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Partial *-algebras of closed operators and their commutants
II. Commutants and bicommutants

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ABSTRACT. — In this second paper on partial Op*-algebras, we present a systematic analysis of commutants and bicommutants, both from the algebraic and the topological point of views, along the lines of the usual theory of W*- and Op*-algebras. In particular we obtain conditions for the validity of the following statements: given a family $\mathfrak{N}$ of unbounded operators, its commutant is a partial Op*-algebra, and/or $\mathfrak{N}$ is dense in its bicommutant for an appropriate topology. We introduce the class of symmetric partial Op*-algebras, which verify those conditions. Finally we compare the commutants of a partial Op*-algebra with those of its canonical extensions to larger domains.

RÉSUMÉ. — Ce second article sur les Op*-algèbres partielles est consacré à une analyse systématique, tant algébrique que topologique, des commutants et bicommutants, dans la ligne de la théorie usuelle des W*- et des Op*-algèbres. On obtient notamment des conditions garantissant la validité des énoncés suivants : étant donné une famille $\mathfrak{N}$ d'opérateurs non bornés, son commutant est une Op*-algèbre partielle, et/ou $\mathfrak{N}$ est dense dans son bicommutant pour une topologie appropriée. On introduit la classe des Op*-algèbres partielles symétriques, qui vérifient lesdites conditions. Enfin, on compare les commutants d'une Op*-algèbre partielle avec ceux de ses extensions canoniques à des domaines plus grands.

1. INTRODUCTION

In the familiar theory of bounded operator algebras, i. e. W* or C*-algebras [1], the notion of commutant plays an essential role. It enters in the very definition of factors and irreducible algebras or representations, it is the basic tool in the decomposition (desintegration) of a given algebra into simpler constituents, factors or irreducible algebras. More ambitiously, it is a cornerstone of the Tomita-Takesaki theory for von Neumann algebras.

Quite naturally then the notion of commutant was extended to unbounded operator algebras, notably Op*-algebras. First Borchers and Yngvason [2] considered *bounded* commutants, of two different types, called respectively strong and weak (the latter goes back to Ruelle's work in axiomatic Quantum Field Theory [3], see also [4]). Next *unbounded* commutants, again strong and weak, were introduced and analyzed by several authors: Gudder and Scruggs [5], Inoue [6], Epifanio and Trapani [7], Mathot [8].

In Part II of this paper we want to extend that analysis to *partial* Op*-algebras, as discussed in detail in Part I [9]. Now, besides the weak and the strong unbounded commutants, two new types appear naturally, i. e. the commutants corresponding to the two kinds of partial multiplications, . and $\square$. We will call these objects *natural commutants.* In addition, one may restrict one's attention to bounded operators and thus one gets four different types of bounded commutants, including the two defined originally by Borchers and Yngvason. Following the standard scenario, the next step is to define *bicommutants*: clearly we get many different types.

Our aim is to make a systematic analysis of all these types of commutants and bicommutants both at the algebraic and at the topological level. The first few steps have been made by Karwowski and one of us [10] but those results require some qualifications (see [10, Add.]). More recently the beginning of a representation theory has been set up for partial Op*-algebras, in collaboration with Lassner (see [11]). There it turns out that, as far as the characterization of irreducibility is concerned, the appropriate object seems to be the bounded natural weak commutant, but the analysis is still preliminary.

The paper is organized as follows. In Section 2 we define the various types of commutants, bounded and unbounded, and derive a few elementary algebraic properties. A natural question is whether the commutant of a set of operators is itself a partial *-algebra. We examine this problem in Section 3, in the case of the weak unbounded commutant [7]. Section 4 is devoted to topological properties. For a *-invariant family $\mathfrak{B}$ of unbounded operators, the basic theorem of von Neumann asserts that the (usual) bicommutant $\mathfrak{B}''$ coincides with the closure of $\mathfrak{B}$ in various topologies, such as the strong or the weak one. What is the corresponding situation for Op*-algebras? For answering that question, we follow closely the strategy of [8]: first consider a set of bounded operators, then a set, or a partial algebra, of unbounded operators that contains a sufficiently large (dense) supply of bounded operators. In Section 5, we extend to partial Op*-algebras the familiar concept of *symmetric* *-algebras. Of course here again several definitions are possible, we compare them and study the properties of commutants and bicommutants of such sets. Finally, Section 6 is a systematic comparison between the various commutants of a given partial Op*-algebra $\mathfrak{M}$ and those of its canonical extensions $\overline{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$ to larger domains, as defined in I. Some additional remarks are summarized in two Appendices.

At this stage, we should mention some related works on unbounded commutants. Araki and Jurzak [*12*] have introduced a special type, which proves useful under some countability assumptions. The object so defined is, in fact, more in the spirit of the theory of operators on partial inner product spaces [*13*]. An analysis of commutants in the latter context has been initiated by Shabani [*14*], but the problem is far from exhausted. In the same vein, Voronin et al. [*15*] and Schmüdgen [*16*] have exploited systematically the notion of intertwining operators. Finally a recent paper by Inoue et al. [*17*] pursues the parallel between Op*-algebras and von Neumann algebras, initiated in [7] and [*8*]. All those works are closely related to the present one, but do not overlap with it.

Obviously one of the main applications of a commutant theory is the study of *abelian* partial algebras. Work on this topic is in progress with W. Karwowski and will reported elsewhere. We thank this author as well as J. Shabani and G. Epifanio, for fruitful discussions.

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2. DEFINITIONS AND ALGEBRAIC PROPERTIES

Let $\mathfrak{N}$ be a $+$-invariant subset of $\mathfrak{C}(\mathscr{D})$. We may consider four different types of unbounded commutants of $\mathfrak{N}$. First we have the *weak unbounded commutant* $\mathfrak{N}'_\sigma$, originally introduced in [7] [8] in the framework of $\mathscr{C}(\mathscr{D}, \mathscr{H})$ (strictly speaking the object defined here is the quotient of the latter by the familiar equivalence relation: $A_1 \sim A_2$ iff $A_1 \upharpoonright \mathscr{D} = A_2 \upharpoonright \mathscr{D}$, as discussed in I, Sec. 3):

$$\mathfrak{N}'_\sigma = \{ X \in \mathfrak{C}(\mathscr{D}) \mid \langle X^* f, Ag \rangle = \langle A^* f, Xg \rangle, \forall f, g \in \mathscr{D}, \forall A \in \mathfrak{N} \} \quad (2.1)$$

Next we have the two commutants corresponding to the two multiplications, $\square$ and $.$, defined on $\mathfrak{C}(\mathcal{D})$: *the weak natural commutant* $\mathfrak{N}'_{\square}$, or commutant in $\mathfrak{C}^w(\mathcal{D})$:

$$\mathfrak{N}'_{\square} = \{ X \in \mathfrak{C}(\mathcal{D}) \mid X \in L^w\mathfrak{N} \cap R^w\mathfrak{N} \text{ and } X \square A = A \square X, \forall A \in \mathfrak{N} \} \quad (2.2)$$

and the *strong natural commutant* $\mathfrak{N}'_.$, in $\mathfrak{C}^s(\mathcal{D})$:

$$\mathfrak{N}'_. = \{ X \in \mathfrak{C}(\mathcal{D}) \mid X \in L^s\mathfrak{N} \cap R^s\mathfrak{N} \text{ and } X.A = A.X, \forall A \in \mathfrak{N} \} \quad (2.3)$$

Both were essentially introduced in [*10*]: the latter was called $\mathfrak{N}'$, whereas $\mathfrak{N}'_{\square}$ is the pull-back to $\mathfrak{C}(\mathcal{D})$ of the *-commutant $\mathfrak{N}'_* \subset \mathfrak{C}^*(\mathcal{D})$. Finally, there is the *strong unbounded commutant*:

$$\mathfrak{N}'_c = \mathfrak{N}'_\sigma \cap \overline{\mathcal{L}^+(\mathcal{D})}. \quad (2.4)$$

The first three of these commutants are $+$-invariant subsets of $\mathfrak{C}(\mathcal{D})$, whereas $\mathfrak{N}'_c$ is an Op*-algebra.

The relations among the four commutants result from the following easy proposition.

PROPOSITION 2.1. — Let $\mathfrak{N} = \mathfrak{N}^+ \subset \mathfrak{C}(\mathcal{D})$. Then one has:

$$\mathfrak{N}'_{\square} = \{ X \in \mathfrak{N}'_\sigma \mid X, X^+ : \mathcal{D} \to \mathcal{D}_*(\mathfrak{N}) \} \quad (2.5)$$

$$\mathfrak{N}'_c = \mathfrak{N}'_{\square} \cap \overline{\mathcal{L}^+(\mathcal{D})}. \quad (2.6) \ \blacksquare$$

COROLLARY 2.2. — Given $\mathfrak{N} = \mathfrak{N}^+ \subset \mathfrak{C}(\mathcal{D})$, the following inclusions hold:

$$\begin{matrix} \mathfrak{N}'_c \subset \\ \mathfrak{N}'_. \subset \end{matrix} \mathfrak{N}'_{\square} \subset \mathfrak{N}'_\sigma. \quad (2.7)$$

If $\mathfrak{N}$ is fully closed, i. e. $\mathcal{D} = \mathcal{D}(\mathfrak{N})$, one gets $\mathfrak{N}'_c = \mathfrak{N}'_.$. (2.8)
If $\mathfrak{N} \subset \mathcal{B}(\mathcal{H})$, then $\mathfrak{N}'_{\square} = \mathfrak{N}'_\sigma$. (2.9)
If $\mathfrak{N} \subset \overline{\mathcal{L}^+(\mathcal{D})}$, the relation (2.7) becomes

$$\mathfrak{N}'_c \subset \mathfrak{N}'_. \subset \mathfrak{N}'_{\square} = \mathfrak{N}'_\sigma. \quad (2.10)$$

Finally, if $\mathfrak{N} \subset \mathcal{B}(\mathcal{H}) \cap \overline{L^+(\mathcal{D})}$, one gets

$$\mathfrak{N}'_c \subset \mathfrak{N}'_. = \mathfrak{N}'_{\square} = \mathfrak{N}'_\sigma. \quad (2.11) \ \blacksquare$$

The proof of all these assertions follows immediately from the definitions and will be omitted.

If $\mathfrak{B} = \mathfrak{B}^+ \subset \mathcal{B}(\mathcal{H})$, the natural commutants are easily described. The weak one, $\mathfrak{B}'_{\square}$, coincides with $\mathfrak{B}'_\sigma$ by (2.9). For the strong one, $\mathfrak{B}'_.$, we obtain another characterization, in the familiar language of von Neumann algebras. Let us consider the following set of operators:

$$\mathfrak{B}_\eta = \{ X \in \mathfrak{C}(\mathcal{A}) \mid X\eta\mathfrak{B}' \} \quad (2.12)$$

where $\mathfrak{B}'$ is the usual bounded commutant in $\mathscr{B}(\mathscr{H})$ and $X\eta\mathfrak{B}'$ means that X is affiliated with the von Neumann algebra $\mathfrak{B}'$ [*1*] [*18*]. Notice that the set $\mathfrak{B}_\eta$ is *-invariant, but *not* +-invariant in general. Its usefulness lies in the following properties.

PROPOSITION 2.3. — Let $\mathfrak{B}$ be a *-algebra of bounded operators. Then:

$$\mathfrak{B}_\eta = \{ X \in L^s\mathfrak{B} \cap R^w\mathfrak{B} \mid X.A = A \,\square\, X, \forall A \in \mathfrak{B} \} \tag{2.13}$$

and

$$\mathfrak{B}'_. = \mathfrak{B}_\eta \cap \mathfrak{B}_\eta^+ = \{ X \in \mathfrak{C}(\mathscr{D}) \mid X, X^+ \eta \mathfrak{B}' \}. \tag{2.14}$$

Proof. — Let $X \in \mathfrak{B}_\eta$. Then $X\eta\mathfrak{B}'$ implies [*1*] that every $A \in \mathfrak{B}$ (even $A \in \mathfrak{B}''$) leaves D(X) invariant and $AXf = XAf$ for any $f \in D(X)$. Therefore $X \in L^s\mathfrak{B}$. Also $X\eta\mathfrak{B}'$ implies $X^*\eta\mathfrak{B}'$, and thus every $A^* \in \mathfrak{B}$ leaves $D(X^*)$ invariant; hence $X \in R^w\mathfrak{B}$. Finally, $AXf = XAf, \forall f \in \mathscr{D}$, i. e. $A \,\square\, X = X.A$.

Conversely, let $X \in L^s\mathfrak{B} \cap R^w\mathfrak{B}$ and $X.A = A \,\square\, X$, $\forall A \in \mathfrak{B}$. Given $f \in D(X)$, there exists a sequence $\{ f_n \} \in \mathscr{D}$ such that $f_n \to f$, $Xf_n \to Xf$. Hence $XAf_n = AXf_n \to AXf$. Since X is closed, $Af \in D(X)$ and $XAf = AXf$. Let now $B \in \mathfrak{B}''$, i. e. B is the strong limit of a net $\{ B_\alpha \} \in \mathfrak{B}$. By the same argument, we get $Bf \in D(X)$ and $XBf = BXf$. Thus $X\eta\mathfrak{B}'$.

Finally the relation (2.14) is immediate. ■

The realization (2.13) of $\mathfrak{B}_\eta$ suggests a possible role for *mixed commutants*, that is, commutants that mix strong and weak products. A systematic study of these is given in Appendix A.

Let $\mathfrak{A}$ be an Op*-algebra. If $\mathfrak{A}$ is closed, hence fully closed, then, by Eq. (2.10) it has only two distinct unbounded commutants: the strong one, $\mathfrak{A}'_c = \mathfrak{A}'_.$, and the weak one, $\mathfrak{A}'_\square = \mathfrak{A}'_\sigma$. If $\mathfrak{A}$ is self-adjoint, all the four commutants coincide.

Since self-adjoint Op*-algebras behave notoriously better than other ones, it is tempting to find a corresponding property for partial Op*-algebras. *Standardness* has been proposed in [*10*] but it seems too strong, and more a property of individual operators rather than a property of the partial algebra as a whole. In the light of Proposition 2.1 and Corollary 2.2, we suggest instead the following notion. We will say that $\mathfrak{N}$, a +-invariant subset of $\mathfrak{C}(\mathscr{D})$, is *normal* if $\mathfrak{N}'_. = \mathfrak{N}'_\square$. This relation means roughly that, between $\mathfrak{N}$ and its commutant, strong and weak products play the same role. The analogy with standardness is obvious. But it is easy to characterize classes of normal subsets. For instance, the following result follows immediately from Corollary 2.2.

LEMMA 2.4. — Let $\mathfrak{O}$ be a +-invariant subset of $\overline{\mathscr{L}^+(\mathscr{D})}$. If $\mathfrak{O}$ is essentially self-adjoint, then it is normal and, moreover, $\mathfrak{O}'_. = \mathfrak{O}'_\square = \mathfrak{O}'_\sigma$. ■

We turn now to *bounded* commutants, denoting as usual the bounded part of a subset $\mathfrak{N} \subset \mathfrak{C}(\mathscr{D})$ by $\mathfrak{N}_b \equiv \mathfrak{N} \cap \mathscr{B}(\mathscr{H})$.

In particular, $\mathfrak{N}'_w \equiv \mathfrak{N}'_{\sigma b}$ and $\mathfrak{N}'_s \equiv \mathfrak{N}'_{cb}$ are the weak, resp. strong, bounded commutant familiar in the theory of algebras of unbounded operators [*2*] [*4*] [*19*]. Then we get immediately from Corollary 2.2:

COROLLARY 2.5. — Given $\mathfrak{N} = \mathfrak{N}^+ \subset \mathfrak{C}(\mathscr{D})$, its bounded commutants obey the following inclusions:

$$\mathfrak{N}'_s \subset \mathfrak{N}'_{\cdot b} \subset \mathfrak{N}'_{\square b} \subset \mathfrak{N}'_w . \qquad (2.15) \quad \blacksquare$$

Here again the situation simplifies if $\mathfrak{N}$ is an Op*-algebra, for then $\mathfrak{N}'_{\square b} = \mathfrak{N}'_w$; if it is closed, we get, in addition, $\mathfrak{N}'_s = \mathfrak{N}'_{\cdot b}$; if it is self-adjoint, all four commutants coincide. It is also worth noticing that the weak natural commutant $\mathfrak{N}'_{\square b}$ appears naturally in the definition of irreducible representations of partial Op*-algebras [*11*].

Next we define bicommutants. From now on we will pay little attention to strong commutants $\mathfrak{N}'_c$: these are Op*-algebras and hence then have been studied in detail in previous publications [*6*] [*8*]. Thus we are left with the three other ones, $\mathfrak{N}'_{\cdot} \subset \mathfrak{N}'_{\square} \subset \mathfrak{N}'_\sigma$ and correspondingly nine possible bicommutants $\mathfrak{N}''_{ij} \equiv (\mathfrak{N}'_i)'_j$, with $i, j = \cdot, \square$ or σ. Since all three notions of commutant reverse order, i. e. $\mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{M}'_i \subset \mathfrak{N}'_i$, the bicommutants obey obvious inclusion relations. We will be interested mostly in the three non-mixed ones: $\mathfrak{N}''_{\cdot\cdot}$, $\mathfrak{N}''_{\square\square}$, $\mathfrak{N}''_{\sigma\sigma}$. In general they are not included into each other, and all three contain $\mathfrak{N}$.

In the case of bounded operators, a *-invariant subset $\mathfrak{B} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra iff it coincides with its (usual) bicommutant, $\mathfrak{B} = \mathfrak{B}''$. The natural extension of this characterization to Op*-algebras is given by the concepts of *V*-algebras* and *SV*-algebras* [*7*]. Indeed V*-algebras play the central role in the algebraic description of complete sets of commuting observables in Quantum Mechanics [*20*].

Now we go one step further. Let $\mathfrak{M}$ be a partial Op*-algebra on $\mathscr{D}$. Then we say:

i) $\mathfrak{M}$ is a *partial V*-algebra* if $\mathfrak{M} = \mathfrak{M}''_{\sigma\sigma}$;
ii) $\mathfrak{M}$ is a *partial SV*-algebra* if $\mathfrak{M} = \mathfrak{M}''_{w\sigma}$.

More generally, a $+$-invariant subset $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$ is called a *V*-set* (resp. a *SV*-set*) if $\mathfrak{N} = \mathfrak{N}''_{\sigma\sigma}$ (resp. $\mathfrak{N} = \mathfrak{N}''_{w\sigma}$). These objects have been introduced in [*21*], in the context of integral decompositions of partial Op*-algebras.

The main question we want to adress in the next section is, under which conditions a given commutant of a subset $\mathfrak{N}$ is a partial Op*-algebra or even a partial (S)V*-algebra.

3. WHEN IS $\mathfrak{N}'_\sigma$ A PARTIAL Op*-ALGEBRA?

If $\mathfrak{B}$ is a *-invariant subset of $\mathscr{B}(\mathscr{H})$, it usual commutant $\mathfrak{B}'$ is a von Neumann algebra. Already for an Op*-algebra $\mathfrak{A}$, one has to distinguish:

the strong unbounded commutant $\mathfrak{A}'_c$ is a *-algebra, but not necessarily the weak one $\mathfrak{A}'_\sigma$. In the present context, of course, the natural question is: given a $+$-invariant subset $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$, which unbounded commutant $\mathfrak{N}'_i$ is a *partial* Op*-algebra, weak or strong, and for what kind of subset $\mathfrak{N}$?

Let $\mathfrak{N}'_i$ be the candidate. Given X, $Y \in \mathfrak{N}'_i$, we have to find under which conditions the product $X \boxdot Y$ ($\boxdot$ stands for . or $\square$), if it exists, belongs to $\mathfrak{N}'_i$ again. Since neither of the products is associative, the only statement that seems reasonable to prove is $X \boxdot Y \in \mathfrak{N}'_\sigma$. On the other hand, one has to express the fact that X and Y commute with the elements of $\mathfrak{N}$, and for this, the condition X, $Y \in \mathfrak{N}'_\sigma$ seems too weak (see the last equality in (3.1), (3.2) below). Thus we need also (at least) X, $Y \in \mathfrak{N}'_\square$, and therefore we assume at the outset $\mathfrak{N}'_\square = \mathfrak{N}'_\sigma$.

Under this condition, we try the . multiplication first. So, let X, $Y \in \mathfrak{N}'_\square = \mathfrak{N}'_\sigma$, such that $X \in L^s(Y)$. Then we compute, for any $A \in \mathfrak{N}$, $f, g \in \mathscr{D}$:

$$\langle (X.Y)f, A^+g \rangle = \langle XYf, A^+g \rangle = \langle Yf, X^*A^+g \rangle = \langle Yf, A^*X^+g \rangle \quad (3.1)$$

and, on the other hand:

$$\langle Af, (X.Y)^+g \rangle = \langle Af, Y^+X^+g \rangle = \langle Y^{+*}Af, X^+g \rangle = \langle A^{+*}Yf, X^+g \rangle . \quad (3.2)$$

Now, for $\mathfrak{N}'_\square = \mathfrak{N}'_\sigma$ to be stable under . means that (3.1) and (3.2) must be equal for every $A \in \mathfrak{N}$, $f, g \in \mathscr{D}$, and this means that we must have $Yf \in D((A^* \upharpoonright X^+\mathscr{D})^*)$ and $(A^* \upharpoonright X^+\mathscr{D})^*Yf = A^{+*}Yf$. Thus we are led to consider the following subset of $\mathscr{H}$ (actually a dense domain containing $\mathscr{D}$), for a fixed $Y \in \mathfrak{N}'_\square = \mathfrak{N}'_\sigma$:

$$\mathscr{K}^s_Y(\mathfrak{N}) = \bigcap_{A \in \mathfrak{N}} D((A^* \upharpoonright [\mathfrak{N}'_\sigma \cap L^s(Y)]^+\mathscr{D})^*) \quad (3.3)$$

Before stating a proposition, we repeat the argument for the $\square$ multiplication. However, for deriving the relations equivalent to (3.1), (3.2), we must impose the conditions $A^+g \in D(X^+)$, $Af \in D(Y)$. Thus we assume $\mathfrak{N}'_\square = \mathfrak{N}'_\sigma$ and take X, $Y \in \mathfrak{N}'_\square \cap L^w(Y)$.

Then for every $A \in \mathfrak{N}$, $f, g \in \mathscr{D}$, we get as above:

$$\langle (X \square Y)f, A^+g \rangle = \langle X^{+*}Yf, A^+g \rangle = \langle Yf, X^+A^+g \rangle = \langle Yf, A^*X^+g \rangle \quad (3.4)$$

$$\langle Af, (X \square Y)^+g \rangle = \langle Af, Y^*X^+g \rangle = \langle YAf, X^+g \rangle = \langle A^{+*}Yf, X^+g \rangle \quad (3.5)$$

and we are led to the same conclusion. Thus we define:

$$\mathscr{K}^w_Y(\mathfrak{N}) = \bigcap_{A \in \mathfrak{N}} D((A^* \upharpoonright [\mathfrak{N}'_\sigma \cap L^w(Y)]^+\mathscr{D})^*) \quad (3.6)$$

These subsets verify very simple inclusions.

LEMMA 3.1. — For every $Y \in \mathfrak{N}'_\sigma$, one has:

$$\mathscr{D}(\mathfrak{N}) \subset \mathscr{K}^{w}_{Y}(\mathfrak{N}) \subset \mathscr{K}^{s}_{Y}(\mathfrak{N}) \subset \mathscr{D}_*(\mathfrak{N}). \tag{3.7}$$

Proof. — Since $1 \in \mathfrak{N}'_\sigma \cap L^s(Y)$, one has

$$\mathscr{K}^{s}_{Y}(\mathfrak{N}) \subset \bigcap_{A \in \mathfrak{N}} D((A^* \upharpoonright \mathscr{D})^*) = \mathscr{D}_*(\mathfrak{N}).$$

On the other hand we have $A = A^{**} \subset (A^* \upharpoonright X^+\mathscr{D})^*$, which gives the first inclusion. The central one is obvious. ■

With the help of this lemma, we may now summarize the whole discussion above as follows.

PROPOSITION 3.2. — Let $\mathfrak{N}$ be a $+$-invariant subset of $\mathfrak{C}(\mathscr{D})$, such that $\mathfrak{N}'_\square = \mathfrak{N}'_\sigma$. Then:

i) $\mathfrak{N}'_\sigma$ is stable under the . multiplication iff $Yf \in \mathscr{K}^{s}_{Y}(\mathfrak{N})$ for every $Y \in \mathfrak{N}'_\sigma$.

ii) Assume, in addition, that $\mathfrak{N}'_\square = \mathfrak{N}'_\sigma \subset L^s\mathfrak{N}$; then $\mathfrak{N}'_\sigma$ is a weak partial V*-algebra iff $Yf \in \mathscr{K}^{w}_{Y}(\mathfrak{N})$ for every $Y \in \mathfrak{N}'_\sigma$. ■

To improve on Proposition 3.2, we must add some restriction on $\mathfrak{N}$. First we assume that it leaves $\mathscr{D}$ invariant. This guarantees that the two conditions $\mathfrak{N}'_\square = \mathfrak{N}'_\sigma$ and $\mathfrak{N}'_\square \subset L^s\mathfrak{N}$ are satisfied automatically. Putting together all particular cases, we get the following results, either from Corollary 2.2 or Lemma 2.4,or from Lemma 3.1 and Proposition 3.2.

PROPOSITION 3.3. — Given a $+$-invariant subset $\mathfrak{O} \subset \overline{\mathscr{L}^+(\mathscr{D})}$, consider the following conditions:

i) $\mathfrak{O} \subset \mathscr{B}(\mathscr{H})$;

ii) $\mathfrak{O}$ is essentially self-adjoint, i. e. $\mathscr{D}(\mathfrak{O}) = \mathscr{D}_*(\mathfrak{O})$;

iii) $\mathfrak{O}$ is normal, i. e. $\mathfrak{O}' = \mathfrak{O}'_\square$;

iv) $\mathfrak{O}' = \mathfrak{O}'_\square = \mathfrak{O}'_\sigma$;

v) $\mathfrak{O}'_\sigma$ is a weak partial V*-algebra and it is stable under the . multiplication.

Then the following implications hold:

$$i) \Rightarrow ii) \Rightarrow iii) \Leftrightarrow iv) \Rightarrow v).$$ ■

COROLLARY 3.4. — If $\mathfrak{O} = \mathfrak{O}^+ \subset \overline{\mathscr{L}^+(\mathscr{D})}$ is self-adjoint then, in addition, $\mathfrak{O}'_c = \mathfrak{O}' = \mathfrak{O}'_\square = \mathfrak{O}'_\sigma$ and $\mathfrak{O}''_{\square\square} = \mathfrak{O}''_{\sigma\sigma}$. ■

(The last result follows from the fact that $\mathfrak{O}'_c \subset \overline{\mathscr{L}^+(\mathscr{D})}$).

All these results apply in particular to Op*-algebras, for which many explicit examples are known, such as polynomial algebras [*22*] [*23*] or tensor algebras [24-26]. We will get more information about bicommutants in the next section.

If $\mathfrak{N}$ does *not* leave $\mathcal{D}$ invariant, we need stronger conditions to guarantee the stability of $\mathfrak{N}'_\sigma$ under . or $\square$. As it turns out, the crucial condition in Proposition 3.3 is *iv*). Indeed:

PROPOSITION 3.5. — Let $\mathfrak{N}$ be a $+$-invariant subset of $\mathfrak{C}(\mathcal{D})$. If one has $\mathfrak{N}' = \mathfrak{N}'_\square = \mathfrak{N}'_\sigma$, then this commutant is a weak partial V*-algebra and it is stable under the . multiplication.

Proof. — $Y \in \mathfrak{N}'$ implies $Y \in L^s\mathfrak{N}$ and $Yf \in \mathcal{D}(\mathfrak{N})$. Thus Proposition 3.2 applies. ■

4. TOPOLOGICAL PROPERTIES

We turn now to the topological properties of commutants and bicommutants. First, we ask for which topology each of them is closed in $\mathfrak{C}(\mathcal{D})$ or in a set of multipliers. For unbounded commutants the answer is summarized in Proposition 4.1 below, where we refer to the various topologies defined in I [9, Sec. 5]. For the convenience of the reader we recall here the most important topologies on the commutants $\mathfrak{N}'_i$ of a given $+$-invariant subset $\mathfrak{N}$ of $\mathfrak{C}(\mathcal{D})$:

. *the quasi-uniform topologies* $\tau_*(\mathfrak{N})$, $\tau_f(\mathfrak{N})$ on $\mathfrak{N}'_\square$, given by the seminorms:

$$\| B \|^{A,\mathcal{M}} = \sup_{f \in \mathcal{M}} \{ \| (B \square A)f \| + \| (A^+ \square B^+)f \| \} \tag{4.1}$$

where $A \in \mathfrak{N}$ and $\mathcal{M} \subset \mathcal{D}$ is a bounded subset in the case of $\tau_*(\mathfrak{N})$, a finite subset in the case of $\tau_f(\mathfrak{N})$.

. *the strong*-topology* s^* on $\mathfrak{N}'_\sigma$, with seminorms:

$$\| B \|_f^* = \| Bf \| + \| B^+ f \| \quad (f \in \mathcal{D}). \tag{4.2}$$

It is worth remembering that the quasi-uniform topologies $\tau_{*,f}(\mathfrak{N})$ are defined only on the space of weak multipliers $M^w\mathfrak{N} = L^w\mathfrak{N} \cap R^w\mathfrak{N}$, not on the whole of $\mathfrak{C}(\mathcal{D})$ as it is the case for s^* and all the weak topologies defined in [9].

We collect now the closure properties of the various unbounded commutants $\mathfrak{N}'_i$ of a given subset $\mathfrak{N} = \mathfrak{N}^+$ of $\mathfrak{C}(\mathcal{D})$. The proof of those results may be found in [8] or [10], or derived from the proof of [10, Proposition 5.7].

PROPOSITION 4.1. — Let $\mathfrak{N}$ be a $+$-invariant subset of $\mathfrak{C}(\mathcal{D})$. Then its unbounded commutants have the following properties:

i) The weak unbounded commutant $\mathfrak{N}'_\sigma$ is closed in the $\mathfrak{N}$-weak*-topology and *a fortiori* in the quasi-weak* and the strong*-topology; for the latter $\mathfrak{N}'_\sigma$ is complete;

ii) If $\mathfrak{N} \subset \overline{\mathscr{L}^+(\mathscr{D})}$, then $\mathfrak{N}'_\sigma$ is weakly closed;

iii) The weak natural commutant $\mathfrak{N}'_\square$ is complete for the quasi-uniform topologies $\tau_{*,f}(\mathfrak{N})$;

iv) The strong natural commutant $\mathfrak{N}'_{\cdot}$ need not be complete for $\tau_{*,f}(\mathfrak{N})$: one has only $\tilde{\mathfrak{N}}'_{\cdot} \subset \mathfrak{N}'_\square$. However $\mathfrak{N}'_{\cdot}$ is closed in $\mathrm{M}^s\mathfrak{N}$. ■

Notice also that the natural commutants $\mathfrak{N}'_\square$, $\mathfrak{N}'_{\cdot}$ need not be s^*-closed in $\mathfrak{C}(\mathscr{D})$. Indeed, if $\mathrm{X} = s^*\text{-lim } \mathrm{X}_\alpha$ with $\mathrm{X}_\alpha \in \mathfrak{N}'_\square$ or $\mathfrak{N}'_{\cdot}$, we can conclude that X belongs to $\mathfrak{N}'_\sigma$, but X need not be a two-sided multiplier of $\mathfrak{N}$.

The closure properties of the bicommutants follow trivially from Proposition 4.1:

PROPOSITION 4.2. — Let $\mathfrak{N} = \mathfrak{N}^+ \subset \mathfrak{C}(\mathscr{D})$ and $i = .$, $\square$ or σ. Then one has, for all i:

i) $\mathfrak{N}''_{i\sigma}$ is closed in $\mathfrak{C}(\mathscr{D})$ for the $\mathfrak{N}'_i$-weak*-topology, *a fortiori* it is qw^*-and s^*-closed;

ii) $\mathfrak{N}''_{i\square}$ is complete for the topologies $\tau_{*,f}(\mathfrak{N}'_i)$;

iii) $\mathfrak{N}''_{i\cdot}$ is closed in $\mathrm{M}^s(\mathfrak{N}'_i)$ for $\tau_{*,f}(\mathfrak{N}'_i)$. ■

Of course the same results hold true for the corresponding commutants of $\mathfrak{N}'_c$ or of the bounded ones $\mathfrak{N}'_{ib}$; in addition, $\mathfrak{N}''_{c\sigma}$ is weakly closed.

Proposition 4.2 yields in particular the following inclusions:

$$\mathfrak{N} \subset \overline{\mathfrak{N}[s^*]} \subset \mathfrak{N}''_{\sigma\sigma} \tag{4.3}$$

$$\mathfrak{N} \subset \overline{\mathfrak{N}[\tau_*(\mathfrak{N}'_\square)]} \subset \overline{\mathfrak{N}[\tau_f(\mathfrak{N}'_\square)]} \subset \mathfrak{N}''_{\square\square} \tag{4.4}$$

$$\mathfrak{N} \subset \overline{\mathfrak{N}[\tau_*(\mathfrak{N}'_{\cdot})]}^s \subset \overline{\mathfrak{N}[\tau_f(\mathfrak{N}'_{\cdot})]}^s \subset \mathfrak{N}''_{\cdot\cdot} \tag{4.5}$$

where $\overline{\mathfrak{N}}^s$ denotes the closure of $\mathfrak{N}$ *in* $\mathrm{M}^s(\mathfrak{N}'_{\cdot})$, for $\tau_{*,f}(\mathfrak{N}'_{\cdot})$.

The main question we want to address in the sequel is: under which conditions are the inclusions in Eqs. (4.3)-(4.5) in fact equalities? To answer it, we will follow closely the strategy of [*8*], that is, consider first a *-algebra $\mathfrak{B}$ consisting of bounded operators, then partial Op*-algebras containing sufficiently many bounded operators.

For the first step the results are summarized in the next proposition.

PROPOSITION 4.3. — Let $\mathfrak{B}$ be a *-algebra of bounded operators containing 1, $\mathscr{D}$ any dense domain in $\mathscr{H}$. Then the unbounded bicommutants of $\mathfrak{B}$ with respect to $\mathscr{D}$ may be characterized as follows:

i) $\mathfrak{B}''_{i\sigma} = \overline{\mathfrak{B}[s^*]} = \overline{\mathfrak{B}[\mathfrak{B}'_i\text{-}w^*]} = \mathfrak{B}''_{w\sigma} = (\mathfrak{B}')'_\sigma$, for $i = .$, $\square$ and σ;

ii) $\mathfrak{B}''_{\cdot\cdot} = \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\cdot})]}^s$, the closure of $\mathfrak{B}$ in $\mathrm{M}^s(\mathfrak{B}'_{\cdot})$;

iii) $\mathfrak{B}''_{\cdot\cdot} \cap \mathfrak{B}''_{\square\square} = \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\cdot})]}^s \cap \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_\square)]}$ and $\mathfrak{B}$ is $\tau_f(\mathfrak{B}'_\square)$-dense in it.

First we prove a technical lemma.

LEMMA 4.4. — Let $\mathfrak{B}$ be as in Proposition 4.3. Then:

i) $\mathfrak{B}$ is s^*-dense in $(\mathfrak{B}')'_\sigma$;

ii) For $j = .$ and $\square$, $\mathfrak{B}$ is dense in $L^w(\mathfrak{B}'_j) \cap (\mathfrak{B}')'_.$ for $\tau_f(\mathfrak{B}'_j)$.

Proof. — We treat all three cases simultaneously, using the argument developed in [8, Prop. 5]. The seminorms defining the topologies in question are all of the generic form (see Eqs. (4.1), (4.2)):

$$\| Y \|^{C,f_1,\dots f_k} = \sup_{1 \leq j \leq k} \{ \| (Y \square C) f_j \| + \| (C^+ \square Y^+) f_j \| \} ;$$

the operator C runs over $\mathfrak{B}'_j$ for $\tau_f(\mathfrak{B}'_j)$, and $C = 1$ for s^* (with $k = 1$). For fixed C and $f \in \mathscr{D}$, we denote by P the orthogonal projection on the (norm)-closed subspace $\overline{\mathfrak{B}Cf}$ of $\mathscr{H}$. Then [8], $PB = BP$ for every $B \in \mathfrak{B}$ and so $P \in \mathfrak{B}' = \mathfrak{B}'_b$, the usual bounded commutant. For $C=1$ and $Y \in (\mathfrak{B}')'_\sigma$, we have $(f, g \in \mathscr{D})$:

$$\langle Yf, (1-P)g \rangle = \langle (1-P)f, Y^+g \rangle = 0,$$

since $f = Pf$. Similarly, for $Y \in M^w(C) \cap (\mathfrak{B}')'_.$, we get:

$$\begin{aligned} \langle (Y \square C)f, (1 - P)g \rangle &= \langle Y^{+*}Cf, (1 - P)g \rangle \\ &= \langle Cf, Y^+(1 - P)g \rangle = \langle Cf, (1 - P)Y^+g \rangle = 0, \end{aligned}$$

since $Cf = PCf$. Thus, in both cases, $(Y \square C)f \in \overline{\mathfrak{B}Cf}$ and one may repeat step by step the argument of [8, Prop. 9].

The only modification is that the mixed product between the extended algebras $\hat{\mathfrak{B}}'_j$ and $\hat{\mathfrak{B}}''_{jj}$ must be defined as follows [1]:

$$\begin{pmatrix} X_1 & O \\ O & X_2 \end{pmatrix} \circ \begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix} = \begin{pmatrix} X_1 \square Z_1 & O \\ O & X_2 \square Z_2 \end{pmatrix}$$

(both $\hat{\mathfrak{B}}'_j$ and $\hat{\mathfrak{B}}''_{jj}$ consist of *diagonal* matrices only; this follows from the inclusion $\hat{\mathfrak{B}}'_j \subset \hat{\mathfrak{B}}'_\sigma$ and the results of [8]).

Exactly as in [8], the conclusion is that every neighbourhood of Y, for the topology defined by the operators $\{ C \}$, contains an element of $\mathfrak{B}$.

For $C = 1$, we obtain:

$$\mathfrak{B} \subset (\mathfrak{B}')'_\sigma \subset \overline{\mathfrak{B}[s^*]} \tag{4.6}$$

since the σ-commutant is s^*-closed. By Eq. (4.3), this proves *i*).

For $C \in \mathfrak{B}'_j$ ($j = .$ or $\square$), we obtain the statement of *ii*), since $\mathfrak{B} \subset M^w(\mathfrak{B}'_j) \cap (\mathfrak{B}')'_.$. However, the latter is in general not closed in $M^w(\mathfrak{B}'_j)$ for the topology $\tau_f(\mathfrak{B}'_j)$. ■

(1) Here, and only here, the symbol $\hat{\mathfrak{B}}$ denotes as in [8] the extended algebra $\mathfrak{B} \oplus \mathfrak{B}$, and has nothing to do with the fully closed extension of $\mathfrak{B}$ discussed in I and in Sec. 5 below.

Proof of Proposition 4.3. — Since $\mathfrak{B} \subset \mathscr{B}(\mathscr{H})$, one has:

$$\mathfrak{B}'_{\cdot} \subset \mathfrak{B}'_{\square} = \mathfrak{B}'_{\sigma}. \qquad (4.7)$$

For proving *i*), observe that Eq. (4.4) and the results of I, Sec. 5 yield the following inclusions:

$$\mathfrak{B} \subset \overline{\mathfrak{B}[s^*]} \subset \overline{\mathfrak{B}[\mathfrak{B}'_{\sigma}\text{-}w]} \begin{matrix} \subseteq \\ \subseteq \end{matrix} \begin{matrix} \overline{\mathfrak{B}[\mathfrak{B}'_{\cdot}\text{-}w^*]} \\ \mathfrak{B}''_{\sigma\sigma} \end{matrix} \begin{matrix} \subseteq \\ \subset \end{matrix} \mathfrak{B}''_{\cdot\sigma} \subset (\mathfrak{B}')'_{\square} = (\mathfrak{B}')'_{\sigma} \qquad (4.8)$$

Then, comparing (4.6) and (4.8), we get the result.

As for *ii*), we observe that $\mathfrak{B}''_{\cdot\cdot}$ is contained in

$$(\mathrm{M}^w(\mathfrak{B}'_{\cdot}) \cap (\mathfrak{B}')'_{\cdot}) \cap \mathrm{M}^s(\mathfrak{B}'_{\cdot}) = \mathrm{M}^s(\mathfrak{B}'_{\cdot}) \cap (\mathfrak{B}')'_{\cdot},$$

thus $\mathfrak{B}$ is dense in $\mathfrak{B}''_{\cdot\cdot}$ for the topology $\tau_f(\mathfrak{B}'_{\cdot})$ by Lemma 4.4. Hence $\mathfrak{B}''_{\cdot\cdot} \subset \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\cdot})]}^s$. By Eq. (4.5), this inclusion is in fact an equality.

Finally, $\mathfrak{B}''_{\cdot\cdot} \cap \mathfrak{B}''_{\square\square} \subset (\mathfrak{B}')'_{\cdot} \cap \mathrm{M}^w(\mathfrak{B}'_{\cdot})$, so that $\mathfrak{B}$ is $\tau_f(\mathfrak{B}'_{\square})$-dense in $\mathfrak{B}''_{\cdot\cdot} \cap \mathfrak{B}''_{\square\square}$, by Lemma 4.4. Hence:

$$\mathfrak{B}''_{\cdot\cdot} \cap \mathfrak{B}''_{\square\square} \subset \mathfrak{B}''_{\cdot\cdot} \cap \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\square})]}.$$

Using *ii*) and Eq. (4.4), we obtain *iii*). ■

REMARKS 4.5. — *a*) The argument of Lemma 4.4 works under more general circumstances. For instance:

. left multipliers $\mathrm{L}^w(\mathfrak{B}'_i)$ instead of $\mathrm{M}^w(\mathfrak{B}'_i)$: $\mathfrak{B}$ is dense in $\mathrm{L}^w(\mathfrak{B}'_i) \cap (\mathfrak{B}')'_{\cdot}$ in the topology $\tau^r_f(\mathfrak{B}'_i)$.

. $\mathrm{C} \in \mathfrak{B}'_{\cdot}$, $\mathrm{Y} \in \mathrm{M}^s(\mathfrak{B}'_{\cdot}) \cap (\mathfrak{B}')'_{\square}$, but since

$$\mathrm{M}^s(\mathfrak{B}'_{\cdot}) \cap (\mathfrak{B}')'_{\square} = \mathrm{M}^s(\mathfrak{B}'_{\cdot}) \cap (\mathfrak{B}')'_{\cdot} \subset \mathrm{M}^w(\mathfrak{B}'_{\cdot}) \cap (\mathfrak{B}')'_{\cdot},$$

we get nothing more.

. $\mathrm{C} \in \mathfrak{B}'_c$, $\mathrm{Y} \in \mathfrak{B}''_{\cdot\square}$: this yields the relation $\mathfrak{B}''_{\cdot\square} \subset \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_c)]}$, but no further conclusion may be drawn from that.

b) The proofs given above apply only for the topologies $\tau_f(\mathfrak{B}'_i)$. In fact the analogous statements do *not* hold for the full quasi-uniform topologies $\tau_*(\mathfrak{B}'_i)$. For instance, when $\mathscr{D} = \mathscr{H}$, $\mathfrak{C}(\mathscr{D}) = \mathscr{B}(\mathscr{H})$, $\mathfrak{B}'_{\cdot} = \mathfrak{B}'$ and $\mathfrak{B}''_{\cdot\cdot} = \mathfrak{B}''$, so that $\mathfrak{B}''_{\cdot\cdot}$ cannot be the closure of $\mathfrak{B}$ in $\tau_*(\mathfrak{B}'_{\cdot})$, since the latter is the operator norm topology!

Before going on, we would like to argue that the set $\mathfrak{B}''_{\cdot\cdot} \cap \mathfrak{B}''_{\square\square}$ is in fact the natural bicommutant of $\mathfrak{B}$. Indeed, an operator Y belongs to it iff Y commutes strongly with every $\mathrm{A} \in \mathfrak{B}'_{\cdot}$, Y.A = A.Y, and still commutes, but weakly, with every B in the larger set $\mathfrak{B}'_{\square}$.

Our next task is to extend the analysis to sets $\mathfrak{N}$ which are no longer contained in $\mathscr{B}(\mathscr{H})$. The outcome, patterned after Proposition 4.3, follows closely the results of [*8*]: If a set $\mathfrak{N}$ contains a *-algebra $\mathfrak{B}$ of bounded operators, suitably dense, then the bicommutants of $\mathfrak{N}$ have the properties described in Proposition 4.3.

PROPOSITION 4.6. — Let $\mathfrak{N}$ be a $\#$-invariant subset of $\mathfrak{C}(\mathcal{D})$ containing a *-algebra $\mathfrak{B}$ of bounded operators, with $1 \in \mathfrak{B}$. Then:

i) $\mathfrak{B}$ is s^*-dense in $\mathfrak{N} \Leftrightarrow \mathfrak{B}'_\sigma = \mathfrak{N}'_\sigma \Rightarrow \mathfrak{N}''_{w\sigma} = \mathfrak{N}''_{\sigma\sigma} = \overline{\mathfrak{N}[s^*]}$;

ii) $\mathfrak{N} \subset \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\cdot})]}^{M^s(\mathfrak{B}'_{\cdot})} = \mathfrak{B}''_{\cdot\cdot} \Leftrightarrow \mathfrak{B}'_{\cdot} = \mathfrak{N}'_{\cdot} \Rightarrow \mathfrak{N}''_{\cdot\cdot} = \overline{\mathfrak{N}[\tau_f(\mathfrak{N}'_{\cdot})]}^{M^s(\mathfrak{N}'_{\cdot})}$ where $\overline{\mathfrak{N}[\tau]}^{\mathfrak{M}}$ denotes the τ-closure of $\mathfrak{N}$ in $\mathfrak{M}$;

iii) $\mathfrak{N} \subset \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\cdot})]}^{M^s(\mathfrak{B}'_{\cdot})} \cap \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\square})]} = \mathfrak{B}''_{\cdot\cdot} \cap \mathfrak{B}''_{\square\square}$

$$\Leftrightarrow \mathfrak{B}'_{\cdot} = \mathfrak{N}'_{\cdot} \quad \textit{and} \quad \mathfrak{B}'_{\square} = \mathfrak{N}'_{\square}$$
$$\Leftrightarrow \mathfrak{B}''_{\cdot\cdot} \cap \mathfrak{B}''_{\square\square} = \mathfrak{N}''_{\cdot\cdot} \cap \mathfrak{N}''_{\square\square}$$
$$\Rightarrow \mathfrak{N}''_{\cdot\cdot} \cap \mathfrak{N}''_{\square\square} = \overline{\mathfrak{N}[\tau_f(\mathfrak{N}'_{\cdot})]}^{M^s(\mathfrak{N}'_{\cdot})} \cap \overline{\mathfrak{N}[\tau_f(\mathfrak{N}'_{\square})]}.$$

Proof. — The proof of all the assertions is almost immediate.

Ad i) : in general, the following inclusions hold:

$$\mathfrak{B} \subset \mathfrak{B}''_{w\sigma} = \mathfrak{B}''_{\sigma\sigma} = \overline{\mathfrak{B}[s^*]} \subset \overline{\mathfrak{N}[s^*]} \subset \mathfrak{N}''_{\sigma\sigma} \subset \mathfrak{N}''_{w\sigma}.$$

Let $\mathfrak{B}$ be s^*-dense in $\mathfrak{N}$; this means $\mathfrak{N} \subset \overline{\mathfrak{B}[s^*]} = \overline{\mathfrak{N}[s^*]}$, i. e. $\mathfrak{B} \subset \mathfrak{N} \subset \mathfrak{B}''_{\sigma\sigma}$. This implies $\mathfrak{B}'_\sigma = \mathfrak{N}'_\sigma$. All the other implications are shown in the same way.

Ad ii) : All three implications follow from the chain of inclusions:

$$\mathfrak{B} \subset \mathfrak{B}''_{\cdot\cdot} = \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\cdot})]}^{M^s(\mathfrak{B}'_{\cdot})} \subset \overline{\mathfrak{B}[\tau_f(\mathfrak{N}'_{\cdot})]}^{M^s(\mathfrak{B}'_{\cdot})}$$
$$\subset \overline{\mathfrak{B}[\tau_f(\mathfrak{N}'_{\cdot})]}^{M^s(\mathfrak{N}'_{\cdot})} \subset \overline{\mathfrak{N}[\tau_f(\mathfrak{N}'_{\cdot})]}^{M^s(\mathfrak{N}'_{\cdot})} \subset \mathfrak{N}''_{\cdot\cdot}.$$

Ad iii) : Same reasoning, treating $\mathfrak{B}''_{\cdot\cdot}$ and $\mathfrak{B}''_{\square\square}$ separately (for the latter, all closures may be taken in $\mathfrak{N}''_{\square\square}$ which is complete). ■

REMARKS 4.7. — *a)* If $\mathfrak{B}'_{\cdot} = \mathfrak{N}'_{\cdot}$, $\mathfrak{B}$ is dense in $\mathfrak{N}$ for $\tau_f(\mathfrak{N}'_{\cdot})$, but the converse need not hold. The same situation holds in case *iii)*.

b) For *i)*, the equality $\mathfrak{B}'_\sigma = \mathfrak{N}'_\sigma$ may also be shown directly, using the density of $\mathfrak{B}$ in $\mathfrak{N}$, in the standard form: every element of $\mathfrak{N}$ is the s^*-limit of a net $\{ M_\alpha \} \in \mathfrak{B}$. Similar reasonings can be made for *ii)* or *iii)*, and also using the continuity properties of the two partial multiplications (see I, § 4.A).

In general, the three density conditions imply each other: *iii)* $\Rightarrow$ *ii)* $\Rightarrow$ *i)*. For the case of an Op*-algebra $\mathfrak{A}$, we may take $\mathfrak{B} = \mathfrak{A}_b$ in Proposition 4.6. Then we get:

$$\mathfrak{A}'_{\cdot} \subset \mathfrak{A}'_{\square} = \mathfrak{A}'_\sigma$$

and

$$(\mathfrak{A}_b)'_{\cdot} = (\mathfrak{A}_b)'_{\square} = (\mathfrak{A}_b)'_\sigma.$$

So, if we impose *ii)*, $\mathfrak{A}'_{\cdot} = (\mathfrak{A}_b)'_{\cdot}$, which implies *i)*, $\mathfrak{A}'_\sigma = (\mathfrak{A}_b)'_\sigma$, we obtain *iii)*.

Thus for an Op*-algebra *ii)* ⇔ *iii)*, and in that case, all six commutants coincide.

The next point is to characterize classes of partial Op*-algebras for which one or another of the conditions of Proposition 4.6 are satisfied. We shall do this in Sec. 5, which will bring into focus the notion of *symmetric* partial Op*-algebra.

5. SYMMETRIC PARTIAL Op*-ALGEBRAS

As it is well-known, an Op*-algebra $\mathfrak{A}$ is called *symmetric* if, for every $A \in \mathfrak{A}$, $(1 + A^*A)^{-1}$ belongs to $\mathfrak{A}_b$. These Op*-algebras enjoy many interesting properties relevant for our discussion, for instance they verify the relation $\mathfrak{A}'_\sigma = (\mathfrak{A}_b)'_\sigma$ [8] [17]. We want to generalize this concept to partial Op*-algebras. As usual several possibilities occur.

Let $\mathfrak{N}$ be a $+$-invariant subset of $\mathfrak{C}(\mathscr{D})$. For every $A \in \mathfrak{N}$ the usual product A^*A is a positive self-adjoint operator, so that $(1 + A^*A)^{-1}$ is a bounded self-adjoint operator. However it need not belong to $\mathfrak{A}_b$ and A^*A is not necessarily $\mathscr{D}$-minimal. This motivates our first definition.

A $+$-invariant subset $\mathfrak{N}$ of $\mathfrak{C}(\mathscr{D})$ is called **-symmetric* if, for every $A \in \mathfrak{N}$, $(1 + A^*A)^{-1}$ belongs to $\mathfrak{N}_b$. The prime examples of such sets are strong natural commutants.

LEMMA 5.1. — Let $\mathfrak{B}$ be a *-algebra of bounded operators. Then its strong natural commutant $\mathfrak{B}'_s$ is *-symmetric.

Proof. — By Proposition 2.3, $\mathfrak{B}'_s = \{ X \in \mathfrak{C}(\mathscr{D}) \mid X, X^+ \eta \mathfrak{B}' \}$, where $\mathfrak{B}'$ is the usual bounded commutant of $\mathfrak{B}$, hence a von Neumann algebra. Notice that $\mathfrak{B}' = \mathfrak{B}'_{s,b}$. Let $X \in \mathfrak{B}'_s$. Then $X \eta \mathfrak{B}'$, hence $X^* \eta \mathfrak{B}'$ and $X^*X \eta \mathfrak{B}'$ [18], which is equivalent to $(1+X^*X)^{-1} \in \mathfrak{B}' \subset \mathfrak{B}'_s$. Thus $\mathfrak{B}'_s$ is *-symmetric (notice that X^* and X^*X do not necessarily belong to $\mathfrak{C}(\mathscr{D})$). ■

In the sequel we will discuss partial Op*-algebras that are *-symmetric as defined above (for Op*-algebras, this reduces to the usual notion of symmetry). More restrictive concepts will be discussed later on.

If an Op*-algebra $\mathfrak{A}$ is symmetric, it is well-known that $\mathfrak{A}'_\sigma = (\mathfrak{A}_b)'_\sigma$, i. e. $\mathfrak{A}_b$ is s^*-dense in $\mathfrak{A}$. This result extends to *partial* Op*-algebras as well:

PROPOSITION 5.2. — Let $\mathfrak{M}$ be a *-symmetric partial Op*-algebra. Then $\mathfrak{M}'_\sigma = (\mathfrak{M}_b)'_\sigma$ and $\mathfrak{M}_b$ is s^*-dense in $\mathfrak{M}$.

Proof. — For every $A \in \mathfrak{M}$, $(1 + A^*A)^{-1}$ belongs to $\mathfrak{M}_b$ and so does $A(1 + A^*A)^{-1}$: the product is an everywhere defined, bounded operator, hence it belongs to $\mathfrak{M}$ and thus to $\mathfrak{M}_b$, since $\mathfrak{M}$ is a partial Op*-algebra. Similarly, for every $n = 1, 2, \ldots$, $A(1 + n^{-1}A^*A)^{-1} \in \mathfrak{M}_b$ and, for $n \to \infty$,

$T_n \equiv (1 + n^{-1}A^*A)^{-1}$ tends strongly to 1, since $\|T_n\| \leqslant 1$, $\forall n$, and the sequence $\{\langle f, T_n f\rangle\}$ is non-decreasing for every $f \in \mathscr{H}$ (see [27, Theor. 4.28]). We have to show that $(\mathfrak{M}_b)'_\sigma \subset \mathfrak{M}'_\sigma$. Let $C \in (\mathfrak{M}_b)'_\sigma$, so that, for $f, g \in \mathscr{D}$, we have:

$$\langle Cf, A(1 + n^{-1}A^*A)^{-1}g \rangle = \langle (A(1 + n^{-1}A^*A)^{-1})^* f, C^+ g \rangle .$$

Inserting in the l. h. s. the decomposition $A = U(A^*A)^{1/2}$, where U is a partial isometry, this may be rewritten:

$$\langle Cf, U(1 + n^{-1}A^*A)^{-1}(A^*A)^{1/2} g \rangle = \langle (1 + n^{-1}A^*A)^{-1}A^+ f, C^+ g \rangle .$$

Taking the (strong or weak) limit $n \to \infty$, we get:

$$\langle Cf, Ag \rangle = \langle A^+ f, C^+ g \rangle ,$$

that is, $C \in \mathfrak{M}'_\sigma$. Therefore we have proven $(\mathfrak{M}_b)'_\sigma = \mathfrak{M}'_\sigma$, which is equivalent to the s^*-density of $\mathfrak{M}_b$ in $\mathfrak{M}$, by Proposition 4.6 *i*). ■

For the natural commutants, the corresponding result holds for the bounded parts, but apparently not for the commutants themselves.

PROPOSITION 5.3. — Let $\mathfrak{M}$ be a *-symmetric partial Op*-algebra. Then the three unbounded commutants $\mathfrak{M}'_\sigma, \mathfrak{M}'_\square, \mathfrak{M}'_{\cdot}$ have the same bounded part, which is a von Neumann algebra equal to the usual commutant $(\mathfrak{M}_b)'$.

Proof. — *A priori* we have for the bounded parts:

$$\begin{array}{ccccc} & \mathfrak{M}'_{\cdot b} \subset & \mathfrak{M}'_{\square b} \subset & \mathfrak{M}'_w \\ & \cap & \cap & \| \\ (\mathfrak{M}_b)' \equiv & (\mathfrak{M}_b)'_{\cdot b} \subset & (\mathfrak{M}_b)'_{\square b} = & (\mathfrak{M}_b)'_w . \end{array} \tag{5.1}$$

Let $C \in (\mathfrak{M}_b)'$. Thus C $\square$-commutes with $(1 + A^*A)^{-1}$ and $A(1 + A^*A)^{-1}$ for every $A \in \mathfrak{M}$, hence we have, for $f \in \mathscr{D}$:

$$C^{+*}A(1 + A^*A)^{-1}f = A(1 + A^*A)^{-1}Cf = AC^{+*}(1 + A^*A)^{-1}f,$$

which means:

$$C^{+*}Ak = AC^{+*}k, \qquad \forall k \in \mathscr{K} \equiv (1 + A^*A)^{-1}\mathscr{D} . \tag{5.2}$$

In fact, all relations may be extended to $f \in D(C)$, using the closedness of C. As for that set $\mathscr{K}$, we get:

$$\mathscr{K} \equiv (1 + A^*A)^{-1}\mathscr{D} \subset D(A^*A) \subset D(A) \tag{5.3}$$

but, in general, $\mathscr{K}$ need not contain $\mathscr{D}$, although it is of course dense in $\mathscr{H}$. Since C is bounded, Eq. (5.2) gives

$$CAk = ACk, \qquad \forall k \in \mathscr{K} \tag{5.4}$$

and therefore

$$\| ACk \| = \| CAk \| \leqslant \| C \| \, . \, \| Ak \| \leqslant \| C \| \, . \, \| k \|_A$$

i. e. C is bounded in the graph norm $\| \, . \, \|_A$. Thus (5.4) extends to all $k \in D(A)$, in particular $k \in \mathscr{D}$, which means that $C \in (\mathfrak{M}'_{\cdot})_b$. Thus we have shown that $(\mathfrak{M}_b)'_{\square b} \subset \mathfrak{M}'_{\cdot b}$. In view of (5.1), this implies that all six bounded commutants are equal, and equal to $(\mathfrak{M}_b)'$, which is indeed a von Neumann algebra. ∎

Remark. — Although $\mathscr{K}$ need not contain $\mathscr{D}$, it is a core for A. Indeed, let $g \in D(A)$ be orthogonal to every $k = (1 + A^*A)^{-1} f$, $f \in \mathscr{D}$, in the graph inner product:

$$\begin{aligned} 0 = \langle g, (1 + A^*A)^{-1} f \rangle_A &= \langle (1 + A^*A)^{1/2} g, (1+A^*A)^{1/2}(1+A^*A)^{-1} f \rangle \\ &= \langle g, (1 + A^*A)(1 + A^*A)^{-1} f \rangle \quad \text{by} \quad (5.3) \\ &= \langle g, f \rangle \end{aligned}$$

and this implies $g = 0$.

As an application of Proposition 5.2, we derive the link between partial V*-algebras and von Neumann algebras.

PROPOSITION 5.4. — Let $\mathfrak{M}$ be a partial V*-algebra; then its bounded part $\mathfrak{M}_b$ is a von Neumann algebra.

Conversely, for any von Neumann algebra $\mathfrak{A}$, there exists a partial V*-algebra $\mathfrak{M}$ such that $\mathfrak{M}_b = \mathfrak{A}$.

Proof. — Let $\mathfrak{M} = \mathfrak{M}''_{\sigma\sigma}$ be a partial V*-algebra. It is s^*-closed in $\mathfrak{C}(\mathscr{D})$, then so is $\mathfrak{M}_b$ in the s^*-topology induced on $\mathscr{B}(\mathscr{H})$, which is weaker than the s^*-topology on $\mathscr{B}(\mathscr{H})$. Hence $\mathfrak{M}_b$ is a s^*-closed *-algebra of bounded operators, i. e. a von Neumann algebra.

Conversely let $\mathfrak{A} = \mathfrak{A}''$ be any von Neumann algebra. Given an arbitrary dense domain $\mathscr{D}_0$ in $\mathscr{H}$, define the domain $\mathscr{D} = \mathfrak{A}\mathfrak{A}'\mathscr{D}_0 = \mathfrak{A}'\mathfrak{A}\mathscr{D}_0$, which contains $\mathscr{D}_0$. Obviously $\mathfrak{A}$ and $\mathfrak{A}'$ leave $\mathscr{D}$ invariant. Thus we get from Proposition 3.3 and Lemma 5.1:

i) $\quad \mathfrak{A}'_{\cdot} = \mathfrak{A}'_{\square} = \mathfrak{A}'_{\sigma} = \{ X \in \mathfrak{C}(\mathscr{D}) \mid X, X^+ \eta \mathfrak{A}' \}$

ii) $\quad (\mathfrak{A}')'_{\cdot} = (\mathfrak{A}')'_{\square} = (\mathfrak{A}')'_{\sigma} = \{ Y \in \mathfrak{C}(\mathscr{D}) \mid Y, Y^+ \eta \mathfrak{A}'' \}$,

and both are *-symmetric partial V*-algebras, with bounded parts $\mathfrak{A}'$ and $\mathfrak{A}'' = \mathfrak{A}$, respectively.

Now take $\mathfrak{M} = \mathfrak{A}''_{\sigma\sigma}$. Since $\mathfrak{A}'_{\sigma}$ is *-symmetric, we have

$$\mathfrak{A}''_{\sigma\sigma} = (\mathfrak{A}'_{\sigma b})'_{\sigma} = (\mathfrak{A}')'_{\sigma},$$

so that $\mathfrak{M}_b = \mathfrak{A}$. ∎

Obviously there is a large non-uniqueness in the answer, coded into the domain $\mathscr{D}$. In fact, if $\mathfrak{A}$ or $\mathfrak{A}'$ has a cyclic vector f_0, then $\mathscr{D}(f_0) \equiv \mathfrak{A}\mathfrak{A}' f_0$ may be used as well.

We turn now to commutants and bicommutants of Op*-algebras. Let $\mathfrak{A}$ be an arbitrary Op*-algebra on $\mathscr{D}$, $\mathfrak{A}_b$ its bounded part.

Then their various commutants obey the following scheme:

$$\begin{array}{ccccccc} \mathfrak{A}'_c & \subset & \mathfrak{A}'_{\cdot} & \subset & \mathfrak{A}'_{\square} & = & \mathfrak{A}'_\sigma \\ \cap & & \cap & & \cap & & \cap \\ (\mathfrak{A}_b)'_c & \subset & (\mathfrak{A}_b)'_{\cdot} & = & (\mathfrak{A}_b)'_{\square} & = & (\mathfrak{A}_b)'_\sigma \end{array} \qquad (5.5)$$

and $(\mathfrak{A}_b)'_\sigma = \{ X \in \mathfrak{C}(\mathscr{D}) \mid X, X^+ \eta (\mathfrak{A}_b)' \}$ is a *-symmetric partial V*-algebra (Proposition 3.3 and Lemma 5.1).

If we assume that $\mathfrak{A}_0$ is a s^*-dense in $\mathfrak{A}$, then $\mathfrak{A}'_\sigma = (\mathfrak{A}_b)'_\sigma$ and we conclude:

PROPOSITION 5.4. — Let $\mathfrak{A}$ be an Op*-algebra on $\mathscr{D}$, with s^*-dense bounded part. Then $\mathfrak{A}'_\sigma$ is a *-symmetric partial V*-algebra, given by:

$$\mathfrak{A}'_\sigma = \{ X \in \mathfrak{C}(\mathscr{D}) \mid X, X^+ \eta (\mathfrak{A}_b)' \}. \qquad (5.6) \quad ■$$

If $\mathfrak{A}$ is symmetric, it verifies $\mathfrak{A}'_\sigma = (\mathfrak{A}_b)'_\sigma$ [8]. Hence:

COROLLARY 5.5. — The conclusions of Proposition 5.4 hold, in particular, for every symmetric Op*-algebra. ■

Clearly the best situation will be obtained when all eight commutants coincide in Eq. (5.5). This condition leads indeed to a stronger result.

PROPOSITION 5.6. — Let $\mathfrak{A}$ be an Op*-algebra that verifies one of the following conditions:

a) $\mathfrak{A}$ is closed and $(\mathfrak{A}_b)'_{\cdot} = \mathfrak{A}'_{\cdot}$

b) $\mathfrak{A}$ is self-adjoint with s^*-dense bounded part.

Then:

i) $\mathfrak{A}'_\sigma$ is a symmetric SV*-algebra and

$$\mathfrak{A}'_\sigma = \mathfrak{A}'_c = \{ X \in \overline{\mathscr{L}^+(\mathscr{D})} \mid X, X^+ \eta (\mathfrak{A}_b)' \}. \qquad (5.7)$$

ii) $\mathfrak{A}''_{\sigma\sigma}$ is a *-symmetric partial SV*-algebra and

$$\mathfrak{A}''_{\sigma\sigma} = \mathfrak{A}''_{c\sigma} = \mathfrak{A}''_{w\sigma} = \mathfrak{A}''_{s\sigma} = \{ X \in \mathfrak{C}(\mathscr{D}) \mid X, X^+ \eta (\mathfrak{A}_b)'' \}. \qquad (5.8)$$

Proof. — Since a closed Op*-algebra verifies $\mathfrak{A}'_c = \mathfrak{A}'_{\cdot}$, in both cases *a)* and *b)*, the eight commutants coincide in Eq. (5.5). Thus, from Proposition 5.4, $\mathfrak{A}'_\sigma = \mathfrak{A}'_c$ is a symmetric V*-algebra given by Eq. (5.7). By Corollary 5.5, $\mathfrak{A}''_{\sigma\sigma}$ is a *-symmetric partial V*-algebra, consisting, according to Eq. (5.6), of operators affiliated with $(\mathfrak{A}'_w)' = ((\mathfrak{A}_b)'_{\cdot b})' = (\mathfrak{A}_b)''$. It remains to prove the SV* character of $\mathfrak{A}'_\sigma$ and $\mathfrak{A}''_{\sigma\sigma}$. Let $\mathfrak{M} = \mathfrak{A}''_{\sigma\sigma}$. Then $\mathfrak{M}'_\sigma = \mathfrak{A}'_\sigma$, which is symmetric, thus $\mathfrak{M}''_{w\sigma} = \mathfrak{A}''_{w\sigma} = \mathfrak{A}''_{\sigma\sigma} = \mathfrak{M}$, i. e. $\mathfrak{A}''_{\sigma\sigma}$ is a partial SV*-algebra. Similarly $\mathfrak{A}''_{\sigma\sigma}$ being *-symmetric, $\mathfrak{A}'''_{\sigma w\sigma} = \mathfrak{A}'''_{\sigma\sigma\sigma} = \mathfrak{A}'_\sigma$ i. e. $\mathfrak{A}'_\sigma$ is a SV*-algebra. ■

Remembering that a closed symmetric Op*-algebra is automatically self-adjoint with s^*-dense bounded part, we get:

COROLLARY 5.7. — The conclusions of Proposition 5.6 hold, in particular, for every closed symmetric Op*-algebra. ■

As mentioned earlier, the notion of *-symmetric partial Op*-algebra is not the most natural one. For instance, if $\mathfrak{M}$ is one, the bounded operator $(1 + A^*A)^{-1}$ belongs to $\mathfrak{M}_b$ for every $A \in \mathfrak{M}$, but A^*A need not even be defined on $\mathscr{D}$, and the products $A^+ \square A$ or $A^+ . A$ do not necessarily exist. This motivates more restrictive concepts.

Let $\mathfrak{M}$ be a partial Op*-algebra. We will say that:

i) $\mathfrak{M}$ is *weakly symmetric* if, for every $A \in \mathfrak{M}$, $A^+ \in L^w(A)$ and $(1 \hat{+} A^+ \square A)^{-1}$ exists and belongs to $\mathfrak{M}_b$;

ii) $\mathfrak{M}$ is *strongly symmetric* if, for every $A \in \mathfrak{M}$, $A^+ \in L^s(A)$ and $(1 \hat{+} A^+ . A)^{-1}$ exists and belongs to $\mathfrak{M}_b$.

Clearly strongly symmetric implies weakly symmetric. As for *-symmetry, we have the following result:

PROPOSITION 5.8. — Let $\mathfrak{M}$ be a weakly symmetric partial Op*-algebra on $\mathscr{D}$. Then, for every $A \in \mathfrak{M}$, $A^*A = A^+ \square A$ is $\mathscr{D}$-minimal and $\mathfrak{M}$ is *-symmetric.

Proof. — For any $A \in \mathfrak{M}$, the condition $A^+ \in L^w(A)$ means that $A\mathscr{D} \subset D(A^*)$ or equivalently $\mathscr{D} \subset D(A^*A)$. Thus $A^+ \square A \subset A^*A$ and $1 \hat{+} A^+ \square A \subset 1 + A^*A$. Hence $\operatorname{Ker}(1 \hat{+} A^+ \square A) \subset \operatorname{Ker}(1 + A^*A) = \{0\}$, i.e. $1 \hat{+} A^+ \square A$ is invertible. Hence its inverse $(1 \hat{+} A^+ \square A)^{-1}$ is a closed operator, with domain $\operatorname{Ran}(1 \hat{+} A^+ \square A)$, and therefore

$$(1 \hat{+} A^+ \square A)^{-1} \subset (1 + A^*A)^{-1} \tag{5.9}$$

so that $(1 \hat{+} A^+ \square A)^{-1}$ is bounded on (the closure of) its domain. Since $(1 \hat{+} A^+ \square A)^{-1} \in \mathfrak{M}_b$ by assumption, that domain contains $\mathscr{D}$, and therefore the two operators in Eq. (5.9) coincide.

It follows that $1 \hat{+} A^+ \square A = 1 + A^*A$ and $A^+ \square A = A^*A$, hence $\mathfrak{M}$ is *-symmetric. ■

Remark. — This does not imply that $\mathfrak{M}$ is *standard*, i.e. $A^+ = A^*$ [*9*], as it is the case for Op*-algebras; the proof of Inoue [*28*] does not work (see below).

Let now $\mathfrak{M}$ be strongly symmetric. It is *a fortiori* weakly symmetric and $A^+ . A = A^+ \square A = A^*A$ for each $A \in \mathfrak{M}$. But there is more.

PROPOSITION 5.9. — A strongly symmetric partial Op*-algebra is standard.

Proof. — Let $\mathfrak{M}$ be strongly symmetric. We have to show that every symmetric element $A = A^+ \in \mathfrak{M}$ is self-adjoint, $A = A^*$. As shown above,

$$A^*A = A^+ . A = A . A = \overline{A^2 \restriction \mathscr{D}} \subset A^2 .$$

On the other hand, $A \subset A^*$ and thus $A^2 \subset A^*A$. Hence $A^2 = A^*A$ is a positive self-adjoint operator, i. e. $A = A^*$. ∎

This proof shows why the same result does not hold if we assume only that $\mathfrak{M}$ is weakly symmetric. In that case:

$$A^*A = A^+ \,\square\, A = A \,\square\, A = \overline{A^{+*}A \upharpoonright \mathscr{D}},$$

and this operator is not necessarily a restriction of A^2. On the other hand, if we assume from the outset that $\mathfrak{M}$ is standard and weakly symmetric, then $\mathfrak{M}$ is also strongly symmetric. Indeed since $A^* = A^+$, we get $A^+ \in L^s(A)$ iff $A^+ \in L^w(A)$ and $A^+ \,\square\, A = A^+ . A = A^*A$.

With the definitions given above, weakly or strongly partial Op*-algebras are apparently not the realization, in $\mathfrak{C}^s(\mathscr{D})$ or $\mathfrak{C}^w(\mathscr{D})$, of the symmetric partial *-algebras defined abstractly in I. Indeed we have used only usual operator inverses, not inverses with respect to the appropriate product . or □. But in fact, as shown in Appendix B, the two approaches are equivalent.

To conclude this section, let us come back to Proposition 4.6. If $\mathfrak{N}$ is a general partial Op*-algebra, its bounded part $\mathfrak{N}_b$ is a *-algebra, containing 1, hence we may take $\mathfrak{B} = \mathfrak{N}_b$ is all the statements.

Let $\mathfrak{N}$ be *-symmetric; by Proposition 5.2, it verifies $\mathfrak{N}'_\sigma = (\mathfrak{N}_b)'_\sigma$, i. e. the weakest density condition *i*) of Proposition 4.6. So if we assume that $\mathfrak{N}$ is weakly or strongly symmetric, one could expect to prove that it verifies a stronger density statement, *ii*) or *iii*) of Proposition 4.6. But this does not seem to be the case, especially if one looks at the proof of Proposition 5.3 and the remark following it. Weak or strong symmetry implies that $\mathscr{D}$ is a core for A^*A, but still we don't know if $\mathscr{K} \supset \mathscr{D}$. Notice that for an Op*-algebra $\mathfrak{A}$, all three notions of symmetry coincide, and imply $\mathfrak{A}'_\square = (\mathfrak{A}_b)'_\square = \mathfrak{A}'_\sigma = (\mathfrak{A}_b)'_\sigma$, but not necessarily $\mathfrak{A}'_. = (\mathfrak{A}_b)'_.$. So the problem remains open.

6. EXTENSIONS OF PARTIAL Op*-ALGEBRAS AND THEIR COMMUTANTS

Let $\mathfrak{M}$ be a partial Op*-algebra on $\mathscr{D}$. As we saw in I [9, Sec. 4], $\mathfrak{M}$ defines two other partial Op*-algebras, $\overline{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$, with domains $\tilde{\mathscr{D}}$ and $\mathscr{D}(\mathfrak{M})$ respectively. The following inclusions hold:

$$\begin{gathered} \mathscr{D} \subset \tilde{\mathscr{D}} \subset \mathscr{D}(\mathfrak{M}) \\ \mathfrak{C}(\mathscr{D}) \supset \mathfrak{C}(\tilde{\mathscr{D}}) \supset \mathfrak{C}(\mathscr{D}(\mathfrak{M})) \end{gathered} \tag{6.1}$$

but $\mathfrak{M}$, $\overline{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ consist of the same closed operators. We will examine their various commutants in turn.

A. Unbounded commutants.

We begin with weak unbounded commutants. From the inclusions (6.1) we get immediately

$$\hat{\mathfrak{M}}'_\sigma \subset \overline{\mathfrak{M}}'_\sigma \subset \mathfrak{M}'_\sigma . \tag{6.2}$$

Indeed the definition (2.1) of the three commutants may be recast in a unified form:

$$(\mathfrak{M}^{(i)})'_\sigma = \{ X \in \mathfrak{C}(\mathcal{D}^{(i)}) \mid \langle X^* f, Ag \rangle = \langle A^* f, Xg \rangle, \forall f, g \in \mathcal{D}^{(i)}, \forall A \in \mathfrak{M}^{(i)} \} \quad (i = 1, 2, 3) \tag{6.3}$$

where we have introduced the following notation: $(\mathfrak{M}^{(1)}, \mathcal{D}^{(1)}) \equiv (\mathfrak{M}, \mathcal{D})$, $(\mathfrak{M}^{(2)}, \mathcal{D}^{(2)}) \equiv (\overline{\mathfrak{M}}, \tilde{\mathcal{D}})$, $(\mathfrak{M}^{(3)}, \mathcal{D}^{(3)}) \equiv (\hat{\mathfrak{M}}, \mathcal{D}(\mathfrak{M}))$. Notice that $\mathcal{D}_*(\mathfrak{M}^{(i)}) = \mathcal{D}_*(\mathfrak{M})$ for $i = 1, 2, 3$. Hence Proposition 2.1 gives the following representation for the weak natural commutants $(\mathfrak{M}^{(i)})'_\square$:

$$(\mathfrak{M}^{(i)})'_\square = \{ X \in (\mathfrak{M}^{(i)})'_\sigma \mid X, X^* : \mathcal{D}^{(i)} \to \mathcal{D}_*(\mathfrak{M}) \} \quad (i = 1, 2, 3)$$

so that, by Eq. (6.2):

$$\hat{\mathfrak{M}}'_\square \subset \overline{\mathfrak{M}}'_\square \subset \mathfrak{M}'_\square . \tag{6.4}$$

It is worth recalling that the involution $+$ and the multiplication $\square$ are the same whether defined on $\mathcal{D}$, $\tilde{\mathcal{D}}$ or $\mathcal{D}(\mathfrak{M})$ [*9*, Prop. 4.2]. Thus there is no ambiguity of notation in Eq. (6.4).

As for the strong natural commutants $(\mathfrak{M}^{(i)})'_.$, we have obviously $R^s\hat{\mathfrak{M}} \subset R^s\overline{\mathfrak{M}} \subset R^s\mathfrak{M}$ and similarly for L^s. On the other hand, the commutation relation $XAf = AXf$ $(A \in \mathfrak{M})$ is the same for the three commutants, except that f is taken in $\mathcal{D}(\mathfrak{M})$, $\tilde{\mathcal{D}}$ or $\mathcal{D}$, respectively. All together we get:

$$\hat{\mathfrak{M}}'_. \subset \hat{\mathfrak{M}}'_. \subset \mathfrak{M}'_. . \tag{6.5}$$

The case of the strong unbounded commutants $(\mathfrak{M}^{(i)})'_c$ will be discussed below.

We summarize these results in a proposition.

PROPOSITION 6.1. — Let $\mathfrak{M}$ be a partial Op*-algebra, $\overline{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$ its canonical extensions. Then for $j = \sigma$, $\square$ or ., the following inclusions hold among unbounded commutants:

$$\hat{\mathfrak{M}}'_j \subset \overline{\mathfrak{M}}'_j \subset \mathfrak{M}'_j . \quad \blacksquare \tag{6.6}$$

REMARK 6.2. — We may combine the inclusions (6.6) with those between the commutants themselves, Eq. (2.7). Write $\sigma = 1$, $\square = 2$, $. = 3$. Then the inclusion relations between the nine unbounded commutants are the following:

$$(\mathfrak{M}^{(i)})'_j \subseteq (\mathfrak{M}^{(i')})'_{j'} \qquad \text{whenever} \qquad i \geqslant i', \qquad j \geqslant j' .$$

B. Bounded commutants.

If we consider the bounded commutants $(\mathfrak{M}^{(i)})'_{jb} \equiv (\mathfrak{M}^{(i)})'_j \cap \mathscr{B}(\mathscr{H})$, for $j = .\,, \square$ or σ, the inclusions (6.6) actually reduce to equalities.

PROPOSITION 6.3. — In the notation of Proposition 6.1, one has for $j = .\,, \square$ and σ, the relation,

$$\widehat{\mathfrak{M}}'_{jb} = \overline{\mathfrak{M}}'_{jb} = \mathfrak{M}'_{jb}. \tag{6.7}$$

Proof. — *i*) Consider first $j = \sigma$, i. e. the weak bounded commutants ($jb \equiv w$). By (6.2) it is enough to show that $\mathfrak{M}'_w \subset \widehat{\mathfrak{M}}'_w$. Let $A \in \mathfrak{M}$, $X \in \mathfrak{M}'_w$ and $f, g \in \mathscr{D}(\mathfrak{M})$. Then there exists a sequence $\{ g_n \} \in \mathscr{D}$ such that $g_n \to g$ and $Ag_n \to Ag$ (this sequence may depend on A), and also a sequence $\{ f_k \} \in \mathscr{D}$ such that $f_k \to f$ and $A^+ f_k \to A^+ f$. Then, since X and X* are bounded and belong to $\mathfrak{M}'_\sigma$, we get:

$$\begin{aligned} \langle A^+ f, Xg \rangle &= \lim_{k,n} \langle A^+ f_k, Xg_n \rangle \\ &= \lim_{k,n} \langle X^* f_k, Ag_n \rangle \\ &= \langle X^* f, Ag \rangle , \end{aligned}$$

i. e. $X \in \widehat{\mathfrak{M}}'_w$.

ii) We turn to the case of weak natural commutants, $j = \square$. Let again $X \in \mathfrak{M}'_{\square b}$. By (2.5) $X \in \mathfrak{M}'_\sigma \cap \mathscr{B}(\mathscr{H})$ and X, X^+ map $\mathscr{D}$ into $\mathscr{D}_*(\mathfrak{M})$. We show that X, X^+ map $\mathscr{D}(\mathfrak{M})$ into $\mathscr{D}_*(\mathfrak{M})$ and thus $X \in \widehat{\mathfrak{M}}'_{\square b}$. For any $A \in \mathfrak{M}$ and $g \in \mathscr{D}(\mathfrak{M})$, there exists a sequence $\{ g_n \} \in \mathscr{D}$ such that $g_n \to g$ and $Ag_n \to Ag$. Then $Xg_n \to Xg$ and the sequence $\{ A^{+*}Xg_n \}$ converges, since $A \square X = X \square A$ implies $A^{+*}Xg_n = XAg_n \to XAg$. Hence $Xg \in D(A^{+*})$. In the same way, $X^+ f \in D(A^*)$ for every $f \in \mathscr{D}(\mathfrak{M})$ and $A \in \mathfrak{M}$, and the assertion is proved.

iii) For the strong natural commutants, $j = .\,$, the argument is identical, replacing A^{+*}, A^* by A, A^+ respectively. ■

C. Special cases.

The argument in the proof of Proposition 6.3 rests on the existence of the sequence $\{ g_n \}$, for which *simultaneously* $g_n \to g$, $Ag_n \to Ag$ and $Xg_n \to Xg$, and similarly for $f_k \to f$. Let now X be an unbounded element of $\mathfrak{M}'_\sigma$. If $\mathscr{D}(\mathfrak{M})$ or $\widetilde{\mathscr{D}}[t_{\mathfrak{M}}]$ is contained in D(X), and thus is a core for X, there exists sequences $\{ g_n \}$ and $\{ g'_k \}$ tending to g, such that $Ag_n \to Ag$ and $Xg'_k \to Xg$ respectively, but not necessarily a *common* sequence for both X and A. It is interesting to note the analogy with the situation described in the Appendix of I. There too the lack of a common sequence for the operators A^2 and B^2 was the origin of the difficulty leading to the breakdown of associativity.

This objection disappears, for the domain $\tilde{\mathscr{D}}[t_{\mathfrak{M}}]$, in certain particular cases. Indeed:

PROPOSITION 6.4. — Let $j = .\,,\ \square$ or σ. Then:

i) $\overline{\mathfrak{M}}'_j \cap \mathfrak{M} = \mathfrak{M}'_j \cap \mathfrak{M}$; (5.8)

ii) If $\mathscr{D}[t_{\mathfrak{M}}]$ is barrelled, $\overline{\mathfrak{M}}'_j = \mathfrak{M}'_j$. (5.9)

Proof. — By definition of $\tilde{\mathscr{D}}[t_{\mathfrak{M}}]$, every $g \in \tilde{\mathscr{D}}[t_{\mathfrak{M}}]$ is the $t_{\mathfrak{M}}$-limit of a convergent net $\{ g_\alpha \} \in \mathscr{D}$, which means that $Ag_\alpha \to Ag$ for all $A \in \mathfrak{M}$ simultaneously.

If $X \in \mathfrak{M}$, then $A \square X = X \square A \in \mathfrak{M}$ too, when they are defined. Then the argument of Proposition 6.3 works and yields *i*).

Let now $\mathscr{D}[t_{\mathfrak{M}}]$ be barrelled. Then, as shown in I, Lemma 3.2, every $X \in \mathfrak{C}(\mathscr{D})$ maps $\mathscr{D}[t_{\mathfrak{M}}]$ continuously into $\mathscr{H}$, i. e. there exists $A_0 \in \mathfrak{M}$ such that $\| Xf \| \leqslant \| A_0 f \|$, $\forall f \in \mathscr{D}$. Given any $g \in \tilde{\mathscr{D}}[t_{\mathfrak{M}}]$ and a net $g_\alpha \to g$ as above, the net $\{ Xg_\alpha \}$ is Cauchy, hence converges to Xg. Therefore the argument of Proposition 6.3 works again, replacing the sequences $\{ g_n \}$, $\{ f_k \}$ by nets $\{ g_\alpha \}$, $\{ f_\beta \}$. Hence we get *ii*). ■

The statement *i*) of this proposition may also be rephrased as follows: the partial Op*-algebras $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ have the same center, for all three notions of commutants.

Finally we turn to strong commutants. For the unbounded ones, $(\mathfrak{M}^{(i)})'_c$, there is no simple inclusion similar to (6.3)-(6.5), because the corresponding Op*-algebras $\mathscr{L}^+(\mathscr{D}^{(i)})$ are not included into each other. However we do get some relations for particular cases.

PROPOSITION 6.5. — For the strong commutants the following relations hold:

i) For the bounded parts: $\mathfrak{M}'_s \subset \overline{\mathfrak{M}}'_s \subset \hat{\mathfrak{M}}'_s$. (6.10)

ii) For the centers: $\mathfrak{M}'_c \cap \mathfrak{M} \subset \overline{\mathfrak{M}}'_c \cap \mathfrak{M}$. (6.11)

iii) If $\mathscr{D}[t_{\mathfrak{M}}]$ is barrelled: $\mathfrak{M}'_c \subset \overline{\mathfrak{M}}'_c$. (6.12)

Proof. — *i*) The inclusion $\mathfrak{M}'_s \subset \overline{\mathfrak{M}}'_s$ is proved as in [25, Section 1.3]. Given $f \in \tilde{\mathscr{D}}[t_{\mathfrak{M}}]$, there is a net $\{ f_\alpha \} \in \mathscr{D}$ such that $f_\alpha \to f$ and $Af_\alpha \to Af$, $\forall A \in \mathfrak{M}$. For $X \in \mathfrak{M}'_s$, this implies $Xf_\alpha \to Xf$ and $AXf_\alpha = XAf_\alpha \to XAf$. Hence $Xf \in D(A)$ for all $A \in \mathfrak{M}$. Furthermore $\{ Xf_\alpha \}$ is Cauchy in each norm $\| \,.\, \|_A$, i. e. $\{ Xf_\alpha \}$ is Cauchy in the topology $t_{\mathfrak{M}}$, hence $Xf \in \tilde{\mathscr{D}}[t_{\mathfrak{M}}]$. As for the other inclusion, the proof is identical except that the net $\{ f_\alpha \}$ converging to $f \in \mathscr{D}(\mathfrak{M})$ may depend on A and is taken in $\tilde{\mathscr{D}}[t_{\mathfrak{M}}]$. Then the argument shows that, for $X \in \overline{\mathfrak{M}}'_s$, $\{ Xf_\alpha \}$ is Cauchy in D(A); thus $Xf \in \mathscr{D}(\mathfrak{M})$ and $AXf = XAf$, i. e. $X \in \hat{\mathfrak{M}}'_s$.

ii) This is proven as Proposition 6.4 *i*).

iii) If $\mathscr{D}[t_{\mathfrak{M}}]$ is barrelled, $\mathfrak{M}'_\sigma = \overline{\mathfrak{M}}'_\sigma$ and $\mathscr{L}^+(\mathscr{D}) \subset \mathscr{L}^+(\tilde{\mathscr{D}})$ by the closed graph theorem; thus $\mathfrak{M}'_c \subset \overline{\mathfrak{M}}'_c$. ■

To conclude this Section, we notice that $\hat{\mathfrak{M}}$ is fully closed, and therefore $\hat{\mathfrak{M}}'_c = \hat{\mathfrak{M}}'$ and $\hat{\mathfrak{M}}'_s = \hat{\mathfrak{M}}'_{\cdot b}$. On the other hand, if $\mathfrak{A}$ is an Op*-algebra, $\mathfrak{A} \subset \overline{\mathscr{L}^+(\mathscr{D})}$, then $\overline{\mathfrak{A}} = \hat{\mathfrak{A}}$ and $\mathfrak{A}'_\square = \mathfrak{A}'_\sigma$. Hence, in that case, there are only three distinct bounded commutants:

$$\mathfrak{A}'_s \subset \mathfrak{A}'_{\cdot b} = \overline{\mathfrak{A}}'_{\cdot b} = \overline{\mathfrak{A}}'_s \subset \mathfrak{A}'_{\square b} = \overline{\mathfrak{A}}'_{\square b} = \mathfrak{A}'_w = \overline{\mathfrak{A}}'_w \qquad (6.13)$$

APPENDIX A

MIXED COMMUTANTS

As mentioned in Section 2, it might be useful to consider also mixed commutants, i. e. commutants that mix strong and weak products or, better, commutants defined in terms of the various mixed multipliers introduced in [*9*]. For a $\ddagger$-invariant subset $\mathfrak{N} \subset \mathfrak{C}(\mathcal{D})$, four different types arise naturally:

i) $C_L(\mathfrak{N}) = \mathfrak{N}'_\Box \cap L^M\mathfrak{N}$ *iii*) $C_{\tilde{L}}(\mathfrak{N}) = \mathfrak{N}'_\Box \cap \tilde{L}^M\mathfrak{N}$

ii) $C_R(\mathfrak{N}) = \mathfrak{N}'_\Box \cap R^M\mathfrak{N}$ *iv*) $C_{\tilde{R}}(\mathfrak{N}) = \mathfrak{N}'_\Box \cap \tilde{R}^M\mathfrak{N}$.

Obviously these sets are not $\ddagger$-invariant, but instead we get:

$$C_L(\mathfrak{N})^\ddagger = C_{\tilde{R}}(\mathfrak{N}), \qquad C_R(\mathfrak{N})^\ddagger = C_{\tilde{L}}(\mathfrak{N}).$$

Thus we may define two $\ddagger$-invariant mixed commutants:

$$\begin{aligned} \mathfrak{N}'_R &= C_R(\mathfrak{N}) \cap C_{\tilde{L}}(\mathfrak{N}) \\ &= \{ X \in \mathfrak{N}'_\Box \mid X, X^\ddagger : \mathcal{D} \to \mathcal{D}(\mathfrak{N}) \} \\ &= \{ X \in \mathfrak{N}'_\Box \mid X^{\ddagger *}Af = AXf, X^*A^\ddagger f = A^\ddagger X^\ddagger f, \forall f \in \mathcal{D}, \forall A \in \mathfrak{N} \} \end{aligned} \tag{A.1}$$

$$\begin{aligned} \mathfrak{N}'_L &= C_L(\mathfrak{N}) \cap C_{\tilde{R}}(\mathfrak{N}) \\ &= \{ X \in \mathfrak{N}'_\Box \mid \mathfrak{N} : \mathcal{D} \to D(X) \cap D(X^\ddagger) \} \\ &= \{ X \in \mathfrak{N}'_\Box \mid XAf = A^{\ddagger *}Xf, X^\ddagger A^\ddagger f = A^*X^\ddagger f, \forall f \in \mathcal{D}, \forall A \in \mathfrak{N} \}. \end{aligned} \tag{A.2}$$

Consequently we get:

$$\mathfrak{N}'_c \subset \mathfrak{N}' = \mathfrak{N}'_R \cap \mathfrak{N}'_L \begin{matrix} \subset \mathfrak{N}'_R \subset \\ \subset \mathfrak{N}'_L \subset \end{matrix} \mathfrak{N}'_\Box \subset \mathfrak{N}'_\sigma. \tag{A.3}$$

The following properties are straightforward:

i) If $\mathfrak{N} \subset \overline{\mathcal{L}^+(\mathcal{D})}$, one gets:

$$C_{\tilde{R}}(\mathfrak{N}) = C_L(\mathfrak{N}) = \mathfrak{N}'_L = \mathfrak{N}'_\Box = \mathfrak{N}'_\sigma, \qquad \mathfrak{N}'_R = \mathfrak{N}'. \tag{A.4}$$

ii) If $\mathfrak{N} \subset \mathcal{B}(\mathcal{H})$, one has:

$$C_R(\mathfrak{N}) = C_{\tilde{L}}(\mathfrak{N}) = \mathfrak{N}'_R = \mathfrak{N}'_\Box = \mathfrak{N}'_\sigma, \qquad \mathfrak{N}'_L = \mathfrak{N}'. \tag{A.5}$$

iii) Thus if $\mathfrak{N} \subset \mathcal{B}(\mathcal{H}) \cap \overline{\mathcal{L}^+(\mathcal{D})}$, all commutants coincide in Eq. (A.3), except $\mathfrak{N}'_c$ in general.

iv) $C_R(\mathfrak{N})$, $C_{\tilde{L}}(\mathfrak{N})$, $\mathfrak{N}'_R$ are vector subspaces of $\mathfrak{C}(\mathcal{D})$, the others not necessarily (because of the non-distributivity of $\mathfrak{C}(\mathcal{D})$).

v) Using the present notation, Proposition 2.3 may be stated as follows: for $\mathfrak{B} = \mathfrak{B}^\ddagger \subset \mathcal{B}(\mathcal{H})$, $\mathfrak{B}_\eta = C_L(\mathfrak{N}) \cap C_{\tilde{L}}(\mathfrak{N})$; then Eq. (2.14) follows from the relation $\mathfrak{B}' = \mathfrak{B}'_R \cap \mathfrak{B}'_L$.

As for the topological properties of the mixed commutants, the proof of [*10*, Proposition 5.7] shows that:

i) $C_R(\mathfrak{N})$, $C_{\tilde{L}}(\mathfrak{N})$ and $\mathfrak{N}'_R$ are complete for the quasi-uniform topologies $\tau_{*,f}(\mathfrak{N})$;

ii) $C_{\tilde{R}}(\mathfrak{N})$, $C_L(\mathfrak{N})$, $\mathfrak{N}'_L$ need *not* be complete; their completions are contained in $\mathfrak{N}'_\Box$

but each of them is $\tau_{*,f}$-closed in the corresponding space of mixed multipliers: $C_{\tilde{R}}(\mathfrak{N})$ in $\tilde{R}^M\mathfrak{N}$, $C_L(\mathfrak{N})$ in $L^M\mathfrak{N}$, $\mathfrak{N}'_L$ in $L^M\mathfrak{N} \cap \tilde{R}^M\mathfrak{N}$.

Corresponding to the mixed commutants, one may now define mixed bicommutants. One ends up with four bicommutants containing $\mathfrak{N}$, but mutually not comparable: $\mathfrak{N}'_{\cdot}{}'_{\cdot}$, $\mathfrak{N}'_L{}'_R$, $\mathfrak{N}'_R{}'_L$, $\mathfrak{N}'_\square{}'_\square$.

Combining the results above with those of Section 4, we get:

i) $\mathfrak{N}'_L{}'_R$ is complete for $\tau_{*,f}(\mathfrak{N}'_L)$, hence closed in $\mathfrak{N}'_L{}'_\square$;

ii) $\mathfrak{N}'_R{}'_L$ need not be complete in $\tau_{*,f}(\mathfrak{N}'_R)$, $\widetilde{\mathfrak{N}'_R{}'_L} \subset \mathfrak{N}'_R{}'_\square$, but $\mathfrak{N}'_R{}'_L$ is closed in $L^M(\mathfrak{N}'_R) \cap \tilde{R}^M(\mathfrak{N}'_R)$.

Finally, the analysis of Section 4 may be extended to mixed bicommutants. Let again $\mathfrak{B}$ be a *-algebra of bounded operators containing 1, as in Proposition 4.3. First $\mathfrak{B}'_{\cdot}{}'_R = \mathfrak{B}'_L{}'_R$ and $\mathfrak{B}'_\square{}'_\square = \mathfrak{B}'_R{}'_\square$ by (A.6). Next, we have:

$$\mathfrak{B} \subset \mathfrak{B}'_{\cdot}{}'_{\cdot} \subset \mathfrak{B}'_{\cdot}{}'_L \subset M^w(\mathfrak{B}'_{\cdot}) \cap (\mathfrak{B}')'_{\cdot}.$$

Thus, using Lemma 4.4, we get

$$\mathfrak{B}'_{\cdot}{}'_{\cdot} = \mathfrak{B}'_{\cdot}{}'_L = \overline{\mathfrak{B}[\tau_f(\mathfrak{B}'_{\cdot})]}^s \subset \mathfrak{B}'_L{}'_R.$$

The final picture is the following:

$$\mathfrak{B} \subset \mathfrak{B}'_R{}'_L \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} \mathfrak{B}'_{\cdot}{}'_{\cdot} = \mathfrak{B}'_{\cdot}{}'_L \subset \mathfrak{B}'_L{}'_R \\ \mathfrak{B}'_\square{}'_\square \end{array} \begin{array}{c} \subset \\ \subset \end{array} \mathfrak{B}'_{\cdot}{}'_\square \qquad (A.6)$$

If we assume, in addition, that $\mathfrak{B}$ leaves $\mathscr{D}$ invariant, so that $\mathfrak{B}'_{\cdot} = \mathfrak{B}'_\square = \mathfrak{B}'_\sigma$, then Proposition 4.3 and the relation (A.6) give finally:

$$\mathfrak{B} \subset \mathfrak{B}'_R{}'_L = \mathfrak{B}'_{\cdot}{}'_{\cdot} = \overline{\mathfrak{B}[\tau_f]} \subset \mathfrak{B}'_L{}'_R \subset \mathfrak{B}'_\square{}'_\square \subset \overline{\mathfrak{B}[s^*]} \subset \mathfrak{B}'_\sigma{}'_\sigma = (\mathfrak{B}')'_\sigma \qquad (A.7)$$

where $\tau_f \equiv \tau_f(\mathfrak{B}'_{\cdot}) = \tau_f(\mathfrak{B}'_\square)$.

A similar discussion may be given for the general case $\mathfrak{N} \supset \mathfrak{B}$ as in Section 4, but it is straightforward and we shall omit it.

APPENDIX B

INVERTIBILITY OF OPERATORS AND SYMMETRIC PARTIAL Op*-ALGEBRAS

In Section 4 we have defined three different types of symmetric partial Op*-algebras: *-symmetric, weakly symmetric and strongly symmetric. Yet none of these definitions coincides *a priori* with the abstract one given in I [9, Section 2D], because we have used usual operator inverses, and not inverses *within* the given partial Op*-algebra. But, as we shall see now, in fact the two approaches are equivalent.

Given $A \in \mathfrak{C}(\mathscr{D})$, we say that:

i) A is *invertible* (in the usual sense) iff there exists a closed operator $B \equiv A^{-1}$ such that $BA = 1 \upharpoonright D(A)$ and $AB = 1 \upharpoonright D(B)$, where $D(B) = \operatorname{Ran} A$. Notice that D(B) need not contain $\mathscr{D}$.

ii) A is *weakly invertible* if there exists $B \in \{A\}'_{\square}$ such that $B \square A = A \square B = 1$.

iii) A is *strongly invertible* if there exists $B \in \{A\}'_{\cdot}$ such that $B.A = A.B = 1$.

What are the relations between these three notions? One is obvious: if A is strongly invertible, it is also weakly invertible, and the two inverses coincide. We collect the other results in a proposition.

PROPOSITION B.1. — Let $A \in \mathfrak{C}(\mathscr{D})$. Then:

i) If A is strongly invertible, with strong inverse B, then A is invertible and $A^{-1} = B$.

ii) If A is invertible *and* weakly invertible, with weak inverse B, then $A^{-1} \subset B^{+*}$.

iii) If A is invertible, and $A^{-1} \in \mathfrak{C}(\mathscr{D})$, then A is also weakly and strongly invertible, and the three inverses coincide.

iv) The same is true, in particular, if A^{-1} is bounded.

Proof. — *i*) We have $ABf = BAf = f$, $\forall f \in \mathscr{D}$.

Let $g \in D(A)$. There exists a sequence $\{g_n\} \in \mathscr{D}$ such that $g_n \to g$ and $Ag_n \to Ag$. Since $BAg_n = g_n$ converges and B is closed, $Ag \in D(B)$ and $BAg = g$. Similarly $ABh = h$ for all $h \in D(B)$. Hence A is invertible and $A^{-1} = B$.

ii) If A is only weakly invertible, we have $A^{+*}Bf = B^{+*}Af = f$, $\forall f \in \mathscr{D}$, and the preceding argument fails. If we assume in addition that A is invertible, then A^{-1} and B^{+*} coincide on $\operatorname{Ran}(A \upharpoonright \mathscr{D})$. We show this is a core for A^{-1}.

Given $g \in D(A^{-1})$, we have $A^{-1}g \in D(A)$, and there exists a sequence $\{k_n\} \in \mathscr{D}$ such that $k_n \to A^{-1}g$, $Ak_n \to g$, i. e. $Ak_n \to g$ in the graph norm of A^{-1}. Thus we get:

$$A^{-1} = \overline{A^{-1} \upharpoonright \operatorname{Ran}(A \upharpoonright \mathscr{D})} = \overline{B^{+*} \upharpoonright \operatorname{Ran}(A \upharpoonright \mathscr{D})} \subset B^{+*}.$$

iii) Since $A^{-1} \in \mathfrak{C}(\mathscr{D})$, we have $\mathscr{D} \subset D(A) \cap D(A^{-1})$. Thus we get $A^{-1}Af = AA^{-1}f = f$, $\forall f \in \mathscr{D}$, which implies

$$A.A^{-1} = A^{-1}.A = A \square A^{-1} = A^{-1} \square A = 1.$$

iv) Obvious, since $\mathscr{B}(\mathscr{H}) \subset \mathfrak{C}(\mathscr{D})$, for any $\mathscr{D}$.

Let us go back to symmetric partial Op*-algebras. If $\mathfrak{M}$ is a weakly symmetric one, we have seen in the proof of Proposition 5.8 that the condition $A^{+} \in L^{w}(A)$ implies $C \equiv 1 \hat{+} A^{+} \square A = 1 + A^{*}A$. So C is invertible, with bounded inverse C^{-1}. By Proposition B.1 *iv*), it is also weakly and strongly invertible, and all three inverses coincide. The same holds if $\mathfrak{M}$ is strongly symmetric.

In conclusion, in the definition of weakly and strongly symmetric algebras, we may as well use weak, resp. strong, inverses, we get the same objects, which are indeed the symmetric partial Op*-algebras in the sense of the abstract definition of I, Section 2D.

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