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Conformal invariance and time decay for non linear wave equations. I

by

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ABSTRACT. — We study the implications of the approximate conformal invariance of the non linear wave equation

$$\square\varphi + f(\varphi) = 0 \qquad (*)$$

on the time decay of its solutions. We first prove the conformal conservation law in as much generality as possible. We then derive a number of estimates of space-time weighted L^l-norms in terms of the kinetic part of the conformal charge and deduce some decay properties of the solutions from them. Typically for $f(\varphi) = \varphi \mid \varphi \mid^{p-1}$ with $4/(n-1) \leqslant p - 1 < 4/(n-2)$, $n \geqslant 3$, where n is the space dimension, we prove (directly from the conservation law) that all solutions of $(*)$ with finite energy and finite conformal charge at time zero decay in time according to

$$\| \varphi(t) \|_l \leqslant Ct^{-(n-1)(1/2-1/l)}$$

for $2 \leqslant l \leqslant 2n/(n-2)$. We also extend these results to some cases with $p - 1 < 4/(n-1)$.

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RÉSUMÉ. — On étudie les implications de l'invariance conforme approchée de l'équation d'onde non linéaire

$$\square\varphi + f(\varphi) = 0 \qquad (*)$$

sur la décroissance en temps de ses solutions. On démontre d'abord la loi de conservation conforme dans un cas aussi général que possible. On démontre ensuite plusieurs estimations de normes L^l d'espace-temps pondérées en termes de la partie cinétique de la charge conforme et on en déduit des propriétés de décroissance en temps des solutions. Typiquement, pour $f(\varphi) = \varphi \mid \varphi \mid^{p-1}$ avec $4/(n-1) \leqslant p - 1 < 4/(n-2)$ et $n \geqslant 3$, où n est la dimension d'espace, on déduit directement de la loi de conservation que toutes les solutions de (*) d'énergie finie et de charge conforme finie au temps zéro décroissent en temps comme

$$\| \varphi(t) \|_l \leqslant Ct^{-(n-1)(1/2-1/l)}$$

pour $2 \leqslant l \leqslant 2n/(n-2)$. On étend ces résultats à quelques situations où $p - 1 < 4/(n-1)$.

1. INTRODUCTION

A large amount of work has been devoted to the theory of Scattering for the non linear wave equation (or non linear massless Klein-Gordon equation)

$$\square\varphi \equiv \ddot{\varphi} - \Delta\varphi = - f(\varphi) \qquad (1.1)$$

where φ is a complex valued function defined in space-time $\mathbb{R}^{n+1}$, the upper dot denotes the time derivative, Δ is the Laplace operator in $\mathbb{R}^n$ and f is a non linear complex valued function, a typical form of which is

$$f(\varphi) = \lambda\varphi \mid \varphi \mid^{p-1} \qquad (1.2)$$

with $1 \leqslant p < \infty$ *[10]* *[18]* *[19]* *[21]* *[22]* *[27]* *[28]* *[32]*. The main purpose of that theory is to study the asymptotic behaviour in time of the solutions of the equation (1.1) by comparing them with the solutions of the simpler linear wave equation $\square\varphi = 0$. A prerequisite to that study is the existence and uniqueness of global solutions of the Cauchy problem for the equation (1.1) for a reasonably large class of initial data. That result is now available for (arbitrarily large) initial data of finite energy, namely for initial data ($\varphi_0 = \varphi(t_0)$, $\psi_0 = \dot{\varphi}(t_0)$) in the energy space $X_e \equiv H^1 \oplus L^2$. The solutions thereby obtained, hereafter called finite energy solutions, are such that $(\varphi, \dot{\varphi})$ is continuous in time with values in X_e. The assumptions on f are fairly general and natural and amount to the conditions $\lambda \geqslant 0$ and [6] [7]

$$p - 1 < 4/(n - 2) \tag{1.3}$$

in the special case (1.2). The theory of Scattering for the equation (1.1) then gives rise to two main questions. The first one is to prove the existence of dispersive solutions, namely of solutions that behave as solutions of $\Box\varphi = 0$ in a suitable sense at $+\infty$ or $-\infty$ in time, or equivalently to prove the existence of the wave operators. That result is generally proved by solving the Cauchy problem at infinity, namely with large (possibly infinite) initial time, by a contraction method, in a space of functions exhibiting a suitable time decay. A slight variation of the contraction argument, based on the same technical information, yields the existence of global solutions with small initial data [9] [12] [14] [15[[16] [24] without reference to the energy and for that reason without any repulsivity condition on f, namely without the condition $\lambda \geqslant 0$ in the special case (1.2). On the other hand the interaction f has to satisfy a suitable condition of decay at infinity in space, namely of decay for small φ, which takes the form of a lower bound on p in the special case (1.2). Using the available $L^{\cdot}$-$L^{\cdot}$ estimates for $\Box\varphi = 0$ [17] [20] [29], it is easy to implement the previous method under the condition $p > p_1(n)$, where $p_1(n)$ is the larger root of the equation [5] [18] [19] [26]

$$n(n - 1)p^2 - p(n^2 + 3n - 2) + 2 = 0\,. \tag{1.4}$$

One sees easily that

$$1 + 4/n < p_1(n) < 1 + 4/n + 8/n^3\,.$$

That condition is not expected to be optimal, however. On the basis of results available for $n = 2$ [9] and $n = 3$ [12], one expects the method to be applicable under the weaker condition $p > p_0(n)$ where $p_0(n)$ is the larger root of the equation

$$(n - 1)p^2 - (n + 1)p - 2 = 0\,. \tag{1.5}$$

For $n = 2, 3$ that extension requires more complicated space-time weighted norms than simply $L^{\cdot}$-norms. The condition $p > p_0(n)$ is expected to be optimal for the applicability of the method, in view of the existing blow up results for non repulsive interactions and small initial data in the opposite situation $p < p_0(n)$ [8] [12] [13] [25].

The second question which arises in the theory of Scattering is to prove asymptotic completeness, namely the fact that all solutions of (1.1) with initial data in a suitably large space are actually dispersive at $+\infty$ and $-\infty$. The only method available for that purpose is based on the approximate conformal invariance of (1.1) and the associated approximate conservation law [27]: there exists a quantity, hereafter called the conformal charge, which is formally approximately conserved for any solution of (1.1) and which, under suitable assumptions on f, is positive and uniformly bounded in time. The conformal conservation law suggests a natural framework to study the theory of Scattering and in particular the problem of asymptotic completeness. The conformal charge, like the energy, consists of two parts: a kinetic part, coming from the linear part $\square\varphi$ of the equation (1.1), and a potential part coming from the interaction term. There exists a Hilbert space Σ (see definition in Section 2, especially (2.10)-(2.13)) which is essentially the largest space where the kinetic parts of the energy and of the conformal charge are defined. Dispersive solutions can then be defined as solutions having asymptotic states in Σ, and asymptotic completeness reduces to the statement that all solutions of (1.1) with initial data in Σ have asymptotic states in Σ. The proof proceeds by first extracting some preliminary decay from the boundedness of the conformal charge, and possibly by improving that decay by further use of the equation (1.1) or rather of the integral equation associated with it. The positivity and uniform boundedness in time of the conformal charge are easily seen to be ensured by assumptions on f which reduce to the repulsivity condition $\lambda \geqslant 0$ and to

$$p - 1 \geqslant 4/(n - 1) \tag{1.6}$$

in the special case (1.2). Under those assumptions, most of the steps leading to the proof of asymptotic completeness are adaptations of known methods. One possible exception is the direct extraction of preliminary time decay from the boundedness of the conformal charge. So far, only the potential part of the conformal charge has been used [*10*] [*27*], with the drawbacks that on the one hand the decay thereby obtained is rather weak and further use of the equation is required to prove asymptotic completeness, and on the other hand a lower boundedness condition on the interaction is unnaturally required.

The first purpose of the present paper is to improve that situation by extracting directly as much time decay as possible from the kinetic part of the conformal charge. Interestingly enough, the time decay thereby obtained is rather strong. It is stronger in some respects than what one would expect from the $L^{\cdot}$-$L^{\cdot}$ estimates on the linear wave equation. In particular, it is sufficient to implement the construction of dispersive solutions by contraction under the assumption $p > p_1(n)$ for given asymptotic states in Σ. Furthermore, it comes out naturally in terms of space-time weighted norms which are strongly reminiscent of those required for $n = 2$ and 3 to push the contraction argument down to its optimal limit $p_0(n)$ [*9*] [*12*].

It is useful at this point to draw a comparison with the non linear Schrödinger equation [*3*] [*11*] [*30*] [*31*]

$$i\dot{\varphi} = -\frac{1}{2}\Delta\varphi + f(\varphi)\,. \tag{1.7}$$

In that case, under conditions on f which are implied by $\lambda \geqslant 0$ and (1.3), any initial data φ_0 in the energy space H^1 generate a unique global solution which is a bounded continuous H^1-valued function of time. The contraction argument leading to the existence of dispersive solutions and of the wave operators works in a reasonably large space for $p > p_0(n+1)$. The shift from n to $n+1$ when switching from (1.1) to (1.7) is best understood by noticing that the solutions of the linear Schrödinger equation $i\dot{\varphi} = -\frac{1}{2}\Delta\varphi$ exhibit the same time decay as those of the free massive Klein-Gordon equation $(\square + m^2)\varphi = 0$, and that dispersion in the massless case $\square\varphi = 0$ occurs (at least in odd dimensions by Huygens' principle) on $(n-1)$-dimensional submanifolds instead of n-dimensional space. Asymptotic completeness can be proved by a method analogous to the conformal invariance method applied to (1.1), where the conformal charge is replaced by an analogous pseudoconformal charge. From the approximate conservation law, it follows immediately that the latter is positive and uniformly bounded in time under assumptions that reduce to $\lambda \geqslant 0$ and

$$p - 1 \geqslant 4/n \tag{1.8}$$

instead of (1.6) in the special case (1.2) (note again the shift from n to $n+1$). The extraction of relevant time decay from the kinetic part of the pseudoconformal charge is now an elementary lemma (as a preliminary version thereof, see Lemma 1.3 of [*3*], which however unnecessarily wastes a limiting case). One of the main results of the present paper is the analogue of that lemma in the case of the equation (1.1).

Remarkably enough, however, the story does not end with the condition (1.8) in the case of the equation (1.7). Actually, by combining the pseudoconformal conservation law with the equation itself, one can prove that asymptotic completeness holds in the space of initial data with finite energy and finite pseudoconformal charge down to $p_0(n+1)$ (limit excluded) [*11*] [*30*] [*31*]. This leads to the question whether (optimistically the conjecture that) asymptotic completeness holds in Σ for the equation (1.1) down to $p_0(n)$ (limit excluded). The second purpose of the present paper is to take a first step toward answering that question by showing that the kinetic part of the conformal charge remains uniformly bounded in time (for $n = 3$) or at most logarithmically bounded in time (for $n = 2$) down to values of p which are strictly lower than the limiting case of (1.6). For that purpose, one first extracts some time decay from the conformal conservation law by using the potential part of the conformal charge, and one exploits that decay by a method adapted from the treatment of the equation (1.7) given in [*11*]. In the present case, for a dimensional reason that will be explained in due course, that method has a natural limit to interactions satisfying conditions that reduce to $\lambda \geqslant 0$ and $p > p_2(n)$ in the special case (1.2), where $p_2(n)$ is the larger root of the equation

$$(n-1)p^2 - (n+2)p - 1 = 0. \tag{1.9}$$

One sees easily that

$$p_1(n) < p_2(n) < 1 + 4/n + 16/n^3.$$

Our results cover the expected range $p > p_2(2) = 2 + \sqrt{5}$ for $n = 2$. For $n = 3$, we require a stronger lower bound on p of no special significance, for technical reasons.

This paper is organized as follows. In Section 2, we derive the conformal conservation law in full generality for any space dimension, for finite energy solutions of the equation (1.1) with finite conformal charge. We first give a brief heuristic discussion connecting the conservation law to the transformation properties of the equation (1.1) under conformal transformations. We then proceed to the actual proof, following that of the analogous pseudoconformal conservation law of the non linear Schrödinger equation. We first regularize the equation (1.1) by introducing two cut-offs, we then derive the conservation law for the solutions of the regularized equation, and finally we remove the cut-offs by a limiting procedure. The final result is stated as Proposition 2.3.

In Section 3, we derive a number of estimates of (possibly weighted) $L^{\cdot}$-norms of a function φ of the space variable x in terms of the conformal charge and of related quantities. That section is logically independent of the previous one and can be read independently. It does not make any reference to the equation (1.1). The proof of the main estimates is an adaptation to the present situation of the usual elementary proof of the Sobolev inequalities. We restrict our attention to space dimension $n \geqslant 2$. For $n \geqslant 3$, the main result is Proposition 3.3, which provides our strongest decay results when applied to solutions of (1.1). For $n = 2$, the corresponding result is given in Proposition 3.4, but the situation is more complicated and the result is usefully complemented by additional information given in Propositions 3.1 and 3.2.

In Section 4, we exploit the results of Sections 2 and 3 to derive decay estimates of solutions of (1.1). We prove in particular (see Proposition 4.1 for a more general and more detailed statement) that for $n \geqslant 3$, any solution of (1.1) with interaction (1.2) satisfying (1.3) and (1.6), with initial data (φ_0, ψ_0) in Σ, namely such that φ_0, $\nabla\varphi_0$, ψ_0, $x\nabla\varphi_0$ and $x\psi_0$ all belong to L^2, satisfies the decay estimate

$$\| \varphi(t) \|_l \leqslant Ct^{-(n-1)(1/2-1/l)}$$

for $2 \leqslant l \leqslant 2n/(n-2)$. We then present the method used to extend the results of Proposition 4.1 to lower values of p and apply it to the case $n = 2$ in Proposition 4.2, where we cover the expected range $p > p_2(2)$, and to the case $n = 3$ in Proposition 4.3, where we obtain only a preliminary result for that case.

We conclude this section by giving the main notation used in this paper. For any l, $1 \leqslant l \leqslant \infty$, we denote by $\| . \|_l$ the norm in $L^l \equiv L^l(\mathbb{R}^n)$. With each l, $1 \leqslant l \leqslant \infty$ we associate the variables $\alpha(l)$, $\gamma(l)$ and $\delta(l)$ defined by

$$\alpha(l) = \gamma(l)/(n-1) = \delta(l)/n = 1/2 - 1/l .$$

For $n = 2$, $\alpha(l) = \gamma(l)$ and we shall use only $\alpha(l)$ when treating the case $n = 2$ by itself. For any integer $n \geqslant 3$, we define $2^* = 2n/(n-2)$. For any integer k, we denote by $H^k \equiv H^k(\mathbb{R}^n)$ the usual Sobolev spaces. For any interval I of $\mathbb{R}$ (possibly $\mathbb{R}$ itself), for any Banach space B, we denote by $\mathscr{C}(I, B)$ (resp. $\mathscr{C}^1(I, B)$, resp. $L^\infty(I, B)$) the space of continuous (resp. continuously differentiable, resp. essentially bounded) functions from I to B. For any open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathscr{C}_0^\infty(\Omega)$ the space of infinitely differentiable functions with compact support contained in Ω. In particular, we define $\mathscr{D}_0 \equiv \mathscr{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$. For any $\lambda \in \mathbb{R}^+$, we define

$$\mathrm{Log}_+ \lambda = \mathrm{Max}\,(0, \mathrm{Log}\,\lambda) .$$

For any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we use the notation $r = |x|$, $\hat{x} = r^{-1}x$ and $\theta = (t^2 + r^2)^{1/2}$. We shall use the operators $\omega = (-\Delta)^{1/2}$, $k = -i\nabla$, $\hat{k} = \omega^{-1}k$, $K(t) = \omega^{-1}\sin\omega t$ and $\dot{K}(t) = \cos\omega t$. We denote by $\langle .,. \rangle$ the scalar product in L^2 and we define the L^2-norm squared of an L^2 vector valued function as the sum of the L^2-norms squared of its components: for instance

$$\| x\nabla\varphi \|_2^2 = \sum_{i,j} \| x_i \nabla_j \varphi \|_2^2 .$$

Finally, the basic space Σ will be defined in Section 2 (See (2.10)-(2.13)).

2. THE CONFORMAL CONSERVATION LAW

In this section we derive the conservation law associated with the approximate conformal invariance of the equation (1.1). In all the section we assume the interaction f to satisfy the following assumptions:

(A.1) $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$ and $f(0) = 0$. If $n \geqslant 2$, f satisfies the estimate

$$| f'(z) | \equiv \mathrm{Max}\left(\left| \frac{\partial f}{\partial z} \right|, \left| \frac{\partial f}{\partial \bar{z}} \right| \right) \leqslant C(1 + |z|^{p-1}) \tag{2.1}$$

for some p, $1 \leqslant p < 1 + 4/(n-2)$, and all $z \in \mathbb{C}$.

(A.2) *a*) There exists a function $V \in \mathscr{C}^1(\mathbb{C}, \mathbb{R})$ such that $V(0) = 0$, $V(z) = V(|z|)$ for all $z \in \mathbb{C}$ and $f(z) = \partial V/\partial \bar{z}$.

b) V satisfies the estimate

$$V(R) \geqslant - a^2R^2 \tag{2.2}$$

for some $a \geqslant 0$ and all $R \geqslant 0$.

At the formal level the conservation law follows by Noether's theorem from the fact that the equation (1.1) is derived from a variational principle associated with the Lagrangian density [2]

$$\mathcal{L} = \left| \frac{\partial \varphi}{\partial t} \right|^2 - | \nabla \varphi |^2 - V(\varphi),$$

provided f satisfies part *a)* of the assumption (A.2). The conservation law can be conveniently written by using the standard relativistic notation (greek indices range from 0 to n, latin indices from 1 to n, zero is used to denote the time coordinate, the metric tensor $g_{\lambda\mu}$ used to raise and lower the indices is defined by $g_{00} = 1$, $g_{ii} = -1$ and $g_{\lambda\mu} = 0$ for $\lambda \neq \mu$, the space-time derivative is denoted by $\partial_\mu = \frac{\partial}{\partial x^\mu}\Big)$. Any space-time vector a defines an infinitesimal transformation of space-time

$$\delta x^\mu = - 2a^\lambda x_\lambda x^\mu + x^\lambda x_\lambda a^\mu$$

and an infinitesimal transformation of the field

$$\delta\varphi = \delta x^\lambda \partial_\lambda \varphi + (n - 1)a^\lambda x_\lambda \varphi .$$

With that transformation is associated an almost conserved current

$$J^\mu = 2a^\lambda x_\lambda x^\rho T^\mu{}_\rho - x^\lambda x_\lambda a^\rho T^\mu{}_\rho + (n-1)a^\lambda x_\lambda \partial^\mu | \varphi |^2 - (n-1)a^\mu | \varphi |^2 \tag{2.3}$$

where $T^\mu{}_\rho$ is the energy momentum tensor

$$T^\mu{}_\rho = 2 \operatorname{Re} (\partial^\mu \bar{\varphi} \partial_\rho \varphi) - \delta^\mu_\rho \mathcal{L} .$$

An elementary computation shows that, for solutions of (1.1),

$$\partial_\mu J^\mu = 2a^\lambda x_\lambda W \tag{2.4}$$

where

$$W(\varphi) = (n + 1)V(\varphi) - (n - 1) \operatorname{Re} (\bar{\varphi} f(\varphi)) . \tag{2.5}$$

Taking $a = (1, 0, 0 \ldots)$ and integrating (2.4) on an hyperplane $t =$ constant yields the conservation law

$$\frac{d}{dt} Q(t, \varphi, \dot{\varphi}) = 2t \int dx W(\varphi) \tag{2.6}$$

where $Q(t, \varphi, \dot{\varphi})$ is the conformal charge defined by

$$Q(t, \varphi, \psi) = Q_0(t, \varphi, \psi) + Q_1(t, \varphi) \tag{2.7}$$

with

$$Q_0(t, \varphi, \psi) = \| r\psi \|_2^2 + \| t\psi \|_2^2 + \| r\nabla\varphi \|_2^2 + \| t\nabla\varphi \|_2^2 \\ + 2t \langle \psi, 2x.\nabla\varphi + (n-1)\varphi \rangle - (n-1) \| \varphi \|_2^2 \quad (2.8)$$

and

$$Q_1(t, \varphi) = \int dx (t^2 + r^2) V(\varphi). \quad (2.9)$$

The natural condition on φ and ψ for $Q_0(t, \varphi, \psi)$ to be defined is that $(\varphi, \psi) \in \Sigma \equiv \Sigma_1 \oplus \Sigma_0$ where

$$\Sigma_1 = H^1 \cap \{ \varphi : r\nabla\varphi \in L^2 \} \quad (2.10)$$

and

$$\Sigma_0 = L^2 \cap \{ \psi : r\psi \in L^2 \}. \quad (2.11)$$

Both Σ_1 and Σ_0 are Hilbert spaces with norms defined by

$$\| \varphi ; \Sigma_1 \|^2 = \| \varphi \|_2^2 + \| \nabla\varphi \|_2^2 + \| r\nabla\varphi \|_2^2 \quad (2.12)$$

and

$$\| \psi ; \Sigma_0 \|^2 = \| \psi \|_2^2 + \| r\psi \|_2^2 \quad (2.13)$$

respectively.

By recombining the first and fourth norm in (2.8) with part of the scalar product and the second and third norm with the rest of the scalar product we obtain, for $(\varphi, \psi) \in \Sigma$, the following equivalent form for Q_0

$$Q_0(t, \varphi, \psi) = \| x\psi + t\nabla\varphi \|_2^2 + \| t\hat{x}\psi + r\nabla\varphi + (n-1)\hat{x}\varphi \|_2^2. \quad (2.14)$$

The actual proof of the conservation law (2.6) is similar to that of the conservation of energy given in [6]. We first regularize the equation (1.1). This allows us to derive the conservation law for the solutions of the regularized equation. We then remove the regularization by a limiting procedure.

The regularization uses a local regularity cutt-off h as in the proof of energy conservation and in addition a space cut-off g at large distances. The cut-off h is taken as a non negative even function in $\mathscr{C}_0^\infty(\mathbb{R}^n)$, such that $\| h \|_1 = 1$, and the cut-off g as a function in $\mathscr{C}^1(\mathbb{R}^n)$ with compact support and such that $0 \leqslant g \leqslant 1$ and $g = 1$ in some region around the origin. Because of the finite propagation speed for the equation (1.1), the space cut-off can be introduced either in the initial data or in the interaction. We choose the second possibility because of the intrinsic interest of localized interactions. We shall eventually let h tend to a δ function and g tend to 1. Those limits will be understood in the following sense. We choose fixed h_1 and g_1 as described above, and for any positive integer j we define $h_j(x) = j^n h_1(jx)$, $g_j(x) = g_1(x/j)$. We shall then take $h = h_j$ and $g = g_k$ and let j and k tend to infinity. We define the regularized interactions with cut-offs h and g by

$$f_{hg}(\varphi) = h * gf(h * \varphi) \quad (2.15)$$

where $*$ denotes convolution in $\mathbb{R}^n$, and

$$f_g(\varphi) = gf(\varphi). \tag{2.16}$$

Under the assumptions (A.1) and (A.2) it is known that, for any initial data (φ_0, ψ_0) in the energy space $X_e \equiv H^1 \oplus L^2$, the Cauchy problem for the equation (1.1) has a unique global solution $(\varphi, \dot{\varphi})$ in $\mathscr{C}(\mathbb{R}, X_e)$, satisfying the energy conservation

$$E(\varphi(t), \dot{\varphi}(t)) = E(\varphi_0, \psi_0) \tag{2.17}$$

where the energy is defined by

$$E(\varphi, \psi) = \| \psi \|_2^2 + \| \nabla\varphi \|_2^2 + \int dx V(\varphi). \tag{2.18}$$

(See Proposition 3.2 of [*6*] and Proposition 3.2 of [*7*]).

We now approximate that solution by solutions of regularized equations.

LEMMA 2.1. — Let f satisfy (A.1) and (A.2), let $(\varphi_0, \psi_0) \in X_e$, let $t_0 \in \mathbb{R}$. Then

(1) The Cauchy problem for the equation

$$\Box\varphi + f_{hg}(\varphi) = 0 \tag{2.19}$$

with initial data $(h * \varphi_0, h * \psi_0)$ at time t_0 has a unique solution φ_{hg} in $\mathscr{C}(\mathbb{R}, H^1)$ and the Cauchy problem for the equation

$$\Box\varphi + f_g(\varphi) = 0 \tag{2.20}$$

with initial data (φ_0, ψ_0) at time t_0 has a unique solution φ_g in $\mathscr{C}(\mathbb{R}, H^1)$. Furthermore $(\varphi_{hg}, \dot{\varphi}_{hg})$ and $(\varphi_g, \dot{\varphi}_g)$ are bounded in X_e locally in t uniformly in h and g.

(2) For any non negative integer k, $(\varphi_{hg}, \dot{\varphi}_{hg}) \in \mathscr{C}^1(\mathbb{R}, H^{k+1} \oplus H^k)$. Let in addition $r\nabla\varphi_0$ and $r\psi_0$ be in L^2. Then $r\nabla\varphi_{hg}$ and $r\dot{\varphi}_{hg}$ belong to $\mathscr{C}^1(\mathbb{R}, H^k)$.

Let φ be the solution of (1.1) in $\mathscr{C}(\mathbb{R}, H^1)$ with initial data (φ_0, ψ_0) at time t_0. Then

(3) The approximate solution φ_{hg} converges to φ when $h \to \delta$ and $g \to 1$ in $\mathscr{C}(I, L^l)$ for any bounded interval I and any $l \in [2, 2^*)$, and in H^1 pointwise in t, while $\dot{\varphi}_{hg}$ converges to $\dot{\varphi}$ in L^2 pointwise in t.

Similarly φ_{hg} converges to φ_g when $h \to \delta$ and φ_g converges to φ when $g \to 1$, in the same sense.

Indication of proof. — Part (1) as well as the first statement of Part (2) follow by standard methods using the integral equation associated with (2.19) [*23*]. The second statement of Part (2) follows by an elementary computation using the fact that

$$[x, \omega] = i\hat{k}.$$

Part (3) is proved as in the proof of Proposition 3.2 of [*6*] by using the estimates from Proposition 3.1 of [6] or equivalently the estimates of Lemma 2.1 of [7]. The proof given in [6] covers the case where there is only a local cut-off h and is easily extended to the present case.

Q. E. D.

We now derive the conformal conservation law for the regularized equation. We recall the notation $\theta^2 = t^2 + r^2$.

PROPOSITION 2.1. — Let f satisfy (A.1) and (A.2), let $(\varphi_0, \psi_0) \in \Sigma$, let $t_0 \in \mathbb{R}$ and let φ be the solution of (2.19) with initial data $(h * \varphi_0, h * \psi_0)$ at time t_0 as described in Lemma 2.1 part (1). Then, for all $t \in \mathbb{R}$, φ satisfies the identity

$$\begin{aligned}
&\frac{d}{dt}\left\{ Q_0(t, \varphi, \dot\varphi) + \int dx \theta^2 g V(h * \varphi) \right\} \\
&\quad = 2t\left\{ (n+1)\int dx g V(h*\varphi) - (n-1)\operatorname{Re}\langle h*\varphi, g f(h*\varphi)\rangle \right\} \\
&\quad + 2t \int dx (x.\nabla g) V(h*\varphi) \\
&\quad + 2\operatorname{Re}\{ \langle \theta(h*\dot\varphi), \theta g f(h*\varphi)\rangle - \langle \theta\dot\varphi, \theta(h*(gf(h*\varphi)))\rangle \} \\
&\quad + 4t\operatorname{Re}\{ \langle x.(h*\nabla\varphi), g f(h*\varphi)\rangle - \langle x.\nabla\varphi, h*(gf(h*\varphi))\rangle \}. \qquad (2.21)
\end{aligned}$$

Proof. — Using Lemma 2.1 part (2) we can compute

$$\begin{aligned}
\frac{d}{dt}Q_0(t, \varphi, \dot\varphi) &= 2t(\|\nabla\varphi\|_2^2 + \|\dot\varphi\|_2^2) + 2\operatorname{Re}\langle \theta\nabla\varphi, \theta\nabla\dot\varphi\rangle \\
&+ 2\operatorname{Re}\langle \theta\dot\varphi, \theta\ddot\varphi\rangle + 2\operatorname{Re}\langle \dot\varphi, 2x.\nabla\varphi + (n-1)\varphi\rangle \\
&+ 2t\operatorname{Re}\langle \ddot\varphi, 2x.\nabla\varphi + (n-1)\varphi\rangle \\
&+ 2t\operatorname{Re}\langle \dot\varphi, 2x.\nabla\dot\varphi + (n-1)\dot\varphi\rangle - 2(n-1)\operatorname{Re}\langle \varphi, \dot\varphi\rangle \\
&= 2\operatorname{Re}\langle \theta\dot\varphi, \theta\Box\varphi\rangle + 2t\operatorname{Re}\langle 2x.\nabla\varphi + (n-1)\varphi, \Box\varphi\rangle. \qquad (2.22)
\end{aligned}$$

On the other hand, using again Lemma 2.1 part (2) we can compute

$$\frac{d}{dt}\int dx\theta^2 g V(h*\varphi) = 2t\int dx g V(h*\varphi) + 2\operatorname{Re}\langle \theta(h*\dot\varphi), \theta g f(h*\varphi)\rangle. \qquad (2.23)$$

Adding (2.22) and (2.23), using the identity

$$\begin{aligned}
0 &= \int dx \nabla.(x g V(h*\varphi)) = n\int dx g V(h*\varphi) \\
&+ \int dx(x.\nabla g)V(h*\varphi) + 2\operatorname{Re}\langle x.(h*\nabla\varphi), g f(h*\varphi)\rangle
\end{aligned}$$

and the equation (2.19) we obtain (2.21) after an elementary computation.

Q. E. D.

The next step consists in taking the limit $h \to \delta$ in the previous identity.

PROPOSITION 2.2. — Let f satisfy (A.1) and (A.2), let $(\varphi_0, \psi_0) \in \Sigma$, let $t_0 \in \mathbb{R}$ and let φ be the solution of (2.20) with initial data (φ_0, ψ_0) at time t_0, as described in Lemma 2.1 part (1). Then, $(\varphi, \dot{\varphi}) \in \mathscr{C}(\mathbb{R}, \Sigma)$ and for all $s, t \in \mathbb{R}$, the following identity holds

$$\begin{aligned} Q_0(t, \varphi(t), \dot{\varphi}(t)) + \int dx(t^2 + r^2)gV(\varphi(t)) \\ = Q_0(s, \varphi(s), \dot{\varphi}(s)) + \int dx(s^2 + r^2)gV(\varphi(s)) \\ + \int_s^t 2\tau d\tau \int dx \{ gW(\varphi(\tau)) + (x.\nabla g)V(\varphi(\tau)) \} . \quad (2.24) \end{aligned}$$

Proof. — Let φ_h be the solution of (2.19) with initial data $(h * \varphi_0, h * \psi_0)$ at time t_0. The function φ_h satisfies the identities (2.21). It follows from Lemma 2.1 part (3) that the sum of the first two terms in the right-hand side of (2.21) tends to

$$2t \int dxgW(\varphi) + 2t \int dx(x.\nabla g)V(\varphi)$$

and, by estimates almost identical with those in the proof of Proposition 3.4 of [*3*], that the last two terms tend to zero when $h \to \delta$ uniformly with respect to t in bounded intervals. Furthermore the term with V in the left-hand side of (2.21) converges to

$$\int dx\theta^2 gV(\varphi)$$

for all t, and $Q_0(t_0, h * \varphi_0, h * \psi_0)$ converges to $Q_0(t_0, \varphi_0, \psi_0)$. Integrating (2.21) between t_0 and t, taking the limit $h \to \delta$ and using the previous remarks one obtains the existence of the limit

$$\begin{aligned} \lim_{h \to \delta} Q_0(t, \varphi_h(t), \dot{\varphi}_h(t)) = Q_0(t_0, \varphi_0, \psi_0) \\ + \int dx(t_0^2 + r^2)gV(\varphi_0) - \int dx(t^2 + r^2)gV(\varphi(t)) \\ + \int_{t_0}^t 2\tau d\tau \int dx \{ gW(\varphi(\tau)) + (x.\nabla g)V(\varphi(\tau)) \} \equiv \Lambda . \quad (2.25) \end{aligned}$$

We now introduce in Q_0 an additional space cut-off g', of the same type as g, which will eventually tend to 1 in the same sense as g, namely along the sequence g_j described previously (note however that in the present proof g is kept fixed). We rewrite Q_0 as a sum of positive terms according to (2.14) or, more concisely,

$$Q_0 = \sum_i \| A_i \varphi \|_2^2 \equiv \| A\varphi \|_2^2$$

where $A = (A_i)$ is a finite set of first order time dependent differential operators in space-time. It follows from Lemma 2.1 part (3) and from (2.25) that, for any t,

$$\| g'A\varphi \|_2^2 = \lim_{h\to\delta} \| g'A\varphi_h \|_2^2 \leqslant \underline{\lim}_{h\to\delta} \| A\varphi_h \|_2^2 = \Lambda .$$

Taking the limit $g' \to 1$ shows that

$$Q_0(t, \varphi, \dot{\varphi}) \leqslant \Lambda \tag{2.26}$$

which in particular implies that $(\varphi(t), \dot{\varphi}(t)) \in \Sigma$ for all $t \in \mathbb{R}$. Furthermore the inequality (2.26) together with the fact that $(\varphi, \dot{\varphi}) \in \mathscr{C}(\mathbb{R}, X_e)$ implies that $(\varphi, \dot{\varphi})$ is weakly continuous in Σ with respect to time.

Exchanging the role of t and t_0 and using the fact that the equation (1.1) is time reversal invariant yields (2.24) with $s = t_0$ and therefore for all s.

The identity (2.24) yields the continuity of the norm of $(\varphi, \dot{\varphi})$ in Σ as a function of time since the potential terms are continuous. This fact together with weak continuity implies strong continuity of $(\varphi, \dot{\varphi})$ in Σ.

Q. E. D.

The final step consists in taking the limit $g \to 1$ in the identity (2.24). For that purpose we need an additional assumption on V.

(A.3) The function V (defined in the assumption (A.2)) satisfies the estimate

$$V(R) \geqslant -CR^{2+4/n} \tag{2.27}$$

for some $C \geqslant 0$ and all $R \geqslant 0$.

It follows from (A.3) that V can be decomposed as $V = V_+ - V_-$ where $V_\pm \in \mathscr{C}^1(\mathbb{C}, \mathbb{R})$, $V_\pm \geqslant 0$ and V_- satisfies an estimate

$$V_-(R) \leqslant CR^{2+4/n} \tag{2.28}$$

for all $R \geqslant 0$, with a constant C possibly larger than that in (A.3).

The assumption (A.3) will be used to derive the following result.

LEMMA 2.2. — Let V satisfy (A.3). Let $\varphi \in \Sigma_1$, let g and g' be two space cut-offs and let χ be the characteristic function of the support of $1 - g'$. Then

$$\int dxg(1 - g')^2 r^2 V_-(\varphi) \leqslant C\{ \| r\nabla\varphi \|_2^2 \| \chi\varphi \|_2^{4/n} + (1 + \| r\nabla g' \|_\infty) \| \chi\varphi \|_2^{2+4/n} \} \tag{2.29}$$

where the constant C depends only on that in (2.28).

Proof. — We introduce an additional cut-off g'' such that $g'' = 1$ on the support of g. Then, by (A.3),

$$\int dxg(1 - g')^2 r^2 V_-(\varphi) \leqslant C \| g''(1 - g')r \,|\, \varphi \,|^{1+2/n} \|_2^2 . \tag{2.30}$$

The last norm is estimated for $n \geqslant 2$ by the Sobolev inequalities as

$$\begin{aligned} \| \cdot \| &\leqslant C \| \nabla (g''(1-g')r|\varphi|^{1+2/n}) \|_{2n/(n+2)} \\ &\leqslant C \{ \| (1-g')r\nabla|\varphi|^{1+2/n} \|_{2n/(n+2)} + \| ((1-g') + |r\nabla g'|)|\varphi|^{1+2/n} \|_{2n/(n+2)} \} \end{aligned}$$

after letting g'' tend to 1. The result (2.29) for $n \geqslant 2$ follows by Hölder's inequality. For $n = 1$ the norm in (2.30) is estimated by

$$\| \cdot \| \leqslant (1/2) \| \chi\varphi \|_2 \| \nabla (g''(1-g')r|\varphi|^2) \|_1$$

from which the result follows as before. Q. E. D.

We now derive the conservation law in its final form.

PROPOSITION 2.3. — Let f satisfy (A.1), (A.2) and (A.3). Let $(\varphi_0, \psi_0) \in \Sigma$. Assume that

$$\int dxr^2 |V(\varphi_0)| < \infty . \tag{2.31}$$

Let $t_0 \in \mathbb{R}$ and let φ be the solution of (1.1) in $\mathscr{C}(\mathbb{R}, H^1)$ with initial data (φ_0, ψ_0) at time t_0. Then

(1) $(\varphi, \dot\varphi) \in \mathscr{C}(\mathbb{R}, \Sigma)$ and

$$\int dxr^2 V(\varphi)$$

is finite for each $t \in \mathbb{R}$ and continuous with respect to t.

(2) For all $s, t \in \mathbb{R}$, the following identity holds

$$\begin{aligned} Q_0(t, \varphi(t), \dot\varphi(t)) &+ \int dx(t^2 + r^2)V(\varphi(t)) \\ &= Q_0(s, \varphi(s), \dot\varphi(s)) + \int dx(s^2 + r^2)V(\varphi(s)) + \int_s^t 2\tau d\tau \int dx W(\varphi(\tau)). \end{aligned} \tag{2.32}$$

Proof. — Let φ_g be the solution of (2.20) with initial data (φ_0, ψ_0) at time t_0. The function φ_g satisfies the identity (2.24) with $s = t_0$. We decompose V as $V = V_+ - V_-$ in the left-hand side of (2.24), we introduce a second space cut-off g' in the term with V_-, we apply Lemma 2.2 and the fact that by (2.14)

$$\| r\nabla\varphi \|_2^2 \leqslant 3 \{ Q_0(t, \varphi, \dot\varphi) + t^2 \| \dot\varphi \|_2^2 + (n-1)^2 \| \varphi \|_2^2 \}$$

to obtain

$$\begin{aligned} Q_0(t, \varphi_g, \dot\varphi_g)\{1 - C\|\chi\varphi_g\|_2^{4/n}\} &+ \int dxg\{\theta^2 V_+(\varphi_g) - (t^2 + r^2(2g' - g'^2))V_-(\varphi_g)\} \\ &\leqslant Q_0(t_0, \varphi_0, \psi_0) + \int dxg(t_0^2 + r^2)V(\varphi_0) \\ &\quad \int_{t_0}^t 2\tau d\tau \int dx\{gW(\varphi_g(\tau)) + (x \cdot \nabla g)V(\varphi_g(\tau))\} \\ &\quad + C\|\chi\varphi_g\|_2^{4/n}\{t^2\|\dot\varphi_g\|_2^2 + \|\varphi_g\|_2^2(1 + \|r\nabla g'\|_\infty)\}. \end{aligned} \tag{2.33}$$

Using Lemma 2.1 part (3), we can choose g' such that

$$1 - C \| \chi\varphi_g \|_2^{4/n} \geqslant 1/2$$

uniformly in g (and uniformly in t for t in a bounded interval). We now take the limit $g \to 1$. By Lemma 2.1 part (3), all terms in (2.33) converge to obvious limits except possibly the terms with Q_0 and V_+ in the left-hand side. We obtain therefore

$$\overline{\lim_{g\to 1}} \left\{ Q_0(t, \varphi_g, \dot\varphi_g)[1 - C\|\chi\varphi\|_2^{4/n}] + \int dx\theta^2 g V_+(\varphi_g) \right\}$$

$$\leqslant \int dx(t^2 + r^2(2g' - g'^2))V_-(\varphi) + Q(t_0, \varphi_0, \psi_0)$$

$$+ \int_{t_0}^{t} 2\tau d\tau \int dx W(\varphi(\tau)) + C\|\chi\varphi\|_2^{4/n}\{t^2\|\dot\varphi\|_2^2 + \|\varphi\|_2^2(1 + \|r\nabla g'\|_\infty)\} \equiv \Lambda(g') \tag{2.34}$$

By the same argument as given in the proof of Proposition 2.2 for the term with Q_0 and a similar argument for the term with V_+, we conclude that $(\varphi(t), \dot\varphi(t)) \in \Sigma$ for all $t \in \mathbb{R}$, that

$$\int dx\theta^2 V_+(\varphi(t)) < \infty$$

and that

$$Q_0(t, \varphi, \dot\varphi)\{1 - C\|\chi\varphi\|_2^{4/n}\} + \int dx\theta^2 V_+(\varphi(t)) \leqslant \Lambda(g'). \tag{2.35}$$

That inequality together with the fact that $(\varphi, \dot\varphi) \in \mathscr{C}(\mathbb{R}, X_e)$ implies that $(\varphi, \dot\varphi)$ is weakly continuous in Σ with respect to time. We then let g' tend to 1. The term with V_- in the right-hand side of (2.35) converges to the obvious limit by the same estimates as in the proof of Lemma 2.2, while $\|\chi\varphi\|_2$ tends to 0. We obtain thereby

$$Q(t, \varphi, \dot\varphi) \leqslant Q(t_0, \varphi_0, \psi_0) + \int_{t_0}^{t} 2\tau d\tau \int dx W(\varphi(\tau)).$$

Exchanging the role of t and t_0 and using the fact that the equation (1.1) is time reversal invariant yields (2.32) with $s = t_0$ and therefore for all s.

Finally we prove that $Q_0(t, \varphi, \dot\varphi)$ and $\int dx\theta^2 V_\pm(\varphi)$ are separately continuous functions of t. By using Lemma 2.2 one sees directly that $\int dx\theta^2 V_-(\varphi)$ is a continuous function of t. On the other hand, from the fact that

$$Q_0(t, \varphi, \dot\varphi) = \|A\varphi\|_2^2 = \lim_{g\to 1} \|gA\varphi\|_2^2, \tag{2.36}$$

$$\int dx\theta^2 V_+(\varphi) = \lim_{g\to 1} \int dx g\theta^2 V_+(\varphi), \tag{2.37}$$

and that the expressions in the right-hand sides of (2.36) and (2.37) (before taking the limit $g \to 1$) are continuous in t since $(\varphi, \dot{\varphi}) \in \mathscr{C}(\mathbb{R}, X_e)$, it follows that $Q_0(t, \varphi, \dot{\varphi})$ and $\int dx \theta^2 V_+(\varphi)$ are lower semicontinuous functions of t. Their continuity follows from their lower semicontinuity and from the fact that their sum is continuous because of the conservation law (2.32).

The continuity of Q_0 together with the weak continuity of $(\varphi, \dot{\varphi})$ in Σ implies the strong continuity of $(\varphi, \dot{\varphi})$ in Σ. Q. E. D.

REMARK 2.1. — We remark that, because of the assumption (A.3), for any $\varphi_0 \in \Sigma_1$, the conditions (2.31) and

$$\int dx r^2 V_+(\varphi_0) < \infty$$

are equivalent by Lemma 2.2. Furthermore that condition is superfluous if V_+ satisfies the estimate

$$V_+(R) \leqslant CR^{2+4/n}$$

for $R \leqslant 1$. If $V \geqslant 0$, the assumption (A.3) is trivially satisfied and the proof of Proposition 2.3 is much simpler.

3. CONFORMAL ESTIMATES

In this section, we derive a number of estimates that can be expressed in terms of the free conformal charge. We restrict our attention to space dimension $n \geqslant 2$. The estimates will involve a positive parameter t and either a generic element $\varphi \in \Sigma_1$ or a generic element $(\varphi, \psi) \in \Sigma$. Except where explicitly stated (namely in Proposition 3.2), we do not regard φ as a function of t and ψ as its time derivative. The estimates will be proved first for φ and ψ in $\mathscr{D}_0 \equiv \mathscr{C}_0^\infty(\mathbb{R}^n \backslash \{0\})$ and then extended by continuity to $(\varphi, \psi) \in \Sigma$ by using the fact that $\mathscr{D}_0$ is dense both in Σ_0 and in Σ_1 (see the Appendix). Most of the proofs for φ and ψ in $\mathscr{D}_0$ will reduce to formal computations, which will be performed without additional comments.

We define the angular momentum operators

$$L_{ij} = x_i \partial/\partial x_j - x_j \partial/\partial x_i, \tag{3.1}$$

$i, j = 1, \ldots, n$. For any $\varphi \in \Sigma_1$ and any sufficiently regular function g, the following identity holds

$$\| gx\nabla\varphi \|_2^2 \equiv \sum_{i,j} \| gx_i \partial\varphi/\partial x_j \|_2^2 = \| gx.\nabla\varphi \|_2^2 + \| gL\varphi \|_2^2 \tag{3.2}$$

where

$$\| gL\varphi \|_2^2 \equiv \sum_{i<j} \| gL_{ij}\varphi \|_2^2 . \tag{3.3}$$

We shall need the following identities.

LEMMA 3.1. — Let $\varphi \in L^2$ and $r\nabla\varphi \in L^2$. Then the following identities hold

$$\| r\nabla\varphi \|_2^2 = \| x\nabla\varphi \|_2^2 = \| \nabla x\varphi \|_2^2 = \| \omega x\varphi \|_2^2 , \tag{3.4}$$

$$\| \omega r\varphi \|_2^2 = \| \nabla r\varphi \|_2^2 = \| x\omega\varphi \|_2^2 = \| r\omega\varphi \|_2^2 , \tag{3.5}$$

$$\| r\nabla\varphi \|_2^2 = \| \nabla r\varphi \|_2^2 + (n-1) \| \varphi \|_2^2 . \tag{3.6}$$

Proof. — The first and last equalities in (3.4) and (3.5) follow immediately from the definitions of r and ω. We shall prove the second equality in (3.4) and the equality (3.6). The second equality in (3.5) will then follow from the second one in (3.4) and from (3.6) by Fourier transform.

In order to prove (3.4), we apply the operator identity

$$\nabla_i x_j = x_j \nabla_i + \delta_{ij}$$

which yields

$$\| \nabla x\varphi \|_2^2 = \| x\nabla\varphi \|_2^2 + n \| \varphi \|_2^2 + 2\,\mathrm{Re} \langle \varphi, x.\nabla\varphi \rangle .$$

The sum of the last two terms is zero, by integration by parts.

The proof of (3.6) results from the following operator identity

$$\begin{aligned} -\nabla_i r^2 \nabla_i &= -\left(r\nabla_i + \frac{x_i}{r} \right)\left(\nabla_i r - \frac{x_i}{r} \right) \\ &= -r\nabla^2 r + 1 + [\nabla_i, x_i] - 2\frac{x_i}{r}\frac{x_i}{r} \\ &= -r\nabla^2 r + n - 1 . \end{aligned}$$

Q. E. D.

The free conformal charge (2.8) can be written in several different forms, which will be used later. We have already used the equivalent form

$$Q_0 = \| x\psi + t\nabla\varphi \|_2^2 + \| t\hat{x}\psi + r\nabla\varphi + (n-1)\hat{x}\varphi \|_2^2 . \tag{2.14}$$

Separating out the angular momentum operators in the second norm in (2.14) by using (3.2), we obtain

$$Q_0 = \| x\psi + t\nabla\varphi \|_2^2 + \| L\varphi \|_2^2 + \| t\psi + x.\nabla\varphi + (n-1)\varphi \|_2^2 . \tag{3.7}$$

On the other hand, from the relation

$$x.k - i(n-1) = \hat{k}.x\omega \tag{3.8}$$

we obtain

$$\langle \psi, x.\nabla\varphi + (n-1)\varphi \rangle = \langle -i\hat{k}\psi, x\omega\varphi \rangle . \tag{3.9}$$

Using (3.5), (3.6) and (3.9), we can rewrite Q_0 as

$$Q_0 = \| x\psi + t\nabla\varphi \|_2^2 + \| it\hat{k}\psi - x\omega\varphi \|_2^2 . \tag{3.10}$$

That expression can be further transformed by introducing the functions

$$\varphi_\pm = (1/2)(\varphi \pm i\omega^{-1}\psi) \tag{3.11}$$

which would correspond to the decomposition of φ into positive and negative frequency parts if φ were a solution of the free wave equation and ψ its time derivative. One can then rewrite

$$Q_0 \equiv Q_0(t, \varphi, \psi) = Q_+(t, \varphi_+) + Q_-(t, \varphi_-) \tag{3.12}$$

where $Q_\pm$ are defined by

$$Q_\pm(t, \varphi) = 2 \| (x\omega \mp tk)\varphi \|_2^2 = 2 \| x\omega \exp(\pm i\omega t)\varphi \|_2^2 \tag{3.13}$$

and the last equality in (3.13) follows from the commutation relation

$$\exp(\pm i\omega t)\, x \exp(\mp i\omega t) = x \pm t\hat{k} . \tag{3.14}$$

We now derive a first set of decay estimates by combining the representation (3.12)-(3.13) with the following classical estimate of the solutions of the free wave equation [*17*] [*20*] [*29*].

LEMMA 3.2. — Let $n \geqslant 2$, let l and s satisfy

$$1 + \delta(s) \leqslant \delta(l) \leqslant \operatorname{Min}(1 + \alpha(s), n(1 + \delta(s))) ,$$

and $2 < l < \infty$ if $n = 2$. Then the following estimate holds

$$\| \varphi \|_l \leqslant Ct^{1-\delta(l)+\delta(s)} \| \omega e^{\pm i\omega t}\varphi \|_s . \tag{3.15}$$

We also need the following elementary result.

LEMMA 3.3. — Let $0 \leqslant \mu < 1$ and $\delta(s) = -\mu$. Then

$$\| \varphi \|_s \leqslant \sqrt{2}\, \| (1 + x^2)^{-1/2} \|_{n/\mu} \| x\varphi \|_2^\mu \| \varphi \|_2^{1-\mu} . \tag{3.16}$$

Proof. — For any $a > 0$

$$\| \varphi \|_s \leqslant \| (a^2 + x^2)^{-1/2} \|_{n/\mu} \| (a^2 + x^2)^{1/2}\varphi \|_2$$
$$= a^{\mu-1} \| (1 + x^2)^{-1/2} \|_{n/\mu} \{ a^2 \| \varphi \|_2^2 + \| x\varphi \|_2^2 \}^{1/2}$$

from which (3.16) follows by taking $a = \| x\varphi \|_2 / \| \varphi \|_2$. Q. E. D.

We can then prove

PROPOSITION 3.1. — Let $(\varphi, \psi) \in \Sigma$ and $2 < l \leqslant l_S \equiv 2(n + 1)/(n - 1)$. Then φ and $\omega^{-1}\psi$ belong to L^l and satisfy the estimate

$$\| \varphi \|_l, \| \omega^{-1}\psi \|_l \leqslant Ct^{-\gamma(l)} E_0(\varphi, \psi)^{\alpha(l)/2} Q_0(t, \varphi, \psi)^{(1-\alpha(l))/2} \tag{3.17}$$

where

$$E_0(\varphi, \psi) = \| \nabla\varphi \|_2^2 + \| \psi \|_2^2 .$$

Proof. — We express φ and ψ in terms of $\varphi_\pm$ by using (3.11). We estimate $\varphi_\pm$ in L^l by (3.15) with $\alpha(l) = 1 + \delta(s)$ and (3.16), so that

$$\| \varphi_\pm \|_l \leqslant Ct^{-\gamma(l)} \| \omega\varphi_\pm \|_2^{\alpha(l)} \| x\omega \exp(\pm i\omega t)\varphi_\pm \|_2^{1-\alpha(l)} . \quad (3.18)$$

The result now follows from (3.12), (3.13), (3.18) and the relation

$$E_0(\varphi, \psi) = 2 \| \omega\varphi_+ \|_2^2 + 2 \| \omega\varphi_- \|_2^2 .$$

Q. E. D.

We now briefly discuss the restrictions on l in Proposition 3.1. The lower limit $l = 2$ is excluded in dimension $n \geqslant 3$ by the use of Hölder's inequality in Lemma 3.3. It will be recovered, actually in a more general form, by the use of the Hardy inequality to be proved below. In dimension $n = 2$, (3.17) does not hold for $l = 2$. The upper limit $l = l_S$ is not expected to be optimal. Actually the estimate (3.17) will be extended for $n \geqslant 3$ up to $l = 2^* \equiv 2n/(n-2)$ in a more general form (see Proposition 3.3 below), and for $n = 2$, to arbitrary l, but with logarithmic corrections (see Proposition 3.4 below). The upper limit $l = 2^*$ for $n \geqslant 3$ is sharp, as shown by the following argument. For $\varphi = \varphi_+$, (3.17) takes the form

$$\begin{aligned} \| \varphi \|_l &\leqslant Ct^{-\gamma(l)} \| \omega\varphi \|_2^{\alpha(l)} \| (x\omega - tk)\varphi \|_2^{1-\alpha(l)} \\ &= Ct^{1-\delta(l)} \| \omega\varphi \|_2^{\alpha(l)} \| (xt^{-1}\omega - k)\varphi \|_2^{1-\alpha(l)} \end{aligned}$$

which for $\varphi \in \mathscr{D}_0$ implies $\| \varphi \|_l = 0$ and therefore $\varphi = 0$ if $\delta(l) > 1$, by letting t tend to infinity.

We now begin the proof of the main estimates, which generalize Proposition 3.1. The starting point is a generalization of (3.7) where we introduce an arbitrary real function h.

LEMMA 3.4. — Let $(\varphi, \psi) \in \Sigma$, let $h \in \mathscr{C}^1(\mathbb{R}^+ \setminus \{0\})$ with h and rdh/dr in $L^\infty(\mathbb{R}^+)$ and in addition

$$| h | \operatorname{Log}_+ (1/r) + r \,| dh/dr | (\operatorname{Log}_+ (1/r))^2 \in L^\infty(\mathbb{R}^+)$$

if $n = 2$. Then

$$\begin{aligned} Q_0(t, \varphi, \psi) = &\, \| x\psi + t\nabla\varphi + txr^{-2}h\varphi \|_2^2 + \| L\varphi \|_2^2 \\ &+ \| t\psi + x.\nabla\varphi + (n-1-h)\varphi \|_2^2 \\ &+ \langle \varphi, \{ (1+t^2r^{-2})h(n-2-h) + (t^2r^{-2}-1)rdh/dr \} \varphi \rangle . \quad (3.19) \end{aligned}$$

Proof. — We give the proof only for φ, ψ in $\mathscr{D}_0$. The extension to general (φ, ψ) in Σ will follow from the Hardy inequality to be proved below. The assumptions made on h together with that inequality ensure that the terms containing h in (3.19) are continuous in the norm of Σ. We rewrite (3.7) as

$$\begin{aligned} Q_0 = &\, \| x\psi + t\nabla\varphi + txr^{-2}h\varphi \|_2^2 + \| L\varphi \|_2^2 + \| t\psi + x.\nabla\varphi + (n-1-h)\varphi \|_2^2 \\ &+ \langle \varphi, \{ 2(n-1)h - (1+t^2r^{-2})h^2 \} \varphi \rangle + 2 \operatorname{Re} \langle x.\nabla\varphi, (1-t^2r^{-2})h\varphi \rangle \end{aligned}$$

which yields (3.19) by an elementary computation using the identity

$$2\,\mathrm{Re}\langle x.\nabla\varphi, g\varphi\rangle = -\langle \varphi, (ng + x.\nabla g)\varphi\rangle. \tag{3.20}$$

Q. E. D.

We next compute the minimum of $Q_0(t, \varphi, \psi)$ over ψ for fixed t and φ. We recall the notation $\theta^2 = t^2 + r^2$.

LEMMA 3.5. — Let $\varphi \in \Sigma_1$ and let h be as in Lemma 3.4. Then

$$\begin{aligned} Q_m(t, \varphi) \equiv & \operatorname*{Inf}_{\psi\in\Sigma_0} Q_0(t, \varphi, \psi) \\ & = \|\theta r^{-1}L\varphi\|_2^2 + \|\theta^{-1}\{(t^2-r^2)\hat{x}.\nabla\varphi - (n-1-\theta^2 r^{-2}h)r\varphi\}\|_2^2 \\ & + \langle \varphi, \{\theta^2 r^{-2}h(n-2-h) + (t^2 r^{-2}-1)r dh/dr\}\varphi\rangle. \end{aligned} \tag{3.21}$$

Proof. — Q_0 is a quadratic form in ψ (see (2.8)) and takes its minimum for

$$\psi = -t\theta^{-2}(2x.\nabla\varphi + (n-1)\varphi). \tag{3.22}$$

The first norm in (3.19) can be rewritten as

$$\|x\psi + t\nabla\varphi + txr^{-2}h\varphi\|_2^2 = \|tr^{-1}L\varphi\|_2^2 + \|r\psi + tr^{-1}(x.\nabla\varphi + h\varphi)\|_2^2 \tag{3.23}$$

by using (3.2). We substitute (3.22) into (3.19) and (3.23), and obtain

$$r\psi + tr^{-1}(x.\nabla\varphi + h\varphi) = tr^{-1}\theta^{-2}\{(t^2-r^2)x.\nabla\varphi - [(n-1)r^2 - h\theta^2]\varphi\} \tag{3.24}$$

and

$$t\psi + x.\nabla\varphi + (n-1-h)\varphi = -\theta^{-2}\{(t^2-r^2)x.\nabla\varphi - [(n-1)r^2 - h\theta^2]\varphi\} \tag{3.25}$$

from which the result follows immediately. Q. E. D.

We now give some applications of Lemma 3.5 corresponding to special choices of the function h. For $h = 0$, (3.21) reduces to

$$Q_m(t, \varphi) = \|\theta r^{-1}L\varphi\|_2^2 + \|\theta^{-1}\{(t^2 - r^2)\hat{x}.\nabla\varphi - (n-1)r\varphi\}\|_2^2. \tag{3.26}$$

For $h = (n-2)/2$, (3.21) reduces to

$$\begin{aligned} Q_m(t, \varphi) = & \|\theta r^{-1}L\varphi\|_2^2 \\ & + \|\theta^{-1}\{(t^2 - r^2)\hat{x}.\nabla\varphi - (n - 1 - \theta^2 r^{-2}(n-2)/2)r\varphi\}\|_2^2 \\ & + (1/4)(n-2)^2\|\theta r^{-1}\varphi\|_2^2. \end{aligned} \tag{3.27}$$

In particular, for $n \geqslant 3$, the L^2-norm of $\theta r^{-1}\varphi$ and *a fortiori* the L^2-norm of φ, is estimated in terms of $Q_m(t, \varphi)$ and *a fortiori* in terms of the Σ_1-norm of φ. This fact justifies the extension of the conclusions of Lemmas 3.4 and 3.5 from $\mathscr{D}_0$ to Σ_1 by continuity for $n \geqslant 3$ under the assumptions made on h.

Finally we obtain an explicit estimate for the radial derivative

$d\varphi/dr = \hat{x}.\nabla\varphi$ by choosing $h = (n-1)r^2\theta^{-2}$. For that choice, (3.21) reduces to

$$Q_m(t,\varphi) = \| \theta r^{-1} L\varphi \|_2^2 + \| \theta^{-1}(t^2 - r^2)\hat{x}.\nabla\varphi \|_2^2 \\ - (n-1)\| \varphi \|_2^2 + (n-1)\langle \varphi, \theta^{-4}t^2\{(n+1)t^2 + (n-3)r^2\}\varphi \rangle \quad (3.28)$$

where the last term is positive for $n \geqslant 3$. In particular for $n \geqslant 3$, (3.27) and (3.28) yield

$$\| \theta r^{-1} L\varphi \|_2^2 + \| \theta^{-1}(t^2 - r^2)\hat{x}.\nabla\varphi \|_2^2 \leqslant [n/(n-2)]^2 Q_m(t,\varphi). \quad (3.29)$$

In space dimension $n = 2$, it is easy to see that the L^2 norm of φ and *a fortiori* the L^2-norm of $\theta r^{-1}\varphi$ is not estimated in terms of $Q_m(t,\varphi)$. Nevertheless one can prove the following weaker result.

LEMMA 3.6. — Let $n = 2$ and $\varphi \in \Sigma_1$. Then

$$\| \theta r^{-1}(1 + \mathrm{Log}_+ t/r)^{-1}\varphi \|_2 \leqslant 2(\| \varphi \|_2 + Q_m(t,\varphi)^{1/2}). \quad (3.30)$$

Proof. — We give the proof in the special case where φ is radial and belongs to $\mathscr{D}_0$. The case of non radial φ is obtained by applying the result for radial φ to the angular square average of φ, and the general case follows from the density of $\mathscr{D}_0$ in Σ_1.

Let h be a real function, $h \in \mathscr{C}^1((0,t))$ and let $\lambda > 1$. From the inequality

$$0 \leqslant \int_0^{t/\lambda} rdr \left| \theta^{-1}\left[(t^2 - r^2)\frac{d\varphi}{dr} - r\varphi\right] + \theta r^{-1}h\varphi \right|^2$$

we obtain by integrating by parts

$$\int_0^{t/\lambda} rdr \,|\varphi|^2 \left\{ (t^2 - r^2)r^{-1}\frac{dh}{dr} - \theta^2 r^{-2}h^2 \right\} \\ \leqslant \int_0^{t/\lambda} rdr \left| \theta^{-1}\left[(t^2 - r^2)\frac{d\varphi}{dr} - r\varphi\right]\right|^2 + h\,|\varphi|^2(t^2 - r^2)\,|_{r=t/\lambda}\,.$$

We choose for h a solution of the equation

$$(t^2 - r^2)\frac{dh}{dr} = 2\theta^2 r^{-1}h^2$$

namely

$$h(r) = (1/2)(1 + \mathrm{Log}\,((t^2 - r^2)/rt))^{-1}\,.$$

Furthermore, we choose from now on $\lambda = (1+\sqrt{5})/2$, so that $\lambda - \lambda^{-1} = 1$ and therefore $h(t/\lambda) = 1/2$. We obtain

$$\int_0^{t/\lambda} rdr\,|\theta r^{-1}\varphi|^2(1 + \mathrm{Log}\,((t^2 - r^2)/rt))^{-2} \leqslant 4Q_m(t,\varphi) + 2\,|\varphi|^2(t^2 - r^2)\,|_{r=t/\lambda}\,.$$

Now

$$
\begin{aligned}
|\varphi|^2(t^2 - r^2)|_{r=t/\lambda} &= 2\,\mathrm{Re}\int_{t/\lambda}^{\infty} rdr\theta r^{-1}\overline{\varphi}\theta^{-1}\left\{(r^2 - t^2)\frac{d\varphi}{dr} + r\varphi\right\} \\
&\leqslant 2\mathrm{Q}_m(t,\varphi)^{1/2}\left\{\int_{t/\lambda}^{\infty} rdr\,|\,\theta r^{-1}\varphi\,|^2\right\}^{1/2} \\
&\leqslant 2\mathrm{Q}_m(t,\varphi)^{1/2}(1+\lambda^2)^{1/2}\,\|\,\varphi\,\|_2\,.
\end{aligned}
$$

Furthermore, for $r \leqslant t/\lambda$

$$
\mathrm{Log}\,((t^2 - r^2)/rt) \leqslant \mathrm{Log}_+\,(t/r)
$$

so that

$$
\int_0^{t/\lambda} rdr\,|\,\theta r^{-1}\varphi\,|^2(1 + \mathrm{Log}_+\,t/r)^{-2} \leqslant 4\mathrm{Q}_m + 4\mathrm{Q}_m^{1/2}(1+\lambda^2)^{1/2}\,\|\,\varphi\,\|_2\,.
$$

On the other hand

$$
\int_{t/\lambda}^{\infty} rdr\,|\,\theta r^{-1}\varphi\,|^2(1 + \mathrm{Log}_+\,t/r)^{-2} \leqslant (1+\lambda^2)\,\|\,\varphi\,\|_2^2
$$

and therefore

$$
\|\,\theta r^{-1}(1 + \mathrm{Log}_+\,t/r)^{-1}\varphi\,\|_2 \leqslant 2\mathrm{Q}_m^{1/2} + (1+\lambda^2)^{1/2}\,\|\,\varphi\,\|_2\,,
$$

from which (3.30) follows since $1 + \lambda^2 < 4$. Q. E. D.

It follows from (3.30) that the norm in the left-hand side is estimated in terms of the norm of φ in Σ_1. This fact justifies the extension of the conclusions of Lemmas 3.4 and 3.5 from $\mathscr{D}_0$ to Σ_1 by continuity for $n = 2$ under the assumptions made on h.

Although for $n = 2$ the L^2-norm of φ cannot be estimated in terms of $\mathrm{Q}_m(t, \varphi)$, it can still be estimated in terms of $\mathrm{Q}_0(t, \varphi, \psi)$ in the case where φ is a function of t and $\psi = \dot{\varphi}$.

PROPOSITION 3.2. — Let $n = 2$, let $\varphi \in \mathscr{C}((0, \mathrm{T}), \Sigma_1) \cap \mathscr{C}^1((0, \mathrm{T}), \Sigma_0)$. Then for any s and t with $0 < s \leqslant t \leqslant \mathrm{T}$, φ satisfies the estimate

$$
\|\,\varphi(t)\,\|_2 \leqslant \|\,\varphi(s)\,\|_2 + \overline{\mathrm{Q}}_0^{1/2}\,\mathrm{Log}\,t/s \tag{3.31}
$$

where

$$
\overline{\mathrm{Q}}_0 = \sup_{0 \leqslant t \leqslant \mathrm{T}} \mathrm{Q}_0(t, \varphi(t), \dot{\varphi}(t))\,. \tag{3.32}
$$

Proof. — Let h be a function of t. From (3.7) with $\psi = \dot{\varphi}$, we obtain

$$
\begin{aligned}
\mathrm{Q}_0(t,\varphi,\dot{\varphi}) &\geqslant \|\,t\dot{\varphi} + x\,.\nabla\varphi + \varphi\,\|_2^2 \\
&= \|\,t\dot{\varphi} + x\,.\nabla\varphi + (1-h)\varphi\,\|_2^2 + h^2\,\|\,\varphi\,\|_2^2 \\
&\quad + 2h\,\mathrm{Re}\,\langle\, t\dot{\varphi} + x\,.\nabla\varphi + (1-h)\varphi, \varphi\,\rangle \\
&\geqslant 2ht\,\mathrm{Re}\,\langle\,\varphi, \dot{\varphi}\,\rangle - h^2\,\|\,\varphi\,\|_2^2\,.
\end{aligned} \tag{3.33}
$$

Let now $t = s \exp \tau$, $y(t) = \| \varphi(t) \|_2^2$, $a > 0$ and $h = (a + \tau)^{-1}$. Then (3.33) implies

$$h \frac{dy}{d\tau} - h^2 y = \frac{d}{d\tau}(hy) \leqslant Q_0(t, \varphi, \dot{\varphi}) \leqslant \overline{Q}_0$$

for $0 \leqslant \tau \leqslant T$, so that by integration

$$y(t) \leqslant y(s)(1 + \tau/a) + \overline{Q}_0 \tau(a + \tau).$$

Taking $a^2 = y(s)\overline{Q}_0^{-1}$, we obtain (3.31). Q. E. D.

We now turn to the proof of the main result of this section, namely the estimate of suitably weighted L^l-norms in terms of Q_m. The proof follows closely the elementary proof of the Sobolev inequalities. We introduce polar coordinates $r = |x|$ and $\xi = (\xi_1, \ldots, \xi_n) \in S^{n-1}$ by $x = r\xi$, so that $\sum_{j=1}^{n} \xi_j^2 = 1$. We first estimate functions supported in a half-space.

LEMMA 3.7. — Let $n \geqslant 2$, let $1/2 \leqslant \delta(l) \leqslant 1$ and $l < \infty$ if $n = 2$. Let h and j be continuous non negative functions of $r \in (0, \infty)$, with j strictly positive. Assume that $h(r)r^{-\mu}$ is non increasing in $(0, a)$ and non decreasing in (a, ∞) for some $\mu \geqslant 0$ and some $a \geqslant 0$ (possibly $a = 0$ or $a = \infty$). Let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n \cap \{ x : x_n > 0 \})$. Then φ satisfies the estimate

$$\| j^{2/l} h^{2\delta(l)/l} \varphi \|_l \leqslant (2^{-(n-1)/n} l(n-1)/n)^{\delta(l)} \| j\varphi \|_2^{1-\delta(l)} \prod_{i=1}^{n-1} \| j L_{ni} \varphi \|_2^{\alpha(l)}$$
$$\times \left\| \xi_n^{2-n} r^{1-n} j^{-1} h^{n-2\delta(l)} \left(\frac{d\varphi}{dr} + \frac{n\mu}{lr} \varphi \right) \right\|_2^{\alpha(l)}. \quad (3.34)$$

Proof. — In the half-space $x_n > 0$, one can use $(r, \xi_1, \ldots, \xi_{n-1})$ as coordinates and regard ξ_n as a function of $(\xi_1, \ldots, \xi_{n-1})$. With that choice of variables, the partial derivative of a function ψ with respect to ξ_i, $1 \leqslant i \leqslant n-1$, can be written as

$$\frac{\partial \psi}{\partial \xi_i} = r \frac{\partial \psi}{\partial x_i} + r \frac{\partial \xi_n}{\partial \xi_i} \frac{\partial \psi}{\partial x_n} = r \frac{\partial \psi}{\partial x_i} - r \frac{\xi_i}{\xi_n} \frac{\partial \psi}{\partial x_n} = \frac{1}{\xi_n} L_{ni} \psi.$$

Therefore, for smooth ψ supported in the region $x_n > 0$,

$$2 | \psi | \leqslant \int d\xi_i' \, | \xi_n^{-1} L_{ni} \psi | \quad (3.35)$$

where the left-hand side is taken at $x = r\xi$, and in the integrand in the right-hand side, the variable ξ_i is replaced by ξ'_i. Let now h satisfy the assumptions of the lemma, and ψ be as above. Then

$$h\,|\,\psi\,| \leqslant \int_0^\infty dr'h \left| \frac{d\psi}{dr'} + \frac{\mu}{r'}\psi \right| \tag{3.36}$$

where the integrand is taken at the point $r'\xi$. In fact, for $r \leqslant a$

$$\begin{aligned} h(r)\,|\,\psi(r\xi)\,| &= h(r)r^{-\mu} \left| \int_0^r dr' \frac{d}{dr'}(r'^{\mu}\psi(r'\xi)) \right| \\ &\leqslant \int_0^r dr'h(r') \left| \frac{d}{dr'}\psi(r'\xi) + \frac{\mu}{r'}\psi(r'\xi) \right| \end{aligned}$$

and similarly for $r \geqslant a$,

$$\begin{aligned} h(r)\,|\,\psi(r\xi)\,| &= h(r)r^{-\mu} \left| \int_r^\infty dr' \frac{d}{dr'}(r'^{\mu}\psi(r'\xi)) \right| \\ &\leqslant \int_r^\infty dr'h(r') \left| \frac{d}{dr'}\psi(r'\xi) + \frac{\mu}{r'}\psi(r'\xi) \right| . \end{aligned}$$

Let now g be a continuous non negative function of $r \in (0, \infty)$. We multiply (3.35) by g for $i = 1, \dots, n-1$, we multiply (3.36) with h replaced by h^{n-1} by ξ_n^{1-n}, multiply the resulting equations together and take the power $1/(n-1)$ to obtain

$$\begin{aligned} 2\xi_n^{-1}gh\,|\,\psi\,|^{n/(n-1)} &\leqslant \prod_{i=1}^{n-1} \left\{ \int d\xi'_i g\xi_n^{-1}\,|\,\mathrm{L}_{ni}\psi\,| \right\}^{1/(n-1)} \\ &\times \left\{ \int_0^\infty dr'\xi_n^{1-n}h^{n-1} \left| \frac{d\psi}{dr'} + \frac{(n-1)\mu}{r'}\psi \right| \right\}^{1/(n-1)} . \end{aligned}$$

Applying Hölder's inequality n times as in the usual proof of the Sobolev inequalities [1], we obtain

$$\begin{aligned} \|\, gh\,|\,\psi\,|^{n/(n-1)} \|_1^{n-1} &\leqslant 2^{1-n} \prod_{i=1}^{n-1} \|\, g\mathrm{L}_{ni}\psi\,\|_1 \\ &\times \left\| \xi_n^{2-n} r^{1-n} h^{n-1} \left(\frac{d\psi}{dr} + \frac{(n-1)\mu}{r}\psi \right) \right\|_1 \end{aligned} \tag{3.37}$$

where we have used the fact that the volume element in $\mathbb{R}^n$ is $d^nx = \xi_n^{-1}r^{n-1}drd\xi_1 \dots d\xi_{n-1}$. We now apply (3.37) to $\psi = |\,\varphi\,|^p$ with $p = l(n-1)/n$ and use the Schwarz inequality to obtain

$$\begin{aligned} \|\,(gh)^{1/l}\varphi\,\|_l^p &\leqslant 2^{(1-n)/n}p\,\|\, gj^{-1}\,|\,\varphi\,|^{p-1}\,\|_2 \prod_{i=1}^{n-1} \|\, j\mathrm{L}_{ni}\varphi\,\|_2^{1/n} \\ &\times \left\| \xi_n^{2-n} r^{1-n} g^{-1} j h^{n-1} \left(\frac{d\varphi}{dr} + \frac{(n-1)\mu}{pr}\varphi \right) \right\|_2^{1/n} . \end{aligned} \tag{3.38}$$

We finally interpolate by Hölder's inequality

$$\| gj^{-1} |\varphi|^{p-1} \|_2 \leqslant \| (gh)^{1/l}\varphi \|_l^{p-\nu} \| j\varphi \|_2^{\nu-1} \tag{3.39}$$

with $\nu\delta(l) = 1$, $1 \leqslant \nu \leqslant 2$ and

$$g = h^{2\delta(l)-1} j^2 .$$

Substituting (3.39) into (3.38) with that choice of g yields (3.34) by an elementary computation. Q. E. D.

The next step consists in removing the restriction on the support of φ by using a partition of unity in the angular variables.

LEMMA 3.8. — Let $n \geqslant 2$, let $1/2 \leqslant \delta(l) \leqslant 1$ and $l < \infty$ if $n = 2$. Let h and j be as in Lemma 3.7. Let $\varphi \in \mathscr{D}_0$. Then φ satisfies the estimate

$$\| j^{2/l} h^{2\delta(l)/l} \varphi \|_l \leqslant (\mathrm{C}l)^{\delta(l)} \| j\varphi \|_2^{1-\delta(l)} \{ \| j\mathrm{L}\varphi \|_2^2 + \| j\varphi \|_2^2 \}^{\gamma(l)/2} \times \left\| r^{1-n} j^{-1} h^{n-2\delta(l)} \left(\frac{d\varphi}{dr} + \frac{n\mu}{lr}\varphi \right) \right\|_2^{\alpha(l)} \tag{3.40}$$

where $\| j\mathrm{L}\varphi \|_2$ is defined by (3.3) and the constant C depends only on n.

Proof. — We introduce a partition of unity on S^{n-1} consisting of a finite number of non negative $\mathscr{C}^\infty$ functions χ_ν such that

$$\sum_\nu \chi_\nu^2 = 1$$

and such that for each ν, there exists $\eta_\nu \in \mathrm{S}^{n-1}$ such that

$$\mathrm{Supp}\, \chi_\nu \subset \{ \xi : \xi . \eta_\nu \geqslant 1/2 \} .$$

We remark that

$$|\varphi| = |\varphi| \sum_\nu \chi_\nu^2 \leqslant \sum_\nu \chi_\nu |\varphi|$$

and we apply Lemma 3.7 to each of the functions $\chi_\nu\varphi$ with the η axis in the direction of η_ν. The factor ξ_n^{2-n} in the last norm of (3.34) is then estimated by 2^{n-2} by the support property of χ_ν. The angular momentum terms are estimated by

$$\prod_{i=1}^{n-1} \| j\mathrm{L}_{ni}\chi_\nu\varphi \|_2 \leqslant \left\{ (n-1)^{-1} \sum_{i=1}^{n-1} \| j\mathrm{L}_{ni}\chi_\nu\varphi \|_2^2 \right\}^{(n-1)/2}$$
$$\leqslant (n-1)^{-(n-1)/2} \| j\mathrm{L}\chi_\nu\varphi \|_2^{n-1}$$

and

$$\sum_\nu \| j\mathrm{L}\chi_\nu\varphi \|_2^2 = \| j\mathrm{L}\varphi \|_2^2 + \| jk\varphi \|_2^2$$

where

$$k^2 = \sum_{i<j} \sum_{\nu} | L_{ij}\chi_\nu |^2$$

so that $k \in \mathscr{C}^\infty(S^{n-1})$.

Collecting the estimates for the functions $\chi_\nu \varphi$ and using the previous remarks yields (3.40). Q. E. D.

The estimates of interest will be obtained from (3.40) for suitable choices of j and h. In particular, we shall choose j and h in such a way that the norms that appear in the right-hand side of (3.40) are controlled by the conformal charge. We consider first the case of dimension $n \geqslant 3$.

PROPOSITION 3.3. — Let $n \geqslant 3$, let $2 \leqslant l \leqslant 2^*$, $\tilde{l} = \mathrm{Max}\,(l, 2n/(n-1))$, let $a \geqslant 0$, let $t > 0$, and $\varphi \in \Sigma_1$. Then φ satisfies the following estimate

$$\begin{aligned} &\| (\theta r^{-1})^{1-\delta(l)} \theta^{\gamma(l)} (\theta^{-1} | t^2 - r^2 | + a)^{\alpha(l)} \varphi \|_l \\ &\quad \leqslant C_0^{\delta(l)} \| \theta r^{-1} \varphi \|_2^{1-\delta(l)} \{ \| \theta r^{-1} L\varphi \|_2^2 + \| \theta r^{-1} \varphi \|_2^2 \}^{\gamma(l)/2} \\ &\quad\quad \times \left\{ \left\| \theta^{-1} | t^2 - r^2 | \left(\frac{d\varphi}{dr} + \frac{n}{\tilde{l} r} \varphi \right) \right\|_2^2 + a^2 \left\| \frac{d\varphi}{dr} \right\|_2^2 \right\}^{\alpha(l)/2} \\ &\quad \leqslant C_1 Q_m(t, \varphi)^{(1-\alpha(l))/2} \left(Q_m(t, \varphi) + a^2 \left\| \frac{d\varphi}{dr} \right\|_2^2 \right)^{\alpha(l)/2} \end{aligned} \tag{3.41}$$

where the constants C_0 and C_1 depend only on n.

Proof. — We first prove the result for $l \geqslant 2n/(n-1)$ and $\varphi \in \mathscr{D}_0$. For that purpose, we apply (3.40) with $j = \theta r^{-1}$ and two different choices of h. The choice

$$h^{n-2\delta(l)} = r^{n-2} | t^2 - r^2 |$$

yields the first inequality in (3.41) with $a = 0$ by an elementary computation. In that case the assumptions of Lemma 3.7 on h are easily seen to be satisfied with $\mu = 1$. The second choice

$$h^{n-2\delta(l)} = r^{n-2}\theta$$

yields the terms with a in the first inequality in (3.41). The assumptions on h are satisfied with $\mu = 0$ (actually h is monotonous).

The first inequality in (3.41) is then extended to $2 \leqslant l \leqslant 2n/(n-1)$ by interpolation by noticing that the left-hand side is log-convex and the right-hand side is log-linear in $\delta(l)$ for $0 \leqslant \delta(l) \leqslant 1/2$. The second inequality in (3.41) follows from (3.26) and (3.27). Finally the extension from $\mathscr{D}_0$ to general $\varphi \in \Sigma_1$ follows by continuity of the last member in the Σ_1-norm and density of $\mathscr{D}_0$ in Σ_1. Q. E. D.

The case of space dimension $n = 2$ presents special difficulties.

PROPOSITION 3.4. — Let $n = 2$, $2 \leqslant l < \infty$, $\tilde{l} = \mathrm{Max}\,(l, 4)$, let $a \geqslant 0$,

$t > 0$ and $\varphi \in \Sigma_1$. Let $\Lambda \equiv \Lambda(r) = 1 + \text{Log}_+ (t/r)$. Then φ satisfies the following estimate.

$$
\begin{aligned}
\| (\theta r^{-1})^{2/l} \Lambda^{\alpha(l)-1} \theta^{\alpha(l)} (\theta^{-1}\Lambda^{-1} | t^2 - r^2 | + a)^{\alpha(l)} \varphi \|_l \\
\leqslant (C_0 \tilde{l})^{\delta(l)} \| \theta r^{-1} \Lambda^{-1} \varphi \|_2^{2/l} \{ \| \theta r^{-1} \Lambda^{-1} L\varphi \|_2^2 + \| \theta r^{-1} \Lambda^{-1} \varphi \|_2^2 \}^{\alpha(l)/2} \\
\times \left\{ \left\| \theta^{-1} \Lambda^{-1} | t^2 - r^2 | \left(\frac{d\varphi}{dr} + \frac{1}{r} \varphi \right) \right\|_2^2 + a^2 \left\| \frac{d\varphi}{dr} \right\|_2^2 \right\}^{\alpha(l)/2} \\
\leqslant \tilde{l}^{\delta(l)} C_1 (Q_m(t, \varphi)^{1/2} + \| \varphi \|_2)^{1-\alpha(l)} \\
\times \left\{ Q_m(t, \varphi)^{1/2} + \| \varphi \|_2 + a \left\| \frac{d\varphi}{dr} \right\|_2 \right\}^{\alpha(l)} . \qquad (3.42)
\end{aligned}
$$

Proof. — We proceed as in the proof of Proposition 3.3. We first prove the result for $l \geqslant 4$ and $\varphi \in \mathcal{D}_0$ by applying (3.40) with $j = \theta r^{-1}\Lambda^{-1}$ and with two choices of h. The choice

$$h^{4/l} = \Lambda^{-2} | t^2 - r^2 |$$

yields the first inequality in (3.42) with $a = 0$ by an elementary computation. In this case the assumptions of Lemma 3.7 on h can be seen to be satisfied with $\mu = l/2$. The choice

$$h^{4/l} = \theta \Lambda^{-1}$$

yields the terms with a in the first inequality in (3.42). The assumptions on h are satisfied with $\mu = 0$. The first inequality in (3.42) is then extended to $2 \leqslant l \leqslant 4$ by interpolation as before. The second inequality in (3.42) follows from (3.26) and (3.30), and the extension from $\mathcal{D}_0$ to Σ_1 follows by continuity and density. Q. E. D.

The additional logarithmic factor Λ^{-1}, which appears in the various norms in (3.42) is needed to allow for an estimate of $\theta r^{-1}\Lambda^{-1}\varphi$ in L^2 in terms of $Q_m(t, \varphi)$. This factor is not necessary to estimate the angular momentum term and the radial derivative of φ if the latter occurs alone. However additional terms with the L^2 norm of $\theta r^{-1}\varphi$ arise from the angular momentum term when one recombines the various angular sectors (see the proof of Lemma 3.8) and from the radial derivative term because we are unable to estimate it by using an h fulfilling the assumptions of Lemma 7 with $\mu = 0$. This forces us to introduce the factor Λ^{-1} in all the norms to be controlled by Q_m.

We now compare the estimates of Propositions 3.3 and 3.4 with the earlier estimates of Proposition 3.1. For $n \geqslant 3$, the estimate (3.41) for φ is stronger than the estimate (3.17) in every respect. In fact, the allowed values of l range over the interval $[2, 2^*]$ for (3.41) instead of the interval $(2, l_S]$ for (3.17). In the left-hand side of (3.41), the L^l norm is improved by various additional factors: the factor $(\theta r^{-1})^{1-\delta(l)}$ yields the Hardy inequality, the factor $\theta^{\gamma(l)}$ yields the time decay $t^{-\gamma(l)}$ and an additional decrease at infinity in space; finally the factor $(\theta^{-1} | t^2 - r^2 |)^{\alpha(l)}$ yields an additional decay away from the light cone. In the case $n = 2$, the same comments apply, with the additional complication that the Hardy inequality now contains a logarithmic correction. Moreover the right-hand side of (3.42) contains the L^2 norm of φ in addition to the conformal charge and the energy. In the applications, this fact will result in an additional factor $\operatorname{Log} t$ through the use of Proposition 3.2, whereas no such factor occurs in (3.17).

It is interesting to remark that the norms that appear in the left-hand side of (3.41) and (3.42) have some similarity with the norm used in [*9*] for $n = 2$ and in [*12*] for $n = 3$ to prove the existence of global solutions for small data or equivalently the existence of dispersive solutions of the equation (1.1) down to the optimal values $p_0(2) = (3 + \sqrt{17})/2$ and $p_0(3) = 1 + \sqrt{2}$. That norm is (see (4.9) *a*) of [*12*])

$$\| (t + r)(1 + | t - r |)^{p-2} \varphi \|_\infty$$

for $n = 3$ and a similar but more complicated one (see (11) of [*9*]) for $n = 2$.

We finally remark that most of the estimates of this section still hold if φ is a scalar field in the presence of an external Yang-Mills potential, with the ordinary derivatives replaced by covariant derivatives. In particular this is true for the estimates of Propositions 3.3 and 3.4. It fails to hold only in those cases where the operator ω plays an essential role, namely in the middle equality in (3.5), in the rewriting of $Q_0(t, \varphi, \psi)$ in the form (3.10) and (3.12), in Lemma 3.2 and in Proposition 3.1. That property follows from the fact that for any function v with values in the space of Yang-Mills potentials or in the space relevant for the coexisting scalar fields, the following inequality holds

$$\partial_\mu | v | \leqslant | D_\mu v | ,$$

where $| \ . \ |$ denotes the norm in the relevant space, and D_μ the covariant derivative corresponding to ∂_μ (see [*4*], especially the Appendix).

4. TIME DECAY

In this section we apply the results of Sections 2 and 3 to derive some decay estimates of the solutions of the equation (1.1). For that purpose we need a repulsivity condition on the interaction of the following form

(A.4) The function V defined in the assumption (A.2) satisfies

$$0 \leqslant (p_1 + 1)V(z) \leqslant 2 \operatorname{Re} \bar{z} f(z) \tag{4.1}$$

for some p_1 with $0 \leqslant p_1 - 1 < 4/(n - 2)$ and for all $z \in \mathbb{C}$.

The situation is all the more favourable as p_1 is larger. The best case is that where $p_1 \geqslant 1 + 4/(n-1)$, in which case (A.4) implies that W, as defined in (2.5), is non positive. In that case we obtain the following result.

PROPOSITION 4.1. — Let $n \geqslant 2$, let f satisfy (A.1), (A.2) and (A.4) with $p_1 \geqslant 1 + 4/(n-1)$. Let $(\varphi_0, \psi_0) \in \Sigma$, let $t_0 \in \mathbb{R}$ and let $(\varphi, \dot\varphi)$ be the solution of (1.1) in $\mathscr{C}(\mathbb{R}, \Sigma)$ with initial data (φ_0, ψ_0) at time t_0, as described in Proposition 2.3. Then $Q_0(t, \varphi, \dot\varphi)$ is uniformly bounded in time and φ satisfies the following estimates.

For $n \geqslant 3$ and for all l with $2 \leqslant l \leqslant 2^*$

$$\| (\theta r^{-1})^{1-\delta(l)} \theta^{\gamma(l)} [\theta^{-1} | t^2 - r^2 | + 1]^{\alpha(l)} \varphi \|_l \leqslant C. \tag{4.2}$$

For $n = 2$ and for all l with $2 < l \leqslant 6$,

$$\| \varphi \|_l \leqslant C | t |^{-\alpha(l)} \tag{4.3}$$

and for all l, $2 \leqslant l < \infty$

$$\| (\theta r^{-1})^{2/l} \Lambda^{\alpha(l)-1} \theta^{\alpha(l)} [\theta^{-1} \Lambda^{-1} | t^2 - r^2 | + 1]^{\alpha(l)} \varphi \|_l \leqslant C l^{\delta(l)} (1 + \mathrm{Log}_+ | t |), \tag{4.4}$$

where $\Lambda = 1 + \mathrm{Log}_+ (| t | / r)$.

Proof. — It follows from Proposition 2.3 with $V \geqslant 0$ and $W \leqslant 0$ that $Q_0(t, \varphi, \dot\varphi)$ is uniformly bounded in time. The estimate (4.2) then follows from Proposition 3.3 and the estimates (4.3) and (4.4) follow from Propositions 3.1, 3.2 and 3.4. Q. E. D.

We now turn to the more difficult case where $p_1 < 1 + 4/(n-1)$. In that situation one can still derive some time decay of the solutions of the equation (1.1) from the conformal conservation law (2.32) (see also [*10*] [*21*]).

LEMMA 4.1. — Let f satisfy (A.1), (A.2) and (A.4) with $p_1 \leqslant 1 + 4(n-1)$ and let

$$\mu = 2 - (p_1 - 1)(n - 1)/2.$$

Let $(\varphi_0, \psi_0) \in \Sigma$ and let (2.31) be satisfied (if not already implied by $\varphi_0 \in \Sigma_1$). Let $(\varphi, \dot\varphi)$ be the solution of (1.1) in $\mathscr{C}(\mathbb{R}, \Sigma)$ with initial data (φ_0, ψ_0) at time 0, as described in Proposition 2.3. Let

$$m(t) = Q(t, \varphi, \dot\varphi) + E(\varphi, \dot\varphi).$$

Then, for all $t \in \mathbb{R}$, the following inequalities hold

$$m(t) \leqslant m(0)(1 + t^2)^{\mu} \tag{4.5}$$

and

$$\int dx V(\varphi) \leqslant m(0)(1 + t^2)^{\mu - 1}. \tag{4.6}$$

Assume in addition that

$$\lim_{R \to 0} V(R) R^{-(p_1 + 1)} > 0. \tag{4.7}$$

Then

$$\| \varphi \|_{p_1+1} \leqslant Cm(0)^{1/(p_1+1)}(1 + |t|)^{1-2\delta(p_1+1)} . \tag{4.8}$$

Proof. — It follows from the assumption (A.4) that

$$W \leqslant \mu V$$

so that the conservation law (2.32) together with the conservation of the energy (2.17) implies

$$\begin{aligned} m(t) &\leqslant m(0) + \mu \int_0^t 2\tau d\tau \int dx V(\varphi(\tau)) \\ &\leqslant m(0) + \mu \int_0^t 2\tau d\tau (1 + \tau^2)^{-1} m(\tau) \end{aligned}$$

from which (4.5) follows by integration. (4.6) is an immediate consequence of (4.5). The assumption (A.4) implies that $V(R)R^{-(p_1+1)}$ is an increasing function of R, and therefore, together with the condition (4.7), implies that for all $R \geqslant 0$,

$$V(R) \geqslant CR^{p_1+1} . \tag{4.9}$$

Then (4.8) follows from (4.6) and (4.9) after noticing that

$$2(\mu - 1)(p_1 + 1)^{-1} = 1 - 2\delta(p_1 + 1) .$$

Q. E. D.

REMARK 4.1. — By combining (4.5) with Proposition 3.3 we obtain for $n \geqslant 3$

$$\| \varphi \|_l \leqslant C(1 + |t|)^{-\gamma(l) + \mu(1-\alpha(l))} \tag{4.10}$$

for all l, $2 \leqslant l \leqslant 2^*$. Similarly, for $n = 2$, from (4.5) and Propositions 3.2 and 3.4 we obtain

$$\| \varphi \|_l \leqslant Cl^{\delta(l)}(1 + |t|)^{-\alpha(l) + \mu(1-\alpha(l))}(1 + \mathrm{Log}_+ |t|)^{1-\alpha(l)} \tag{4.11}$$

for all l, $2 \leqslant l < \infty$. The power of the $\mathrm{Log}_+ |t|$ in (4.11) is not optimal. In particular for $2 < l \leqslant 6$, Proposition 3.1 yields (4.11) without any logarithm. For $l = p_1 + 1$, the estimates (4.10) and (4.11) are weaker than (4.8).

We next show that by combining the decay estimate of Lemma 4.1 with the equation (1.1) itself one can extend the conclusions of Proposition 4.1, up to additional logarithmic corrections for $n = 2$, to values of p_1 strictly smaller than $1 + 4/(n - 1)$. We restrict our attention to space dimension $n \geqslant 2$. The general method is inspired by that applied in [*11*] to treat the same problem for the non linear Schrödinger equation.

The free conformal charge Q_0 as given by (3.7) can be rewritten as

$$Q_0 = \| M\varphi \|_2^2 + \| L\varphi \|_2^2 + \| D\varphi \|_2^2 \tag{4.12}$$

where M and D are the first order differential operators

$$M = x\frac{\partial}{\partial t} + t\nabla \tag{4.13}$$

$$D = t\frac{\partial}{\partial t} + x.\nabla + n - 1 \tag{4.14}$$

and L is defined by (3.1). M is the generator of pure Lorentz transformations, and D is the generator of dilations. The method consists in estimating the functions $\Phi_A \equiv A\varphi$ in L^2 uniformly in time for A = M, L and D. For that purpose we derive evolution equations for the functions $A\varphi$ from the equation (1.1). From the commutation relations

$$[\Box, A] = 0 \qquad \text{for} \quad A = L, M$$

and

$$[\Box, D] = 2\Box$$

and from (1.1) it follows that the functions Φ_A satisfy the equations

$$\Box\Phi_A + f'(\varphi)\Phi_A = 0, \tag{4.15}$$

for A = L, M, and

$$\Box\Phi_D + f'(\varphi)\Phi_D + (n+1)f(\varphi) - (n-1)f'(\varphi)\varphi = 0 \tag{4.16}$$

where

$$f'(\varphi)\Phi \equiv \frac{\partial f}{\partial \varphi}\Phi + \frac{\partial f}{\partial \overline{\varphi}}\overline{\Phi}.$$

The differential equations (4.15) and (4.16) can be combined with the initial conditions at time t_0, which we take for simplicity to be zero, to yield the integral equations

$$\Phi_A(t) = \Phi_A^{(0)}(t) - \int_0^t d\tau K(t-\tau)f'(\varphi(\tau))\Phi_A(\tau) \tag{4.17}$$

for A = L, M, and

$$\Phi_D(t) = \Phi_D^{(0)}(t) - \int_0^t d\tau K(t-\tau)\{ f'(\varphi(\tau))\Phi_D(\tau) + (n+1)f(\varphi(\tau)) - (n-1)f'(\varphi(\tau))\varphi(\tau)\}. \tag{4.18}$$

The corresponding integral equation for φ is

$$\varphi(t) = \varphi^{(0)}(t) - \int_0^t d\tau K(t-\tau)f(\varphi(\tau)). \tag{4.19}$$

The free terms in the equations (4.17), (4.18) and (4.19) are

$$\Phi_A^{(0)}(t) = \dot{K}(t)\Phi_A(0) + K(t)\dot{\Phi}_A(0)$$

and

$$\varphi^{(0)}(t) = \dot{K}(t)\varphi_0 + K(t)\psi_0,$$

where (φ_0, ψ_0) are the initial data for (1.1) at time zero. In particular $\varphi^{(0)} \in L^\infty(\mathbb{R}, L^2)$ for $n \geqslant 3$ as can be seen either by applying the usual Hardy inequality in Fourier transform to ψ_0 or by applying the conformal invariance to the free equation (see for instance (3.27)). For $n = 2$ we obtain only

$$\| \varphi^{(0)}(t) \|_2 \leqslant C(1 + \mathrm{Log}_+ | t |) \tag{4.20}$$

by Proposition 3.2. The initial data for the functions Φ_A are easily obtained from the definitions of the operators A. For $A = L$ we find

$$\Phi_L(0) = L\varphi_0 , \qquad \dot{\Phi}_L(0) = L\psi_0 ,$$

so that $\Phi_L(0) \in L^2$ and $\omega^{-1}\dot{\Phi}_L(0) = \omega^{-1}(\nabla \times x)\psi_0 \in L^2$ since $x\psi_0 \in L^2$ and therefore $\Phi_L^{(0)} \in L^\infty(\mathbb{R}, L^2)$. For $A = D$ we find $\Phi_D(0) = x.\nabla\varphi_0 + (n-1)\varphi_0 \in L^2$ and $\omega^{-1}\dot{\Phi}_D(0) = \omega^{-1}(n\psi_0 + x.\nabla\psi_0) = \omega^{-1}\nabla(x\psi_0) \in L^2$ so that $\Phi_D^{(0)} \in L^\infty(\mathbb{R}, L^2)$. For $A = M$ we find $\Phi_M(0) = x\psi_0 \in L^2$ and

$$\begin{aligned} \omega^{-1}\dot{\Phi}_M(0) &= \omega^{-1}\nabla\varphi_0 + \omega^{-1}x(\Delta\varphi_0 - f(\varphi_0)) \\ &= \omega^{-1}\nabla_j(x\nabla_j\varphi_0) - \omega^{-1}xf(\varphi_0) . \end{aligned} \tag{4.21}$$

The first term in the right-hand side of (4.21) belongs to L^2, while the second term requires a special treatment, which we give in the following Lemma.

LEMMA 4.2. — Let $n \geqslant 2$, let $f \in \mathscr{C}(\mathbb{C}, \mathbb{C})$ satisfy

$$| f(z) | \leqslant C(| z |^{p_1} + | z |^{p_2}) \tag{4.22}$$

for all $z \in \mathbb{C}$, with

$$4/n \leqslant p_1 - 1 \leqslant p_2 - 1 \leqslant 4/(n-2) \tag{4.23}$$

and in addition $p_2 < \infty$ for $n = 2$. Let $\varphi_0 \in \Sigma_1$ and let $\Phi^{(0)} = K(t)xf(\varphi_0)$. Then

(1) For $n \geqslant 3$, $\omega^{-1}xf(\varphi_0) \in L^2$ and $\Phi^{(0)} \in L^\infty(\mathbb{R}, L^2)$.
(2) For $n = 2$,

$$(1 + \mathrm{Log}_+ | t |)^{-1}\Phi^{(0)} \in L^\infty(\mathbb{R}, L^2) . \tag{4.24}$$

Proof. — It is sufficient to consider the case of one single power $p = p_1 = p_2$ in (4.22). For $n \geqslant 3$, we estimate

$$\begin{aligned} \| \omega^{-1}xf(\varphi_0) \|_2 &\leqslant C \| xf(\varphi_0) \|_{2n/(n+2)} \leqslant C \| x\varphi_0 \|_{2*} \| \varphi_0 \|_{(p-1)n/2}^{p-1} \\ &\leqslant C \| \nabla x\varphi_0 \|_2 \| \varphi_0 ; H^1 \|^{p-1} , \end{aligned} \tag{4.25}$$

where we have used the Sobolev inequalities twice. This proves that $\omega^{-1}xf(\varphi_0) \in L^2$ and therefore that $\Phi^{(0)} \in L^\infty(\mathbb{R}, L^2)$ for $n \geqslant 3$. For $n = 2$, we regard the function $K(t)xf(\varphi_0)$ as the solution of the free wave equation $\Box\varphi = 0$ with initial data $(0, xf(\varphi_0))$ at time zero. We apply Proposition 3.2 to that solution, and use the fact that its energy and conformal charge are constant and equal to $\| xf(\varphi_0) \|_2^2$ and $\| x^2 f(\varphi_0) \|_2^2$ respectively, so that by (3.31) with $s = 1$

$$\| K(t) x f(\varphi_0) \|_2 \leqslant \| x f(\varphi_0) \|_2 + \| x^2 f(\varphi_0) \|_2 \operatorname{Log}_+ | t | . \tag{4.26}$$

We next estimate

$$\begin{aligned} \| x f(\varphi_0) \|_2 &\leqslant C \| \nabla (r f(\varphi_0)) \|_1 \leqslant C \{ \| \nabla r \varphi_0 \|_2 + \| \varphi_0 \|_2 \} \| \varphi_0 \|_{2(p-1)}^{p-1} \\ &\leqslant C \| \varphi_0 ; \Sigma_1 \| \, \| \varphi_0 ; H^1 \|^{p-1} \end{aligned} \tag{4.27}$$

by the Sobolev and Schwarz inequalities, provided $p \geqslant 2$. Similarly

$$\begin{aligned} \| x^2 f(\varphi_0) \|_2 &\leqslant C \| r | \varphi_0 |^{p/2} \|_4^2 \leqslant C \| \nabla (r | \varphi_0 |^{p/2}) \|_{4/3}^2 \\ &\leqslant C \{ \| \nabla r \varphi_0 \|_2 + \| \varphi_0 \|_2 \}^2 \| | \varphi_0 |^{p/2-1} \|_4^2 \leqslant C \| \varphi_0 ; \Sigma_1 \|^2 \| \varphi_0 ; H^1 \|^{p-2} \end{aligned} \tag{4.28}$$

provided $p \geqslant 3$. (4.24) follows then from (4.26)-(4.28). Q. E. D.

In order to derive the boundedness of $Q_0(t, \varphi, \dot{\varphi})$ (uniformly in time for $n \geqslant 3$, logarithmically for $n = 2$) we now proceed as follows. We first insert the available time decay (4.8) for φ into the integral equation (4.19) and derive additional decay properties of φ therefrom. We then insert that decay into the integral equations (4.17) and (4.18) to derive boundedness properties of Φ_A ($A = L, M, D$) in L^2. As indicated in the introduction, that method can be expected to work only for $p_1 > p_2(n)$ (see (1.9)), for a simple dimensional reason that we now explain. We are working with norms of φ of the type

$$\operatorname{Sup}_t | t |^{\nu} \| \varphi(t) \|_l \tag{4.29}$$

and similar norms for the functions Φ_A. (The argument would not be changed if we used more complicated norms involving homogeneous Sobolev or Besov spaces). Under a space-time dilation $(t, x) \rightarrow (at, ax)$, such a norm changes by a factor a^d where the dimension d is defined by

$$d = \nu + n/l = \nu + n/2 - \delta(l) .$$

In particular, the estimate (4.8) which we use as a starting point yields a norm of dimension

$$d_1 = 2\delta(p_1 + 1) - 1 + n/(p_1 + 1) = n/2 - 1 + \delta(p_1 + 1) . \tag{4.30}$$

We insert that information into the integral equation (4.19) in order to estimate new norms of φ. In order to allow for the use of Gronwall's inequality, the latter must appear at most linearly (for an optimal result almost linearly) in an estimate of the right-hand side of the equation (4.19). Now the integral operator $\int d\tau K(t - \tau)$ has dimension 2. Assuming for definiteness that the interaction f consists of a single power (1.2) with

$p = p_1$, we find that an improved estimate for φ can result only if $(p_1 - 1)d_1 > 2$, or equivalently

$$(p_1 - 1)(n - 1) - (n + 2) + 2n/(p_1 + 1) > 0,$$

which is equivalent to (1.9) with $p = p_1$, namely $p_1 > p_2(n)$. The same argument yields the same conclusion if we try to insert (4.8) directly into the integral equations (4.17), (4.18) in order to estimate Φ_A. It is worth noting that a similar dimensional argument yields the same lower bound on p when one tries to solve the Cauchy problem at infinity by using the basic estimate (3.15) in the special case $1/s + 1/l = 1$ (see Theorem 14 in [*28*]). In that case, the time decay appearing in the right-hand side of (3.15) is again $t^{1-2\delta(l)}$ and the natural condition on p (in the special case (1.2)) comes out as

$$p(2\delta(p + 1) - 1) > 1 \tag{4.31}$$

which is again equivalent to $p > p_2(n)$.

We now come back to the task of estimating the functions Φ_A. In space dimension $n = 2$, we can cover the expected range $p > p_2(2) = 2 + \sqrt{5}$ by elementary methods, and we treat that case first. For that purpose, we need an additional estimate on the operator $K(t)$.

LEMMA 4.3. — Let $n = 2$, let $0 < \varepsilon \leqslant 1$. Then for all $h \in L^1 \cap L^{1/(1-\varepsilon/2)}$, the following estimate holds

$$\begin{aligned} \| K(t)h \|_2^2 &\leqslant (3/4\pi)t^2(t^2 - 1)^{-1} \operatorname{Log} |t| \, \| h \|_1^2 + \| (1 - \Delta)^{-1/2} h \|_2^2 \\ &\leqslant (3/4\pi)t^2(t^2 - 1)^{-1} \operatorname{Log} |t| \, \| h \|_1^2 + \operatorname{Max}(1, (4\pi\varepsilon)^{-1}) \, \| h \|_{1/(1-\varepsilon/2)}^2 . \end{aligned} \tag{4.32}$$

Proof. — Let

$$\hat{h}(\xi) = (2\pi)^{-1} \int d\xi e^{-ix\cdot\xi} h(x)$$

be the Fourier transform of h. We estimate

$$\begin{aligned} \| K(t)h \|_2^2 &= \int d\xi \, |\xi|^{-2} \sin^2(t|\xi|) \, |\hat{h}(\xi)|^2 \\ &= \int d\xi \, |\xi|^{-2}(1 + \xi^2)^{-1} \sin^2(t|\xi|) \, |\hat{h}(\xi)|^2 \\ &\quad + \int d\xi (1 + \xi^2)^{-1} \sin^2(t|\xi|) \, |\hat{h}(\xi)|^2 \\ &\leqslant I_1 \| h \|_1^2 + \| (1 - \Delta)^{-1/2} h \|_2^2 \end{aligned} \tag{4.33}$$

by the Young inequality, and with

$$I_1 = (2\pi)^{-2} \int d\xi \, |\xi|^{-2}(1 + \xi^2)^{-1} \sin^2(t|\xi|).$$

Using the inequality $\sin^2 y \leqslant (3/2)y^2(1 + y^2)^{-1}$, we obtain

$$I_1 \leqslant (3/4\pi)t^2(t^2 - 1)^{-1} \operatorname{Log}|t| \tag{4.34}$$

by an elementary computation. The first inequality in (4.32) follows from (4.33) and (4.34). In order to prove the second inequality, we estimate

$$\| (1 - \Delta)^{-1/2} h \|_2^2 \leqslant \| \hat{h} \|_{2/\varepsilon}^2 \left\{ \int d\xi (1 + \xi^2)^{-1/(1-\varepsilon)} \right\}^{1-\varepsilon}$$
$$\leqslant [(1 - \varepsilon)/(4\pi\varepsilon)]^{1-\varepsilon} \| h \|_{1/(1-\varepsilon/2)}^2$$

by the Hölder and Young inequalities and an elementary computation.

Q. E. D.

In order to estimate the interaction in the integral equations (4.17)-(4.19), we need an additional assumption on f which reinforces (A.1) and which we state as follows

(A.1′) $f \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$, $f(0) = 0$ and f satisfies the estimate

$$| f'(z) | \leqslant C(| z |^{p_1 - 1} + | z |^{p_2 - 1}) \tag{4.35}$$

for some p_1, p_2 with $1 \leqslant p_1 \leqslant p_2 < 1 + 4/(n - 2)$, $p_2 < \infty$ for $n = 2$, and for all $z \in \mathbb{C}$.

We are now in a position to state our result for $n = 2$.

Proposition 4.2. — Let $n = 2$. Let f satisfy the assumptions (A.1′), (A.2), (A.4) and (4.7) with $5 \geqslant p_1 > p_2(2) \equiv 2 + \sqrt{5} \simeq 4.236$. Let $(\varphi_0, \psi_0) \in \Sigma$ and let $(\varphi, \dot{\varphi})$ be the solution of the equation (1.1) in $\mathscr{C}(\mathbb{R}, \Sigma)$ with initial data (φ_0, ψ_0) at time zero, as described in Proposition 2.3. Then $Q_0(t, \varphi, \dot{\varphi})$ is estimated by

$$Q_0(t, \varphi, \dot{\varphi}) \leqslant C(1 + \operatorname{Log}_+ |t|)^2 \tag{4.36}$$

and for all l, $2 \leqslant l < \infty$, φ satisfies the estimate

$$\| (\theta r^{-1})^{2/l} \Lambda^{\alpha(l)-1} \theta^{\alpha(l)} [\theta^{-1} \Lambda^{-1} | t^2 - r^2 | + 1]^{\alpha(l)} \varphi \|_l$$
$$\leqslant C l^{\delta(l)} (1 + \operatorname{Log}_+ |t|). \tag{4.4}$$

Remark. — We have taken the same p_1 in the assumptions (A.1′) and (A.4) for simplicity. We have also restricted our attention to the case $p_1 \leqslant 5$, since the case $p_1 \geqslant 5$ is already covered under more economical assumptions by the simpler proposition 4.1. Note also that since $p_2(n) > 1 + 4/n$, the assumption (A.1′) with $p_1 > p_2(n)$ implies (4.22), (4.23).

Proof of the proposition. — We concentrate on the main estimates, and leave aside part of the abstract details. The proof proceeds in two steps.

First step. — We substitute the available decay (4.8) into the integral equation (4.19) to obtain additional decay of φ. From Propositions 3.2 and 3.4 and from the conservation of the energy and of the conformal charge for the equation $\Box\varphi = 0$, it follows that the term $\varphi^{(0)}$ in (4.19) is estimated as

$$\| \varphi^{(0)}(t) \|_l \leqslant C l^{\delta(l)} (1 + |t|)^{-\alpha(l)} (1 + \mathrm{Log}_+ |t|) \tag{4.37}$$

for all l, $2 \leqslant l < \infty$. We next estimate the integral in (4.19) by using (3.15) with $s = (p_1 + 1)/p_1$ and l defined by

$$\delta(l) = 1 + \alpha(s) = 1 - \alpha(p_1 + 1) \tag{4.38}$$

so that

$$1 - \delta(l) - \delta(p_1 + 1) = - \alpha(p_1 + 1) \tag{4.39}$$

(for $2 + \sqrt{5} < p_1 \leqslant 5$, l satisfies $6 \leqslant l < 2 + 2\sqrt{5}$). We obtain

$$\| K(t - \tau) f(\varphi) \|_l \leqslant C | t - \tau |^{-\alpha(p_1 + 1)} \| f(\varphi) \|_{(p_1 + 1)/p_1} . \tag{4.40}$$

Using the assumption (A.1′) in the integrated form (4.22), we estimate the last norm as

$$\| f(\varphi) \|_{(p_1 + 1)/p_1} \leqslant C \left(\| \varphi \|_{p_1 + 1}^{p_1} + \| \varphi \|_{p_2(p_1 + 1)/p_1}^{p_2} \right) . \tag{4.41}$$

The first norm in the right-hand side is estimated by (4.8), while the second norm is estimated by interpolation between the first one and $\| \varphi \|_m$ for some large m. The latter norm is then estimated by (4.11), with a stronger time decay than given by (4.8) for $\| \varphi \|_{p_1 + 1}$. In fact

$$\begin{aligned} &\alpha(m) - \mu(1 - \alpha(m)) - 2\delta(p_1 + 1) + 1 \\ &\quad = (1 - \alpha(m))(1 - \mu - \delta(p_1 + 1)) + (1 + \alpha(m)) \left(\frac{\delta(m)}{1 + \alpha(m)} - \delta(p_1 + 1) \right) \end{aligned}$$

is easily seen to be positive for large m, since the condition $p_1 > p_2(n)$ is equivalent to $1 - \mu - \delta(p_1 + 1) > 0$. Therefore the second norm in the right-hand side of (4.41) has a better time decay than the first one, so that

$$\| f(\varphi) \|_{(p_1 + 1)/p_1} \leqslant C(1 + \tau)^{-p_1(2\delta(p_1 + 1) - 1)} . \tag{4.42}$$

Now $p_1 > p_2(n)$ is equivalent to $p_1(2\delta(p_1 + 1) - 1) > 1$, as remarked before, so that the right-hand side of (4.42) is integrable in time. It then follows from (4.19), (4.38), (4.40) and (4.42) that φ satisfies the decay estimate

$$\| \varphi(t) \|_l \leqslant C(1 + |t|)^{-\alpha(p_1 + 1)} \tag{4.43}$$

for l defined by (4.39).

Second step. — We insert the decay just obtained in the equations (4.17) and (4.18) to obtain estimates of Φ_A. We consider only the equation (4.17) which covers the cases $A = L$ and $A = M$. The additional terms in (4.18) for $A = D$ can be estimated in a similar way. Using Lemma 4.3, we estimate the integrand in (4.17) for $0 \leqslant \tau \leqslant t$ by

$$\begin{aligned} \| K(t-\tau) \Phi_A f'(\varphi) \|_2 &\leqslant C(1 + \mathrm{Log}_+ (t-\tau)) \| \Phi_A f'(\varphi) \|_1 + C \| \Phi_A f'(\varphi) \|_{1/(1-\varepsilon/2)} \\ &\leqslant C(1 + \mathrm{Log}_+ t) \left(\| \Phi_A f'(\varphi) \|_1 + \| \Phi_A f'(\varphi) \|_{1/(1-\varepsilon/2)} \right) . \end{aligned} \tag{4.44}$$

We estimate only the second norm in the last member of (4.44). The first norm is estimated by taking $\varepsilon = 0$ in the result thereby obtained. Using the assumption (A.1′) we obtain

$$\| \Phi_A f'(\varphi) \|_{1/(1-\varepsilon/2)} \leqslant \| \Phi_A \|_2 \| f'(\varphi) \|_{2/(1-\varepsilon)} \leqslant C \| \Phi_A \|_2 \sum_{i=1,2} \| \varphi \|^{p_i - 1}_{2(p_i - 1)/(1-\varepsilon)} . \quad (4.45)$$

We estimate the last norm by interpolation as

$$\| \varphi \|_{2(p_i - 1)/(1-\varepsilon)} \leqslant \| \varphi \|^{\sigma_i}_{l/\varepsilon'} \| \varphi \|^{1-\sigma_i}_{l} \quad (4.46)$$

with l defined by (4.39), ε' small and positive, and σ_i given by homogeneity, namely, after using the fact that $2/l = \alpha(p_1 + 1)$,

$$1 - \varepsilon = (p_i - 1)(1 - \sigma_i(1 - \varepsilon'))\alpha(p_1 + 1) . \quad (4.47)$$

We estimate the first norm in the right-hand side of (4.46) by (4.11) and the second norm by (4.43), thereby obtaining

$$\| \varphi \|^{p_i - 1}_{2(p_i - 1)/(1-\varepsilon)} \leqslant C(1 + \mathrm{Log}_+ \tau)^{(p_i - 1)\sigma_i}(1 + \tau)^{-\nu_i} \quad (4.48)$$

with

$$\nu_i = (p_i - 1) \{ \sigma_i [\alpha(l/\varepsilon') - \mu(1 - \alpha(l/\varepsilon'))] + (1 - \sigma_i)\alpha(p_1 + 1) \} . \quad (4.49)$$

It is easy to see that ν is increasing in p, namely $\nu_2 > \nu_1$, and we concentrate on ν_1, dropping the subscript 1 for brevity in σ_1, ν_1. Eliminating σ partly by using (4.47) and eliminating l by using (4.39) in the form $2/l = \alpha(p_1 + 1)$, we obtain

$$\begin{aligned} \nu &= 1 - \varepsilon + (p_1 - 1)\sigma [\alpha(l/\varepsilon') - \mu(1 - \alpha(l/\varepsilon')) - \varepsilon'\alpha(p_1 + 1)] \\ &= 1 - \varepsilon + (p_1 - 1)(\sigma/2) [1 - \mu - \varepsilon'(3 + \mu)\alpha(p_1 + 1)] . \end{aligned} \quad (4.50)$$

We next rewrite the condition $p_1 > p_2(2)$ in the form

$$\eta \equiv 1 - \mu - \delta(p_1 + 1) \equiv (p_1 - 1)\alpha(p_1 + 1) - 1 > 0$$

and obtain from (4.50)

$$\begin{aligned} \nu &= 1 - \varepsilon + \sigma \{ (1 + \eta)(1 - (3 + \mu)\varepsilon'/2) + (p_1 - 1)\eta/2 \} \\ &> 1 - \varepsilon + \sigma(1 + \eta)(1 - 2\varepsilon') . \end{aligned}$$

Now (4.47) can be rewritten as

$$\sigma(1 + \eta)(1 - \varepsilon') = \eta + \varepsilon$$

so that finally

$$\nu_2 \geqslant \nu_1 \equiv \nu > 1 + \eta - \varepsilon'(\eta + \varepsilon)/(1 - \varepsilon') \geqslant 1 + \eta/2 \quad (4.51)$$

for ε' sufficiently small, namely $\varepsilon'/(1 - \varepsilon') \leqslant \eta/[2(\eta + \varepsilon)]$. Substituting (4.44), (4.45), (4.48), (4.51) into (4.17) and estimating the free term $\Phi_A^{(0)}$

as indicated in the previous discussion and in Lemma 4.2, we obtain

$$\| \Phi_A(t) \|_2 \leqslant C(1 + \mathrm{Log}_+ t)\left(1 + \int_0^t d\tau (1 + \mathrm{Log}_+ \tau)^\zeta (1+\tau)^{-(1+\eta/2)} \| \Phi_A(\tau) \|_2 \right) \tag{4.52}$$

with $\zeta = \mathrm{Max}_i \, (p_i - 1)\sigma_i$, so that by Gronwall's inequality

$$\| \Phi_A(t) \|_2 \leqslant C(1 + \mathrm{Log}_+ | t |). \tag{4.53}$$

The estimate (4.36) now follows from (4.12) and (4.53). In order to prove (4.4) in the present case, we remark that by applying the same argument and estimates to the equation (4.19) as just given for the equation (4.17), and using in particular the property (4.20) of the free term $\varphi^{(0)}$, we can prove that

$$\| \varphi(t) \|_2 \leqslant C(1 + \mathrm{Log}_+ | t |). \tag{4.54}$$

The estimate (4.4) then follows from (4.53), (4.54) and Proposition 3.4 (see (3.42)). Q. E. D.

In space dimensions $n \geqslant 3$, the situation is more complicated, and we are not able to extend the results of Proposition 4.1 to the whole range $p > p_2(n)$. Here we present only a preliminary result for $n = 3$, which can be obtained by the same type of estimates as used previously. Better but still incomplete results can be obtained by more refined estimates involving homogeneous Besov spaces. Since the result presented here is very preliminary, we restrict our attention to the case of a single power interaction of the form (1.2).

PROPOSITION 4.3. — Let $n = 3$. Let $f(\varphi) = \lambda\varphi | \varphi |^{p-1}$ with $\lambda > 0$ and $3 \geqslant p > (11 + \sqrt{137})/8 \simeq 2.838$. Let $(\varphi_0, \psi_0) \in \Sigma$ and let $(\varphi, \dot\varphi)$ be the solution of the equation (1.1) in $\mathscr{C}(\mathbb{R}, \Sigma)$ with initial data (φ_0, ψ_0) at time zero, as described in Proposition 2.3. Then $Q_0(t, \varphi, \dot\varphi)$ is bounded uniformly in time and φ satisfies the estimate (4.2).

Proof. — The proof is very similar to that of Proposition 4.2 and consists of the same two steps. We shall therefore present it more briefly. Although it applies only to $n = 3$, we shall keep n general until the end of the argument, for clarity.

First step. — We substitute the available decay (4.8) into the equation (4.19) to obtain additional decay for φ. By Proposition 3.3, $\varphi^{(0)}$ is estimated as

$$\| \varphi^{(0)}(t) \|_l \leqslant C(1 + | t |)^{-\gamma(l)} \tag{4.55}$$

for $2 \leqslant l \leqslant 2^*$. We estimate the integral in (4.19) by (3.15) with $s = (p+1)/p$ and l defined by (4.38), so that

$$\begin{aligned} \| K(t - \tau) f(\varphi) \|_l &\leqslant C \, | t - \tau |^{-\gamma(p+1)} \| \varphi \|_{p+1}^p \\ &\leqslant C \, | t - \tau |^{-\gamma(p+1)} (1 + \tau)^{p(1 - 2\delta(p+1))} \end{aligned} \tag{4.56}$$

where the last inequality follows from (4.8). Since $p > p_2(3)$, the last factor in (4.56) is integrable, and we obtain from (4.19), (4.55) and (4.56)

$$\| \varphi(t) \|_l \leqslant C(1 + |t|)^{-\gamma(p+1)}. \tag{4.57}$$

Note that this result holds for any $n \geqslant 3$ under the condition $p_2(n) < p \leqslant 1 + 4/(n-1)$.

Second step. — We insert the decay just obtained into the equations (4.17) and (4.18) to estimate Φ_A. As above, we consider only (4.17). Using (3.15) with $l = 2$, $s = 2n/(n+2)$, we estimate

$$\begin{aligned} \| K(t-\tau)\Phi_A f'(\varphi) \|_2 &\leqslant C \| \Phi_A f'(\varphi) \|_{2n/(n+2)} \\ &\leqslant C \| \Phi_A \|_2 \| \varphi \|_{(p-1)n}^{p-1}. \end{aligned} \tag{4.58}$$

We estimate the last norm by interpolation as

$$\| \varphi \|_{(p-1)n} \leqslant \| \varphi \|_{2^*}^{\sigma} \| \varphi \|_l^{1-\sigma} \tag{4.59}$$

which is allowed provided $p - 1 \leqslant 2/(n-2)$, namely $p \leqslant 3$ for $n = 3$ (at this point, the method breaks down for $n \geqslant 4$). We estimate the last two norms by (4.10) with $l = 2^*$ and by (4.57), thereby obtaining

$$\| \varphi \|_{(p-1)n} \leqslant C(1 + \tau)^{-\nu((p-1)n)} \tag{4.60}$$

with $\nu(.)$ defined by

$$\nu(m) = \{ \gamma(p+1)(1-\delta(m)+(1-\mu)(1-1/n)(\delta(m)-1+\alpha(p+1)) \} \alpha(p+1)^{-1}. \tag{4.61}$$

Uniform boundedness of Φ_A in L^2 then follows by Gronwall's inequality from the uniform boundedness of $\Phi_A^{(0)}$ (see Lemma 4.2 and the preceding discussion), from (4.17), (4.58) and (4.60), provided

$$(p-1)\nu((p-1)n) > 1. \tag{4.62}$$

We now remark that

$$(p-1)(n-1)(1-\delta((p-1)n)) - 1 = -(n-2)(1-\mu)$$

so that (4.62) can be simplified by dividing by $(1-\mu)$ to yield

$$(p-1)(n-1)\{ \delta((p-1)n) - 1 + \alpha(p+1) \} > (n-2)\delta(p+1)$$

or equivalently

$$(n-1)^2(p-1)^2 + ((n-3)^2 - 3)(p-1) - 4(n-1) > 0 \tag{4.63}$$

after an elementary computation. For $n = 3$, (4.63) reduces to

$$4(p-1)^2 - 3(p-1) - 8 > 0$$

which (for $p \geqslant 1$) is equivalent to the assumed lower bound on p. Proposition 4.3 then follows from the uniform boundedness of Φ_A in L^2, from (4.12) and from Proposition 3.3. Q. E. D.

APPENDIX

LEMMA A.1. — (1) For any n, $\mathscr{D}_0$ is dense in Σ_0.
(2) For $n \geqslant 2$, $\mathscr{D}_0$ is dense in Σ_1.

Proof. — (1) For $\varphi \in \Sigma_0$, from

$$\langle \varphi, \psi \rangle + \langle r\varphi, r\psi \rangle = 0$$

for any $\psi \in \mathscr{D}_0$, it follows that $(1 + r^2)\varphi$ has support in $\{ 0 \}$ which is incompatible with $\varphi \in L^2$ unless $\varphi = 0$.

(2) By cutting and regularizing one shows directly by standard methods that $\mathscr{C}_0^\infty(\mathbb{R}^n)$ is dense in Σ_1 for any n. On the other hand $\mathscr{D}_0$ is dense in H^1 for $n \geqslant 2$. In fact, for $\varphi \in H^1$, from

$$\langle \varphi, \psi \rangle + \langle \nabla\varphi, \nabla\psi \rangle = 0$$

for all $\psi \in \mathscr{D}_0$, it follows that $\varphi - \Delta\varphi$ has support in $\{ 0 \}$ which implies $\varphi = 0$, as is easily seen in Fourier transform.

Let now χ be a fixed real function in $\mathscr{C}^\infty(\mathbb{R}^n)$ with compact support and equal to 1 in a neighborhood of the origin. Let $\varphi \in \Sigma_1$, let $\psi_1 \in \mathscr{D}_0$ and let $\psi_2 \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. Then $\chi\psi_1 + (1 - \chi)\psi_2$ belongs to $\mathscr{D}_0$ and

$$\| \varphi - (\chi\psi_1 + (1 - \chi)\psi_2) ; \Sigma_1 \| \leqslant \| \chi(\varphi - \psi_1) ; \Sigma_1 \| + \| (1 - \chi)(\varphi - \psi_2) ; \Sigma_1 \| . \quad \text{(A.1)}$$

The norms in the right-hand side of (A.1) can be estimated as follows:

$$\| \chi(\varphi - \psi_1) ; \Sigma_1 \| \leqslant C_0 \| \chi(\varphi - \psi_1) ; H^1 \| \leqslant C_1 \| \varphi - \psi_1 ; H^1 \|$$

and

$$\| (1 - \chi)(\varphi - \psi_2) ; \Sigma_1 \| \leqslant C_1 \| \varphi - \psi_2 ; \Sigma_1 \|$$

where the constant C_1 depends only on χ. Letting $\psi_1 \to \varphi$ in H^1 and $\psi_2 \to \varphi$ in Σ_1 yields $\chi\psi_1 + (1 - \chi)\psi_2 \to \varphi$ in Σ_1. Q. E. D.

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