L^2-spaces by using commutator techniques. These estimates are used to show that, for potentials $V(\underline{x})$ decaying more rapidly than $|\underline{x}|^{-2}$, the scattering operator leaves a certain dense subset $\mathcal{D}$ of the domain of definition of $\underline{Q}^2$ invariant. This implies the existence of the global time delay (defined in terms of sojourn times) on $\mathcal{D}$ if $V(\underline{x}) = 0(|\underline{x}|^{-2-\varepsilon})$ at infinity, $\varepsilon > 0$.

RÉSUMÉ. — En utilisant des techniques de commutateurs, nous démontrons des estimations concernant la décroissance temporelle pour des produits de groupes d'évolution de Schrödinger considérés entre des espaces L^2 à poids. Nous utilisons ces estimations pour montrer que, pour des potentiels $V(\underline{x})$ décroissant plus vite que $|\underline{x}|^{-2}$, l'opérateur de diffusion laisse invariant un certain sous-ensemble dense $\mathcal{D}$ du domaine de définition de $\underline{Q}^2$. Ceci implique l'existence du temps de retard global (défini en termes de temps de séjour) sur $\mathcal{D}$ si $V(\underline{x}) = 0(|\underline{x}|^{-2-\varepsilon})$ à l'infini, $\varepsilon > 0$.

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1. INTRODUCTION

Recently ([1]-[3]) there has been some renewed interest in the problem of proving the existence of time delay, defined as the limit as $r \to \infty$ of the difference of the sojourn times of a scattering state and of the associated asymptotic free state in a ball B_r of radius r centered at the origin of configuration space. Mathematically this problem amounts essentially to that of extracting the finite part from the difference of two diverging quantities, and physicists expect this finite part to be the Eisenbud-Wigner time delay (the energy derivative of the phase of the S-matrix).

In [1] and [2] the existence of the time delay was obtained under very weak decay assumptions on the potential $V(\underline{x})$ (it suffices for V to decay as $|\underline{x}|^{-\alpha}$ with $\alpha > 1$), but at the expense of replacing the sojourn time in a ball B_r by a slightly different quantity (a « weighted » sojourn time, with the weight of localization at a point $\underline{x}$ depending on the distance of $\underline{x}$ from the origin). The paper [3] on the other hand deals with the case where all points $\underline{x}$ in B_r have the same weight; this has the most satisfactory physical interpretation within the framework of quantum mechanics, since it can be expressed in terms of projections. However, the decay assumption on the potential made in [3] was rather strong ($\alpha > 4$).

The principal purpose of the present note is to show that the above-mentioned rather strong decay condition of [3] can be considerably relaxed ($\alpha > 2$ is sufficient). We recall that the paper [3] contains a general theorem on the existence of time delay under certain regularity assumptions on the S-matrix. It will be shown below that these regularity conditions are satisfied for potentials decaying as $|\underline{x}|^{-\alpha}$ with $\alpha > 2$. The proof relies on commutator methods (similar to those used in [4]-[6] and in [1]); these are more powerful than the Hilbert-Schmidt estimates used in [3].

The organization of the paper is as follows. In Section 2 we recall the general theorem from Ref. [3] and introduce the notations that we shall use. Section 3 contains some preliminary commutator estimates involving the Hamiltonian. In Section 4 we derive the principal inequality for the time evolution group, and in Section 5 we use this inequality to verify the hypotheses of the general theorem of Section 2.

2. NOTATIONS AND BASIC THEOREM

As in [3], we denote by $\underline{Q} = (Q_1, \ldots, Q_n)$ and $\underline{P} = (P_1, \ldots, P_n)$ the n-component position and momentum operator respectively in the complex Hilbert space $\mathscr{H} \equiv L^2(\mathbb{R}^n)$. The free Hamiltonian is $H_0 = \underline{P}^2 = \Sigma_{j=1}^n P_j^2$ and the total Hamiltonian has the form $H = H_0 + V(\underline{Q})$. Throughout the paper, the potential $V(\underline{x})$ is assumed to satisfy the following condition:

(H1) V *is a real-valued function defined on* $\mathbb{R}^n$ *of the form*

$$V(\underline{x}) = (1 + \underline{x}^2)^{-\alpha/2}[W_1(\underline{x}) + W_2(\underline{x})] \tag{1}$$

with $\alpha > 0$, $W_1 \in L^\infty(\mathbb{R}^n)$ *and* $W_2 \in L^q(\mathbb{R}^n)$ *for some* q *satisfying* $q \geq 2$ *and* $q > n/2$.

In Sections 4 and 5 we shall need stronger assumptions on W_1 and W_2, and the results of Section 4 will hold for $\alpha > 1$, whereas those of Section 5 will require $\alpha > 2$. We denote by $U_t \equiv \exp(-iHt)$ and $U_t^0 \equiv \exp(-iH_0 t)$ the unitary time evolution operators associated with H and H_0 respectively.

If V satisfies (H1), then H is self-adjoint with domain $D(H) = D(H_0) \subseteq D(V)$, and $\| (W_1 + W_2)(H_0 + a)^{-1} \| < \infty$ for each $a > 0$. If $\alpha > 1$, then the wave operators $\Omega_\pm = s - \lim U_{-t} U_t^0$ as $t \to \pm\infty$ exist and are complete, and the scattering operator $S \equiv \Omega_+^* \Omega_-$ is unitary. If F_r denotes the orthogonal projection in $\mathcal{H}$ onto the subspace of all state vectors localized in the ball $B_r = \{ \underline{x} \in \mathbb{R}^n \mid |\underline{x}| \leq r \}$ in configuration space, then the local time delay of a state vector $f \in \mathcal{H}$ is defined as

$$\tau_r(f) = \int_{-\infty}^{\infty} (\| F_r U_t \Omega_- f \|^2 - \| F_r U_t^0 f \|^2) dt. \tag{2}$$

The following general theorem on the existence of the global time delay (i. e. the limit of $\tau_r(f)$ as $r \to \infty$) was given in [3]:

PROPOSITION 1. — *a) Assume that* $f \in L^2(\mathbb{R}^n)$ *is such that i) its Fourier transform* $\tilde{f}$ *has compact support in* $\mathbb{R}^n \setminus \{ 0 \}$, *ii)* $\underline{Q}^2 f \in L^2(\mathbb{R}^n)$, *iii)* $\underline{Q}^2 S f \in L^2(\mathbb{R}^n)$,

iv) $\| U_t \Omega_- f - U_t^0 f \| \in L^1((-\infty, 0]; dt)$

and v) $\| U_t \Omega_- f - U_t^0 S f \| \in L^1([0, \infty); dt)$.

Then the sequence $\{ \tau_r(f) \}$ *converges as* $r \to \infty$.

b) Assume in addition that the S-*matrix* $S(\lambda)$ *at energy* λ *is continuously differentiable with respect to* λ *on some interval* $(a, b) \subset (0, \infty)$ *and that the support of* $\tilde{f}(\underline{k})$ *lies in* $\{ \underline{k} \in \mathbb{R}^n \mid a < \underline{k}^2 < b \}$. *Then*

$$\lim_{r \to \infty} \tau_r(f) = (f, Tf), \tag{3}$$

where T *is the Eisenbud-Wigner time delay operator (i. e.* T *is a decomposable operator in the spectral representation of* $H_0 : T = \{ T(\lambda) \}_{\lambda > 0}$, *with* $T(\lambda) = -iS(\lambda)^* dS(\lambda)/d\lambda$).

We denote by $\sigma_p^+(H) \equiv \sigma_p^+$ the set of all positive eigenvalues of H (if any) and by ω a positive number such that $H + \omega > I$ (I denotes the identity operator, and the existence of ω is guaranteed by the fact that H is bounded from below if V satisfies (H1)). We set $L = (H + \omega)^{-1}$ and $L_0 = (H_0 + \omega)^{-1}$ and have $\| L \| \leq 1$ and $\| L_0 \| \leq 1$. We denote by A the self-adjoint infinitesimal generator of the dilation group and define $\tilde{V}$ by

$$\tilde{V} = V + \frac{i}{2}[A, V], \tag{4}$$

with

$$A = \frac{1}{2}(\underline{P}\cdot\underline{Q} + \underline{Q}\cdot\underline{P}). \tag{5}$$

We set $|\underline{Q}| = (\Sigma_{j=1}^{n} Q_j^2)^{1/2}$, $\langle Q \rangle = (1 + \underline{Q}^2)^{1/2}$ and $\langle Q \rangle_\varepsilon = (1 + \varepsilon\underline{Q}^2)^{1/2}$, where $\varepsilon > 0$.

If B is a linear operator in $\mathscr{H}$, we write $B \in \mathscr{B}(\mathscr{H})$ if its domain D(B) is dense in $\mathscr{H}$ and B is bounded on D(B) or if one can associate to it a densely defined bounded sesquilinear form. We shall then denote its closure by the same letter B. We define $\mathscr{W}^\infty$ to be the set of all functions $\psi : \mathbb{R}^n \to \mathbb{C}$ such that ψ and all its partial derivatives (of any order) are in $L^\infty(\mathbb{R}^n)$. The letter c will be used for various constants that need not be identical even within the same proof.

Finally we mention some commutator identities that will frequently be used (z denotes a complex number):

$$[B, C_1C_2] = [B, C_1]C_2 + C_1[B, C_2], \tag{6}$$

$$[B, (C + z)^{-1}] = (C + z)^{-1}[C, B](C + z)^{-1}, \tag{7}$$

$$[B, e^{-itD}] = -i\int_0^t e^{-iD(t-\zeta)}[B, D]e^{-iD\zeta}d\zeta \tag{8}$$

if D is self-adjoint.

3. PRELIMINARY COMMUTATOR RELATIONS

In this section we show that certain operators are bounded between weighted L^2-spaces. In general we shall indicate the proof only for the case where the weights are integer powers of $1 + |\underline{x}|$; the corresponding result for non-integer powers then follows by interpolation. We shall use the fact that $D(|\underline{Q}|) = \cap_{j=1}^{n} D(Q_j)$ and that $\| |\underline{Q}| f \|^2 = \Sigma_{j=1}^{n} \| Q_j f \|^2$ (see Lemma 5 of [3]). For Lemmas 1-7 we assume that V satisfies (H1).

LEMMA 1. — *Let $\varphi \in \mathscr{W}^\infty$. Then $\langle Q \rangle^\mu \varphi(\underline{P}) \langle Q \rangle^{-\mu} \in \mathscr{B}(\mathscr{H})$ for each real number $\mu \geq 0$.*

Proof. — Let $\mu > 0$ be an integer. It suffices to prove that, for each n-tuple $(\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that $\Sigma_{j=1}^{n} \alpha_j = \mu$, the operator $Z \equiv \Pi_{j=1}^{n} Q_j^{\alpha_j} \varphi(\underline{P}) \langle Q \rangle^{-\mu}$ is in $\mathscr{B}(\mathscr{H})$. This is easily checked by commuting the powers of $Q_1, \ldots, Q_n$ through $\varphi(\underline{P})$, using the relation $Q_j\varphi(\underline{P}) = \varphi(\underline{P})Q_j + i(\text{grad}\,\varphi)_j(\underline{P})$, which expresses Z as a finite sum of bounded everywhere defined operators. ■

LEMMA 2. — *Let* $\psi \in C_0^\infty(\mathbb{R})$ *or let* ψ *be the function* $\psi(\lambda)=(\lambda+\omega)^{-1}$. *Furthermore, let* W *be a* H_0*-bounded operator commuting with* $\underline{Q}$. *Then one has for each* $\mu \in \mathbb{R}$ *and* $j = 1, \ldots, n$:

a) $$\langle Q \rangle^\mu \psi(H_0) \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H}),$$

b) $$\langle Q \rangle^\mu P_j \psi(H_0) \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H}),$$

c) $$\langle Q \rangle^{\mu-1} A \psi(H_0) \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H}),$$

d) $$\langle Q \rangle^\mu \psi(H_0) W \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H}).$$

Proof. — *a)* and *b)* follow from Lemma 1, since the functions ψ_j, defined as $\psi_0(\underline{\xi}) = (\underline{\xi}^2 + \omega)^{-1}$ and $\psi_k(\underline{\xi}) = \xi_k(\underline{\xi}^2 + \omega)^{-1}$ $(k = 1, \ldots, n)$ belong to $\mathcal{W}^\infty$ for each j. *c)* follows from *b)* by noticing that $A = \underline{Q}\cdot\underline{P} - in/2 = \underline{P}\cdot\underline{Q} + in/2$.

To derive *d)* for $\psi(\lambda) = (\lambda + \omega)^{-1}$, one proceeds as in the proof of Lemma 1 by noticing for instance that $\Pi_{j=1}^n Q_j^{\gamma_j}(H_0 + \omega)^{-1} W \langle Q \rangle^{-\mu}$ is a sum of terms of the form $\varphi_1(\underline{P})(H_0 + \omega)^{-1} W \varphi_2(\underline{Q})$ with $\varphi_1, \varphi_2 \in L^\infty(\mathbb{R}^n)$ if $\Sigma_{j=1}^n \gamma_j \leq \mu$ since, for any n-tuple $(\beta_1, \ldots, \beta_n)$ of non-negative integers:

$$\left(\frac{\partial}{\partial \xi_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{\beta_n} (\underline{\xi}^2 + \omega)^{-1} = (\underline{\xi}^2 + \omega)^{-1}\theta(\underline{\xi})$$

with $\theta \in L^\infty(\mathbb{R}^n)$. Next, if $\psi \in C_0^\infty(\mathbb{R})$, we have for example:

$$\langle Q \rangle^\mu \psi(H_0) W \langle Q \rangle^{-\mu} = \langle Q \rangle^\mu (H_0 + \omega)^{-1} W \langle Q \rangle^{-\mu} \cdot \langle Q \rangle^\mu \{ (H_0 + \omega)\psi(H_0) \} \langle Q \rangle^{-\mu},$$

and each of the two factors on the r. h. s. has already been shown to be in $B(\mathcal{H})$. ■

LEMMA 3. — *Let* W *be as in Lemma 2. Then one has for each* $\mu \in \mathbb{R}$: $\langle Q \rangle^\mu (H+\omega)^{-1} W \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$, $\langle Q \rangle^\mu (H+\omega)^{-1} \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$ and $\langle Q \rangle^\mu P_j (H + \omega)^{-1} \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$ $(j = 1, \ldots, n)$.

Proof. — We prove for example that $\langle Q \rangle^\mu (H + \omega)^{-1} W \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$ for $\mu \geq 0$. Since W^* is also H-bounded, this clearly holds for $\mu = 0$. By the second resolvent equation and (1) one has

$$\langle Q \rangle^\mu (H+\omega)^{-1} W \langle Q \rangle^{-\mu} = \langle Q \rangle^\mu (H_0+\omega)^{-1} W \langle Q \rangle^{-\mu} - \\ - \langle Q \rangle^\mu (H_0+\omega)^{-1}(W_1+W_2) \langle Q \rangle^{-\mu} \cdot \langle Q \rangle^{\mu-\alpha}(H+\omega)^{-1} W \langle Q \rangle^{-(\mu-\alpha)} \cdot \langle Q \rangle^{-\alpha}. \quad (9)$$

We take $\mu = \alpha$ and find from Lemma 2 *d)* that each term on the r. h. s. of (9) is in $\mathcal{B}(\mathcal{H})$. Hence, by interpolation, $\langle Q \rangle^\mu (H+\omega)^{-1} W \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$ for all $\mu \in [0, \alpha]$. Next we choose $\mu \in (\alpha, 2\alpha]$ and obtain, by using the preceding result and (9), that $\langle Q \rangle^\mu (H + \omega)^{-1} W \langle Q \rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$ for these values of μ. By iteration (take $\mu \in (2\alpha, 3\alpha]$, then $\mu \in (3\alpha, 4\alpha]$, etc.) one obtains the desired result for all $\mu > 0$. ■

LEMMA 4. — *Let* $L = (H + \omega)^{-1}$. *Then for each* $\mu \geq 0$ *there is a constant* c_μ *such that for all* $\tau \in \mathbb{R}$:

$$\| \langle Q \rangle^{\mu} e^{-iL\tau} \langle Q \rangle^{-\mu} \| \leq c_\mu (1 + |\tau|)^{\mu}. \tag{10}$$

Proof. — *i*) Let m be an integer, $m \geq 2$, and let $\varepsilon > 0$. Then, since V commutes with $\underline{Q}$, it is not difficult to calculate the following commutator (first on the Schwartz space $S(\mathbb{R}^n)$ of infinitely differentiable function of rapid decay, then by approximation on $D(H_0)$), by writing $H_0 = -\Sigma_{j=1}^n (\partial/\partial x_j)^2$:

$$\left[H, \frac{|\underline{Q}|^m}{\langle Q \rangle_\varepsilon^m} \right] = m(m + n - 2) \frac{|\underline{Q}|^{m-2}}{\langle Q \rangle_\varepsilon^m} - m(m + n) \frac{\varepsilon |\underline{Q}|^m}{\langle Q \rangle_\varepsilon^{m+2}}$$
$$+ m(m + 2) \frac{\varepsilon^2 |\underline{Q}|^{m+2}}{\langle Q \rangle_\varepsilon^{m+4}} - 2im\underline{P} \cdot \underline{Q} \frac{|\underline{Q}|^{m-2}}{\langle Q \rangle_\varepsilon^m} + 2im\underline{P} \cdot \underline{Q} \frac{\varepsilon |\underline{Q}|^m}{\langle Q \rangle_\varepsilon^{m+2}}.$$

Similarly one finds that

$$\left[H, \frac{Q_k}{\langle Q \rangle_\varepsilon} \right] = -(n+2) \frac{\varepsilon Q_k}{\langle Q \rangle_\varepsilon^3} + 3 \frac{\varepsilon^2 Q_k \underline{Q}^2}{\langle Q \rangle_\varepsilon^5} - 2iP_k \frac{1}{\langle Q \rangle_\varepsilon} + i\underline{P} \cdot \underline{Q} \frac{\varepsilon Q_k}{\langle Q \rangle_\varepsilon^3} .$$

For later use we observe that these commutators have the following structure:

$$\left[H, \frac{Q_k}{\langle Q \rangle_\varepsilon} \right] = \varphi_0^{(k)}(\varepsilon, \underline{Q}) + \sum_{j=1}^n P_j \varphi_j^{(k)}(\varepsilon, \underline{Q}) \tag{11}$$

and

$$\left[H, \frac{|\underline{Q}|^m}{\langle Q \rangle_\varepsilon^m} \right] = |\underline{Q}|^{m-2} \varphi_{m,0}(\varepsilon, \underline{Q}) + \sum_{j=1}^n P_j Q_j |\underline{Q}|^{m-2} \varphi_{m,1}(\varepsilon, \underline{Q}) \tag{12}$$

with

$$\sup_{\varepsilon \in (0,1]} \| \varphi_j^{(k)}(\varepsilon, \underline{Q}) \| \leq \kappa_1 < \infty \tag{13}$$

and

$$\sup_{\varepsilon \in (0,1]} \| \varphi_{l,v}(\varepsilon, \underline{Q}) \| \leq \kappa_l < \infty , \tag{14}$$

for all $k = 1, \ldots, n$, all $j = 0, \ldots, n$, all $l \geq 2$ and $v = 0, 1$.

ii) Now let $\mu = 1$, $f \in \mathscr{H}$ and $g \in D(|\underline{Q}|)$. Then, by using (8) and (7), we obtain that, for $k = 1, \ldots, n$:

$$(Q_k g, e^{-iL\tau} \langle Q \rangle^{-1} f) = \lim_{\varepsilon \to 0} \left(\frac{Q_k}{\langle Q \rangle_\varepsilon} g, e^{-iL\tau} \langle Q \rangle^{-1} f \right)$$
$$= \lim_{\varepsilon \to 0} (g, e^{-iL\tau} Q_k \langle Q \rangle^{-1} \langle Q \rangle_\varepsilon^{-1} f) -$$
$$- \lim_{\varepsilon \to 0} i \int_0^\tau \left(g, e^{-iL(\tau-\zeta)} L \left[H, \frac{Q_k}{\langle Q \rangle_\varepsilon} \right] L e^{-iL\zeta} \langle Q \rangle^{-1} f \right) d\zeta . \tag{15}$$

The absolute value of the first scalar product on the r. h. s. is less than $\| f \| \| g \|$ for all $\varepsilon > 0$. By virtue of (11) and (13), the absolute value of the integral in the second term on the r. h. s. is majorized, for all $\varepsilon \in (0, 1)$, by $c \| f \| \| g \| \int_0^\tau (\| L \|^2 + \| L \| n \| \, |\underline{P}| \, L \|) d\zeta$. Hence (15) implies that

$$\| Q_k e^{-iL\tau} \langle Q \rangle^{-1} \| \leq c_1 (1 + |\tau|),$$

which proves (10) for $\mu = 1$.

iii) We now let μ be an integer m, $m \geq 2$, and proceed by induction. As above we have

$$|\underline{Q}|^m e^{-iL\tau} \langle Q \rangle^{-m}$$
$$= \lim_{\varepsilon \to 0} \left[e^{-iL\tau} \frac{|\underline{Q}|^m}{\langle Q \rangle_\varepsilon^m} \langle Q \rangle^{-m} - i \int_0^\tau e^{-iL(\tau - \zeta)} L \left[H, \frac{|\underline{Q}|^m}{\langle Q \rangle_\varepsilon^m} \right] L e^{-iL\zeta} \langle Q \rangle^{-m} d\zeta \right].$$

The norm of the first term in the square bracket is less than 1 for all $\varepsilon > 0$, whereas the norm of the second term is majorized by

$$\{ \| L \varphi_{m,0}(\varepsilon, \underline{Q}) \| \, \| \, |\underline{Q}|^{m-2} (H + \omega)^{-1} \langle Q \rangle^{-m+1} \| +$$
$$+ \sum_{j=1}^n \| L P_j \| \, \| \varphi_{m,1}(\varepsilon, \underline{Q}) \| \, \| Q_j |\underline{Q}|^{m-2} (H + \omega)^{-1} \langle Q \rangle^{-m+1} \| \} .$$
$$\cdot \int_0^\tau \| \langle Q \rangle^{m-1} e^{-iL\zeta} \langle Q \rangle^{-m} \| d\zeta .$$

The curly bracket is seen to be bounded by a constant independent of $\varepsilon \in (0, 1)$ by virtue of (14) and Lemma 3, and the norm under the integral is bounded by $c_{m-1}(1 + |\zeta|)^{m-1} \leq c_{m-1}(1 + |\tau|)^{m-1}$ from the induction hypothesis. ■

LEMMA 5. — *Let W be as in Lemma 2, let $\varphi \in C_0^\infty(\mathbb{R})$ and let α be the number appearing in (1). Let $\beta, \gamma \in \mathbb{R}$ be such that $\beta + \gamma \leq \alpha$. Then one has*

$$\langle Q \rangle^{\beta - 1} A \{ \varphi(H) - \varphi(H_0) \} \langle Q \rangle^\gamma \in \mathscr{B}(\mathscr{H}),$$
$$\langle Q \rangle^\beta \{ \varphi(H) - \varphi(H_0) \} \langle Q \rangle^\gamma \in \mathscr{B}(\mathscr{H})$$

and

$$\langle Q \rangle^\beta \{ \varphi(H) - \varphi(H_0) \} W \langle Q \rangle^\gamma \in \mathscr{B}(\mathscr{H}).$$

Proof. — Without loss of generality we may assume that $\varphi \in C_0^\infty((-\omega, \infty))$, where ω is such that $H + \omega > I$. Let θ be defined by $\theta(\lambda) = \varphi(\lambda^{-1} - \omega)$ and observe that $\theta \in C_0^\infty(\mathbb{R})$ and that supp $\theta \subset (0, \infty)$. We have, with $L = (H + \omega)^{-1}$:

$$\varphi(H) = \theta(L) = (2\pi)^{-1/2} \int_{-\infty}^\infty \tilde{\theta}(t) e^{iLt} dt , \tag{16}$$

and similarly $\varphi(H_0) = \theta(L_0)$.

We indicate the proof of the last assertion of the lemma. The proof of the first two assertions is similar. By the second resolvent equation we have

$$I - e^{-iLt}e^{iL_0t} = i\int_0^t e^{-iLs}(L - L_0)e^{iL_0s}ds = -i\int_0^t e^{-iLs}LVL_0e^{iL_0s}ds.$$

Hence

$$(2\pi)^{1/2}\,\|\langle Q\rangle^\beta\{\varphi(H) - \varphi(H_0)\}W\langle Q\rangle^\gamma\|$$
$$= \left\|\int_{-\infty}^{\infty}\tilde\theta(t)\langle Q\rangle^\beta e^{iLt}(I - e^{-iLt}e^{iL_0t})W\langle Q\rangle^\gamma dt\right\|$$
$$\le \int_{-\infty}^{\infty}dt\,|\tilde\Theta(t)|\left|\int_0^t ds\,\|\langle Q\rangle^\beta e^{iL(t-s)}\langle Q\rangle^{-\beta}\|\cdot\right.$$
$$\cdot\;\|\langle Q\rangle^\beta(H+\omega)^{-1}(W_1+W_2)\langle Q\rangle^{-\beta}\|\cdot\|\langle Q\rangle^{\beta-\alpha}e^{iL_0s}\langle Q\rangle^{\alpha-\beta}\|\cdot$$
$$\cdot\;\|\langle Q\rangle^{-(\alpha-\beta)}(H_0+\omega)^{-1}W\langle Q\rangle^\gamma\|\,|. \tag{17}$$

The second and the fourth norm in the integrand are finite by Lemma 3, since $\alpha - \beta \ge \gamma$, and the product of the remaining two norms is majorized by $c(1+|t|)^{\alpha+2|\beta|}$ by virtue of Lemma 4. Since $\theta \in C_0^\infty(\mathbb{R})$, the double integral is finite. ■

LEMMA 6. — *Let* W *be as in Lemma 2, and let* $\varphi \in C_0^\infty(\mathbb{R})$. *Then one has for each* $\mu \in \mathbb{R}$: $\langle Q\rangle^\mu\varphi(H)W\langle Q\rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$ *and* $\langle Q\rangle^{\mu-1}A\varphi(H)\langle Q\rangle^{-\mu} \in \mathcal{B}(\mathcal{H})$.

Proof. — This follows immediately from Lemma 2 *c*, *d*) and Lemma 5. ■

LEMMA 7. — *Let* $\varphi, \chi \in C_0^\infty(\mathbb{R})$ *and* $m = 1, 2, \ldots$. *Then: a) the functions* $\tau \mapsto \langle Q\rangle^{m-1}AU_\tau\varphi(H)\langle Q\rangle^{-m}$ *and* $\tau \mapsto \langle Q\rangle^m U_\tau\varphi(H)\langle Q\rangle^{-m}$ *are continuous in operator norm. b) One has (as forms on* $\mathcal{H}$*)*

$$\langle Q\rangle^{-1}\chi(H)[A, U_t]\varphi(H)\langle Q\rangle^{-1} = -i\int_0^t\langle Q\rangle^{-1}\chi(H)U_{t-\zeta}[A, H]U_\zeta\varphi(H)\langle Q\rangle^{-1}d\zeta \tag{18}$$

and

$$\langle Q\rangle^{-2}\chi(H)[\underline{Q}^2, U_t]\varphi(H)\langle Q\rangle^{-2} = -i\int_0^t\langle Q\rangle^{-2}\chi(H)U_{t-\zeta}[\underline{Q}^2, H]U_\zeta\varphi(H)\langle Q\rangle^{-2}d\zeta. \tag{19}$$

Proof. — *a*) It suffices to prove continuity at $\tau = 0$. First, because all partial derivatives of the function $\underline{k} \mapsto [\exp(-i\underline{k}^2\tau) - 1]\varphi(\underline{k}^2)$ converge to zero in $L^\infty(\mathbb{R}^n)$ as $\tau \to 0$, it follows from the proof of Lemmas 1 and 2 that $\langle Q\rangle^m U_\tau^0\varphi(H_0)\langle Q\rangle^{-m}$ and $\langle Q\rangle^{m-1}AU_\tau^0\varphi(H_0)\langle Q\rangle^{-m}$ are continuous at $\tau = 0$. Next we proceed as in the proof of Lemma 5 to show that $\langle Q\rangle^m[U_\tau\varphi(H) - U_\tau^0\varphi(H_0)]\langle Q\rangle^{-m}$ and $\langle Q\rangle^{m-1}A[U_\tau\varphi(H) - U_\tau^0\varphi(H_0)]\langle Q\rangle^{-m}$

are continuous at $\tau=0$. We may again assume that $\varphi \in C_0^\infty((-\omega, \infty))$, and we set $\theta_\tau(\lambda)=\exp(-i\tau/\lambda+i\omega\tau)\varphi(\lambda^{-1}-\omega)$. We observe that $d^k/d\lambda^k[\theta_\tau(\lambda)-\theta_0(\lambda)]$ converges to zero in $L^1(\mathbb{R}; d\lambda)$ as $\tau \to 0$, for each $k=0, 1, 2, \ldots$, which implies that $[\tilde{\theta}_\tau(t)-\tilde{\theta}_0(t)](1+|t|)^l$ converges to zero in $L^1(\mathbb{R}; dt)$ as $\tau \to 0$ for each $l \geq 0$. Then for example the first of the claimed results follows from (17) in which $\tilde{\theta}(t)$ is replaced by $\tilde{\theta}_\tau(t) - \tilde{\theta}_0(t)$.

b) The result of *a)* implies for instance that the operator-valued function

$$\zeta \mapsto \langle Q \rangle^{-2}\chi(H)U_\zeta^* \underline{Q}^2 U_\zeta \varphi(H) \langle Q \rangle^{-2} \tag{20}$$

is strongly continuously differentiable, so that (19) is a special case of (8), obtained by integrating the derivative of (20). ■

For the remainder of the paper we need a somewhat stronger assumption on V than (H1), viz. the following condition:

(H2) *the function* $V: \mathbb{R}^n \to \mathbb{R}$ *has the form (1) with* $\alpha > 0$, $W_1 \in L^\infty(\mathbb{R}^n)$, $\underline{x} \cdot \mathrm{grad}\, W_1 \in L^\infty(\mathbb{R}^n) + L^{q_1}(\mathbb{R}^n)$ *and* $(1 + |\underline{x}|)W_2 \in L^\infty(\mathbb{R}^n) + L^{q_2}(\mathbb{R}^n)$, *where* q_1 *and* q_2 *satisfy* $q_j \geq 2$ *and* $q_j > n/2$.

LEMMA 8. — *Assume that* V *satisfies* (H2). *Let* $\beta, \gamma \in \mathbb{R}$ *be such that* $\beta+\gamma \leq \alpha$ *and let* $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R})$. *Then the operator* $\langle Q \rangle^\beta \varphi_1(H)\tilde{V}\varphi_2(H)\langle Q \rangle^\gamma$ *belongs to* $\mathcal{B}(\mathcal{H})$.

Proof. — Notice that $\tilde{V}$ has the form

$$\tilde{V} \equiv V + \frac{i}{2}[A, V] = \langle Q \rangle^{-\alpha} W(\underline{Q}) + \frac{i}{2} A W_2 \langle Q \rangle^{-\alpha} - \frac{i}{2} \langle Q \rangle^{-\alpha} W_2 A, \tag{21}$$

with

$$W(\underline{x}) = W_1(\underline{x}) + W_2(\underline{x}) - \frac{\alpha \underline{x}^2}{1+\underline{x}^2} W_1(\underline{x}) + \underline{x} \cdot \mathrm{grad}\, W_1(\underline{x}) \in L^\infty(\mathbb{R}^n) + L^{q_1}(\mathbb{R}^n) + L^{q_2}(\mathbb{R}^n).$$

The contribution of the first term in (21) to the norm of

$$\langle Q \rangle^\beta \varphi_1(H)\tilde{V}\varphi_2(H)\langle Q \rangle^\gamma$$

is majorized by

$$\| \langle Q \rangle^\beta \varphi_1(H) \langle Q \rangle^{-\beta} \| \, \| \langle Q \rangle^{-\alpha+\beta+\gamma} \| \, \| \langle Q \rangle^{-\gamma} W \varphi_2(H) \langle Q \rangle^\gamma \|,$$

which is finite by Lemma 6. The contribution from the second term (and similarly that from the third term) in (21) to this norm does not exceed the number

$$\| \langle Q \rangle^\beta \varphi_1(H) A \langle Q \rangle^{-\beta-1} \| \, \| \langle Q \rangle^{-\alpha+\beta+\mu} \| \, \| \langle Q \rangle^{-\mu} (\langle Q \rangle W_2) \varphi_2(H) \langle Q \rangle^\mu \|,$$

which is again finite by Lemma 6. ■

4. TIME DECAY FOR EVOLUTION GROUPS

The main purpose of this section is to show that, if V satisfies (H2) with $\alpha > 2$, the function $t \mapsto \|(1 + |\underline{Q}|)^{-\alpha}\varphi(H)U_t\Omega_\pm \psi(H_0)f\|$ decays faster than $|t|^{-2}$ for suitable φ, ψ and for $f \in D(|Q|^{\alpha'})$ with $\alpha' > 2$. This result will be an easy consequence of the following lemma.

LEMMA 9. — *Let* V *satisfy* (H2) *with* $\alpha > 1$. *Let* $\varphi \in C_0^\infty((0, \infty)\backslash\sigma_p^+)$ *and* $\psi \in C_0^\infty((0, \infty))$. *Then, for each real number* $\kappa \in [0, \alpha]$ *and each* $\varepsilon > 0$, *there is a constant c such that for all* $s, t \in \mathbb{R}$:

a) $$\| \langle Q \rangle^{-\kappa} \varphi(H) U_t \langle Q \rangle^{-\kappa} \| \leqslant c(1 + |t|)^{-\kappa+\varepsilon}, \tag{22}$$

b) $$\| \langle Q \rangle^{-\kappa} \varphi(H) U_{t-s} U_s^0 \psi(H_0) \langle Q \rangle^{-\kappa} \| \leqslant c(1 + |t|)^{-\kappa+\varepsilon}. \tag{23}$$

Proof. — We observe that the l. h. s. of (22) can be obtained from that of (23) by setting $s = 0$ and $\psi \equiv 1$. Since the proofs of (22) and (23) are similar and can be done together, we also admit in the equations and inequalities below the function $\psi \equiv 1$. We refer to Case *a)* if $s = 0$ and $\psi \equiv 1$ and to Case *b)* if $\psi \in C_0^\infty((0, \infty))$, and our task is then to prove (23) in Case *a)* and in Case *b)*.

ii) We fix a function $\theta \in C_0^\infty((0, \infty)\backslash\sigma_p^+)$ such that $\theta\varphi = \varphi$ and write (for the moment formally):

$$[A, U_{t-s}\theta(H)U_s^0] = [A, U_{t-s}]\theta(H)U_s^0 + U_{t-s}[A, \theta(H)]U_s^0 + U_{t-s}\theta(H)[A, U_s^0]. \tag{24}$$

By (8) and the fact that $[A, H_0] = 2iH_0$, this leads to

$$\begin{aligned}[A, U_{t-s}\theta(H)U_s^0] = {} & 2(t - s)HU_{t-s}\theta(H)U_s^0 - \\ & - 2\int_0^{t-s} U_{t-s-\zeta}\tilde{V}U_\zeta\theta(H)U_s^0 d\zeta + U_{t-s}[A, \theta(H_0)]U_s^0 + \\ & + U_{t-s}[A, \theta(H) - \theta(H_0)]U_s^0 + 2sU_{t-s}\theta(H)(H - V)U_s^0.\end{aligned} \tag{25}$$

We observe that, by virtue of Lemmas 6 and 7, the identities (24) and (25) are meaningful when sandwiched between $\langle Q \rangle^{-1}\chi(H)$ and $\psi(H_0)\langle Q \rangle^{-1}$ if $\chi, \psi \in C_0^\infty(\mathbb{R})$.

Let $\varphi_1 \in C_0^\infty((0, \infty)\backslash\sigma_p^+)$ be defined by $\varphi_1(\lambda) = \lambda^{-1}\varphi(\lambda)$. Upon premultiplying and postmultiplying (25) by $\langle Q \rangle^{-\kappa}\varphi_1(H)$ and $\psi(H_0)\langle Q \rangle^{-\kappa}$ respectively ($\kappa \geq 1$) and by noticing that $[A, \theta(H_0)] = 2iH_0\theta'(H_0) \equiv \psi_1(H_0)$, one finds that

$$(1 + |t|) \| \langle Q \rangle^{-\kappa} \varphi(H) U_{t-s} U_s^0 \psi(H_0) \langle Q \rangle^{-\kappa} \| \leq \sum_{l=1}^{7} N_{\kappa,l}(s, t), \tag{26}$$

with

$$N_{\kappa,1}(s, t) = \| \langle Q \rangle^{-\kappa} \varphi(H) U_{t-s} U_s^0 \psi(H_0) \langle Q \rangle^{-\kappa} \| + \| \langle Q \rangle^{-\kappa} \varphi_1(H) U_{t-s} U_s^0 \psi_1(H_0) \psi(H_0) \langle Q \rangle^{-\kappa} \|,$$

$$N_{\kappa,2}(s, t) = \| \langle Q \rangle^{-\kappa} \varphi_1(H) A U_{t-s} \theta(H) U_s^0 \psi(H_0) \langle Q \rangle^{-\kappa} \|,$$

$$N_{\kappa,3}(s, t) = \| \langle Q \rangle^{-\kappa} \varphi_1(H) U_{t-s} U_s^0 A \psi(H_0) \langle Q \rangle^{-\kappa} \|,$$

$$N_{\kappa,4}(s,t)=\left\| \int_s^t d\tau \langle Q \rangle^{-\kappa}\varphi_1(H)U_{t-\tau}\tilde{V}U_{\tau-s}\theta(H)U_s^0\psi(H_0)\langle Q \rangle^{-\kappa} \right\|,$$

$$N_{\kappa,5}(s,t)=\| \langle Q \rangle^{-\kappa}\varphi_1(H)U_{t-s}A\{\theta(H)-\theta(H_0)\}U_s^0\psi(H_0)\langle Q \rangle^{-\kappa}\|,$$

$$N_{\kappa,6}(s,t)=\| \langle Q \rangle^{-\kappa}\varphi_1(H)U_{t-s}\{\theta(H)-\theta(H_0)\}AU_s^0\psi(H_0)\langle Q \rangle^{-\kappa}\|,$$

$$N_{\kappa,7}(s,t)=|s|\,\| \langle Q \rangle^{-\kappa}\varphi_1(H)U_{t-s}VU_s^0\psi(H_0)\langle Q \rangle^{-\kappa}\|.$$

ii) We first let $\kappa = 1$. To prove (22) and (23) for $\kappa = 1$, it suffices to show that each $N_{1,l}(s,t)$ is bounded (in Case *a*) and in Case *b*)) by a constant independent of s and t. For $l = 1$ this is evident. For $l = 2$ it follows from Lemma 6 (with $\mu = 1$) and for $l = 3$ from Lemma 2*c*) in Case *b*) and from the fact that $\varphi_1(H)P_j \in \mathscr{B}(\mathscr{H})$ in Case *a*). To treat $N_{1,4}$ one can use Lemma 8 with β, $\gamma > 1/2$ and the fact that $(1+|Q|)^{-\delta}$ is locally H-smooth on $(0,\infty)\backslash\sigma_p^+$ if $\delta > 1/2$ (see for instance [7] or Lemma 2.3 of [4]). For $l = 5$ and $l=6$ the result follows from Lemma 5 with $\beta=1$ and $\gamma=0$, since $\alpha > 1$.

All the above estimates hold in Case *a*) and in Case *b*). In Case *a*), $N_{1,7}=0$, hence (22) holds for $\kappa = 1$, with $\varepsilon = 0$. This result, with H replaced by H_0 (i. e. for $V = 0$, with $\sigma_p^+(H_0) = \emptyset$), can be used to estimate $N_{1,7}$ in Case *b*). It suffices to observe that $\varphi_1(H)(W_1 + W_2)\langle Q \rangle^{-\alpha+1} \in \mathscr{B}(\mathscr{H})$ and $|s|\,\|\langle Q \rangle^{-1}U_s^0\psi(H_0)\langle Q \rangle^{-1}\| \leq$ const. for all $s \in \mathbb{R}$ by (22).

iii) By interpolation between $\kappa = 0$ and $\kappa = 1$, it follows that (22) and (23) hold (with $\varepsilon = 0$) for each $\kappa \in [0,1]$, each $\varphi \in C_0^\infty((0,\infty)\backslash\sigma_p^+)$ and each $\psi \in C_0^\infty((0,\infty))$ (the constant c depends on φ and ψ).

iv) As in the proof of Lemma 3, we now derive (22) and (23) for general κ by a recursion procedure. It suffices to prove the following: Let $\nu \in (1,\alpha]$ and assume that (22) and (23) hold for $\kappa = \nu - 1$; then these inequalities are true also for $\kappa = \nu$. This follows from (26) provided that we can show that, for any $\eta \in (0, \nu - 1)$, one has

$$N_{\nu,l}(s,t) \leqslant c_\eta(1+|t|)^{-(\nu-1)+\eta} \qquad (l = 1, \ldots, 7). \tag{27}$$

This inequality is immediate for $l=1$ (in Case *a*) and in Case *b*)) by the recursion hypothesis, and similarly for $l=2$ since $\langle Q \rangle^{-\nu}\varphi_1(H)A\langle Q \rangle^{\nu-1} \in \mathscr{B}(\mathscr{H})$ by Lemma 6. To deal with $N_{\nu,3}$ in Case *a*), we write

$$\varphi_1(H)U_tA = \{\varphi_1(H)U_t\langle Q \rangle^{-(\nu-1)}\}\cdot\{\langle Q \rangle^{\nu-1}\theta(H)A\}$$

and use the recursion hypothesis and Lemma 6. In Case *b*) we choose a function $\psi_0 \in C_0^\infty((0,\infty))$ such that $\psi_0\psi = \psi$, write

$$A\psi(H_0)=\{\psi(H_0)\langle Q \rangle^{-(\nu-1)}\}\cdot\{\langle Q \rangle^{\nu-1}\psi_0(H_0)A\}+2iH_0\psi'(H_0)$$

and then proceed as in Case *a*).

Next we majorize the norm of the integrand in $N_{\nu,4}$ as follows (θ_1 is a function in $C_0^\infty((0,\infty)\backslash\sigma_p^+)$ such that $\theta_1\theta = \theta$):

$$\|\langle Q \rangle^{-\nu}\varphi_1(H)U_{t-\tau}\langle Q \rangle^{-\beta}\|\cdot\|\langle Q \rangle^{\beta}\theta_1(H)\tilde{V}\theta_1(H)\langle Q \rangle^{\gamma}\|\cdot$$
$$\cdot\|\langle Q \rangle^{-\gamma}\theta(H)U_{\tau-s}U_s^0\psi(H_0)\langle Q \rangle^{-\nu}\|. \tag{28}$$

We extend the domain of integration to all of $\mathbb{R}$ and choose β, γ as follows: (1) if $\tau \in [t/2, 3t/2]$: $\beta = 1, \gamma = \nu - 1$, (2) if $\tau \notin [t/2, 3t/2]$: $\beta = \nu - 1, \gamma = 1$. The second norm in (28) is then finite by Lemma 8, and the other two norms can be estimated by using (22) and (23) for $\kappa = 1$ and for $\kappa = \nu - 1$. This leads to

$$\mathrm{N}_{\nu,4}(s, t) \leqslant c_\eta \left| \int_{t/2}^{3t/2} (1 + |t - \tau|)^{-1}(1 + |\tau|)^{-(\nu-1)+\eta/2} d\tau \right| + c_\eta \int_{\mathbb{R} \setminus [t/2, 3t/2]} (1 + |t - \tau|)^{-(\nu-1)+\eta/4}(1 + |\tau|)^{-1} d\tau,$$

where c_η is a finite constant depending on η, ν, φ_1, θ_1, θ and V. We observe that $|\tau| \geq |t|/2$ in the domain of the first integral and $|t - \tau| \geq |t|/2$ in that of the second integral. Hence

$$\mathrm{N}_{\nu,4}(s, t) \leqslant c_\eta \left(1 + \frac{1}{2}|t|\right)^{-(\nu-1)+\eta/2} 2 \log\left(1 + \frac{1}{2}|t|\right) + c_\eta \left(1 + \frac{1}{2}|t|\right)^{-(\nu-1)+\eta/2} \int_{\mathbb{R}} \left(\frac{1+|\tau|}{1+|t-\tau|}\right)^{\eta/4} (1+|\tau|)^{-1-\eta/4} d\tau .$$

Since $(1 + |\tau|)(1 + |t - \tau|)^{-1} \leq 1 + |t|$ for all $t, \tau \in \mathbb{R}$, this implies (27) for $l = 4$.

For $l = 5$ we write

$$\mathrm{A}\{\theta(\mathrm{H}) - \theta(\mathrm{H}_0)\} = \langle \mathrm{Q} \rangle^{-\delta} \cdot \langle \mathrm{Q} \rangle^{\delta} \mathrm{A}\{\theta(\mathrm{H}) - \theta(\mathrm{H}_0)\} \langle \mathrm{Q} \rangle^{\gamma} \cdot \langle \mathrm{Q} \rangle^{-\gamma}. \quad (29)$$

We first take $\delta = \nu - 1$ and $\gamma = 0$ and obtain from Lemma 5 and the recursion hypothesis that $\mathrm{N}_{\nu,5}(s, t) \leq c(1 + |t - s|)^{-\nu+1+\eta}$. This implies (27) for $l = 5$ in Case *a*) as well as in Case *b*) under the additional assumption $|t - s| \geq |t|/2$. On the other hand, if $|t - s| \leq |t|/2$, we set $\delta = 0$ and $\gamma = \nu - 1$ in (29) and obtain that $\mathrm{N}_{\nu,5}(s, t) \leq c(1 + |s|)^{-\nu+1+\eta}$, which implies (27) for these values of s, t because $|s| \geq |t|/2$. For $l = 6$ we proceed similarly.

It remains to treat $\mathrm{N}_{\nu,7}$. In Case *a*) this term is zero and the proof is complete in this case. More precisely, if (22) and (23) hold for $\kappa = \nu - 1$, then (22) holds also for $\kappa = \nu$. One can now use this result to show that (23) also holds for $\kappa = \nu$. It suffices to verify that $\mathrm{N}_{\nu,7}$ satisfies (27) in Case *b*). This is done by writing

$$\mathrm{N}_{\nu,7}(s, t) \leqslant \| \langle \mathrm{Q} \rangle^{-\nu} \varphi_1(\mathrm{H}) \mathrm{U}_{t-s} \langle \mathrm{Q} \rangle^{-\beta} \| \cdot \cdot \| \langle \mathrm{Q} \rangle^{\beta} \theta(\mathrm{H})(\mathrm{W}_1 + \mathrm{W}_2) \langle \mathrm{Q} \rangle^{-\alpha+\gamma} \| \cdot |s| \, \| \langle \mathrm{Q} \rangle^{-\gamma} \mathrm{U}_s^0 \psi(\mathrm{H}_0) \langle \mathrm{Q} \rangle^{-\nu} \| .$$

If $|t - s| \geq |t|/2$ we take $\beta = \nu - 1$ and $\gamma = 1$. The first factor is then bounded by $c(1 + |t - s|)^{-(\nu-1)+\eta} \leq c(1 + |t|/2)^{-(\nu-1)+\eta}$, the second one is finite by Lemma 6 and the third one is bounded by a constant independent of s

by (22) with $\kappa=1$. If $|t-s| < |t|/2$, we have $|s| \geq |t|/2$, and we choose $\beta=0$, $\gamma=\nu$ and use (22) for $\kappa=\nu$ to obtain that

$$N_{\nu,7}(s,t) \leq c|s|(1+|s|)^{-\nu+\eta} \leq c(1+|t|/2)^{-\nu+1+\eta}. \qquad \blacksquare$$

PROPOSITION 2. — *Assume that* V *satisfies* (H2) *with* $\alpha > 1$. *Let* $f \in \mathcal{H}$ *be such that* $\tilde{f}$ *has compact support in* $\mathbb{R}^n \setminus \{0\}$ *and* $f \in D(|\underline{Q}|^\rho)$ *for some* $\rho > 1$. *Let* $\delta < \min(\alpha, \rho)$ *and let* $\varphi \in C_0^\infty((0,\infty)\setminus\sigma_p^+)$. *Then there is a constant c such that for all* $t \in \mathbb{R}$:

$$\|\langle Q\rangle^{-\alpha}\varphi(H)U_t\Omega_\pm f\| \leqslant c(1+|t|)^{-\delta}. \qquad (30)$$

Proof. — Choose $\psi \in C_0^\infty((0,\infty))$ such that $\psi(H_0)f = f$ and apply Lemma 9 *b*) with $\kappa = \min(\alpha,\rho)$, $\varepsilon = \kappa - \delta$ and with $s \to \pm\infty$. $\blacksquare$

COROLLARY. — *In addition to the hypotheses of Proposition 2, assume that* $\alpha > 2$ *and* $\rho > 2$. *Then there is a constant c and a number* $\eta > 0$ *such that for all* $t \in \mathbb{R}$:

$$\|\langle Q\rangle^{-\alpha}\varphi(H)U_t\Omega_\pm f\| \leqslant c(1+|t|)^{-2-\eta}. \qquad (31)$$

We refer to [*8*] and to the references cited there for other results on time decay similar to Lemma 9*a*).

5. EXISTENCE OF TIME DELAY

We now prove the existence of time delay for potentials decaying as $|\underline{x}|^{-\alpha}$ with $\alpha > 2$. For $\rho > 0$, we define $\mathcal{D}_\rho$ to be the following dense subset of $\mathcal{H}$:

$$\mathcal{D}_\rho = \{ g \in L^2(\mathbb{R}^n) \mid g \in D(|\underline{Q}|^\rho),\ g = \psi(H_0)g \text{ for some } \psi \in C_0^\infty((0,\infty)\setminus\sigma_p^+(H)) \}.$$

We then have the following result:

PROPOSITION 3. — *Assume that* V *satisfies* (H2) *with* $\alpha > 2$, *and let* $f \in \mathcal{D}_\rho$ *for some* $\rho > 2$. *Then f satisfies all hypotheses of Proposition 1. Hence* $\lim \tau_r(f)$ *as* $r \to \infty$ *exists and is given by (3).*

Remark. — The assumptions on V imply that $S(\lambda)$ is continuously differentiable (in operator norm) on $(0,\infty)\setminus\sigma_p^+$ [*4*].

Proof. — We must show that the assumptions *iii*), *iv*) and *v*) of Proposition 1 are satisfied. The validity of *iii*) will be given as a separate result, see Proposition 4 below.

To verify assumption *iv*) of Proposition 1, we choose $\varphi \in C_0^\infty((0,\infty)\setminus\sigma_p^+)$ such that $\varphi(H_0)f = f$ and write

$$\|U_t\Omega_- f - U_t^0 f\| = \|\varphi(H)(U_t\Omega_- f - U_t^0 f) + (\varphi(H)-\varphi(H_0))U_t^0 f\|$$
$$\leqslant \left\| \int_{-\infty}^t ds U_s^* \varphi(H) V U_s^0 f \right\| + \|(\varphi(H)-\varphi(H_0))\langle Q\rangle^\alpha\| \, \|\langle Q\rangle^{-\alpha}\varphi(H_0)U_t^0 f\|.$$

The last term is bounded by $c(1+|t|)^{-2-\eta}$ with $\eta > 0$, by Lemma 5 (with $\beta=0$ and $\gamma=\alpha$) and (31) (for $V=0$). Similarly, for $t \leq 0$:

$$\left\| \int_{-\infty}^{t} ds U_s^* \varphi(H) V U_s^0 f \right\| \leqslant \| \varphi(H)(W_1 + W_2) \| \int_{-\infty}^{t} ds\, c(1 + |s|)^{-2-\eta} \leqslant c'(1 + |t|)^{-1-\eta}.$$

These estimates show that $\| U_t \Omega_- f - U_t^0 f \| \in L^1((-\infty, 0]; dt)$, so that assumption *iv*) of Proposition 1 is satisfied.

The verification of assumption *v*) is rather similar. One writes

$$\begin{aligned} \| U_t \Omega_- f - U_t^0 S f \| &= \| \varphi(H_0)(U_t - U_t^0 \Omega_+^*) \Omega_- f + (\varphi(H) - \varphi(H_0)) U_t \Omega_- f \| \\ &\leqslant \left\| \int_t^{\infty} ds U_s^{0*} \varphi(H_0) V U_s \Omega_- f \right\| \\ &\quad + \| (\varphi(H) - \varphi(H_0)) \langle Q \rangle^{\alpha} \| \cdot \| \langle Q \rangle^{-\alpha} U_t \varphi(H) \Omega_- f \| \end{aligned}$$

and proceeds as before by using Lemma 5 and (31). ■

PROPOSITION 4. — *Assume that* V *and* f *are as in Proposition 3. Then* $\Omega_\pm f \in D(\underline{Q}^2)$ *and* $Sf \in D(\underline{Q}^2)$.

Proof. — *i*) Let $f \in \mathscr{D}_\rho$, $\rho > 2$, and choose $\psi, \theta \in C_0^\infty((0, \infty) \backslash \sigma_p^+)$ such that $\psi(H_0) f = f$ and $\theta\psi = \psi$. Observe that $\theta'\psi = \theta''\psi = 0$; in particular, since $\underline{Q} = i\underline{\nabla}_p$:

$$[\underline{Q}^2, \theta(H_0)] \psi(H_0) = [A, \theta(H_0)] \psi(H_0) = 0. \tag{32}$$

We first proceed as in part *i*) of the proof of Lemma 9. By Lemmas 6 and 7, the following formal identities are correct when sandwiched between $\langle Q \rangle^{-2} \theta(H)$ and $\psi(H_0) \langle Q \rangle^{-2}$:

$$\begin{aligned} [\underline{Q}^2, U_t^* \theta(H) U_t^0] = [\underline{Q}^2, U_t^*] \theta(H) U_t^0 + U_t^* [\underline{Q}^2, \theta(H) - \theta(H_0)] U_t^0 \\ + U_t^* [\underline{Q}^2, \theta(H_0)] U_t^0 + U_t^* \theta(H) [\underline{Q}^2, U_t^0]. \end{aligned} \tag{33}$$

Now

$$[\underline{Q}^2, U_t^0] = (4tA - 4t^2 H_0) U_t^0.$$

Furthermore, by using first (19) and the relation $[\underline{Q}^2, H_0] = 4iA$, and then (18) and $[A, H] = 2i(H - \tilde{V})$, one obtains that

$$\begin{aligned} [\underline{Q}^2, U_t^*] &= -i \int_0^{-t} U_{t+\zeta}^* [\underline{Q}^2, H_0] U_\zeta d\zeta = \\ &= 4 \int_0^{-t} d\zeta U_{t+\zeta}^* \left(U_\zeta A - i \int_0^{\zeta} d\tau U_{\zeta-\tau} [A, H] U_\tau \right) \\ &= -4t U_t^* A + 4t^2 U_t^* H - 8 \int_0^{-t} d\zeta \int_0^{\zeta} d\tau U_{t+\tau}^* \tilde{V} U_\tau. \end{aligned}$$

Upon inserting these relations into (33) and by writing

$$[A, \theta(H)] = [A, \theta(H) - \theta(H_0)] + [A, \theta(H_0)]$$

and using (32), one finds that for each $h \in D(\underline{Q}^2)$:

$$\begin{aligned}(\underline{Q}^2\theta(H)h, U_t^*\theta(H)U_t^0 f) &= (h, \theta^2(H)U_t^*U_t^0\underline{Q}^2 f) - \\ &- 4t(h, \theta(H)U_t^*[A, \theta(H) - \theta(H_0)]U_t^0\psi(H_0)f) + \\ &+ 4t^2(h, U_t^*\theta^2(H)VU_t^0\psi(H_0)f) - \\ &- 8\int_0^{-t} d\zeta \int_0^{\zeta} d\tau(h, U_{t+\tau}^*\theta(H)\tilde{V}\theta(H)U_\tau\theta_1(H)U_t^0\psi(H_0)f) + \\ &+ (h, \theta(H)U_t^*[\underline{Q}^2, \theta(H) - \theta(H_0)]U_t^0\psi(H_0)f),\end{aligned} \quad (34)$$

where $\theta_1 \in C_0^\infty((0,\infty)\backslash\sigma_p^+)$ is such that $\theta_1\theta = \theta$.

ii) We now let $t \to \pm\infty$ in (34). We observe that the second term on the r. h. s. converges to zero by Lemmas 5 and 9, since

$$[A, \theta(H) - \theta(H_0)] \langle Q \rangle^{\alpha-1} \in \mathscr{B}(\mathscr{H})$$

and

$$|t| \, \| \langle Q \rangle^{-\alpha+1} U_t^0\psi(H_0)f \| \le c_\varepsilon |t| (1 + |t|)^{-\alpha+1+\varepsilon},$$

which tends to zero as $|t| \to \infty$ if ε is chosen in $(0, \alpha - 2)$. Similarly one finds, by using (1), (22) and Lemma 5, that the third and the fifth term on the r. h. s. converge to zero. Since the l. h. s. and the first term on the r. h. s. have limits, the fourth term on the r. h. s. is also convergent; in fact the double integral of the vector-valued function given by the second factor in the scalar product defines a vector $g(t)$ in $\mathscr{H}$, for any $t \in [-\infty, +\infty]$, because one has by Lemma 8 (with $\beta = 0$, $\gamma = \alpha$) and (23) that

$$\| U_{t+\tau}^*\theta(H)\tilde{V}\theta(H)U_\tau\theta_1(H)U_t^0\psi(H_0)f \| \leqslant c \, \| \langle Q \rangle^\rho f \| (1 + |t+\tau|)^{-\delta}, \quad (35)$$

where $2 < \delta < \min(\alpha, \rho)$. Thus (34) implies, together with the fact that $\theta(H)\Omega_\pm f = \Omega_\pm f$, that for each $h \in D(\underline{Q}^2)$:

$$(\underline{Q}^2\theta(H)h, \Omega_\pm f) = (h, \Omega_\pm\underline{Q}^2 f) - 8(h, g(\pm\infty)). \quad (36)$$

Since $\Omega_\pm f = \psi(H)\Omega_\pm f$, we have

$$(\underline{Q}^2 h, \Omega_\pm f) = (\psi(H)\underline{Q}^2 h - \psi(H)\underline{Q}^2\theta(H)h, \Omega_\pm f) + (\underline{Q}^2\theta(H)h, \Omega_\pm f). \quad (37)$$

Now the following identity is easily checked by simple algebraic manipulations and by using (32):

$$\begin{aligned}\psi(H)\underline{Q}^2 - \psi(H)\underline{Q}^2\theta(H) = \{ \psi(H) - \psi(H_0) \} \, \underline{Q}^2 \, \{ I - \theta(H) \} - \\ - \psi(H_0)\underline{Q}^2 \{ \theta(H) - \theta(H_0) \}.\end{aligned}$$

Together with (37), Lemma 5 and (36), this shows that, for each $f \in \mathscr{D}_\rho$ with $\rho > 2$, there are two vectors $f_\pm$ in $\mathscr{H}$ such that $(\underline{Q}^2 h, \Omega_\pm f) = (h, f_\pm)$, for all $h \in D(\underline{Q}^2)$. Thus $\Omega_\pm f \in D(\underline{Q}^2)$ for each $f \in \mathscr{D}_\rho$, $\rho > 2$.

iii) Next we fix a vector $\hat{f} \in \mathscr{D}_\rho$, $\rho > 2$, and choose real-valued functions φ, χ, ψ, $\theta \in C_0^\infty((0, \infty)\backslash\sigma_p^+)$ such that $\varphi(H_0)\hat{f} = \hat{f}$, $\chi\varphi = \varphi$, $\psi\chi = \chi$ and $\theta\psi = \psi$, and we let $g \in D(|\underline{Q}|^3)$. Then, since $\varphi(H_0)[\underline{Q}^2, \chi(H_0)] = 0$ as in (32) and because $S\varphi(H_0) = \varphi(H_0)S$, we have

$$(S\hat{f}, \underline{Q}^2 g) = (S\hat{f}, \varphi(H_0)\underline{Q}^2\chi(H_0)g) = (S\hat{f}, \underline{Q}^2\chi(H_0)g).$$

Also $f \equiv \chi(H_0)g$ belongs to $\mathcal{D}_3$, since $\psi(H_0)f = f$. Hence we obtain from (34) or (36), with $f = \chi(H_0)g$ and $h \equiv \Omega_-\hat{f} = \theta(H)\Omega_-\hat{f}$, that

$$(S\hat{f}, \underline{Q}^2 g) = (S\hat{f}, \underline{Q}^2 f) = (h, \Omega_+\underline{Q}^2 f) = (\chi(H_0)\Omega_+^*\underline{Q}^2\Omega_-\hat{f}, g) + \\ + 8 \lim_{t\to+\infty} \int_0^{-t} d\zeta \int_0^{\zeta} d\tau(\chi(H_0)U_t^{0*}\theta_1(H)U_\tau^*\theta(H)\tilde{V}\theta(H)U_{t+\tau}\Omega_-\hat{f}, g), \quad (38)$$

with θ_1 as in (34). As in (35) one sees that the double integral of the vector-valued function in the first factor of the scalar product defines a vector in $\mathcal{H}$ for each $t \in [-\infty, +\infty]$, since by (31)

$$\| \langle Q \rangle^{-\alpha}\theta(H)U_{t+\tau}\Omega_- f \| \leqslant c(1 + |t + \tau|)^{-2-\eta}$$

for some $\eta > 0$. Thus (38) implies the existence of a vector e in $\mathcal{H}$ such that $(S\hat{f}, \underline{Q}^2 g) = (e, g)$ for all $g \in D(|\underline{Q}|^3)$, so that $S\hat{f} \in D(\underline{Q}^2)$ because the operator $\underline{Q}^2$ is essentially self-adjoint on $D(|\underline{Q}|^3)$. ■

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