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A phase cell approach to Yang-Mills theory. — VI. Non-Abelian lattice-continuum duality

by

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ABSTRACT. — A one to one correspondence is exhibited between continuum Yang-Mills fields configurations (that are sufficiently smooth) and lattice configurations (of group elements associated to lattice bonds) on an infinite sequence of lattices whose configurations are related by block spin transformations. The lattice configurations and lattice Wilson actions approach the continuum fields and continuum action as the lattices get finer and finer (under suitable smoothness and fall off conditions on the continuum field).

RÉSUMÉ. — On exhibe une correspondance biunivoque entre les configurations de champs de Yang Mills (suffisamment régulières) dans le continu et des configurations sur réseau (d'éléments du groupe interne associés aux liens du réseau), sur une suite infinie de réseaux dont les configurations sont reliées par des transformations de blocs de spins. Les configurations sur réseau et les actions de Wilson sur réseau approchent les champs et l'action dans le continu lorsque le réseau devient de plus en plus fin (sous des hypothèses convenables de régularité et de décroissance des champs dans le continu).

We consider an infinite sequence of finer and finer lattices. To each (oriented) bond of each lattice there is assigned a group element, lying in

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a compact simple Lie group, G. There is a compatibility requirement to these assignments; the assignments to any one of the lattices are determined in terms of the assignments to any finer lattice, by an averaging procedure due to Balaban (in small field regions, with a suitable continuous extension to large field regions). Given a continuously differentiable gauge potential, $A_\mu(x)$ (a vector field with values in the Lie Algebra of G), one can define compatible assignments to the lattices, as above, such that, in a suitable sense the lattice « fields » approach $A_\mu(x)$ as one passes to finer and finer lattices. The lattice actions likewise approach the continuum action $\frac{1}{2}\int(dA + A \wedge A)^2$ (at least if $A_\mu(x)$ and its first partials fall off rapidly enough at infinity).

We denote the configuration (group assignments) on $\mathscr{L}^r$, the *rth* lattice of edge size $\ell_r = 1/2^r$, by $\mathscr{F}^r$. The block spin transformation BS_r carries $\mathscr{F}^r$ into a configuration $\mathscr{F}^{r-1}$ on $\mathscr{L}^{r-1}$. We let $\hat{\mathscr{F}}^r$ be the set of all configurations, $\{ \mathscr{F}^r \}$.

$$BS_r : \hat{\mathscr{F}}^r \to \hat{\mathscr{F}}^{r-1}. \tag{1}$$

A set of compatible assignments to the lattices, as considered above, may be viewed as a point in the *inverse limit* [*1*] of the system of spaces and mappings in (1). The lattice-continuum duality, of the first paragraph, is, more abstractly, an injection of the space of continuously differentiable gauge potentials into the inverse limit of the system (1).

THE BLOCK SPIN TRANSFORMATION

We follow the discussion in I (we use Roman numerals to refer to papers in this series). We let e be an edge in $\mathscr{L}^r$; and the two blocks, associated to the vertices of e in $\mathscr{L}^{r+1}$, be as pictured in Figure 2 of I. There one has defined 2^4 paths Γ_x in $\mathscr{L}^{r+1}$. To any path in a lattice, Γ, a group element is defined, g_Γ.

$$g_\Gamma = g(e_1)\cdot g(e_2)\cdot\cdot g(e_n) \tag{2}$$

where Γ is composed of the compatibly oriented edges, $e_1, \ldots, e_n$ in the indicated order; and $g(e_i)$, of course, is the group element associated to e_i. $g(e)$ will be a function of the g_{Γ_x}. Specifying this function determines the block spin transformation.

For the purposes of this paper we need only the following properties of this function (defining the group element $g(e)$ as above). (The detailed specification of the function is given in III. We should actually work with edge size $1/N^4$ instead of $1/2^4$; this would make trivial changes in the present paper.)

1) The function is continuous.

2) There is an $\varepsilon > 0$, such that if

$$| g_{\Gamma_x} | < \varepsilon, \quad \text{all} \quad 2^4\, x \tag{3}$$

then $g(e)$ is defined by minimizing

$$\sum_x d^2(g(e), g_{\Gamma_x}). \tag{4}$$

Here we have let $d(g_1, g_2)$ be the invariant distance on G arising from an invariant metric. If $u(g) = e^A$ with A « small » we may set

$$d^2(\mathrm{I}d, g) = -\operatorname{Tr} A^2 \tag{5}$$

for a unitary representations $u(g)$. (We will often write $g = e^A$, meaning $u(g) = e^A$.) We also use the notation

$$| g | = d(\mathrm{I}d, g). \tag{6}$$

All this notation follows III.

The function as specified for « small » g in (2) above, yields « Balaban averaging. » (This definition actually differs slightly from that used originally by Balaban.) We use the same function to define all such transformations, and thus all the BS_r.

RESULTS

Given an edge, e, and a continuum potential, $A_\mu(x)$, we let $g_0(e, A_\mu)$ be the group element obtained by exponentiating

$$\int_e \vec{A} \cdot \vec{ds} \tag{7}$$

where the line integral is along a straight line joining the vertices of e in the proper orientation.

DEFINITION. — Let $A_\mu(x)$ be a continuously differentiable gauge potential. We say a compatible set of lattice assignments $\{ g(e_\alpha) \}$ is *associated* to $A_\mu(x)$ if there is a $\sigma(r) > 0$ satisfying

$$\lim_{r \to \infty} \sigma(r) = 0 \tag{8}$$

and a continuous function on R^4, $m(x) > 0$, such that for all e_α

$$d(g(e_\alpha), g_0(e_\alpha, A_\mu)) < \frac{1}{2^{r(\alpha)}} \sigma(r(\alpha)) m(c(e_\alpha)) \tag{9}$$

where e_α is in $\mathscr{L}^{r(\alpha)}$ and $c(e)$ is the center of edge e.

THEOREM 1. — There is a unique set of compatible lattice assignments associated to each continuously differentiable $A_\mu(x)$.

We assume the lattice action on $\mathscr{L}^r$ is given by

$$S_0^r = \frac{1}{4} \sum_p | g_{\partial p} |^2 \tag{10}$$

where the sum is over plaquettes in $\mathscr{L}^r$. (We actually need this to be the expression for the action only when all $| g_{\partial p} |$ are sufficiently small.)

THEOREM 2. — Assume $A_\mu(x)$ is continuously differentiable, and that $A_\mu(x)$ and its first partials fall off at infinity faster than $1/x^{2+\varepsilon}$ (i. e. $| | x |^{2+\varepsilon} A_\mu(x) | < c$ and likewise for $DA_\mu(x)$). Then, for the compatible set of lattice assignments associated to $A_\mu(x)$, the corresponding lattice actions, S_0^r, converge to the continuum action as $r \to \infty$.

We do not know if the bounds on $A_\mu(x)$ and $DA_\mu(x)$ are necessary; they were not for the Abelian case. We do not pursue this here.

We note that the construction of a compatible assignment to the lattices, for a given $A_\mu(x)$, of I does satisfy (9) above. The construction of I is model to our present work.

CONSTRUCTION OF THE LATTICE ASSIGNMENTS

We follow I. For a fixed r_0, we construct assignments on $\mathscr{L}^{r_0}$ by setting

$$g(e, r_0) = g_0(e, A_\mu) \tag{11}$$

for e in $\mathscr{L}^{r_0}$. The $g(e, r_0)$ for all e with level $e < r_0$ are then determined by the block spin transformations (yielding assignments on $\mathscr{L}^r$ given assignments on $\mathscr{L}^{r+1}$). We then define the lattice assignments associated to $A_\mu(x)$ by

$$g(e) = \lim_{r_0 \to \infty} g(e, r_0) \tag{12}$$

We must of course prove this limit exists. In fact, in I we used a different prescription than (11). Either prescription may be used, here and there; one gets the same limit.

ANALYTICITY AND ESTIMATES

We consider an edge e in $\mathscr{L}^r$ and the edges $\{ e_\alpha \}$ in $\mathscr{L}^{r+1}$ involved in the block spin transformation determination of $g(e)$; i. e. $g(e)$ is a function of the 2^4 g_{Γ_x} as mentioned before, and Γ_x are built up of the edges $\{ e_\alpha \}$.

We consider the situation when the $g(e_\alpha)$ are all close to the identity. We write

$$g(e) = e^{A(e)}$$
$$g(e_\alpha) = e^{A(e_\alpha)} \tag{13}$$

with the $A(e_\alpha)$, $A(e)$ small. If G has rank r, we view each $A(e_\alpha)$ as a homogeneous linear function of r complex variables (real for actual values), and likewise $A(e)$. Then the transformation from the $g(e_\alpha)$ to $g(e)$ may be viewed as given by r complex analytic functions (for the $A(e_\alpha)$ small enough). If there are M elements in $\{ e_\alpha \}$, each of these r functions is an infinite polynomial in Mr variables. Thus the block spin transformation, as it yields $g(e)$, is given as

$$w_i = f_i(z_1, \ldots, z_{Mr}) \qquad i = 1, \ldots, r \tag{14}$$

with

$$A(e) = \sum_i w_i E^i \tag{15}$$

$$A(e_\alpha) = \sum_j z_{i_\alpha(j)} E^j . \tag{16}$$

E^i generators of the Lie Algebra. The f_i are analytic for

$$| z_i | < \varepsilon' . \tag{17}$$

The functions f_i are the same for all such transformations. We also know that the f_i have no constant terms, and that the linear terms are the same as in the Abelian theory treated in I. We write

$$A(e) = F(\{ A(e_\alpha) \}) = F^L(\{ A(e_\alpha) \}) + F'(\{ A(e_\alpha) \}) \tag{18}$$

where F^L is exactly the linear portion of the transformation, made up of the linear terms in (14).

This analytic structure above will be important to us in other situations. Here we only need the following estimates that follow from this.

We write

$$\| A(e_\alpha) \| = \operatorname*{Sup}_\alpha | A(e_\alpha) | \tag{19}$$

with

$$| A |^2 = - \mathrm{Tr}\,(A^2) \tag{20}$$

Then we have

$$| F' | \leq c \, \| A(e_\alpha) \|^2 \tag{21}$$

$$| DF' | \leq c \, \| A(e_\alpha) \|, \; | DDF' | \leq c \tag{22}$$

both for

$$\| A(e_\alpha) \| < \varepsilon'' . \tag{23}$$

We have written D for a derivative of F' with respect to any z_i on which it depends.

FINAL INDUCTIVE BOUNDS

The estimates of this section are straightforward, easier to work out oneself than to read about; we will be brief. The basic idea is to use the fact that the linear part of the transformations (F^L of (18)) has desired convergence properties. This was non-trivial, but shown in I, whose bounds we will use; the non-trivial aspects of the present proof all are from I. We must show that the non-linear terms, from F′ of (18), lead to small controllable effects.

A basic reduction we use is the observation that we may work only with lattices $\mathscr{L}^r$, for $r > \bar{r}$, for any fixed $\bar{r}$, and the results for all the lattices follow, if they hold for $r > \bar{r}$. Using bounds on the $A_\mu(x)$ we will be able to select an $\bar{r}$ such that for $r > \bar{r}$ the BS_r all operate only in a region where (21) and (22) hold.

We now take a given configuration at level r, and consider the configuration at level s ($s < r$) as determined by the block spin transformations. (In all that follows we work in a region where (21) and (22) hold. We must constantly check that all operations keep us in this region.) We write the schematic equation

$$A(s) = L^{s,r}A(r) + L^{s,r-1}N^{r-1}(A(r)) + L^{s,r-2}N^{r-2}(A(r-1)) + \dots + N^s(A(s+1)). \quad (24)$$

$A(r')$ stands for the $A(e_i)$, e_i at level r'. L^{r_1,r_2} is the linear transformation from $A(r_2)$ to $A(r_1)$ as given by the linear terms, the F^L. $N^r(A(r+1))$ are the nonlinear terms developed in BS_{r+1}, a portion of $A(r)$ as a non-linear function of the $A(r+1)$. We write $|A(r)|$ for the $\operatorname{Sup}_{e_i \in \mathscr{L}^r} |A(e_i)|$ and obtain from (24), path averaging considerations of I, and (21)

$$|A(s)| \leq c2^{r-s}|A(r)| + c2^{r-s-1}|A(r)|^2 + c2^{r-s-2}|A(r-1)|^2 + \dots + c|A(s+1)|^2. \quad (25)$$

We set $|A(r)| = 2^{-r}a$. Then if s_0 is large enough, depending only on a and c (of (25)), we find

$$|A(r')| \leq 2c2^{-r'}a \quad (26)$$

for $s_0 < r' < r$, by inductive use of (25). In particular one needs

$$2 \geq [1 + c^2 a 2^{-s_0+1}]. \quad (27)$$

From this we deduce

$$|A(e, r_0)| \leq c' 2^{-r(e)} \quad (28)$$

for $r(e)$ greater than some $\bar{r}$, and all r_0. We have here also used the uniform bounds on $|A_\mu(x)|$. (28) plays the role of (2.6) in I.

We need now study the convergence of the $A(e, r_0)$ as $r_0 \to \infty$, at the same time getting the uniqueness aspect of Theorem 1. We consider two sets of assignments at level r, $A_1(r)$ and $A_2(r)$. We assume

$$|A_i(r)| \leq 2^{-r}a \qquad i = 1, 2 \tag{29}$$

and (with $b < 1$)

$$|\delta A(r)| \leq 2^{-r}ba \tag{30}$$

where

$$|\delta A(r)| = \operatorname*{Sup}_{e_\alpha \in \mathscr{L}^r} |A_1(e_\alpha) - A_2(e_\alpha)|.$$

We then inductively use (25), (21), and (22) to find

$$|\delta A(r')| \leq 2c2^{-r'}ba \tag{31}$$

for $\bar{s}_0 < r' < r$. ($\bar{s}_0$ depends only on a and c.) This result (along with the uniform bounds on $|DA_\mu(x)|$ and $|A_\mu(x)|$) yields convergence of the $A(e, r_0)$ and the uniqueness property of Theorem 1.

We will not detail the proof of Theorem 2. The careful reader may be puzzled at the truth of Theorem 2, when the estimate in (9) is not strong enough to ensure that two fields $\{ g(e_\alpha) \}$ and $(g_0(e_\alpha, A_\mu) \}$ for e_α at level $r(\alpha)$ have actions $S_0^{r(\alpha)}$ close in value. But we need by virtue of the uniqueness in Theorem 1 study only the limit

$$\lim_{r \to \infty} \lim_{s \to \infty} S_0^r(\{ A(., s) \}) \tag{32}$$

and the $A(e, r_0)$ converge with better bounds as $r_0 \to \infty$.

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