ANNALES DE L'I. H. P., SECTION A

MARIA HOFFMANN-OSTENHOF
THOMAS HOFFMANN-OSTENHOF
JÖRG SWETINA

Asymptotics and continuity properties near infinity of solutions of Schrödinger equations in exterior domains

Annales de l'I. H. P., section A, tome 46, nº 3 (1987), p. 247-280
<http://www.numdam.org/item?id=AIHPA_1987__46_3_247_0>

Ann. Inst. Henri Poincaré,
Vol. 46, n° 3, 1987, p. 247-280. *Physique théorique*

Asymptotics and continuity properties near infinity of solutions of Schrödinger equations in exterior domains

by

Maria HOFFMANN-OSTENHOF (*)
Institut für Mathematik, Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria
and
Institut für Theoretische Physik, Universität Wien,
Boltzmanngasse 5, A-1090 Wien, Austria

and

Thomas HOFFMANN-OSTENHOF and Jörg SWETINA (**)
Institut für Theoretische Chemie und Strahlenchemie,
Universität Wien, Währingerstr. 17, A-1090 Wien, Austria

ABSTRACT. — Let $(-\Delta + V - E)\psi = 0$ in $\Omega_R = \{ x \in \mathbb{R}^n \mid |x| > R \}$, $\psi \in L^2(\Omega_R)$, where $E < 0$ and $V = V_1(|x|) + V_2(x)$ with V_1, V_2 tending to zero for $|x| \to \infty$ and satisfying suitable regularity assumptions. Further let $(-\Delta + V_1(|x|) - E)v(|x|) = 0$ for $|x| > R$ where $v > 0$ and $v \to 0$ for $|x| \to \infty$.

Previous results on the asymptotics on ψ/v for $n = 2$ are here extended to the n-dimensional case: It is shown that $\frac{\psi}{v}(|x|.x/|x|)$ satisfies certain regularity properties uniformly for $|x| \to \infty$ as a map from S^{n-1} to $\mathbb{R}$. Furthermore using a certain scaling it is shown that the asymptotic behaviour of ψ/v can be characterized by eigenfunctions of the isotropic $(n-1)$-dimensional harmonic oscillator.

(*) Partially supported by Bundesministerium für Wissenschaft und Forschung, Austria.
(**) Supported by « Fonds zur Förderung der wissenschaftlichen Forschung in Osterreich », Project no. P4829, and supported by Stiftung Volkswagenwerk.

RÉSUMÉ. — Soit $(-\Delta + V - E)\psi = 0$ dans $\Omega_R = \{ x \in \mathbb{R}^n \| x | > R \}$, $\psi \in L^2(\Omega_R)$, où $E < 0$ et $V = V_1(|x|) + V_2(x)$ avec V_1, V_2 tendant vers zero pour $|x| \to \infty$ et satisfaisant des conditions de régularité convenables. Soit en outre $(-\Delta + V_1(|x|) - E)v(|x|) = 0$ pour $|x| > R$ où $v > 0$ et $v \to 0$ pour $|x| \to \infty$. On étend ici au cas de la dimension n des résultats antérieurs sur le comportement asymptotique de ψ/v pour $n = 2$. On montre que $\frac{\psi}{v}(|x| \cdot x/|x|)$ satisfait certaines propriétés de régularité comme application de S^{n-1} dans $\mathbb{R}$ uniformément pour $|x| \to \infty$. En outre, en utilisant un certain changement d'échelle, on montre que le comportement asymptotique de ψ/v peut être caractérisé au moyen des fonctions propres de l'oscillateur harmonique isotrope à $n-1$ dimensions.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In [*13*] we considered the 2-dimensional Schrödinger equation in exterior domains. Here we shall derive similar results for the n-dimensional case. Our results will cover also some examples of physical relevance, for instance eigenstates of a Hamiltonian, that describes a one electron molecule with fixed nuclei. The case $n \geq 3$ forces us to develop some new techniques but we shall rely on results obtained in [*13*]. This paper is not self-contained and we shall frequently refer to [*13*]. See also [*13*] for motivation.

We start by describing the problem in the n-dimensional setting: We consider real valued $W^{2,2}$-solutions $\psi(x)$ of

$$(-\Delta + V - E)\psi = 0 \quad \text{for} \quad x \in \Omega_R \tag{1.1}$$
$$\Omega_R = \{ x \in \mathbb{R}^n : |x| = r > R \}, \quad R > 0, \quad n \geq 2.$$

Here the Sobolev space $W^{2,2}(\Omega_R)$ is defined as in [*7*]. Throughout the paper we assume that

$$E < 0 \tag{1.2}$$

and that $V(x)$ satisfies the following assumptions in Ω_R:

$$V(x) \text{ is real valued and continuous} \tag{A.1}$$

$$\lim_{|x| \to \infty} V(x) = 0. \tag{A.2}$$

(1.2), (A.1) and (A.2) imply that we can choose R so that

$$\inf_{x \in \Omega_R} \left(V(x) + \frac{(n-1)(n-3)}{4r^2} - E \right) > 0. \tag{A.3}$$

The conditions on V imply that there is a unique selfadjoint operator associated to $-\Delta+V-E$ whose corresponding quadratic form is positive definite with form core $C_0^\infty(\Omega_R)$. This guarantees that the Dirichlet problem in Ω_R is uniquely solvable given ψ on $\partial\Omega_R$. Furthermore our assumptions imply that ψ has continuous derivatives in Ω_R [7].

We split $V(x)$ such that

$$V(x) = V_1(r) + V_2(x) \tag{1.3}$$

and assume that V_1 and V_2 satisfy the assumptions (A) separately. As in [13] we consider the radial comparison problem

$$(-\Delta + V_1(r) - E)v(r) = 0 \quad \text{for} \quad r > R. \tag{1.4}$$

Since V_1 satisfies (A) there exists $v(r) > 0$ such that $v \in L^2(\Omega_R)$ and $v(R) > 0$. In the following we shall investigate the regularity—and asymptotic properties of the function

$$u = \frac{\psi}{v}. \tag{1.5}$$

We start with a result given essentially in [12], see also Thoe [17] who obtained related results with different methods independently.

THEOREM 1. — Let $V_1(r)$ and $V_2(x)$ satisfy the assumption (A) and assume that in Ω_R

i) $V_1(r)$ is continuously differentiable and

$$\left|\frac{dV_1}{dr}\right| \leq cr^{-\varepsilon-1} \quad \text{for some } c, \varepsilon > 0.$$

ii) $|V_2| \leq c_0 r^{-1-\gamma_0}$ for some $c_0, \gamma_0 > 0$.

Assume that ψ and v satisfy (1.1) and (1.4) respectively, then for some $0 < c_- \leq c_+ < \infty$ $|u| \leq c_+$ and $\left(\int_{S^{n-1}} u^2 d\sigma\right)^{1/2} \geq c_-$ for $r \geq R$. Here $d\sigma$ denotes normalized integration over the unit sphere S^{n-1}.

Proof. — The proof of the lower bound was given under less restrictive conditions in [12] and the proof of the upper bound given for the 2-dimensional case in [13] carries over. □

This result tells us that in an averaged sense ψ and v have the same asymptotics. To obtain more detailed informations on the properties of u we consider u as a map from $(R, \infty) \times S^{n-1}$ to $\mathbb{R}$, write $u = u(ry)$ where $y = x/r \in S^{n-1}$ and consider the regularity properties of $u(ry)$ for fixed $r > R$ as a map from S^{n-1} to $\mathbb{R}$.

Let us first introduce hyperspherical coordinates in $\mathbb{R}^n$, $n \geq 3$:

$$\begin{aligned} x_1 &= r \cos \theta_1 \\ x_j &= r\left(\prod_{i=1}^{j-1} \sin \theta_i\right) \cos \theta_j, \qquad 2 \leq j \leq n-2, \\ x_{n-1} &= r\left(\prod_{i=1}^{n-2} \sin \theta_i\right) \cos \phi \\ x_n &= r\left(\prod_{i=1}^{n-2} \sin \theta_i\right) \sin \phi \end{aligned} \tag{1.6}$$

with $0 \leq \theta_j \leq \pi$, $j = 1, \ldots, n-2$, $-\pi \leq \phi \leq \pi$. For our purposes it will be advantageous to replace these angles by

$$\xi_i = \theta_i - \frac{\pi}{2}, \qquad 1 \leq i \leq n-2, \qquad \phi = \xi_{n-1}. \tag{1.7}$$

We shall denote by $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1})$ a vector in $\overline{Q}$

$$Q = \left\{ \xi \in \mathbb{R}^{n-1} : -\frac{\pi}{2} < \xi_i < \frac{\pi}{2}, 1 \leq i \leq n-2, -\pi < \xi_{n-1} < \pi \right\}. \tag{1.8}$$

The Laplacian reads in these coordinates

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{L^2}{r^2} \tag{1.9}$$

with the Laplace-Beltrami operator $-L^2$ given by

$$\begin{aligned} -L^2 = &\frac{\partial^2}{\partial \xi_1^2} - (n-2) \operatorname{tg} \xi_1 \frac{\partial}{\partial \xi_1} \\ &+ \sum_{j=1}^{n-2}\left(\prod_{i=1}^{j} \cos \xi_i\right)^{-2}\left(\frac{\partial^2}{\partial \xi_{j+1}^2} - (n-j-2) \operatorname{tg} \xi_{j+1} \frac{\partial}{\partial \xi_{j+1}}\right). \end{aligned} \tag{1.10}$$

In order to state regularity results for $u(ry)$, $y \in S^{n-1}$ we introduce an atlas on S^{n-1} whose charts are C^ω (real analytic) compatible. Let Q be given by (1.8) and define for $\xi \in Q$

$$\begin{aligned} \Phi^{-1}(\xi) = &\cos\left(\xi_1 + \frac{\pi}{2}\right)e_1 + \sum_{j=1}^{n-3}\prod_{i=1}^{j} \sin\left(\xi_i + \frac{\pi}{2}\right) \cos\left(\xi_{j+1} + \frac{\pi}{2}\right)e_{j+1} + \\ &+ \prod_{i=1}^{n-2} \sin\left(\xi_i + \frac{\pi}{2}\right)(\cos \xi_{n-1} e_{n-1} + \sin \xi_{n-1} e_n) \end{aligned}$$

where the e_j are the canonical basis of $\mathbb{R}^n$, i. e.

$$e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, \quad e_n = (0, 0, \ldots, 0, 1).$$

Denoting $\Phi^{-1}(Q) = U$ and noting that $\overline{U} = S^{n-1}$ we obtain from the chart (U, Φ) by rotations the charts (U_i, Φ_i), $i \in I$ some index set, $\bigcup_{i \in I} U_i = S^{n-1}$, and the collection of these charts is our C^ω-atlas. For a function $f: S^{n-1} \to \mathbb{R}$ we say then as usually $f \in C^k(S^{n-1})$ if $\forall i \in I$, $(f \circ \Phi_i^{-1})(\xi) \in C^k(Q)$. We also need Hölder continuity and say for $f: S^{n-1} \to \mathbb{R}$, $f \in C^{k,\alpha}(S^{n-1})$, for some $\alpha \in (0, 1]$ if $\forall i \in I$, $(f \circ \Phi_i^{-1})(\xi) \in C^{k,\alpha}(Q)$. Here C^k, $C^{k,\alpha}$ have the usual meaning [7]. If $\alpha = 1$ we talk of Lipschitz continuity. Analogously we define $C^\infty(S^{n-1})$ and $C^\omega(S^{n-1})$ (real analyticity).

In order to describe the uniform behaviour with respect to r of the regularity properties of $u(ry)$ we introduce the following definitions:

DEFINITION 1.1. — Let $g: \Omega_R \to \mathbb{R}$ be continuous. $g \in C^{k,\alpha}_{u,R}(S^{n-1})$ means that $\forall R_0 > R$

$$\{ g(r\Phi_i^{-1}) \mid r \geq R_0, i \in I \}$$

is a uniformly bounded set of $C^{k,\alpha}(Q)$ functions. Analogously $C^\infty_{u,R}(S^{n-1})$ and $C^\omega_{u,R}(S^{n-1})$ is defined.

Obviously we could have chosen any other equivalent C^ω-atlas on S^{n-1} to describe the regularity properties of u, for instance we could have used stereographic projection and in fact in section 2 we will use some kind of stereographic projection.

After these definitions we can state our regularity results for u in the n-dimensional case:

THEOREM 2. — Let V_1, V_2 and V satisfy the assumptions of Theorem 1.

a) If $\gamma_0 > \frac{1}{2}$ set $\delta = 1$, while if $\gamma_0 \leq \frac{1}{2}$ take $\delta \in (0, 2\gamma_0)$, then $u \in C^{0,\delta}_{u,R}(S^{n-1})$ and $\lim_{r \to \infty} u(ry) \equiv A(y)$ exists $\forall y \in S^{n-1}$, and $r^{\gamma_0} \mid u - A \mid$ is bounded in Ω_R. Furthermore $A \in C^{0,\delta}(S^{n-1})$.

b) Suppose that for $0 \leq k \leq m$

$$r^{1+\gamma_k} V_2 \in C^k_{u,R}(S^{n-1})$$

where $\gamma_k \geq \frac{1}{2} + \nu$ (with some $\nu > 0$) for $0 \leq k \leq m-1$ and $\gamma_m > 0$. Let $\delta_k = 1$ for $k \leq m-1$ and $0 < \delta_m < 2\gamma_m$ if $\gamma_m \leq 1/2$, otherwise $\delta_m = 1$, then $u \in C^{m,\delta_m}_{u,R}(S^{n-1})$.

Let ∂^k_ξ denote any partial derivative of order k, then for $k \leq m$ $\lim_{r \to \infty} \partial^k_\xi u(r\Phi_i^{-1}(\xi))$ exists $\forall \xi \in Q$, $\forall i \in I$ and $A \in C^{m,\delta_m}(S^{n-1})$. In addition for $0 \leq k \leq m-2$, $\mu_k = \min_{1 \leq i \leq k} (1, \gamma_1, \ldots, \gamma_k)$

$$\mid \partial^k_\xi (u(r\Phi_i^{-1}(\xi)) - A(\Phi_i^{-1}(\xi))) \mid r^{\mu_k} \leq C_m \tag{1.11}$$

$\forall \xi \in Q$, $\forall i \in I$, $\forall r \geq R_0 > R$ with some $0 < C_m < \infty$. In particular if for some $\nu > 0$, $r^{\nu+3/2}V_2 \in C^{\infty}_{u,R}(S^{n-1})$, then $u \in C^{\infty}_{u,R}(S^{n-1})$ and $A \in C^{\infty}(S^{n-1})$.

c) If for some $\alpha > 1/2$, $r^{1+\alpha}V_2 \in C^{\omega}_{u,R}(S^{n-1})$, then $u \in C^{\omega}_{u,R}(S^{n-1})$ and $A \in C^{\omega}(S^{n-1})$. Furthermore $\forall k \in \mathbb{N}$

$$| \partial^k_\xi(u(r\Phi_i^{-1}(\xi)) - A(\Phi_i^{-1}(\xi))) | r^a \leq C_k, \qquad a = \min(1, \alpha) \qquad (1.12)$$

$\forall \xi \in Q$, $\forall i \in I$, $\forall r \geq R_0 > R$ with some $0 < C_k < \infty$.

Remark 1.1. — As noted already in [*13*] these results reflect the fact that we consider for $r \to \infty$ a non-uniformly elliptic problem as is obvious if we write (1.1) or the corresponding equation for u in the spherical coordinates (r, ξ) so that with (1.9)

$$\left(- \frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{L^2}{r^2} + V - E \right)\psi(r\Phi^{-1}(\xi)) = 0. \qquad (1.13)$$

That part *a)* of Theorem 2 is not far from optimal is demonstrated in [*13*] for the two-dimensional case with an explicit example.

Next we consider the asymptotics of u in more detail. We assume that the conditions of part *c)* of Theorem 2 hold. Hence $u \in C^{\omega}_{u,R}(S^{n-1})$ and $A \in C^{\omega}(S^{n-1})$. For each $y \in S^{n-1}$ we consider asymptotic regions $\left\{ x \in \Omega_R \,\middle\|\, \left| y - \frac{x}{r} \right| < Kr^{-\beta}, r > \overline{R} \right\}$ where $\beta \in (0, 1/2]$, $K \in (0, \infty)$ and $\overline{R}$ large. Now we rotate our coordinate system so that $y = e_{n-1} = (0, \ldots, 0, 1, 0)$ and consider $A(\Phi^{-1}(\xi))$ in a neighbourhood of $\xi = 0$. For simplicity we shall often write $A(\xi)$ resp. $u(r, \xi)$ instead of $u(r\Phi^{-1}(\xi))$. $A(\xi)$ is then because of part *c)* of Theorem 2 real analytic and we have near $\xi = 0$

$$A(\xi) = P_M(\xi) + 0(|\xi|^{M+1}) \qquad (1.14)$$

where P_M, the first nonvanishing term in the Taylor expansion of $A(\xi)$, is a homogeneous polynomial of degree M, M a nonnegative integer and

$$P_M = \sum_{l_1 + l_2 + \ldots l_{n-1} = M} a_{l_1, l_2 \ldots l_{n-1}} \xi_1^{l_1} \xi_2^{l_2} \ldots \xi_{n-1}^{l_{n-1}}. \qquad (1.15)$$

The following theorem describes how the asymptotics of u in the domains

$$D_\beta = \{ r\Phi^{-1}(\xi) \in \Omega_{\overline{R}} \,\|\, \xi | < r^{-\beta} \} \qquad \beta \in (0, 1/2), \ \overline{R} \text{ large}$$
$$D^k_{1/2} = \{ r\Phi^{-1}(\xi) \in \Omega_{\overline{R}} \,\|\, \xi | < kr^{-1/2} \} \qquad k < \infty \qquad (1.16)$$

is related to the assumptions (1.14) and (1.15) on A.

THEOREM 3. — Let V, V_1 and V_2 satisfy the conditions of part *c)* of

Theorem 2. Suppose $A(\xi)$ satisfies (1.14) with P_M given by (1.15). Then in D_β defined by (1.16) for some $\varepsilon > 0$

$$u(r, \xi_1, \xi_2 \dots \xi_{n-1}) =$$
$$= (2\sqrt{|E|}\, r)^{-M/2} \sum_{l_1+l_2+\dots l_{n-1}=M} a_{l_1,l_2,\dots l_{n-1}} \prod_{j=1}^{n-1} H_{l_j}(b r^{1/2} \xi_j)\, .$$
$$.(1 + 0(r^{-\varepsilon})) + 0(r^{-\beta M - \min(a-1/2,\beta)}) \quad (1.17)$$

where

$$b = 2^{-1/2} |E|^{1/4} \quad (1.18)$$

and the H_l are the usual Hermite polynomials of degree l:

$$H_l(z) = \sum_{k=0}^{[l/2]} b_{k,l} (2z)^{l-2k}, \quad (1.19)$$

with $[l/2]$ integer part of $l/2$ and

$$b_{k,l} = (-1)^k \frac{l!}{k!\,(l-2k)!}. \quad (1.20)$$

(1.17) implies (see section 3).

COROLLARY 1. — Suppose that A is given by (1.14) then for

$$\sum_{i=1}^{n-1} z_i^2 \le k^2 < \infty, \lim_{r\to\infty} r^{M/2} u(r, z_1/b\sqrt{r}, z_2/b\sqrt{r}, \dots, z_{n-1}/b\sqrt{r}) =$$
$$= \left(\frac{1}{2b}\right)^M \sum_{l_1+l_2+\dots l_{n-1}=M} a_{l_1,l_2\dots l_{n-1}} \prod_{j=1}^{n-1} H_{l_j}(z_j) \quad (1.21)$$

and $(u - A) r^{M/2}$ is bounded in $D_{1/2}^{k/b}$.

Let us discuss the results given in Theorem 3. The corresponding 2-dimensional results have been given in [*13*]. For this case the r. h. s. of (1.17) reduces to a single Hermite polynomial and we could therefore draw in a direct way conclusions about the asymptotics of nodal lines and nodal domains since Hermite polynomials have separated nondegenerate zeros. The situation is a lot more complicated for $n \ge 3$ since then with $z = (z_1, z_2 \dots z_{n-1})$

$$\mathscr{H}_M(z) \equiv \sum_{l_1+l_2+\dots l_{n-1}=M} a_{l_1,l_2,\dots l_{n-1}} H_{l_1}(z_1) \dots H_{l_{n-1}}(z_{n-1}) \quad (1.22)$$

might itself have a complicated nodal structure. Some observations are however straightforward: We have with

$$|z| = \left(\sum_{i=1}^{n-1} z_i^2\right)^{1/2} \quad \text{and} \quad \Delta = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial z_i^2}$$

$$(-\Delta + |z|^2)e^{-|z|^2/2}\mathscr{H}_M(z) = (2M + n - 1)\mathscr{H}_M e^{-|z|^2/2}. \quad (1.23)$$

Hence $\mathscr{H}_M e^{-|z|^2/2}$ is an eigenfunction of the $(n-1)$-dimensional quantum mechanical isotropic harmonic oscillator. Although a detailed analysis of the nodes of such $\mathscr{H}_M$'s is not available the fact that $\mathscr{H}_M$ satisfies (1.23) implies (see [*1*] or [*11*]) that $\mathscr{H}_M$ changes sign in every ball containing z_0 with $\mathscr{H}_M(z_0) = 0$. The same is true in Ω_R for ψ and hence for u itself but in general for ψ or u restricted to some hypersurface e. g. the surface of a ball in $\mathbb{R}^n$ this will not be true.

As mentioned in [*13*] Theorem 3 is nontrivial even if we consider (1.1) with $V_2 \equiv 0$. To illustrate this we consider

$$(-\Delta + V_1(r) - E)\psi = 0 \quad \text{in} \quad \Omega_R \quad (1.24)$$

where ψ is assumed to be nonradial. Certainly if we set

$$\psi(ry) = \sum_{l=0}^{M} c_l f_l(r) Y^{(l)}(y), \qquad y \in S^{n-1} \quad (1.25)$$

where each $f_l \in L^2(\Omega_R)$ satisfies

$$\left(-\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr} + \frac{l(l+n-2)}{r^2} + V_1(r) - E\right)f_l(r) = 0$$

and where the $Y^{(l)}$ are the usual surface harmonics (see section 2) then ψ will satisfy (1.24). If we pick $f_0(r)$ positive (this corresponds to $v(r)$ of (1.4)) we can demonstrate the results of Theorem 3 by choosing M and the c_l's appropriately and by selecting the $Y^{(l)}$ so that

$$A(y) = \lim_{r\to\infty} \frac{1}{f_0(r)} \sum_{l=0}^{M} c_l f_l Y^{(l)}(y)$$

has zeros of prescribed order. A physical relevant example is for instance the nonrelativistic Hydrogen atom where one can explicitly simulate the findings of Theorem 3 for eigenfunctions due to the well-known degeneracies of excited eigenvalues. (See any textbook of quantum mechanics.)

More on the explicit structure of nodes and nodal domains will be given in a forthcoming paper [*10*] where a suitable generalization of

Corollary 1 of [13] on the growth properties of nodal domains is established. For the 2-dimensional case it is shown in [9] (by using the scaling given in Corollary 1) that the nodal lines of ψ look, roughly speaking, asymptotically either like straight lines or like branches of parabolas.

Most of the relevant literature on asymptotics of solutions of exterior problems of Schrödinger type has been cited in [13]. The recent results of Herbst [8] and Froese and Herbst [6] provide some new insights in the exponential decay of such solutions in cones for a wide class of potentials. Local properties of nodes have been investigated recently by Cafarelli and Friedman [3] who generalize results of Bers [3] and Cheng [4].

In section 2 we shall prove Theorem 2. The major steps will involve some estimates which enable us to use the results of [13] together with some rather involved arguments to set up the iterations which are necessary for the proofs of part *b*) and *c*) of Theorem 2.

In section 3 we prove Theorem 3. Similarly as in [13] we analyse iterations of an integro-differential equation for $u(r, \xi)$. Here again the case $n > 2$ is more involved than the two-dimensional case.

We have profitted during the beginning of our work on this subject from many discussions with K. Yajima. We gratefully acknowledge helpful comments by P. Michor and W. Thirring. Some ideas that appear in this paper stem from thesis of J. S. (1984, unpublished).

2. PROOF OF THEOREM 2

Our assumptions on V imply via standard techniques for elliptic PDE's [7] that ψ is $C^1(\overline{\Omega}_{R_0})$ for any $R_0 > R$. Since $v(r) > 0$ in Ω_R this implies also that u is locally differentiable in Ω_{R_0} and hence that for each finite fixed r, $u(r.) \in C^1(S^{n-1})$. Hence the problem is to establish uniformity with respect to r. The main strategy will be to reduce the problem to the two-dimensional case for which the results derived in [13] are available.

First we shall derive a lemma which permits us to draw conclusions on the regularity of a continuous function $f: S^{n-1} \to \mathbb{R}$ once the regularity along geodesics in S^{n-1}, i. e. « great circles » is known. As is easily seen, each great circle in S^{n-1} can be expressed by a parametrized curve γ_{v_1,v_2} in S^{n-1} of the form

$$\gamma_{v_1,v_2}(t) = v_1 \cos t + v_2 \sin t \quad \text{with} \quad t \in [-\pi, \pi] \tag{2.1}$$

where v_1 and v_2 are orthogonal unit vectors in $\mathbb{R}^n$. By $\mathcal{F}$ we denote this family of unit speed geodesics

$$\mathcal{F} = \{ \gamma_{v_1,v_2} : v_1, v_2 \in \mathbb{R}^n, (v_1, v_2) = 0, | v_1 | = | v_2 | = 1 \} .$$

LEMMA 2.1. — Let $f: S^{n-1} \to \mathbb{R}$ be continuous.

a) Suppose that $\forall \gamma \in \mathcal{F}$, $f \circ \gamma \in C^k(-\pi, \pi)$ for some $k \in \mathbb{N} \cup \{0\}$, then $f \in C^k(S^{n-1})$.

In addition if for some $\alpha \in (0, 1]$ and $\forall \gamma \in \mathcal{F}$,

$$\left| \frac{d^k}{dt^k} f(\gamma(t_0)) - \frac{d^k}{dt^k} f(\gamma(t_1)) \right| \leq C_{k,\alpha} | t_0 - t_1 |^\alpha \quad \text{for} \quad t_0, t_1 \in [-\pi, \pi] \quad (2.2)$$

then $f \in C^{k,\alpha}(S^{n-1})$.

b) Suppose there exist constants $C, \delta > 0$ so that for all $k \in \mathbb{N}$ and $\forall \gamma \in \mathcal{F}$

$$\left| \frac{d^k}{dt^k} f(\gamma(t)) \right| \leq C k!\, \delta^{-k} \quad (2.3)$$

then

$$f \in C^\omega(S^{n-1}).$$

Proof. — It will be convenient to use some kind of stereographic projection: Let $U = \{ x \in S^{n-1} : x_n < -d \}$, $0 < d < 1$, and let $E^{(n-1)} = \{ x \in \mathbb{R}^n : x_n = -1 \}$. Obviously $E^{(n-1)}$ is the tangent plane of S^{n-1} in the point $(0, 0, \ldots 0, -1)$. Let

$$(x_1, x_2, \ldots, x_{n-1}) \in X = \left\{ x \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i^2 < d^{-2} - 1 \right\}$$

and define

$$\phi^{-1} : X \to U \subset S^{n-1} \quad \text{by} \quad \phi^{-1}(x_1, \ldots, x_{n-1}) = \left(\sum_{i=1}^{n-1} x_i^2 + 1 \right)^{-1/2} (x_1, \ldots, x_{n-1}, -1)$$

then it is straightforward to see that if we draw a ray from the origin to a point $(x_1, \ldots, x_{n-1}, -1) \in E^{(n-1)}$ with $\sum_{i=1}^{n-1} x_i^2 < d^{-2} - 1$ that this ray hits S^{n-1} at the point $\phi^{-1}(x_1, x_2, \ldots, x_{n-1})$. Furthermore ϕ maps a geodesic in $U \subset S^{n-1}$ into a straight line in X. By rotations we get $U_i \subset S^{n-1}$, $i \in I$, I some index set, such that $\bigcup_{i \in I} U_i = S^{n-1}$ and $\phi_i^{-1} : X \to U_i \subset S^{n-1}$. The charts (U_i, ϕ_i), $i \in I$ constitute a C^ω-atlas (ω-equivalent to the one of the preceding section).

To verify *a*) of Lemma 2.1 we first show that $\forall i \in I$, $f \circ \phi_i^{-1} \in C^k(X)$. Let $a \in X$, then $\forall b \in \mathbb{R}^{n-1}$ with $|b| = 1$ we have $a + bt \in X$ for $|t| \leq \varepsilon(a)$, where $\varepsilon(a)$ depends only on the distance of a from ∂X. For any ϕ_i and a given as above there exists a $\gamma \in \mathcal{F}$ such that $\phi_i^{-1}(a + bt) = \gamma(t)$ $\forall |t| \leq \varepsilon(a)$.

Since $f \circ \gamma \in C^k$ and $(f \circ \gamma)(t) = (f \circ \phi_i^{-1})(a + bt)$ the directional derivatives

$$D_b^k(f \circ \phi_i^{-1})(a) \equiv \lim_{t \to 0} \frac{d^k}{dt^k} (f \circ \phi_i^{-1})(a + bt)$$

exist, are bounded $\forall a \in X$ and $\forall b \in \mathbb{R}^{n-1}$, $|b| = 1$ and furthermore $D_b^k(f \circ \phi_i^{-1})$ is continuous in the direction b. Therefrom it follows via some results of Boman (see [*2*]: Lemmata 4,5) that $f \circ \phi_i^{-1} \in C^k(X)$ $\forall i \in I$, implying $f \in C^k(S^{n-1})$.

If in addition for some $\alpha \in (0, 1]$ (2.2) holds, then obviously

$$|D_b^k(f \circ \phi_i^{-1})(a + bt) - D_b^k(f \circ \phi_i^{-1})(a)| \leq C_{k,\alpha} |t|^\alpha \ \forall |t| \leq \varepsilon(a),$$

$\forall a \in X$ and $\forall b \in \mathbb{R}^{n-1}$, $|b| = 1$. Now again we make use of Boman's results: If $\alpha = 1$ we conclude from the above via Lemma 6 in [*2*] that $f \circ \phi_i^{-1} \in C^{k,1}(X)$; whereas for the case $0 < \alpha < 1$, $f \circ \phi_i^{-1} \in C^{k,\alpha}(X)$ follows with the help of Lemma 7 by proceeding in the same way as for the proof of Theorem 2 in [*2*].

To show *b*) we proceed as before and note that (2.3) implies for $k \in \mathbb{N} \cup \{0\}$

$$\left| \left(\frac{d^k}{dt^k} (f \circ \phi_i^{-1}) \right)(a + bt) \right| \leq Ck!\,\delta^{-k} \tag{2.4}$$

for $|t| \leq \varepsilon(a)$, $\forall a \in X$ and $\forall b \in \mathbb{R}^{n-1}$ with $|b| = 1$. Again Boman's results [*2*] imply $f \circ \phi_i^{-1} \in C^\infty(X)$ $\forall i \in I$. Therefore

$$\left(\frac{d^k}{dt^k} (f \circ \phi_i^{-1}) \right)(a + bt) = (b \cdot \nabla)^k((f \circ \phi_i^{-1})(a + bt))$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_{n-1}} \right)$. But due to (2.4)

$$\sum_{k=0}^{\infty} ((b \cdot \nabla)^k((f \circ \phi_i^{-1})(a + bt)))\,|_{t=0} \frac{t^k}{k!}$$

converges absolutely for every $|t| \leq \delta_1 < \delta$. (Note that

$$P_k(bt) = ((b \cdot \nabla)^k \cdot ((f \circ \phi_i^{-1})(a + bt)))\,|_{t=0} t^k/k!$$

is a homogeneous polynomial of degree k in the variables $b_1 t, b_2 t, \dots, b_{n-1} t$.) Applying a result given e. g. in Mujica ([*14*], Proposition 4.6) it follows that for $|t| < \delta_1/e$

$$\sum_{k=0}^{\infty} ((b \cdot \nabla)^k (f \circ \phi_i^{-1}))(a + bt)\,|_{t=0} \frac{t^k}{k!} =$$

$$= \sum_{\alpha_1, \dots, \alpha_{n-1}=0}^{\infty} t^k b_1^{\alpha_1} \dots b_{n-1}^{\alpha_{n-1}} \frac{|\alpha|!}{\alpha_1! \dots \alpha_{n-1}!} \frac{\partial^{|\alpha|}(f \circ \phi_i^{-1})(a)}{\partial x_1^{\alpha_1} \dots \partial x_{n-1}^{\alpha_{n-1}}}$$

where $\alpha=(\alpha_1, \ldots, \alpha_{n-1})$ denotes the multi-index with $\sum_{i=1}^{n-1} \alpha_i = |\alpha|$. Both series converge absolutely and uniformly. The above result holds $\forall b \in \mathbb{R}^{n-1}$ with $|b| = 1$ and therefore b) is proved. □

We start now the proof of Theorem 2 as in [*13*] and split ψ so that

$$\psi = \psi_0 + \eta \tag{2.5}$$

where

$$\begin{aligned} (-\Delta + V_1 - E)\psi_0 = 0 \quad &\text{for} \quad x \in \Omega_{R_0} \\ \psi_0 = \psi \quad \text{on} \quad \partial\Omega_{R_0}, \quad &\psi_0 \in L^2(\Omega_{R_0}) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} (-\Delta + V_1 - E)\eta = -V_2\psi \quad &\text{for} \quad x \in \Omega_{R_0} \\ \eta = 0 \quad \text{for} \quad &x \in \partial\Omega_{R_0}. \end{aligned} \tag{2.7}$$

First we investigate the regularity properties of ψ_0/v.

LEMMA 2.2. — Under the assumptions of Theorem 1 on V_1 and E, $\psi_0/v \in C^{\omega}_{u,R}(S^{n-1})$. Specifically if $\gamma(t)$ is a geodesic given as in Lemma 2.1 there are constants d_0, $d > 0$ such that for $r \geq R_0 > R$

$$\left| \frac{d^k}{dt^k} \frac{\psi_0(r\gamma(t))}{v(r)} \right| \leq d_0 k! \left[d\left(\frac{1}{R_0} - \frac{1}{r} \right) \right]^{-k-1} \quad \forall t \in [-\pi, \pi]. \tag{2.8}$$

Proof of Lemma 2.2. — We first show that $\frac{\psi_0}{v}(r, .) \in C^\infty(S^{n-1})$ for $r > R_0$. Let $\tilde{\psi}_0 = r^{(n-1)/2}\psi_0$. We know that for fixed r, $\tilde{\psi}_0(ry) \in L^2(S^{n-1})$. We express $\tilde{\psi}_0$ in a series of surface harmonics $Y^{(k)}$ so that

$$\tilde{\psi}_0 = \sum_{k=0}^{\infty} a_k f_k(r) Y^{(k)}\left(\frac{x}{r}\right) \tag{2.9}$$

where $Y^{(k)} \in \mathcal{H}_k$ and $\mathcal{H}_k$ denotes the linear span of the surface harmonics $Y^{(k)}$ of degree k, i. e. the restriction of the real valued homogeneous harmonic polynomials in $\mathbb{R}^n$ of degree k to S^{n-1}. Then with L^2 given by (1.10) we have

$$L^2 Y^{(k)} = k(k + n - 2)Y^{(k)}. \tag{2.10}$$

We also assume the $Y^{(k)}$ to be normalized and real valued so that

$$\int_{S^{n-1}} Y^{(k)} Y^{(l)} d\sigma = \delta_{kl} \tag{2.11}$$

where δ_{kl} is the Kronecker delta. Obviously the coefficients a_k are determined by

$$\int_{S^{n-1}} Y^{(l)}\tilde{\psi}_0 d\sigma = \sum_{k=0}^{\infty} a_k f_k(r) \int_{S^{n-1}} Y^{(k)}Y^{(l)} d\sigma = a_l f_l(r)$$

and if we set $f_k(R_0) = 1\ \forall k$ then $\sum_{l=0}^{\infty} a_l Y^{(l)}(\Phi^{-1}(\xi)) = \tilde{\psi}(R_0\Phi^{-1}(\xi)) = \tilde{\psi}(R_0, \xi)$.

From (2.10) and (2.11) we infer that

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+n-2)}{r^2} + \tilde{V}_1(r) \right) f_l(r) = 0 \qquad \forall l \in \mathbb{N} \cup \{0\} \quad (2.12)$$

where

$$\tilde{V}_1 = V_1(r) - E + \frac{(n-1)(n-3)}{4r^2}. \qquad (2.13)$$

If we set $L^2(l) = l(l+n-2)$, then with the aid of Lemma 2.3 of [*13*]

$$f_l(r) \leq \frac{\tilde{v}(r)}{\tilde{v}(R_0)} \exp\left[-bL(l)\left(\frac{1}{R_0} - \frac{1}{r}\right)\right] \qquad (2.14)$$

for some $0 < b < \infty$ and where $\tilde{v}(r) = r^{(n-1)/2}v(r)$. This implies that

$$\left(\frac{\psi_0}{v}\right)(ry) = \sum_{l=0}^{\infty} c_l(r) Y^{(l)}(y)$$

where the $|c_l(r)| \leq e^{-dl}$ for some $d > 0$ and for $r \geq R_0 + \varepsilon$, $\varepsilon > 0$. By a result given in Stein [*16*] this implies that $\psi_0(r.) \in C^\infty(S^{n-1})\ \forall r > R_0$.

Next we show that $\psi_0/v \in C^{\omega}_{u,R}(S^{n-1})$. First we derive an upper bound to $\sup_{y \in S^{n-1}} |Y^{(l)}|$: We have [*5*] $\forall l \geq 0$ that $Y^{(l)} = \sum_{m=1}^{h(l)} b_{l,m} Y_{l,m}$ where for fixed l the $Y_{l,m}$ are orthonormal on S^{n-1} and

$$h(l) = \frac{(2l+n-2)(l+n-3)!}{(n-2)!\,l!} \quad \text{for} \quad l \geq 1.$$

From (2.11) it follows that $\sum_{m=1}^{h(l)} |b_{l,m}|^2 = 1$. From [*5*] we also know that $\sum_{m=1}^{h(l)} |Y_{l,m}|^2 = c(n)h(l)$ for some constant $c(n)$, implying

$$\sum_{m=1}^{h(l)} |Y_{l,m}|^2 \leq C(n)^2 l^{n-2}. \qquad (2.15)$$

Therefore

$$|Y^{(l)}| = \left|\sum_{m=1}^{h(l)} b_{l,m}Y_{l,m}\right| \leq \left(\sum_{m=1}^{h(l)} |b_{l,m}^2|\right)^{1/2}\left(\sum_{m=1}^{h(l)} Y_{l,m}^2\right)^{1/2} \leq C(n)l^{(n/2)-1}. \tag{2.16}$$

Let $P_l(x) = |x|^l Y^{(l)}\left(\frac{x}{r}\right)$ for $x \in \mathbb{R}^n\backslash\{0\}$, then P_l is a harmonic polynomial, homogeneous of degree l. Let e_{n-1} and e_n be the unit vectors $e_{n-1} = (0, \ldots, 1, 0)$ and $e_n = (0, \ldots, 1)$ in $\mathbb{R}^n$. It is easily seen that for $k \in \mathbb{N}$, $\xi_{n-1} \in [-\pi, \pi]$

$$\left.\frac{\partial^k P_l(x)}{\partial x_n^k}\right|_{x=\cos\xi_{n-1}e_{n-1}+\sin\xi_{n-1}e_n} = \frac{\partial^k}{\partial\xi_{n-1}^k} Y^{(l)}(\Phi^{-1}(\xi))|_{\xi=(0,\ldots 0,\xi_{n-1})}. \tag{2.17}$$

Now let $B_\rho=\{x\in\mathbb{R}^n: |x|<\rho\}$. Since P_l is harmonic for $x\in D_\delta\equiv B_{1+\delta}\backslash\overline{B}_{1-\delta}$ for any small δ we can use an estimate for harmonic functions (see for instance [7], p. 23), namely

$$\sup_{|x|=1}\left|\frac{\partial^k P_l(x)}{\partial x_n^k}\right| \leq \left(\frac{nk}{\delta}\right)^k \sup_{x\in D_\delta}|P_l(x)| \tag{2.18}$$

for any small δ and $l, k \in \mathbb{N}$. But (2.16) implies

$$\sup_{x\in D_\delta}|P_l(x)| \leq (1+\delta)^l l^{(n/2)-1}C(n). \tag{2.19}$$

Combining (2.17), (2.18) and (2.19) we get $\forall l, k \in \mathbb{N}$ and for every small $\delta>0$

$$\left|\left.\frac{\partial^k}{\partial\xi_{n-1}^k} Y^{(l)}(\Phi^{-1}(\xi))\right|_{\xi=(0,\ldots,0,\xi_{n-1})}\right| \leq C(n)\left|\frac{nk}{\delta}\right|^k(1+\delta)^l l^{(n/2)-1} \tag{2.20}$$

for $\xi_{n-1} \in [-\pi, \pi]$. Now we consider $\left(\frac{\partial}{\partial\xi_{n-1}}\right)^k \frac{\psi_0}{v}(r, \xi)$ for $\xi=(0,\ldots, 0, \xi_{n-1})$, where we again denote

$$\frac{\psi_0}{v}(r\Phi^{-1}(\xi)) = \frac{\psi_0}{v}(r, \xi).$$

From (2.9), (2.14) and (2.20) we get

$$\left|\left.\frac{\partial^k}{\partial\xi_{n-1}^k}\frac{\psi_0}{v}(r, \xi)\right|_{\xi=(0,\ldots,0,\xi_{n-1})}\right| \leq$$

$$\leq C_1\sum_{l=0}^{\infty}\exp\left[-bL(l)\left(\frac{1}{R_0}-\frac{1}{r}\right)\right](1+\delta)^l\left(\frac{nk}{\delta}\right)^k l^{(n/2)-1} \leq$$

$$\leq C_2\sum_{l=0}^{\infty}\exp\left[-b'l\left(\frac{1}{R_0}-\frac{1}{r}\right)\right](1+\delta)^l\left(\frac{nk}{\delta}\right)^k \tag{2.21}$$

for some $b' \in (0, b)$. Since $\delta > 0$ in (2.21) is an arbitrary small number we can choose $\delta \equiv \delta(r) = \exp\left[(b'-v)\left(\frac{1}{R_0} - \frac{1}{r}\right)\right] - 1$ for some $v \in (0, b')$ to obtain

$$\text{r. h. s. of (2.21)} \leq C_3 \sum_{l=0}^{\infty} \exp\left[-v\left(\frac{1}{R_0} - \frac{1}{r}\right)l\right]\left(\frac{nk}{\delta(r)}\right)^k$$
$$\leq C_4\left(\frac{1}{R_0} - \frac{1}{r}\right)^{-1}\left(\frac{nk}{\delta(r)}\right)^k.$$

Noting that $\delta(r) \geq C\left(\frac{1}{R_0} - \frac{1}{r}\right)$ for $r \geq R_0$ with a suitable $C > 0$ we obtain via Stirling's formula

$$\left| \frac{\partial^k}{\partial \xi_{n-1}^k} \frac{\psi_0}{v}(r, \xi) \Bigg|_{\xi=(0,\dots,0,\xi_{n-1})} \right| \leq d_0 k!\left(d\left(\frac{1}{R_0} - \frac{1}{r}\right)\right)^{-k-1} \tag{2.22}$$

for $r \geq R_0$, $\xi_{n-1} \in [-\pi, \pi]$, $k \in \mathbb{N} \cup \{0\}$ with some constants $d_0, d > 0$ not depending on k and r. Now $t \to \Phi^{-1}(0, 0, \dots, 0, t) = \cos t e_{n-1} + \sin t e_n$ describes a geodesic on S^{n-1} and $-\Delta + V_1(r) - E$ is invariant with respect to rotations. Hence it follows from (2.22) (now using the notation of Lemma 2.1) that $\forall \gamma \in \mathscr{F}$, (2.8) holds. Lemma 2.1 implies that $\psi/v \in C^{\omega}_{u,R_0}(S^{n-1})$ and since R_0 can be chosen arbitrarily close to R this proves Lemma 2.2. □

We continue with the

Proof of part a) *of Theorem 2.* — We start by investigating the Hölder continuity of $\eta(r\,.)/v$: For short we again write $\eta(r, \xi)$ instead of $\eta(r\Phi^{-1}(\xi))$ and we note that $t \to \Phi^{-1}(t, \xi_2, \dots, \xi_{n-1})$, $t \in (-\pi/2, \pi/2)$ with fixed arbitrary $\xi_2, \dots, \xi_{n-2} \in (-\pi/2, \pi/2)$, $\xi_{n-1} \in (-\pi, \pi)$ describes a geodesic on S^{n-1}. For reasons that will become clear below it suffices to show that with $\xi_2, \dots, \xi_{n-1}$ as above we have

$$\frac{1}{2}|\eta(r, \xi_1, \dots, \xi_{n-1}) - \eta(r, -\xi_1, \xi_2, \dots, \xi_{n-1})| \leq C_\delta(r)v(r)|\xi_1|^\delta$$
$$\forall \xi_1 \in [0, \pi/2) \quad \text{and} \quad \text{for } r > R_0 \quad \text{with} \tag{2.23}$$
$$C_\delta(r) = \frac{4c_0c_+}{\pi}\left(C_1 d(\gamma_0, \delta) + C_2\left(b\left(\frac{1}{R_0} - \frac{1}{r}\right)\right)^{-1}\right)$$

with some suitable constants $C_1, C_2, b, d(\gamma_0, \delta) > 0$ and c_0, γ_0 defined according to our assumptions on V_2 and c_+ given as in Theorem 1. We note that it is necessary to give $C_\delta(r)$ so explicitly because it will enter in the proof of part *b*) and *c*) of Theorem 2.

To verify (2.23) we shall follow partly [*13*]. Let

$$\eta_a(r, \xi) = \frac{1}{2}(\eta(r, \xi_1, \ldots, \xi_{n-1}) - \eta(r, -\xi_1, \xi_2, \ldots, \xi_{n-1})) \quad (2.24)$$

$$W_a(r, \xi) = -\frac{1}{2}((V_2\psi)(r, \xi_1, \ldots, \xi_{n-1}) - (V_2\psi)(r, -\xi_1, \xi_2, \ldots, \xi_{n-1})) \quad (2.25)$$

with $\xi \in Q$ such that $\xi_1 \in [0, \pi/2)$. We have from (2.7)

$$(-\Delta + V_1 - E)\eta_a = W_a \quad \text{in} \quad \Omega_-, \qquad \Omega_- = \{ x \in \Omega_{R_0} : x_1 < 0 \}. \quad (2.26)$$

By Theorem 1 and the assumptions on V_2 we get

$$|W_a| \le c_0 c_+ r^{-1-\gamma_0} v(r) \equiv G_0(r) \quad (2.27)$$

which together with Kato's distributional inequality (see [*13*] for the same argument) leads to

$$(-\Delta + V_1 - E)|\eta_a| \le G_0 \quad \text{in} \quad \Omega_-, \qquad \eta_a = 0 \quad \text{in} \quad \partial\Omega_-. \quad (2.28)$$

By the maximum principle a function $F \in W^{1,2}(\Omega_-)$ which satisfies

$$(-\Delta + V_1 - E)F = G_0 \quad \text{in} \quad \Omega_-, \qquad F = 0 \quad \text{in} \quad \partial\Omega_- \quad (2.29)$$

will obey

$$|\eta_a| \le F \qquad \text{for} \qquad x \in \Omega_-. \quad (2.30)$$

Having in mind the definition of η_a it suffices to find a suitable upper bound to F. We first observe that F is invariant under rotations around the x_1-axis, since in (2.29), Δ, $V_1 - E$, G_0 and the boundary conditions enjoy this property. Therefore $F(r\Phi^{-1}(\xi))$ depends only on r and ξ_1 and we shall write $F(r, \xi_1)$. We could now try to expand F for fixed r in spherical (zonal) harmonics but we shall follow another route and derive the upper bound for F by reducing our problem to a two-dimensional one. First we note that $F \in C^2(\Omega_{R_0})$ by elliptic regularity. Let $\tilde{F}(r, \xi_1) = r^{(n-1)/2}F(r, \xi_1)$, $\tilde{G}_0 = r^{(n-1)/2}G_0$ and let $\tilde{V}_1$ be given by (2.13), then $\tilde{F}(r, \xi_1)$ satisfies

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial \xi_1^2} + \frac{n-2}{2r^2}\,\mathrm{tg}\,\xi_1 \frac{\partial}{\partial \xi_1} + \tilde{V}_1 \right)\tilde{F} = \tilde{G}_0 \quad (2.31)$$

in

$$\tilde{\Omega}_- = \left\{ (r, \xi_1) \in \mathbb{R}^2 : r > R_0,\ 0 < \xi_1 < \frac{\pi}{2} \right\} \quad (2.32)$$

and

$$\tilde{F}(R_0, \xi_1) = 0 \quad \text{for} \quad \xi_1 \in \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad \tilde{F}(r, 0) = 0 \quad \forall r \ge R_0. \quad (2.33)$$

The 2-dimensional problem defined by (2.31), (2.32) and (2.33) leaves $\tilde{F}(r, \xi_1)$ at $\xi_1 = \pi/2$ unspecified. But by the rotational symmetry of $F(x)$

with respect to the x_1-axis $\partial F(r, \xi_1)/\partial\xi_1 = 0$ for $\xi_1 = \pi/2$ and since $F \in C^2(\Omega_{R_0})$ it is easily seen that

$$\lim_{\xi_1 \to \pi/2} \operatorname{tg} \xi_1 \frac{\partial F(r, \xi_1)}{\partial \xi_1} = -\left(\frac{\partial^2}{\partial \xi_1^2} F\right)\left(r, \frac{\pi}{2}\right). \tag{2.34}$$

Hence we can symmetrize our problem in the following way: let

$$\begin{aligned} \tilde{f}(r, \omega) &= \tilde{F}(r, \omega) \qquad \text{for} \quad \omega \in \left(0, \frac{\pi}{2}\right] \\ \tilde{f}(r, \omega) &= \tilde{f}(r, \pi - \omega) \qquad \text{for} \quad \omega \in \left[\frac{\pi}{2}, \pi\right) \end{aligned} \tag{2.35}$$

and set

$$\hat{\Omega} = \{ (r \cos \omega, r \sin \omega) \in \mathbb{R}^2 : r > R_0, 0 < \omega < \pi \}. \tag{2.36}$$

Furthermore define on $\mathbb{R}^2$

$$\mathscr{P} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial \omega^2} + \tilde{V}_1(r) \tag{2.37}$$

then it is easily seen that

$$\left(\mathscr{P} + \frac{n-2}{r^2} \operatorname{tg} \omega \frac{\partial}{\partial \omega}\right)\tilde{f}(r, \omega) = \tilde{G}_0(r) \quad \text{in} \quad \hat{\Omega}, \qquad \tilde{f} = 0 \quad \text{in} \quad \partial\hat{\Omega}. \tag{2.38}$$

Hence we have reduced our problem to a two-dimensional one and it suffices to find a suitable upper bound to $\tilde{f}$.

LEMMA 2.3. — Let $\tilde{f}$ be defined as above and suppose

$$\tilde{g}(r, \omega) > 0, \qquad \int_{R_0}^{\infty} \int_0^{\pi} | \tilde{g}(r, \omega) |^2 dr d\omega < \infty,$$

and

$$\mathscr{P}\tilde{g} = \tilde{G}_0 \quad \text{in} \quad \hat{\Omega}, \qquad \tilde{g} = 0 \quad \text{in} \quad \partial\hat{\Omega} \tag{2.39}$$

then $\tilde{g} \geq \tilde{f}$ in $\hat{\Omega}$.

Proof. — Existence and uniqueness follows immediately by noting that with $r = (x_1^2 + x_2^2)^{1/2}$, $x_1 = r \cos \omega$, $x_2 = r \sin \omega$, $r^{-1/2}\tilde{g}(r, \omega)$ transformed into two-dimensional cartesian coordinates satisfies

$$\left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \tilde{V}_1\right) r^{-1/2}\tilde{g}(x_1, x_2) = r^{-1/2}\tilde{G}_0 \tag{2.40}$$

in $\Omega_2 = \{ x \in \mathbb{R}^2 : r > R_0, x_2 > 0 \}$ and $\tilde{g}(x_1, x_2) = 0$ in $\partial\Omega_2$.
Since $\tilde{V}_1 > 0$ and continuously differentiable, $r^{-1/2}\tilde{G}_0 \in L^2(\Omega_2) \cap C^2(\Omega_2)$, $\tilde{G}_0 > 0$, we have uniqueness and $g > 0$, $g \in L^2(\Omega_2) \cap C^2(\Omega_2)$.

We first show that $\tilde{g}(r, \omega)$ is monotonically increasing in ω for $\omega \in [0, \pi/2]$. By symmetry we observe that $\tilde{g}(r, \omega) = \tilde{g}(r, \pi - \omega)$ for $\omega \in (0, \pi)$. Let $\overline{\omega} \in (0, \pi/2]$ and define $h(r, \omega) = \tilde{g}(r, 2\overline{\omega} - \omega)$. Then for $\omega \in (0, \overline{\omega})$, $r > R_0$,

$\mathcal{P}\tilde{g} = \tilde{G}_0(r)$, $\mathcal{P}h = \tilde{G}_0(r)$ and $\tilde{g}(r,0) \leq h(r,0)$, $\tilde{g}(r,\overline{\omega}) = h(r,\overline{\omega})$ for $r > R_0$ and $h(R_0,\omega) = \tilde{g}(R_0,\omega) = 0\ \forall\omega \in (0,\overline{\omega})$. Hence by the maximum principle $(h - \tilde{g})(r,\omega)$ cannot have a negative minimum for $\omega \in (0,\overline{\omega})$ so that $h \geq g$ for $\omega \in (0,\overline{\omega})$. Therefore for $r > R_0$, $\tilde{g}(r,\omega) \leq \tilde{g}(r, 2\overline{\omega} - \omega)\ \forall\omega\in(0,\overline{\omega}]$ and for arbitrary fixed $\overline{\omega} \in (0,\pi/2]$. This implies $\partial\tilde{g}(r,\omega)/\partial\omega \geq 0$ for $\omega \in (0,\pi/2)$ and by symmetry $\partial\tilde{g}(r,\omega)/\partial\omega \leq 0$ for $\omega \in (\pi/2,\pi)$. This implies that $\operatorname{tg}\omega\,\partial\tilde{g}(r,\omega)/\partial\omega \geq 0$ for $\omega\in[0,\pi]$. Also $\lim_{\omega\to\pi/2} \operatorname{tg}\omega\,\partial\tilde{g}(r,\omega)/\partial\omega = -\partial^2 g\left(r,\frac{\pi}{2}\right)/\partial\omega^2$ corresponding to (2.34) and we find

$$\left(\mathcal{P} + \frac{n-2}{r^2}\operatorname{tg}\omega\frac{\partial}{\partial\omega}\right)\tilde{g}(r,\omega) \geq \tilde{G}_0(r) \quad \text{in} \quad \hat{\Omega}.$$

This differential inequality together with (2.38) implies

$$\left(\mathcal{P} + \frac{n-2}{r^2}\operatorname{tg}\omega\frac{\partial}{\partial\omega}\right)(\tilde{g}(r,\omega) - \tilde{f}(r,\omega)) \geq 0 \quad \text{in} \quad \hat{\Omega}$$

and therefore $(\tilde{g} - \tilde{f})(r,\omega)$ cannot have a negative minimum by the maximum principle proving the assertion of Lemma 2.3. □

We continue with the proof of part *a*) of Theorem 2. Collecting our findings we obtain by (2.30) and the above lemma

$$|\eta_a(r,\xi)| \leq F(r,\xi_1) \leq r^{-(n-1)/2}\tilde{g}(r,\xi_1), \qquad \xi_1 \in \left[0,\frac{\pi}{2}\right] \tag{2.41}$$

with $\tilde{g}$ satisfying (2.39). Now since we have reduced our problem to the 2-dimensional case we can use the results derived in [*13*] to obtain an upper bound to $\tilde{g}$. There we have shown (see eq. (3.21) and (3.32) and identify $\tilde{F}_0$ with $\tilde{g}$) that for $r > R_0$ and $\omega \in (0,\pi)$

$$\frac{\tilde{g}(r,\omega)}{\tilde{v}(r)} \leq \frac{4c_0c_+}{\pi}\left(C_1 d(\gamma_0,\delta)\,|\,\omega\,|^\delta + C_0\,|\,\omega\,|\left(b\left(\frac{1}{R_0} - \frac{1}{r}\right)^{-1}\right)\right) \tag{2.42}$$

with suitable constants C_1, C_0, b, $d(\gamma_0,\delta)$. (2.41) and (2.42) immediately imply (2.23). Now rotating the coordinate system it is easily seen by the assumption on V_2 and equation (2.7) for η that the preceding arguments, particularly the estimates remain unchanged and hence we conclude by (2.23) that

$$\frac{|\,\eta(r\gamma(t_2)) - \eta(r\gamma(t_1))\,|}{v(r)\,|\,t_1 - t_2\,|^\delta} \leq \frac{4c_0c_+}{\pi}\left(C_1 d(\gamma,\delta) + C_2\left(b\left(\frac{1}{R_0} - \frac{1}{r}\right)\right)^{-1}\right) \quad \forall\gamma\in\mathcal{F}. \tag{2.43}$$

On the other hand we have from Lemma 2.2 $\forall\gamma\in\mathcal{F}$

$$\frac{|\,\psi_0(r\gamma(t_2)) - \psi_0(r\gamma(t_1))\,|}{v(r)\,|\,t_2 - t_1\,|} \leq d_0\left(d\left(\frac{1}{R_0} - \frac{1}{r}\right)\right)^{-2}. \tag{2.44}$$

Recalling $\psi = \psi_0 + \eta$ in Ω_{R_0} we obtain $\forall \gamma \in \mathscr{F}$

$$\frac{|\psi(r\gamma(t_2)) - \psi(r\gamma(t_1))|}{v(r)\,|t_2 - t_1|^\delta} \leq \left(\frac{1}{R_0} - \frac{1}{r}\right)^{-2}\left(d_0 d^{-2} + \frac{4c_0 c_+}{\pi}(C_1 d(\gamma, \delta) + C_2 b^{-1})\right). \tag{2.45}$$

This verifies the Hölder continuity of $u = \psi/v$ asserted in part *a*) of Theorem 2 because of Lemma 2.1.

To complete the proof of part *a*) of Theorem 2 we have to show that $\lim_{r\to\infty} u(ry) \equiv A(y)\ \forall y \in S^{n-1}$ exists and that $A \in C^{0,\delta}(S^{n-1})$. Denoting $u(r\Phi^{-1}(\xi))$ by $u(r, \xi)$ and assuming without loss

$$\limsup_{r\to\infty} u(r, 0) > \liminf_{r\to\infty} u(r, 0)$$

we derive a contradiction. For this purpose we proceed as in [*13*] and pick a function $0 \leq \chi$, $\chi \in C_0^\infty(S^{n-1})$ with sufficiently small support so that

$$g(r) \equiv \int_{S^{n-1}} u\chi d\sigma$$

also has no limit, i. e.

$$\limsup_{r\to\infty} g(r) > \liminf_{r\to\infty} g(r). \tag{2.46}$$

Following [*13*] (see the arguments from (3.35) to (3.40)) we see that with a suitable constant C and with $\gamma = \min(1, \gamma_0)$

$$|g'| \leq Cr^{-1-\gamma} \qquad \text{for} \qquad r \geq \overline{R}, \overline{R} \text{ sufficiently large}$$

and this in turn implies via integration that the limit of $g(r)$ for $r \to \infty$ exists contradicting (2.46). The Hölder continuity of A is an immediate consequence of (2.45).

Proof of part b) *and* c) *of Theorem 2.* — First we show by using elliptic regularity as in [*13*] that u satisfies for $0 \leq k \leq m-1$, $u(r, .) \in C^k(S^{n-1})$ and $u(r, .) \in C^{m,\alpha}(S^{n-1})$ for $r \in (R_a, R_b)$ for any R_a, R_b such that $R_0 < R_a < R_b < \infty$ for some $\alpha > 0$. Pick for $1 \leq k \leq 2m$, $\rho_k \in (R_0, R_a)$ increasing and $R_k \in (R_b, \infty)$ decreasing and define $D_k = \{ x \in \Omega_R \mid \rho_k < r < R_k \}$ and set $D_0 = \Omega_{R_0}$. Then $D_{k+1} \subset\subset D_k$ (strictly contained) and $\{ x \in \Omega_{R_0} \mid R_a < |x| < R_b \} \subset\subset D_k\ \forall k$. Let $m = 1$ and consider (1.1) in D_0. We know that the assumptions on V and E imply that $\psi \in W^{2,2}(\Omega_{R_0})$ and that $\partial\psi/\partial x_i$, $i = 1, 2, \ldots n$ belongs to $W^{1,2}(\Omega_{R_1})$. Noting that expressed in cartesian coordinates

$$\frac{\partial}{\partial \xi_{n-1}} \psi(r\Phi^{-1}(\xi)) = \left(-x_n \frac{\partial}{\partial x_{n-1}} + x_{n-1} \frac{\partial}{\partial x_n}\right)\psi(x) \tag{2.47}$$

we obtain by differentiating (1.1) in the distributional sense

$$(-\Delta + V - E)\frac{\partial\psi}{\partial\xi_{n-1}} = -\frac{\partial V}{\partial\xi_{n-1}}\psi. \tag{2.48}$$

Since $\partial V/\partial\xi_{n-1}$ is bounded by assumption $\partial\psi/\partial\xi_{n-1}$ is Hölder continuous in $D_2 \subset\subset D_1$ by Theorem 8.22 of [7]. Now let $m > 1$. We consider (2.48) in D_3. Again following [*13*] we observe that Theorem 8.30 and Theorem 8.9 of [7] yield $\partial^2\psi/\partial\xi_{n-1}^2 \in W^{1,2}(D_3)$. Differentiating (2.48) with respect to ξ_{n-1} we get

$$(-\Delta + V - E)\frac{\partial^2\psi}{\partial\xi_{n-1}^2} = -2\frac{\partial V}{\partial\xi_{n-1}}\frac{\partial\psi}{\partial\xi_{n-1}} - \frac{\partial^2 V}{\partial\xi_{n-1}^2}\psi. \tag{2.49}$$

By the preceding arguments the r. h. s. of (2.49) is bounded in D_3 and we conclude via Theorem 8.22 of [7] that $\partial^2\psi/\partial\xi_{n-1}^2$ is Hölder continuous in D_4. Repeating these arguments we obtain by differentiating (2.49) sufficiently often with respect to ξ_{n-1} that $\partial^m\psi/\partial\xi_{n-1}^m$ is Hölder continuous in D_{2m}. Since $v(r) > 0$ these local properties hold also for u.

Finally we observe that the set

$$\{ y \in S^{n-1} \mid y = \Phi^{-1}(\xi) \text{ with } \xi_1 = \xi_2 = \ldots \xi_{n-2} = 0,\ \xi_{n-1} \in (-\pi, \pi) \}$$

is a great circle on S^{n-1} so that by the rotational invariance of the assumptions in part *b*) of Theorem 2, $\psi(r\gamma(t))$ and hence $u(r\gamma(t))$ have the above mentioned regularity properties $\forall\gamma \in \mathscr{F}$. Thus we conclude via Lemma 2.1 that for $k \leq m - 1$, $u(r, .) \in C^k(S^{n-1})$ for $r \in (R_a, R_b)$ and $u(r, .) \in C^{m,\alpha}(S^{n-1})$ for some $1 \geq \alpha > 0$. In particular we note that if $V_2(r, .) \in C^\infty(S^{n-1})$ for $r \geq R_0$ that $u(r, .) \in C^\infty(S^{n-1})$ for $r > R_0$.

Next we show the uniformity of the regularity properties of u with respect to r for $m \geq 1$. We follow the proof of part *a*) and we shall also use estimates established in [*13*] (see inequality (3.48) there). Due to our assumption in Theorem 2 *b*) we have $r^{1+\gamma_k}V_2 \in C^k_{u,R}(S^{n-1})$ for $0 \leq k \leq m$ which obviously implies the existence of constants $c_k > 0$ such that

$$\left| r^{1+\gamma_k}\frac{\partial^k V_2(r\Phi^{-1}(\xi))}{\partial\xi_{n-1}^k} \right| \leq c_k \tag{2.50}$$

$\forall r \geq R_0$ and $\xi \in Q$. (2.50) remains true if we rotate the coordinate system. Specifically $r^{1+\alpha}V_2 \in C^\omega_{u,R}(S^{n-1})$ implies that for $k = 0, 1, 2, \ldots, c_k \leq c_0 k!\,\delta^{-k}$ for some $\delta > 0$.

Noting that $u(r, .) \in C^k(S^{n-1})$ for $0 \leq k \leq m - 1$ for $r \geq R_0$ we shall demonstrate for $0 \leq k \leq m - 1$ that

for $\xi, \xi' \in Q$ with $\xi = (\xi_1, \ldots, \xi_{n-1})$, $\xi' = (\xi_1, \ldots, \xi_{n-2}, \xi'_{n-1})$

$$\left|\frac{\partial^k}{\partial\xi^k_{n-1}}u(r\Phi^{-1}(\xi))-\frac{\partial^k}{\partial\xi^k_{n-1}}u(r\Phi^{-1}(\xi'))\right|\le|\xi_{n-1}-\xi'_{n-1}|\,.\,M_{k+1}\left(\frac{1}{R_0}-\frac{1}{r}\right)^{-k-2},$$

$$M_{k+1}=M_0\left((k+1)!\,d^{-k-2}+\sum_{j=0}^{k}\binom{k}{j}c_{k-j}M_j\right) \quad (2.51)$$

$$M_0=\max(B,\,c_+B,\,d_0) \quad \text{with} \quad B=\frac{4}{\pi}(C_1 d(\gamma,1)+C_2 b^{-1}),$$

$$\gamma=\min(1,\gamma_1,\gamma_2\,\ldots\,\gamma_m) \quad \text{and} \quad c_j,\ 0\le j\le m \ \text{ given in (2.50).}$$

(Compare (3.48) in [13].) To verify (2.51) we proceed by induction. The case $k=0$ follows from the proof of part *a*) of Theorem 2: Pick any ξ, $\xi'\in Q$ as above, then there exists a $\gamma\in\mathcal{F}$ with $\Phi^{-1}(\xi)=\gamma(t)$, $\Phi^{-1}(\xi')=\gamma(t')$. Further it is easily seen that $|\xi_{n-1}-\xi'_{n-1}|\ge|t-t'|$ and hence it follows from (2.45) that $\forall r\ge R_0$, $\forall\xi,\xi'\in Q$ as above

$$\frac{|u(r\Phi^{-1}(\xi))-u(r\Phi^{-1}(\xi'))|}{|\xi_{n-1}-\xi'_{n-1}|}\le\left(\frac{1}{R_0}-\frac{1}{r}\right)^{-2}M_1 \quad (2.52)$$

verifying (2.51) for $k=0$ and any $m>0$. Now assume that (2.51) holds for $k-1$. To show that it is also true for k we use $\partial^k\psi/\partial\xi^k_{n-1}\in W^{1,2}_{\mathrm{loc}}(\Omega_{R_0})$ for $0\le k\le m-1$ and that in Ω_{R_0}

$$(-\Delta+V-E)\frac{\partial^k\psi}{\partial\xi^k_{n-1}}=-\sum_{j=0}^{k-1}\binom{k}{j}\left(\frac{\partial^{k-j}}{\partial\xi^{k-j}_{n-1}}V\right)\left(\frac{\partial^j\psi}{\partial\xi^j_{n-1}}\right). \quad (2.53)$$

Moreover since for $r\ge R_0$, $\psi(r,.)\in C^k(S^{n-1})$ and $\psi_0(r,.)\in C^\omega(S^{n-1})$, $\partial^k\psi/\partial\xi^k_{n-1}=\partial^k\psi_0/\partial\xi^k_{n-1}+\partial^k\eta/\partial\xi^k_{n-1}$. Defining η_a and W_a as in (2.24), (2.25) we have

$$(-\Delta+V_1-E)\frac{\partial^k}{\partial\xi^k_{n-1}}\eta_a=\frac{\partial^k W_a}{\partial\xi^k_{n-1}} \quad \text{in} \quad \Omega_-=\{x\in\Omega_{R_0}:x_1<0\}. \quad (2.54)$$

This equation corresponds to (2.26) in the proof of part *a*) of Theorem 2 and in the following we shall use the procedure given there to derive an upper bound to $|\eta_a|$ in order to obtain an upper bound to $|\partial^k\eta_a/\partial\xi^k_{n-1}|$. Hence we give only the main steps. Noting that due to our induction hypothesis (see 2.51)

$$\left|\frac{\partial^j\psi}{\partial\xi^j_{n-1}}(r\Phi^{-1}(\xi))\right|\le v(r)M_j\left(\frac{1}{R_0}-\frac{1}{r}\right)^{-j-1} \quad \text{for } r\ge R_0,\ 0\le j\le k,\ \xi\in Q \quad (2.55)$$

and taking into account the assumptions on V_2 (see inequality (2.50)) we have

$$\left|\frac{\partial^k W_a}{\partial\xi^k_{n-1}}\right|\le\sum_{j=0}^{k}\binom{k}{j}\left|\frac{\partial^{k-j}V_2}{\partial\xi^{k-j}_{n-1}}\right|\left|\frac{\partial^j\psi}{\partial\xi^j_{n-1}}\right|\le G_k(r) \quad (2.56)$$

with

$$G_k(r) = \sum_{j=0}^{k} \binom{k}{j} c_{k-j} r^{-1-\gamma_k} M_j (R_0^{-1} - r^{-1})^{-j} v(r) . \tag{2.57}$$

Hence in Ω_- (given as in (2.26))

$$(-\Delta + V_1(r) - E) \left| \frac{\partial^k}{\partial \xi_{n-1}^k} \eta_a \right| \leq G_k(r)$$

$$\left| \frac{\partial^k}{\partial \xi_{n-1}^k} \eta_a \right| = 0 \quad \text{in} \quad \partial\Omega_- . \tag{2.58}$$

Following the arguments given in the proof of part *a*) we arrive at

$$\left| \frac{\partial^k \eta_a}{\partial \xi_{n-1}^k} \right| \leq r^{(-n+1)/2} \tilde{g}_k(r, \xi_1) \quad \text{for} \quad \xi_1 \in \left[0, \frac{\pi}{2} \right) \tag{2.59}$$

where analogously to Lemma 2.3

$$\mathscr{P} \tilde{g}_k = \tilde{G}_k \quad \text{in} \quad \hat{\Omega} \quad \text{with} \quad \tilde{g}_k = 0 \quad \text{in} \quad \partial\hat{\Omega} \tag{2.60}$$

and as in (2.40)

$$\left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \tilde{V}_1 \right) r^{-1/2} \tilde{g}_k = r^{-1/2} \tilde{G}_k \quad \text{in} \quad \Omega_2 .$$

But this equation can be identified with the equation given before (3.49) in [*13*] when V_1, F_k, G_k is replaced by $\tilde{V}_1, r^{-1/2}\tilde{g}_k, r^{-1/2}\tilde{G}_k$. Making use of the upper bound on F_k derived in [*13*] (see the inequality preceding (3.52) and replace F_k by $r^{-1/2}\tilde{g}_k$) we obtain

$$\tilde{g}_k(r, \xi_1) \leq | \xi_1 | \tilde{v} m_{k+1}(r) \quad \text{for} \quad \xi_1 \in \left[0, \frac{\pi}{2} \right), \qquad r \geq R_0 \tag{2.61}$$

$$m_{k+1}(r) = \frac{4}{\pi} \sum_{j=0}^{k} \binom{k}{j} c_{k-j} M_j \left(\frac{1}{R_0} - \frac{1}{r} \right)^{-j-1} \left(C_1 d(\gamma, 1) + C_2 b^{-1} \left(\frac{1}{R_0} - \frac{1}{r} \right)^{-1} \right).$$

Note that we used $\gamma_i > 1/2$ for $1 \leq i \leq m - 1$. Combining (2.59) and (2.61) we get for $r \geq R_0$

$$\left| \frac{\partial^k \eta}{\partial \xi_{n-1}^k} (r\Phi^{-1}(\xi_1, \xi_2, \ldots, \xi_{n-1})) - \frac{\partial^k \eta}{\partial \xi_{n-1}^k} (r\Phi^{-1}(-\xi_1, \xi_2, \ldots, \xi_{n-1})) \right| \leq$$
$$\leq | 2\xi_1 | m_{k+1}(r) v(r) . \tag{2.62}$$

Now note that equation (2.54) is invariant with respect to rotations of the coordinate system in the sense that $-\Delta + V_1(r) - E$ and the upper bounds to $| \partial^k W_a / \partial \xi_{n-1}^k |$ stay invariant. But under rotations a geodesic in S^{n-1}, specifically $t \to \Phi^{-1}(t, \xi_2, \ldots, \xi_{n-1})$ with $\xi_2, \xi_3, \ldots, \xi_{n-1}$ fixed, stays a geodesic and hence (2.62) implies

$$\left| \left(\frac{\partial^k \eta}{\partial \xi_{n-1}^k} \right)(r\gamma(t)) - \left(\frac{\partial^k \eta}{\partial \xi_{n-1}^k} \right)(r\gamma(t')) \right| \leq m_{k+1}(r) \mid t - t' \mid v(r) \qquad (2.63)$$

$\forall \gamma \in \mathscr{F}$, $\forall r \geq R_0$. But for any $\xi, \xi' \in Q$ defined as in (2.51) there is a $\gamma \in \mathscr{F}$ with $\Phi^{-1}(\xi) = \gamma(t)$ and $\Phi^{-1}(\xi') = \gamma(t')$ with $|t - t'| \leq | \xi_{n-1} - \xi'_{n-1} |$ and hence for $r \geq R_0$

$$\left| \frac{\partial^k \eta}{\partial \xi_{n-1}^k}(r\Phi^{-1}(\xi)) - \frac{\partial^k \eta}{\partial \xi_{n-1}^k}(r\Phi^{-1}(\xi')) \right| \leq m_{k+1}(r) \mid \xi_{n-1} - \xi'_{n-1} \mid v(r) . \qquad (2.64)$$

On the other hand we obtain from Lemma 2.2 in an analogous way that $\forall \xi, \xi' \in Q$, $\forall r \geq R_0$

$$\left| \frac{\partial^k}{\partial \xi_{n-1}^k} \psi_0(r\Phi^{-1}(\xi)) - \frac{\partial^k}{\partial \xi_{n-1}^k} \psi_0(r\Phi^{-1}(\xi')) \right|$$
$$\leq d_0(k+1)! \left(d\left(\frac{1}{R_0} - \frac{1}{r} \right) \right)^{-k-2} . \mid \xi_{n-1} - \xi'_{n-1} \mid v(r) . \qquad (2.65)$$

Combining the last two inequalities we arrive at

$$\left| \frac{\partial^k u}{\partial \xi_{n-1}^k}(r\Phi^{-1}(\xi)) - \frac{\partial^k u}{\partial \xi_{n-1}^k}(r\Phi^{-1}(\xi')) \right| \leq | \xi_{n-1} - \xi'_{n-1} | \Bigg(m_{k+1}(r) + d_0(k+1)! .$$
$$\left(d\left(\frac{1}{R_0} - \frac{1}{r} \right) \right)^{-k-2} \Bigg) \leq M_{k+1} \left(\frac{1}{R_0} - \frac{1}{r} \right)^{-k-2} | \xi_{n-1} - \xi'_{n-1} | \qquad (2.66)$$

verifying (2.51) for $0 \leq k \leq m - 1$.

To show uniform Hölder continuity for $\partial^m u / \partial \xi_{n-1}^m$ we proceed as before but start with $k = m$ in equation (2.54). The corresponding estimate is then for $r \geq R_0$, $\forall \xi, \xi' \in Q$ with ξ, ξ' differing only in the ξ_{n-1} component

$$\left| \frac{\partial^m u}{\partial \xi_{n-1}^m}(r\Phi^{-1}(\xi)) - \frac{\partial^m u}{\partial \xi_{n-1}^m}(r\Phi^{-1}(\xi')) \right| \leq | \xi_{n-1} - \xi'_{n-1} |^{\delta_m} \left(\frac{1}{R_0} - \frac{1}{r} \right)^{-m-2} C(\delta_m) \qquad (2.67)$$

with some $0 < C(\delta_m) < \infty$, and where the δ_m is defined in Theorem 2. Specifically it follows from (2.67) that for every $R_1 > R_0$

$$\left| \frac{\partial^m u}{\partial t^m}(r\Phi^{-1}(0, 0, \ldots, 0, t)) - \frac{\partial^m u}{\partial t^m}(r\Phi^{-1}(0, 0, \ldots, 0, t')) \right| \leq C(\delta_m, R_1) \mid t - t' \mid^{\delta_m} \qquad (2.68)$$

for every $r \geq R_1$ and for $t, t' \in (-\pi, \pi)$. Since the geodesics in S^{n-1} can be obtained from rotations of the geodesic $\Phi^{-1}(0, \ldots, 0, t) = e_{n-1} \cos t + e_n \sin t$ and since $-\Delta + V_1(r) - E$ is rotation invariant and since by assumption

$r^{1+\gamma_k}V_2 \in C^k_{u,R}(S^{n-1})$ for $0 \leq k \leq m$ it follows from (2.68) that for $r \geq R_1 > R_0$

$$\left| \frac{\partial^m u}{\partial t^m}(r\gamma(t)) - \frac{\partial^m u}{\partial t^m}(r\gamma(t')) \right| \leq C(\delta_m, R_1) \, | t - t' |^{\delta_m} \qquad \forall \gamma \in \mathcal{F} . \quad (2.69)$$

Now we apply Lemma 2.1 *a*) and since R_0 was arbitrary close to R we obtain

$$u \in C^{m,\delta_m}_{u,R}(S^{n-1}) .$$

If V_2 satisfies the conditions of part *c*) of Theorem 2 then the constants c_k in inequality (2.50) obey $c_k \leq c_0 k! \, \delta^{-k}$ for some $\delta > 0$ and (2.51) implies that for some $M > 0$ and some $\delta > 0$

$$\left| \frac{\partial^k u}{\partial \xi^k_{n-1}}(r\Phi^{-1}(\xi)) \right| \leq \frac{Mk!}{\delta^k} \quad (2.70)$$

$\forall \xi \in Q$, $\forall r \geq R_1$ and $k \in \mathbb{N} \cup \{ 0 \}$. The proof of (2.70) is the same as for the corresponding result in [*13*]. As above we conclude that (2.70) implies that $\forall \gamma \in \mathcal{F}$

$$\left| \frac{\partial^k u}{\partial t^k}(r\gamma(t) \right| \leq \frac{Mk!}{\delta^k} \qquad \forall r \geq R_1 \quad (2.71)$$

thus by Lemma 2.1 $u \in C^{\omega}_{u,R_1}(S^{n-1})$.

Finally we have to show the existence of the limits of u asserted in part *b*) and *c*) of Theorem 2. For $0 \leq k \leq m$ we have with L^2 given by (1.10)

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{\tilde{v}}\frac{\partial \tilde{v}}{\partial r} + \frac{L^2}{r^2} + V_2 \right) \frac{\partial^k u}{\partial \xi^k_{n-1}}(r\Phi^{-1}(\xi)) =$$

$$= - \sum_{j=0}^{k-1} \binom{k}{j} \left(\frac{\partial^{k-j}}{\partial \xi^{k-j}_{n-1}} \right) V_2(r\Phi^{-1}(\xi)) \frac{\partial^j u}{\partial \xi^j_{n-1}}(r\Phi^{-1}(\xi)) . \quad (2.72)$$

Let us write $u^{(k)} = \partial^k u(r\Phi^{-1}(\xi))/\partial \xi^k_{n-1}$. Suppose there is a $\bar{\xi} \in Q$ such that $u^{(k)}(r\Phi^{-1}(\bar{\xi}))$ does not converge for $r \to \infty$. We derive a contradiction as in [*13*] and as in part *a*). We pick $0 \leq \chi \in C^\infty_0(S^{n-1})$ such that $g_k(r) \equiv \int_{S^{n-1}} \chi u^{(k)} d\sigma$ has also no limit. $g_k(r)$ can be shown as in [*13*] to satisfy $| dg_k/dr | \leq dr^{-1-\gamma}$, $\gamma = \min(1, \gamma_1, \ldots, \gamma_m)$ for some d and this implies existence of $\lim_{r\to\infty} g_k$ and hence of $\lim_{r\to\infty} u^{(k)}(r\Phi^{-1}(\bar{\xi}))$. This leads by the usual argument to the existence of the limits of $\partial^k u(r\gamma(t))/\partial t^k$ $\forall \gamma \in \mathcal{F}$ and hence via Lemma 2.1 to the existence of $\lim_{r\to\infty} \partial^m_\xi u$ as asserted. That $\partial^m_\xi u \to \partial^m_\xi A$ follows from an equicontinuity argument as in [*13*]. Such an argument implies also $A \in C^{m,\delta_m}(S^{n-1})$. Analogously we conclude by (2.7) for $k \leqslant m - 2$ that

$$\left| \frac{\partial}{\partial r} \frac{\partial^k u}{\partial \xi^k_{n-1}}(r\Phi^{-1}(\xi)) \right| \leq C_k r^{-1-\mu_k}, \qquad \mu_k = \min(1, \gamma_1, \ldots, \gamma_k)$$

for some C_k and from this follows $\forall\gamma \in \mathcal{F}$ via integration

$$\left| \frac{\partial^k}{\partial t^k} (u(r\gamma(t)) - A(\gamma(t))) \right| r^{\mu_k} \leq C'_k$$

and hence (1.11) via Lemma 2.1. The proof of the corresponding estimates for part *c*) of Theorem 2 is similar.

3. PROOF OF THEOREM 3

To verify Theorem 3 we essentially follow the ideas of the proof of the analogous result in the 2-dimensional case given in [*13*], however $n \geq 3$ requires some new techniques.

We start (as in [*13*]) with an integro-differential equation for $u(r, \xi)$ for $r > R$

$$u(r, \xi) = A(\xi) + \int_r^\infty \int_x^\infty \frac{\tilde{v}^2(y)}{\tilde{v}^2(x)} \left(\frac{L^2}{y^2} + V_2(y, \xi) \right) u(y, \xi) dy dx \qquad (3.1)$$

where the Laplace-Beltrami operator L^2 is given by (1.10). Analogously to the 2-dimensional case [*13*] this identity follows from (1.1), (1.4) and Theorem 2. Iterating this equation we shall work out the asymptotics for $u(r, \xi)$ for $r \to \infty$ and $|\xi|$ small. To make the analysis of these iterations more transparent we simplify the notation and introduce classes of functions:

DEFINITION 3.1. — Let $\varepsilon > 0$ and $B_\varepsilon = \{ \xi \in \mathbb{R}^{n-1} : |\xi| < \varepsilon \}$. We define S_m to be the following family of functions

$$S_m = \{ f : Q \to \mathbb{R} f \text{ real analytic in } B_\varepsilon \quad \text{for some} \quad \varepsilon > 0, \quad f = 0(|\xi|^m) \quad \text{for} \quad |\xi| \to 0 \},$$

PROPOSITION 3.1. — Let $f \in S_m$, $g \in S_n$, then

$$f + g \in S_{\min(m,n)}$$
$$f \cdot g \in S_{m+n}$$
$$\frac{\partial^k}{\partial \xi_1^{k_1} \dots \partial \xi_{n-1}^{k_{n-1}}} f \in S_{\max(0, m-k)}, \qquad \text{where} \quad \sum_{i=1}^{n-1} k_i = k. \qquad (3.2)$$

REMARK 3.1. — We shall denote any member of S_m by the symbol s_m

and we formalize the above statements by introducing operations between the symbols s_m in an evident way, so that (3.2) reads:

$$s_m + s_n = s_{\min(m,n)}, \qquad s_m \cdot s_n = s_{m+n},$$
$$\frac{\partial^k}{\partial \xi_1^{k_1} \dots \partial \xi_{n-1}^{k_{n-1}}} s_m = s_{\max(0,m-k)}.$$

Further for convenience we define for $m \in \mathbb{Z}$

$$s_m \equiv s_{\max(0,m)}.$$

Proof of proposition 3.1. — The first two statements are trivial and the third one follows easily from Taylor's formula with remainder.

We proceed by analysing (3.1). According to (1.10) we have near $\xi = 0$

$$\mathrm{L}^2 = -\Delta + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2} + s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l} \qquad \text{with} \quad \Delta = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \xi_i^2}. \tag{3.3}$$

Thereby we used that for $|\xi| < \varepsilon$, $(\cos \xi_j)^{-1} = 1 + s_2$ and $\operatorname{tg} \xi_j = s_1$ in the notation introduced above. We continue to simplify the analysis of (3.1) by introducing the following notations:

$$\left.\begin{aligned}
&\mathrm{Q} = \frac{\tilde{v}^2(y)}{\tilde{v}^2(x)}, \qquad \mathrm{Q}_i = \frac{\tilde{v}^2(y_i)}{\tilde{v}^2(x_i)}, \qquad (i \in \mathbb{N})\\
&\mathrm{W}_i = \mathrm{V}_2(y_i, \xi), \qquad \mathrm{W} = \mathrm{V}_2(y, \xi)\\
&\mathrm{T}_i = y_i^{-2}\mathrm{L}^2 + \mathrm{W}_i = y_i^{-2}\left(-\Delta + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2} + s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l}\right) + \mathrm{W}_i\\
&\mathrm{T} = y^{-2}\mathrm{L}^2 + \mathrm{W}.
\end{aligned}\right\} \tag{3.4}$$

According to Theorem 2 part *c*) $u, \mathrm{A} \in \mathrm{C}^{\omega}(\mathrm{S}^{n-1})$ and we can iterate (3.1) to obtain for $\mathrm{N} \geq 1$

$$u(r, \xi) = \mathrm{A}(\xi) + \sum_{k=1}^{\mathrm{N}} \left\langle \prod_{i=1}^{k} \mathrm{Q}_i \mathrm{T}_i \mathrm{A} \right\rangle + \left\langle \prod_{i=1}^{\mathrm{N}} \mathrm{Q}_i \mathrm{T}_i (u(y_{\mathrm{N}}, \xi) - \mathrm{A}) \right\rangle \tag{3.5}$$

where as in [*13*] for $1 \leq k \leq \mathrm{N}$

$$\left\langle \prod_{i=1}^{k} \mathrm{Q}_i \mathrm{T}_i \mathrm{A} \right\rangle = \int_r^\infty \int_{x_1}^\infty \int_{y_1}^\infty \dots \int_{x_{k-1}}^\infty \int_{y_{k-1}}^\infty \int_{x_k}^\infty \mathrm{Q}_1 \mathrm{T}_1 \dots \mathrm{Q}_k \mathrm{T}_k \mathrm{A}\, dy_k dx_k \dots dy_1 dx_1 . \tag{3.6}$$

Due to our assumption $\mathrm{A}(\xi)$ satisfies (1.14), i. e., $\mathrm{A} = \mathrm{P}_{\mathrm{M}} + s_{\mathrm{M}+1}$ for some integer $\mathrm{M} \geq 1$ with P_{M} given by (1.15). Let

$$\mathrm{I}_{0,\mathrm{M}}(\xi) = \mathrm{A}(\xi) \tag{3.7}$$

and

$$\mathrm{I}_{k,\mathrm{M}}(r,\xi) = \left\langle \prod_{i=1}^{k} \mathrm{Q}_i \mathrm{T}_i \mathrm{A} \right\rangle \quad \text{for} \quad 1 \leq k \leq \mathrm{N}, \qquad \mathrm{N} \geq \mathrm{M}. \tag{3.8}$$

(3.5) becomes then with $m \equiv [\mathrm{M}/2]$,

$$u(r,\xi) = \sum_{k=0}^{m} \mathrm{I}_{k,\mathrm{M}} + \sum_{k=m+1}^{\mathrm{N}} \mathrm{I}_{k,\mathrm{M}} + \left\langle \prod_{i=1}^{\mathrm{N}} \mathrm{Q}_i \mathrm{T}_i (u(y_{\mathrm{N}},\xi) - \mathrm{A}) \right\rangle. \tag{3.9}$$

The following estimates established in [13] will be frequently used for the analysis of the asymptotics of (3.9):

PROPOSITION 3.2. — Let V_1 satisfy the conditions (A) and suppose

$$-\tilde{v}'' + \left(\mathrm{V}_1 + \frac{(n-1)(n-3)}{4r^2} - \mathrm{E} \right) \tilde{v} = 0 \tag{3.10}$$

in (R, ∞) with $0 < v$, $v \in \mathrm{L}^2(\Omega_\mathrm{R})$, $\tilde{v} = r^{(n-1)/2} v$, then for some $\varepsilon > 0$ and r large

$$\int_r^\infty \tilde{v}^2 dx = \frac{1}{2} \left(\mathrm{V}_1 + \frac{(n-1)(n-3)}{4r^2} - \mathrm{E} \right)^{-1/2} (1 + 0(r^{-\varepsilon})) \tilde{v}^2(r), \tag{3.11}$$

for $\gamma > 0$

$$\int_r^\infty \int_x^\infty \frac{\tilde{v}^2(y)}{\tilde{v}^2(x)} y^{-1-\gamma} dy dx = \frac{1}{2\sqrt{|\mathrm{E}|}\gamma} (1 + 0(r^{-\varepsilon})) r^{-\gamma} \tag{3.12}$$

and with the notation (3.4) and (3.6)

$$\left\langle \prod_{j=1}^{k} \mathrm{Q}_j y_j^{-1-\gamma_j} \right\rangle = 0(r^{-\sum_{j=1}^{k} \gamma_j}) \tag{3.13}$$

where $\gamma_1, \ldots, \gamma_k > 0$, $k \geq 1$.

Let us remember the definition of D_β given in (1.16) and introduce the following class of functions:

DEFINITION 3.2. — Let $m \geq 0$, then

$$\mathcal{G}_m \equiv \left\{ g \in \mathrm{C}^1(\mathrm{D}_\beta) \colon g(r,.) \in \mathrm{C}^\omega(|\xi| \leq r^{-\beta}) \text{ with } \frac{\partial^k g}{\partial \xi_1^{k_1} \ldots \partial \xi_{n-1}^{k_{n-1}}} = 0(r^{-m}),\ k = \sum_{i=1}^{n-1} k_i,\ \forall k \geq 0 \right\}.$$

The proof of the following proposition is immediate (apply Prop. 3.2).

PROPOSITION 3.3. — Let $f \in \mathcal{G}_m$, $g \in \mathcal{G}_k$, then

$$f + g \in \mathcal{G}_{\min(m,k)}, \qquad f \cdot g \in \mathcal{G}_{m+k} \quad \text{for} \quad m, k \geq 0 \tag{3.14}$$
$$\frac{\partial^k f}{\partial \xi_1^{k_1} \dots \partial \xi_{n-1}^{k_{n-1}}} f \in \mathcal{G}_m \qquad \langle Qf \rangle \in \mathcal{G}_{m-1}, \qquad m > 1 .$$

Remark 3.2. — Since for our purpose it suffices to determine the classes $\mathcal{G}_m$ to which the higher order terms in (3.9) belong we shall introduce the following simplified notation (see also [*13*]): We denote any member of $\mathcal{G}_m$ by the symbol G_m and in accordance with Proposition 3.3 the meaning of the following operations with this symbols is evident, so (3.14) reads

$$G_m + G_k = G_{\min(m,k)}, \qquad G_m \cdot G_k = G_{m+k},$$
$$\frac{\partial^k}{\partial \xi_1^{k_1} \dots \partial \xi_{n-1}^{k_{n-1}}} G_m = G_m, \qquad \langle QG_m \rangle = G_{m-1}.$$

We proceed by investigating the asymptotic behaviour of $I_{k,M}$:

LEMMA 3.1. — Let $M \geq 1$, then in D_β for r large

$$I_{k,M}(r, \xi) = \frac{(-1)^k}{k!}(2\sqrt{|E|}r)^{-k}(\Delta^k A)(1 + 0(r^{-\varepsilon})) + \sum_{j=0}^{2k-1} s_{M-j} G_{\delta_j} \tag{3.15}$$

for $k \geq 1$, for some $\varepsilon > 0$ and with some $\delta_0 \geq \frac{1}{2} + \delta$, $\delta_j \geq \left[\frac{j+1}{2}\right] + \delta \ \forall j \geq 1$ and where $\delta = a - 1/2 \equiv \min(\alpha, 1) - 1/2 > 0$, according to the assumption of part *c*) of Theorem 2.

Proof of Lemma 3.1. — We proceed by induction with respect to k. So we have to show first

$$I_{1,M}(r, \xi) = -\frac{1}{2\sqrt{|E|}r}(\Delta A)(1 + 0(r^{-\varepsilon})) + s_M G_{\delta_0} + s_{M-1} G_{\delta_1}. \tag{3.16}$$

But

$$I_{1,M} = \langle QTA \rangle = \langle Qy^{-2} \rangle \left(-\Delta + s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l} + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2} \right) A + \langle QWA \rangle .$$

From Proposition 3.2 we get

$$I_{1,M} = -\frac{1}{2\sqrt{|E|}r}(1 + 0(r^{-\varepsilon}))\left(\Delta + s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l} + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2} \right) A + G_\alpha A .$$

Since A is real analytic we have because of (1.14) $A = s_M$ for $|\xi| \to 0$ and by Proposition 3.1

$$\left(s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l} + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2}\right) A = s_M.$$

This implies

$$I_{1,M} = -\frac{1}{2\sqrt{|E|}\,r}(1 + 0(r^{-\varepsilon}))\Delta A + G_a s_M, \qquad a = \min(\alpha, 1)$$

which verifies (3.16).

Hence we have to verify (3.15) under the induction hypothesis for $k-1$:

$$I_{k,M} = \langle QTI_{k-1,M} \rangle = \left\langle Q\left(y^{-2}\left(-\Delta + s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l} + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2}\right) + W\right) \right. .$$

$$\left. \cdot \left(\frac{(-1)^{k-1}}{(k-1)!\,y^{k-1}}(2\sqrt{|E|})^{-k+1}(\Delta^{k-1}A)(1 + 0(y^{-\varepsilon})) + \sum_{j=0}^{2k-3} s_{M-j} G_{\delta_j}\right)\right\rangle .$$

Application of Proposition 3.2 leads to

$$\begin{aligned} I_{k,M} = {} & \frac{(-1)^k}{k!\,r^k(2\sqrt{|E|})^k}(\Delta^k A)(1 + 0(r^{-\varepsilon})) + \\ & + \langle Qy^{-k-1} \rangle (1 + 0(r^{-\varepsilon}))\left(s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l} + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2}\right)\Delta^{k-1} A + \\ & + \langle QG_{\alpha+k} \rangle \Delta^{k-1} A + \left\langle QT \sum_{j=0}^{2k-3} s_{M-j} G_{\delta_j} \right\rangle . \end{aligned} \tag{3.17}$$

But by Proposition 3.1 we have

$$\Delta^{k-1} A = s_{M-2k+2} \quad \text{and} \quad \left(s_1 \sum_{l=1}^{n-2} \frac{\partial}{\partial \xi_l} + s_2 \sum_{l=2}^{n-1} \frac{\partial^2}{\partial \xi_l^2}\right)\Delta^{k-1} A = s_{M-2k+2} \tag{3.18}$$

and by Proposition 3.3

$$\langle QG_{\alpha+k} \rangle \Delta^{k-1} A = G_{k-1+\alpha} s_{M-2k+2} . \tag{3.19}$$

Noting that by Proposition 3.1 $Ts_i G_j = s_{j-2} G_{j+2} + s_i G_{j+1+\alpha}$ we conclude by Proposition 3.3 that

$$\left\langle QT \sum_{j=0}^{2k-3} s_{M-j} G_{\delta_j} \right\rangle = \sum_{j=0}^{2k-3} s_{M-j-2} G_{\delta_j+1} + \sum_{j=0}^{2k-3} s_{M-j} G_{\delta_j+\alpha} . \tag{3.20}$$

Inserting (3.18) and (3.20) into (3.17) we obtain

$$I_{k,M} = \frac{(-1)^k}{k!} r^{-k}(2\sqrt{|E|})^{-k}(\Delta^k A)(1 + 0(r^{-\varepsilon})) + s_{M-2k+2}G_{k-1+\alpha} + \\ + \sum_{j=0}^{2k-3} (s_{M-j-2}G_{\delta_j+1} + s_{M-j}G_{\delta_j+\alpha}). \quad (3.21)$$

Since $1 + \delta_j \geq \left[\frac{j+3}{2}\right]$ we observe that

$$\sum_{j=0}^{2k-3} s_{M-j-2}G_{\delta_j+1} = \sum_{l=2}^{2k-1} s_{M-l}G_{\delta_l},$$

and further $G_{k-1+\alpha}$ can be replaced by $G_{\delta_{2k-2}}$. Taking this into account equation (3.21) verifies Lemma 3.1. □

Let us now investigate the asymptotics of $I_{k,M}$ for $k \geq \left[\frac{M}{2}\right] + 1 = m+1$: Since $r^{-k}\Delta^k A = s_0 G_k$ Lemma 3.1 implies

$$I_{k,M} = s_0 G_{m+1} = \sum_{j=0}^{2k-1} s_{M-j}G_{\delta_j}$$

and with $\delta = a - 1/2$

$$\sum_{k=m+1}^{N} I_{k,M} = s_0 G_{M/2+a-1/2} + \sum_{j=0}^{2N-1} s_{M-j}G_{\delta_j} = s_0 G_{M/2+a-1/2} + \\ + s_{M-2m}G_{m+\delta} + \sum_{j=0}^{2m-1} s_{M-j}G_{\delta_j}. \quad (3.22)$$

On the other hand we have

$$\sum_{k=0}^{m} I_{k,M} = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{1}{(2\sqrt{|E|r})^k} (\Delta^k A)(1+0(r^{-\varepsilon})) + \sum_{j=0}^{2m-1} s_{M-j}G_{\delta_j}. \quad (3.23)$$

Combining (3.22) and (3.23) we obtain

$$\sum_{k=0}^{N} I_{k,M} = \sum_{k=0}^{m} \frac{(-1)^k}{k!} (2\sqrt{|E|r})^{-k}(\Delta^k A)(1+0(r^{-\varepsilon})) + \sum_{j=0}^{2m-1} s_{M-j}G_{\delta_j} + \\ + s_0 G_{M/2+\delta} + s_{M-2m}G_{m+\delta}. \quad (3.24)$$

But in D_β it is easily seen that for large r

$$s_0 G_{M/2+\delta} + s_{M-2m} G_{m+\delta} + \sum_{j=0}^{2m-1} s_{M-j} G_{\delta_j} = 0(r^{-M/2-\delta+\delta_0(\beta)})$$

where $\delta_0(\beta) = M\left(\frac{1}{2} - \beta\right)$. Next remember that $A = P_M + s_{M+1}$ and observe that

$$\sum_{k=0}^{m} 0(r^{-k})\Delta^k s_{M+1} = 0(r^{-\beta(M+1)})$$

so that we arrive at

$$\sum_{k=0}^{N} I_{k,M} = \sum_{k=0}^{m} \frac{(-1)^k}{k!}(2\sqrt{|E|}r)^{-k}(\Delta^k P_M)(1+0(r^{-\varepsilon})) + 0(r^{-\beta M})0(r^{-\min\left(a-\frac{1}{2},\beta\right)}). \tag{3.25}$$

To investigate the asymptotics of the leading term in (3.25) we need.

LEMMA 3.2. — Denote $m = [M/2]$, then

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \Delta^k \prod_{i=1}^{n-1} \xi_i^{l_i} = \prod_{i=1}^{n-1} H_{l_i}(\xi_i/2) \tag{3.26}$$

where we assume $n \geq 2$, $l_i \in \mathbb{N} \cup \{0\}$ and $\sum_{i=1}^{n-1} l_i = M$ for M a positive integer, and where the Hermite polynomials H_{l_i} have been introduced in section 1.

Proof of Lemma 3.2. — For $n = 2$ (3.26) is proven in [*13*]. We proceed by induction with respect to n. Without loss we assume that $l_{n-1} \equiv l \geq 1$ and we set $Q = \prod_{i=1}^{n-2} \xi_i^{l_i}$. Then by the induction hypothesis

$$\sum_{k=0}^{[(M-l)/2]} \frac{(-1)^k}{k!}\left(\sum_{j=1}^{n-2} \frac{\partial^2}{\partial \xi_j^2}\right)^k Q = \prod_{i=1}^{n-2} H_{l_i}(\xi_i/2). \tag{3.27}$$

Now let $D^2 = \sum_{j=1}^{n-2} \frac{\partial^2}{\partial \xi_j^2}$ and $\partial^2 = \frac{\partial^2}{\partial \xi_{n-1}^2}$. Using the binomial expansion we get

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \Delta^k (Q\xi_n^l) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (D^{2j}Q)(\partial^{2k-2j}\xi_{n-1}^l) =$$

$$= \sum_{i=0}^{[l/2]} \sum_{k=i}^{m} \frac{(-1)^k}{k!} \binom{k}{k-i} D^{2(k-i)} Q \partial^{2i} \xi_{n-1}^l =$$

$$= \sum_{i=0}^{[l/2]} \frac{(-1)^i}{i!} \partial^{2i} \xi_{n-1}^l \sum_{j=0}^{[(M-l)/2]} \frac{(-1)^j}{j!} D^{2j} Q = \prod_{i=1}^{n-2} H_{l_i}\left(\frac{\xi_i}{2}\right) H_l\left(\frac{\xi_{n-1}}{2}\right)$$

where we took into account that the degree of the polynomial Q is $M - l$ and the induction hypothesis (3.27). This proves the lemma. □

In order to apply Lemma 3.2 to equation (3.25) we need the (probably well-known) identity (3.26) in a scaled version, which is easily established:

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \lambda^{-k} \Delta^k \prod_{i=1}^{n-1} \xi_i^{l_i} = \lambda^{-M/2} \prod_{i=1}^{n-1} H_{l_i}\left(\frac{\xi_i}{2}\sqrt{\lambda}\right) \tag{3.28}$$

for $\lambda > 0$. If we set $\lambda = 2\sqrt{|E|}\, r$ we get with $P_M(\xi)$ defined by (1.15) and $\mathscr{H}_M$ given in (1.22)

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{1}{(2\sqrt{|E|}\, r)^k} (\Delta^k P_M)(\xi) = (2\sqrt{|E|}\, r)^{-M/2} \mathscr{H}_M\left((2\sqrt{|E|}\, r)^{1/2} \frac{1}{2} \xi\right). \tag{3.29}$$

Application of (3.29) to (3.25) gives

$$\sum_{k=0}^{N} I_{k,M}(r, \xi) = (2\sqrt{|E|}\, r)^{-M/2} \mathscr{H}_M\left((2\sqrt{|E|}\, r)^{1/2} \frac{1}{2} \xi\right)(1 + 0(r^{-\varepsilon})) + + 0(r^{-\beta M - \min(a - 1/2, \beta)}). \tag{3.30}$$

Next we return to equation (3.9), insert (3.30) into it and obtain for large r in D_β

$$u(r, \xi) = \text{r. h. s. of } (3.30) + \left\langle \prod_{i=1}^{N} Q_i T_i(u(y_N, \xi) - A(\xi)) \right\rangle \tag{3.31}$$

so it remains to investigate the asymptotics of the remainder term in (3.31). According to Theorem 2 we have

$$u(r, \xi) - A(\xi) \in G_a \tag{3.32}$$

and we conclude via Proposition 3.2 (eq. (3.13)) that

$$\left\langle \prod_{i=1}^{N} Q_i T_i(u(y_N, \xi) - A(\xi)) \right\rangle = \left\langle \prod_{i=1}^{N} Q_i y_i^{-1-a} G_a \right\rangle = 0\left(\left\langle \prod_{i=1}^{N} Q_i y_i^{-1-a} y_N^{-a} \right\rangle \right)$$

$$= 0(r^{-(N+1)a}) = 0(r^{-M/2-a}) \qquad \text{for any } N \geq M. \quad (3.33)$$

(3.33) together with (3.31) gives the desired result (1.17)

$$u(r, \xi) = (2\sqrt{|E|}r)^{-M/2} \mathscr{H}_M\left((2\sqrt{|E|}\, r)^{1/2} \frac{1}{2} \xi \right)(1 + 0(r^{-\varepsilon})) + 0(r^{-\beta M - \min\left(a - \frac{1}{2}, \beta\right)})$$

in D_β for large r, proving Theorem 3.

Specifically we have for large r in $D_{1/2}^{k/b}$ for any $k < \infty$

$$r^{M/2} u(r, \xi) = (2\sqrt{|E|})^{-M/2} \mathscr{H}_M\left((2\sqrt{|E|} r)^{1/2} \frac{1}{2} \xi \right)(1 + 0(r^{-\varepsilon})) + 0(r^{-a+1/2})$$

and with $z = b\sqrt{r}\xi$, b given in (1.18), it follows that

$$\lim_{r \to \infty} u\left(r, \frac{z}{b\sqrt{r}}\right) r^{M/2} = (2\sqrt{|E|})^{-M/2} \mathscr{H}_M(z)$$

proving Corollary 1 of section 1.

REFERENCES

[1] L. Bers, Local behaviour of solutions of general elliptic equations. *Comm. Pure Appl. Math.*, t. **8**, 1955, p. 473-496.

[2] J. Boman, Differentiability of a function and of its composition with functions of one variable. *Math. Scand.*, t. **20**, 1967, p. 249-268.

[3] L. A. Cafarelli and A. Friedman, Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations. *J. Differ. Equations*, t. **60**, 1985, p. 420-433.

[4] S. Y. Cheng, Eigenfunctions and nodal sets. *Comment. Math. Helvetici*, t. **51**, 1976, p. 43-55.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Bateman manuscript project: higher transcendental functions*, vol. II, McGraw Hill, New York, Toronto, London, 1953.

[6] R. Froese and I. Herbst, Patterns of exponential decay for solutions to second order elliptic equations in a sector of $\mathbb{R}^2$, to appear in *J. d'Analyse Math.*

[7] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer Verlag, Berlin, Heidelberg, New York, 1977.

[8] I. Herbst, Lower bounds in cones for solutions to the Schrödinger equation, *J. d'Analyse Math.*, t. **47**, 1986, p. 151-174.

[9] M. Hoffmann-Ostenhof, Asymptotics of the nodal lines of solutions of 2-dimensional Schrödinger equations, *submitted for publication.*

[10] M. HOFFMANN-OSTENHOF and T. HOFFMANN-OSTENHOF, On the asymptotics of nodes of L^2-solutions of Schrödinger equations, *in preparation.*

[11] M. HOFFMANN-OSTENHOF, T. HOFFMANN-OSTENHOF and B. SIMON, Brownian motion and a consequence of Harnack's inequality: nodes of quantum wave functions. *Proc. Amer. Math. Soc.*, t. **80**, 1980, p. 301-305.

[12] M. HOFFMANN-OSTENHOF, T. HOFFMANN-OSTENHOF and J. SWETINA, Pointwise bounds on the asymptotics of spherically averaged L^2-solutions of one-body Schrödinger equations. *Ann. Inst. H. Poincaré*, t. **42**, 1985, p. 341-361.

[13] M. HOFFMANN-OSTENHOF, T. HOFFMANN-OSTENHOF and J. SWETINA, Continuity and nodal properties near infinity for solutions of 2-dimensional Schrödinger equations. *Duke Math. J.*, t. **53**, 1986, p. 271-306.

[14] J. MUJICA, *Complex analysis in Banach spaces*, *North-Holland Mathematics Studies*, t. **120**, 1986, chapter 1.

[15] M. H. PROTTER and H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice Hall, Englewood Cliffs, 1967.

[16] E. M. STEIN, *Singular integrals and differentiability of functions*, Princeton University Press, Princeton, NJ, 1970.

[17] D. W. THOE, Lower bounds for solutions of perturbed Helmholtz equations in exterior regions. *J. Math. Anal. Appl.*, t. **102**, 1984, p. 113-122.

(Manuscrit reçu le 15 juillet 1986)