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Scattering theory for Hartree type equations

by

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ABSTRACT. — In this paper we will study the asymptotic behavior in time of the solutions and the scattering theory for the following Hartree type equation

$$(1) \qquad iu_t + \Delta u = \lambda f(|u|^2)u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

$$(2) \qquad u(0,x) = \phi(x), \qquad x \in \mathbb{R}^n,$$

where $f(|u|^2) = |x|^{-\gamma} * |u|^2$, $0 < \gamma < \text{Min}(4, n)$, $n \geqq 2$ and $\lambda \in \mathbb{R}$.

$$\text{Let } \Sigma^{l,m} = \left\{ v \in L^2(\mathbb{R}^n); \| v \|^2_{\Sigma^{l,m}} = \sum_{|\alpha| \leqq l} \| D^\alpha v \|_2^2 + \sum_{|\beta| \leqq m} \| x^\beta v \|_2^2 < \infty \right\}, \ l, m \in \mathbb{N}.$$

We prove that when $(4/3) < \gamma < \text{Min}(4, n)$ and $\lambda > 0$, all solutions of (1)-(2) with $\phi \in \Sigma^{l,m}$ are dispersive in $\Sigma^{l,m}$ and that when $1 < \gamma < \text{Min}(4, n)$ and $\lambda \in \mathbb{R}$, the solutions of (1)-(2) with $\phi \in \Sigma^{l,m}$ and $\| \phi \|_{\Sigma^{1,1}}$ small are dispersive in $\Sigma^{l,m}$. This implies asymptotic completeness in $\Sigma^{l,m}$ of the wave operators for $(4/3) < \gamma < \text{Min}(4, n)$ and $\lambda > 0$. Furthermore when $\lambda > 0$, we show the existence of scattering states in $L^2(\mathbb{R}^n)$ for arbitrary data in $\Sigma^{1,1}$ if $1 < \gamma < (4/3)$ and the non-existence of scattering states in $L^2(\mathbb{R}^n)$ for $0 < \gamma \leqq 1$.

RÉSUMÉ. — On étudie le comportement asymptotique en temps des solutions et la théorie de la diffusion pour l'équation de type Hartree suivante

$$(1) \qquad iu_t + \Delta u = \lambda f(|u|^2)u, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

$$(2) \qquad u(0,x) = \phi(x), \qquad x \in \mathbb{R}^n,$$

où $f(|u|^2) = |x|^{-\gamma} * |u|^2$, $0 < \gamma < \text{Min}(4, n)$, $n \geqq 2$ et $\lambda \in \mathbb{R}$.

Soit $\Sigma^{l,m} = \left\{ v \in L^2(\mathbb{R}^n) ; \|v\|^2_{\Sigma^{l,m}} = \sum_{|\alpha| \leq l} \|D^\alpha v\|_2^2 + \sum_{|\beta| \leq m} \|x^\beta v\|_2^2 < \infty \right\}$, $l, m \in \mathbb{N}$. On montre que lorsque $(4/3) < \gamma < \mathrm{Min}(4, n)$ et $\lambda > 0$, toutes les solutions de (1)-(2) avec $\phi \in \Sigma^{l,m}$ sont dispersives dans $\Sigma^{l,m}$, et que lorsque $1 < \gamma < \mathrm{Min}(4, n)$ et $\lambda \in \mathbb{R}$, les solutions de (1)-(2) avec $\phi \in \Sigma^{l,m}$ et $\|\phi\|_{\Sigma^{1,1}}$ petit sont dispersives dans $\Sigma^{l,m}$. Ceci entraîne la complétude asymptotique des opérateurs d'onde dans $\Sigma^{l,m}$ pour $(4/3) < \gamma < \mathrm{Min}(4, n)$ et $\lambda > 0$. En outre, pour $\lambda > 0$, on montre l'existence d'états de diffusion dans $L^2(\mathbb{R}^n)$ pour des données initiales arbitraires dans $\Sigma^{1,1}$ si $1 < \gamma < (4/3)$ et la non existence de tels états pour $0 < \gamma \leqq 1$.

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1. INTRODUCTION

This paper deals with the asymptotic behavior in time of the solutions and the scattering theory for the following Hartree type equation

$$iu_t + \Delta u = \lambda f(|u|^2)u, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.1}$$

$$u(0, x) = \phi(x), \qquad x \in \mathbb{R}^n, \tag{1.2}$$

where Δ is the n-dimensional Laplacian in x, $\lambda \in \mathbb{R}$ and

$$f(|u|^2) = |x|^{-\gamma} * |u|^2 = \int_{\mathbb{R}^n} |x - y|^{-\gamma} |u(t, y)|^2 dy$$

for $0 < \gamma < n$. Let $U(t)$ be an evolution operator associated with the free Schrödinger equation and $\Sigma^{l,m}$ be the Hilbert space defined by

$$\Sigma^{l,m} = \left\{ \psi \in L^2(\mathbb{R}^n) ; \|\psi\|^2_{\Sigma^{l,m}} = \sum_{|\alpha| \leq l} \|D^\alpha \psi\|_2^2 + \sum_{|\beta| \leq m} \|x^\beta \psi\|_2^2 < \infty \right\}$$

with the inner product

$$(\psi, \psi)_{\Sigma^{l,m}} = \sum_{|\alpha| \leq l} (D^\alpha \psi, D^\alpha \psi) + \sum_{|\beta| \leq m} (x^\beta \psi, x^\beta \psi),$$

where

$$(f, g) = \int_{\mathbb{R}^n} f . \bar{g} dx .$$

When $\lambda > 0$ and $2 < \gamma < \mathrm{Min}(4, n)$, Ginibre-Velo [7] showed that *a*) For any $u_+ \in \Sigma^{l,1}$ ($l \in \mathbb{N}$) there exists a unique $\phi \in \Sigma^{l,1}$ such that

$$\| u_+ - U(-t)u(t) \|_{\Sigma^{l,1}} \to 0 \qquad (t \to +\infty), \tag{1.3}$$

where $u(t)$ is the solution of (1.1)-(1.2) with $U(-t)u(t)$ in $C(\mathbb{R}; \Sigma^{l,1})$.

For any $u_- \in \Sigma^{l,1}$ ($l \in \mathbb{N}$) the same result as above holds valid with $+\infty$ replaced by $-\infty$ in (1.3).

b) For any $\phi \in \Sigma^{l,1}$ there exist unique $u_\pm \in \Sigma^{l,1}$ such that the solution $u(t)$ of (1.1)-(1.2) with $U(-t)u(t)$ in $C(\mathbb{R}; \Sigma^{l,1})$ satisfies

$$\| u_\pm - U(-t)u(t) \|_{\Sigma^{l,1}} \to 0 \qquad (t \to \pm\infty).$$

We note that *a*) and *b*) imply the existence of the wave operators defined in the space $\Sigma^{l,1}$ and their asymptotic completeness (for details see Corollary 5.1). They proved the above results by using the pseudoconformal conservation law

$$\sum_{|\beta|=1} \| x^\beta U(-t)u(t) \|_2^2 + 4t^2 P(u(t)) = \sum_{|\beta|=1} \| x^\beta \phi \|_2^2 + 4(2-\gamma) \int_0^t sP(u(s))ds, \tag{1.4}$$

where
$$P(\phi) = \frac{\lambda}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x)|^2 |\phi(y)|^2}{|x-y|^\gamma} dxdy, \tag{1.5}$$

and $L^p - L^q$ estimates for solutions of the free Schrödinger equation.

By making use of the space-time estimates obtained by Strichartz [*20*] (see also Ginibre-Velo [*9*]) and $L^p - L^q$ estimates, Strauss [*19*] showed *a*) and *b*) in the space $\Sigma^{1,0}$ provided that $2 \leqq \gamma < \mathrm{Min}(4, n)$ and $\| \phi \|_{\Sigma^{1,0}}$ is sufficiently small, and he also showed that if $(4/3) < \gamma < \mathrm{Min}(4, n)$ and $u_- \in L^{4n/(2n+\gamma)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then for sufficiently large $T > 0$ there exists a unique solution $u(t)$ in $C((-\infty, -T]; L^{4n/(2n-\gamma)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$ of (1.1) such that

$$\| u_- - U(-t)u(t) \|_2 \to 0 \quad (t \to -\infty).$$

When $\gamma = 1$, $n = 3$ and $\lambda > 0$, Glassey [*10*] established that for non-zero $\phi \in \mathscr{S}(\mathbb{R}^3)$ there do not exist any $u_\pm \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ satisfying

$$\| u_\pm - U(-t)u(t) \|_2 \to 0 \quad (t \to \pm\infty). \tag{1.6}$$

But his result does not seem to imply the non-existence of any scattering states $u_\pm \in L^2(\mathbb{R}^3)$, since in his proof he essentially uses the fact that $\| U(t)u_\pm \|_\infty = O(|t|^{-n/2})$ as $t \to \pm\infty$.

Our main purpose in this paper is to extend the above results as follows when $n \geqq 2$.

(I). Suppose that $1 < \gamma < \mathrm{Min}(4, n)$ and $\lambda > 0$. Then, for any $\phi \in \Sigma^{1,1}$

there exist unique scattering states $u_\pm \in L^2(\mathbb{R}^n)$ satisfying (1.6) (see section 3).

(II). Suppose that $1 < \gamma < \mathrm{Min}(4, n)$ and $\varepsilon > 0$ is sufficiently small. Let u_+, u_- and $\phi \in \Sigma_\varepsilon^{l,m} = \{ \psi \in \Sigma^{l,m}; \| \psi \|_{\Sigma^{1,1}} \leqq \varepsilon \}$. Then *a)* and *b)* hold valid in the space $\Sigma^{l,m}$ (see section 4).

(III). Suppose that $(4/3) < \gamma < \mathrm{Min}(4, n)$ and $\lambda > 0$. Then *a)* and *b)* hold valid in the space $\Sigma^{l,m}$ (see section 5).

(IV). Suppose that $0 < \gamma \leqq 1$ and $\lambda > 0$. Then, for any non-zero $\phi \in \Sigma^{1,1}$ there do not exist any $u_\pm \in L^2(\mathbb{R}^n)$ satisfying (1.6) (see section 3).

REMARK 1.1. — (I) and (IV) imply that $\gamma = 1$ is a critical value. (II) and (III) not only extend Strauss' result [*19*] and Ginibre-Velo's result [*7*] but also formulate the scattering problem for (1.1)-(1.2) in the more natural space $\Sigma^{l,m}$ than $\Sigma^{l,1}$ used by Ginibre-Velo [*7*].

REMARK 1.2. — In the case of $f(|u|^2)u = |u|^{p-1}u$, there exist the analogous results to (I), (III) and (IV) which were obtained by Y. Tsutsumi-K. Yajima [*23*], Y. Tsutsumi [*21*] [*22*], Ginibre-Velo [*6*] [*9*], Strauss [*17*] and Barab [*1*].

(A) [*23*]. Suppose that $1 + (2/n) < p < \alpha(n)$ and $\lambda > 0$. Then the same result as (I) holds valid, where $\alpha(n) = \infty$ for $n = 1, 2$ and $\alpha(n) = (n+2)/(n-2)$ for $n \geqq 3$.

(B) [*22*]. Suppose that $\beta(n) < p < \alpha(n)$ and $\lambda > 0$. Then the same result as (III) holds valid in the space $\Sigma^{1,1}$, where $\beta(n) = (n+2+\sqrt{n^2+12n+4})/2n$.

(C) [*9*]. Suppose that $1 + (4/n) < p < \alpha(n)$, $n \geqq 3$ and $\lambda > 0$. Then the same result as (III) holds valid in the space $H^{1,2}(\mathbb{R}^n)$ $(= \Sigma^{1,0})$.

(D) [*21*]. Suppose that $1 < p \leqq 1 + (2/n)$ and $\lambda > 0$. Then the same result as (IV) holds valid.

Y. Tsutsumi-K. Yajima [*23*] showed (A) by using the following pseudo-conformal conservation law

$$\sum_{|\beta|=1} \| x^\beta U(-t)u(t) \|_2^2 + (8t^2/(p+1))\lambda \| u(t) \|_{p+1}^{p+1} + \frac{4(np-n-4)}{p+1}\lambda \int_0^t s \| u(s) \|_{p+1}^{p+1} ds = \sum_{|\beta|=1} \| x^\beta \phi \|_2^2, \quad (1.7)$$

and the following transform C

$$u(t, x) = (Cv)(t, x) = (it)^{-n/2} e^{i|x|^2/4t} \overline{v}(1/t, x/t). \quad (1.8)$$

(B) has been proved by Y. Tsutsumi [*22*] by using (1.7), (1.8) and the space-time estimates obtained by Strichartz [20] (see also Ginibre-Velo [9]). Ginibre-Velo [9] have shown (C) by making use of the Morawetz estimate instead of (1.7) (see also Brenner [2] [3]). Y. Tsutsumi has shown (D) by using (1.8) and a contradiction argument (see also Strauss [17] and Barab [1]).

Finally we introduce some notations which will be used in this paper. Let $1 \leqq p \leqq \infty$, $m \in \mathbb{N} \cup \{0\}$, α be a multi-index, $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ and $D^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\partial_j = \partial/\partial x_j$ $(j = 1, 2, \ldots, n)$. $H^{m,p} = H^{m,p}(\mathbb{R}^n)$ denote the usual Sobolev spaces, namely, the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to $\| f \|_{m,p} = \sum_{|\alpha| \leqq m} \| D^\alpha f \|_p$, where

$$\| g \|_p = \left(\int_{\mathbb{R}^n} | g(x) |^p dx \right)^{1/p} \quad (1 \leqq p < \infty), \quad \| g \|_\infty = \sup_{x \in \mathbb{R}^n} | g(x) | .$$

We put

$$J^\alpha = e^{i|x|^2/4t} (2it)^{|\alpha|} D^\alpha e^{-i|x|^2/4t} = U(t) x^\alpha U(-t) \tag{1.9}$$

(see, e. g., [12]). For any interval $I \subset \mathbb{R}$ and any Banach space with the norm $\| \cdot \|_B$, $C(I; B)$ denotes the space of B-valued continuous functions on I. Positive constants will be denoted by C and will change from line to line. If necessary, by $C(*, \ldots, *)$ we denote constants depending on the quantities appearing in parentheses.

2. PRELIMINARIES

We summarize some useful lemmas in this section.

LEMMA 2.1 (the Gagliardo-Nirenberg inequality). — Let q, r be any numbers satisfying $1 \leqq q$, $r \leqq \infty$, and let j, m be any integers satisfying $0 \leqq j < m$. Then for any $u \in H^{m,r} \cap L^q$

$$\sum_{|\alpha| = j} \| D^\alpha u \|_p \leqq M \sum_{|\beta| = m} \| D^\beta u \|_r^a \| u \|_q^{1-a}, \tag{2.1}$$

where $(1/p) = (j/m) + a((1/r) - (m/n)) + (1 - a)/q$ for all a in the interval $(j/m) \leqq a \leqq 1$ with the following exception: if $m - j - (n/r)$ is a nonnegative integer, then (2.1) is asserted for $a = j/m$, and where M is a constant depending only on n, m, j, q, r, a.

For Lemma 2.1 see, e. g., Friedman [5].

LEMMA 2.2. — Let $2 \leqq p \leqq \infty$ and $(1/p) + (1/p') = 1$. Then for any $\phi \in L^{p'}$

$$\| U(t) \phi \|_p \leqq C | t |^{-\delta(p)} \| \phi \|_{p'},$$

where $$\delta(p) = (n/2) - (n/p).$$

LEMME 2.3. — For any $\phi \in L^2$

$$\left(\int_{-\infty}^{\infty} \| U(t)\phi \|_r^q dt\right)^{1/q} \leqq C \| \phi \|_2,$$

where $$0 \leqq (n/2) - (n/r) = (2/q) < 1.$$

For Lemma 2.2 see, e. g., Ginibre-Velo [*6*], and for Lemma 2.3 see, e. g., Strichartz [*20*] and Ginibre-Velo [*9*].

LEMMA 2.4. — Let $1<p<q<\infty$, $0<\gamma<n$ and $(1/q)=(1/p)-(n-\gamma)/n$. Then for any $\phi \in L^p$

$$\| I^\gamma(\phi) \|_q \leqq M \| \phi \|_p,$$

where

$$I_\gamma(\phi)(x) = \int_{\mathbb{R}^n} |x-y|^{-\gamma}\phi(y)dy,$$

and M is a constant depending only on γ, p, q, n.

For Lemma 2.4 see, e. g., Stein [*16*].

LEMMA 2.5. — Let $0 < b < 1, a + b > 1$ and $f(t) \in C(\mathbb{R}^+ ; \mathbb{R}^+)$ satisfying the following inequality

$$f(t) \leqq C(1+t)^{-b} + C\int_0^t (t-s)^{-b}(1+s)^{-a}f(s)ds, \quad \text{for all } t \geqq 0.$$

Then $$f(t) \leqq C(1+t)^{-b} \quad \text{for all } t \geqq 0.$$

For Lemma 2.5 see, e. g., N. Hayashi-M. Tsutsumi [*11*].

LEMMA 2.6. — Let $0 < \gamma < n$, $q = 4n/(2n-\gamma)$ and $q' = 4n/(2n+\gamma)$. Then for any $\phi \in L^q$

$$\| f(|\phi|^2)\phi \|_{q'} \leqq C |P(\phi)|^{1/2} \| \phi \|_q \leqq C \| \phi \|_q^3. \tag{2.2}$$

Proof. — We have by Hölder's inequality and Lemma 2.4

$$\| f(|\phi|^2)\phi \|_{q'}^{q'} \leqq C\int\left(\int \frac{|\phi(y)|^2}{|x-y|^\gamma}dy \cdot |\phi(x)|\right)^{q'} dx$$

$$\leqq C\int\left(\left(\int \frac{|\phi(y)|^2}{|x-y|^\gamma}dy\right)^{1/2}\left(\int \frac{|\phi(y)|^2}{|x-y|^\gamma}dy\right)^{1/2} |\phi(x)|\right)^{q'} dx$$

$$\leqq C\left(\int\left(\int \frac{|\phi(y)|^2}{|x-y|^\gamma}dy\right)|\phi(x)|^2 dx\right)^{q'/2} \times \left(\int\left(\int \frac{|\phi(y)|^2}{|x-y|^\gamma}dy\right)^{q'/(2-q')} dx\right)^{(2-q')/2}$$

$$\leqq C |P(\phi)|^{q'/2} \| \phi \|_q^{q'}. \tag{2.3}$$

We again use Hölder's inequality and Lemma 2.4 to obtain

$$
\begin{aligned}
|P(\phi)| &\leqq C\int\left(\int\frac{|\phi(y)|^2}{|x-y|^\gamma}dy\right)|\phi(x)|^2dx \\
&\leqq C\left(\int\left(\int\frac{|\phi(y)|^2}{|x-y|^\gamma}dy\right)^{2n/\gamma}dx\right)^{\gamma/2n}\left(\int|\phi(x)|^q dx\right)^{2/q} \\
&\leqq C\|\phi\|_q^4 . \qquad (2.4)
\end{aligned}
$$

(2.2) follows from (2.3) and (2.4). Q. E. D.

LEMMA 2.7. — Let $l, m \in \mathbb{N}$, $0 < \gamma < n$, $q = 4n/(2n-\gamma)$, $q' = 4n/(2n+\gamma)$, $r = 2n/(n-\gamma)$ and $\varepsilon > 0$ be sufficiently small. Then

$$
\sum_{|\alpha|=l}\|D^\alpha f(u_1\cdot\bar{u}_2)u_3(t)\|_{q'} \leqq C\prod_{j=1}^{3}\|u_j(t)\|_q^{1-a_j(l)}\cdot\sum_{|\alpha|=l}\|D^\alpha u_j(t)\|_q^{a_j(l)}, \quad \left(\sum_{j=1}^{3}a_j(l)=1\right), \qquad (2.5)
$$

$$
\sum_{|\beta|=m}\|J^\beta f(u_1\cdot\bar{u}_2)u_3(t)\|_{q'} \leqq C\prod_{j=1}^{3}\|u_j(t)\|_q^{1-a_j(m)}\cdot\sum_{|\beta|=m}\|J^\beta u_j(t)\|_q^{a_j(m)}, \quad \left(\sum_{j=1}^{3}a_j(m)=1\right), \qquad (2.6)
$$

$$
\begin{aligned}
\sum_{|\alpha|=l}\|D^\alpha f(u_1.\bar{u}_2)u_3(t)\|_2 &\leqq C\prod_{j=1}^{3}\|u_j(t)\|_r^{1-b_j(l)}\cdot\sum_{|\alpha|=l}\|D^\alpha u_j(t)\|_2^{b_j(l)} \\
&+C\left(\prod_{j=1}^{2}\|u_j(t)\|_{r+\varepsilon}+\prod_{j=1}^{2}\|u_j(t)\|_{r-\varepsilon}\right)\cdot\sum_{|\alpha|=l}\|D^\alpha u_3(t)\|_2, \quad \left(\sum_{j=1}^{3}b_j(l)=1\right), \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{|\beta|=m}\|J^\beta f(u_1\cdot\bar{u}_2)u_3(t)\|_2 &\leqq C\prod_{j=1}^{3}\|u_j(t)\|_r^{1-b_j(m)}\cdot\sum_{|\beta|=m}\|J^\beta u_j(t)\|_2^{b_j(m)} \\
&+C\left(\prod_{j=1}^{2}\|u_j(t)\|_{r+\varepsilon}+\prod_{j=1}^{2}\|u_j(t)\|_{r-\varepsilon}\right)\cdot\sum_{|\beta|=m}\|J^\beta u_3(t)\|_2, \quad \left(\sum_{j=1}^{3}b_j(m)=1\right), \qquad (2.8)
\end{aligned}
$$

for any u_1, $\bar{u}_2$ and u_3 having finite right hand sides.

Proof. — We note (1.9). We put $v_j = e^{-i|x|^2/4t}u_j$. A simple calculation shows

$$\sum_{|\beta|=m} \| J^\beta f(u_1 \cdot \bar{u}_2)u_3 \|_{q'} \leqq C \sum_{|\beta|=m} \| (2it)^m D^\beta f(v_1 \cdot \bar{v}_2)v_3 \|_{q'}$$

$$\leqq C \sum_{\substack{|\beta|=|\beta_1|+|\beta_2|+|\beta_3| \\ 0 \leqq |\beta_1|, |\beta_2|, |\beta_3| \leqq m}} \| (2it)^m f(D^{\beta_1}v_1 \cdot \overline{D^{\beta_2}v_2})D^{\beta_3}v_3 \|_{q'} . \quad (2.9)$$

By virtue of Hölder's inequality, Lemma 2.1 and Lemma 2.4 the R. H. S. of (2.9) is dominated by

$$C|t|^m \sum_{\substack{|\beta|=|\beta_1|+|\beta_2|+|\beta_3| \\ 0 \leqq |\beta_1|, |\beta_2|, |\beta_3| \leqq m}} \prod_{j=1}^{3} \| D^{\beta_j}v_j \|_{p_j}$$

$$\leqq C|t|^m \prod_{j=1}^{3} \| v_j \|_q^{1-a_j(m)} \cdot \sum_{|\beta|=m} \| D^\beta v_j \|_q^{a_j(m)} , \quad (2.10)$$

where p_j and $a_j(m)$ satisfy

$$1/q' = \sum_{j=1}^{3} (1/p_j) - (n-\gamma)/n , \quad (2.11)$$

$$1/p_j = |\beta_j|/n + a_j \cdot ((1/q) - (m/n)) + (1-a_j)/q . \quad (2.12)$$

From (2.11) and (2.12) we see that $\sum_{j=1}^{3} a_j(m) = 1$. Indeed we have from (2.11) and (2.12)

$$1/q' = (2n+\gamma)/4n = m/n + ((1/q)-(m/n)) \cdot \sum_{j=1}^{3} a_j(m) + 3/q - \sum_{j=1}^{3} a_j(m)/q - (n-\gamma)/n$$

$$= m/n + (2n+\gamma)/4n - m \cdot \sum_{j=1}^{3} a_j(m)/n .$$

Hence we have $\sum_{j=1}^{3} a_j(m) = 1$. Therefore (2.10) shows (2.6). The same argument as in the proof of (2.6) yields (2.5). We next prove (2.7) and (2.8).

In the same way as in the proof of (2.6) we have

$$
\begin{aligned}
\sum_{|\beta|=m} \| J^\beta f(u_1 \cdot \bar{u}_2) u_3 \|_2
&\leqq C|t|^m \sum_{\substack{|\beta|=|\beta_1|+|\beta_2|+|\beta_3| \\ 0\leqq|\beta_1|,|\beta_2|\leqq m \\ 0\leqq|\beta_3|\leqq m-1}} \| f(D^{\beta_1}v_1 \cdot \overline{D^{\beta_2}v_2}) D^{\beta_3} v_3 \|_2 \\
&\quad + C|t|^m \sum_{|\beta|=m} \| f(v_1 \cdot \bar{v}_2) D^\beta v_3 \|_2 \\
&\leqq C|t|^m \sum_{\substack{|\beta|=|\beta_1|+|\beta_2|+|\beta_3| \\ 0\leqq|\beta_1|,|\beta_2|\leqq m \\ 0\leqq|\beta_3|\leqq m-1}} \prod_{j=1}^{3} \| D^{\beta_j} v_j \|_{p_j} \\
&\quad + C|t|^m \| f(v_1 \cdot \bar{v}_2) \|_\infty \cdot \sum_{|\beta|=m} \| D^\beta v_3 \|_2 \\
&\leqq C|t|^m \prod_{j=1}^{3} \| vj \|_r^{1-b_j(m)} \cdot \sum_{|\beta|=m} \| D^\beta v_j \|_2^{b_j(m)} \\
&\quad + C|t|^m \left(\prod_{j=1}^{2} \| v_j \|_{r+\varepsilon} + \prod_{j=1}^{2} \| v_j \|_{r-\varepsilon} \right) \cdot \sum_{|\beta|=m} \| D^\beta v_3 \|_2, \qquad (2.13)
\end{aligned}
$$

where p_j and $b_j(m)$satisfy

$$
1/2 = \sum_{j=1}^{3} (1/p_j) - (n-\gamma)/n, \qquad (2.14)
$$

$$
1/p_j = |\beta_j|/n - b_j \cdot ((1/2) - (m/n)) + (1 - b_j)/r. \qquad (2.15)
$$

We see that $\sum_{j=1}^{3} b_j(m) = 1$. Indeed (2.14) and (2.15) show

$$
\begin{aligned}
1/2 &= m/n + ((1/2)-(m/n)) \cdot \sum_{j=1}^{3} b_j(m) + 3/r - \sum_{j=1}^{3} b_j(m)/r - (n-\gamma)/n \\
&= m/n + ((1/2)-(m/n)-(1/r)) \cdot \sum_{j=1}^{3} b_j(m) + 3/r - (n-\gamma)/n \\
&= m/n + 1/r + ((1/2)-(m/n)-(1/r)) \cdot \sum_{j=1}^{3} b_j(m).
\end{aligned}
$$

Hence we have $\sum_{j=1}^{3} b_j(m) = 1$. (2.8) follows from (2.13). The same argument yields (2.7). Q. E. D.

3. EXISTENCE AND NON-EXISTENCE OF SCATTERING STATES

In this section we will consider for what value of γ the solution of (1.1)-(1.2) has the scattering states $u_\pm \in L^2(\mathbb{R}^n)$ satisfying (1.6) and for what value of γ the solution of (1.1)-(1.2) does not.

Our main theorem in this section is as follows.

THEOREM 3.1. — Let $\lambda > 0$.

1) Suppose that $1 < \gamma < \text{Min}(4, n)$. Then, for any $\phi \in \Sigma^{1,1}$ there exist unique scattering states $u_\pm \in L^2$ satisfying

$$\| u_\pm - U(-t)u(t) \|_2 \to 0 \qquad (t \to \pm \infty), \tag{3.1}$$

where $u(t)$ is a solution of (1.1)-(1.2) with $U(-t)u(t)$ in $C(\mathbb{R}; \Sigma^{1,1})$.

2) Suppose that $0 < \gamma \leqq 1$. Then, for any non-zero $\phi \in \Sigma^{1,1}$ there do not exist any scattering states $u_\pm \in L^2$ satisfying (3.1).

REMARK 3.1. — When $\gamma = 1$, $n = 3$ and $\lambda > 0$, Glassey [*10*] showed the non-existence of scattering states $u_\pm$ satisfying $\| U(t)u_\pm \|_\infty = O(|t|^{-n/2})$ $(t \to \pm \infty)$. Theorem 3.1 (2) shows the non-existence of any scattering states in L^2.

Proof of Theorem 3.1. — We prove Theorem 3.1, following [*22*] and [*23*]. By $\hat{f}$ and $\check{f}$ we denote the Fourier transform and the inverse Fourier transform of f, respectively.

Our proof is based on the following observation: Since the asymptotic profile of the free evolution $U(t)f$ is given by $(1/it)^{n/2} \exp(i|x|^2/4t)\hat{f}(x/t)$ and (1.1) is transformed by (1.8) into the new equation

$$iv_t + \Delta v = \lambda |t|^{\gamma-2} f(|v|^2)v, \quad t \in \mathbb{R}\backslash\{0\}, \quad x \in \mathbb{R}^n, \tag{3.2}$$

the relation (3.1) is equivalent to the existence of the strong limits

$$\lim_{t \to \pm\infty} v(t) = v_\pm(0) \qquad \text{in } L^2, \tag{3.3}$$

(see, e. g., [*22*, Lemma 2.8] and [*23*]). We define the operator R by

$$Rg = \int_{\mathbb{R}^n} |x - y|^{-\gamma} g(y) dy. \tag{3.4}$$

We note that $f(|u|^2) = \mathrm{R}\,|u|^2$. Now we consider only the case $t \to +\infty$, since the case $t \to -\infty$ can be treated in the same way.

1) We first prove (1). The calculations below are rather formal, but they can be easily justified by the regularizing technique of Ginibre-Velo [6-9]. We multiply (3.2) by $\bar{v}(t)$ and take the imaginary part to obtain

$$\| v(s) \|_2 = \| v(t) \|_2, \qquad 0 < s \leqq t < +\infty. \tag{3.5}$$

If $0 < \gamma \leqq 2$, we multiply (3.2) by $t^{2-\gamma}\bar{v}_t$ and take the real part. This leads us to

$$s^{2-\gamma}\|\nabla v(s)\|_2^2 + \mathrm{P}(v(s)) \leqq t^{2-\gamma}\|\nabla v(t)\|_2^2 + \mathrm{P}(v(t)), \quad 0<s\leqq t<+\infty, \tag{3.6}$$

if $0 < \gamma \leqq 2$. If $2 < \gamma < \mathrm{Min}(4, n)$, we multiply (3.2) by $\bar{v}_t$ and take the real part. This leads us to

$$\|\nabla v(s)\|_2^2 + s^{\gamma-2}\mathrm{P}(v(s)) = \|\nabla v(t)\|_2^2 + t^{\gamma-2}\mathrm{P}(v(t)), \quad 0<s\leqq t<+\infty, \tag{3.7}$$

if $2 < \gamma < \mathrm{Min}(4, n)$. By (3.5-3.7), Lemma 2.4 and the Sobolev imbedding theorem we conclude that if $0 < \gamma \leqq 2$,

$$t^{1-(\gamma/2)}\| \nabla v(t) \|_2, \quad \| \mathrm{R}^{1/2} | v(t) |^2 \|_2, \quad \| v(t) \|_2 \leqq \mathrm{C} \tag{3.8}$$

for $t \in (0, 1]$ and that if $2 < \gamma < \mathrm{Min}(4, n)$,

$$\| \nabla v(t) \|_2, \quad \| \mathrm{R}^{1/2} | v(t) |^2 \|_2, \quad \| v(t) \|_2 \leqq \mathrm{C} \tag{3.9}$$

for $t \in (0, 1]$, where $\mathrm{C} = \mathrm{C}(n, \gamma, \| v(1) \|_{\Sigma^{1,1}})$. Let $\psi \in \mathrm{H}^{1,2}$. By (3.2) we have

$$\begin{aligned}(v(t)-v(s), \psi) &= \int_s^t (v_\tau(\tau), \psi)d\tau \\ &= -i\int_s^t (\nabla v(\tau), \nabla\psi)d\tau - i\int_s^t \tau^{\gamma-2}(f(|v|^2)v(\tau), \psi)d\tau,\end{aligned} \tag{3.10}$$

for $0 < s, t < +\infty$. Since $\gamma - 2 > -1$ for $1 < \gamma$ and $\mathrm{H}^{1,2}$ is dense in L^2, (3.8-3.10) show that the weak limit

$$w - \lim_{t\to+0} v(t) = v_+(0) \tag{3.11}$$

exists in L^2. Now we choose $\psi = v(t)$ in (3.10). Then,

$$\begin{aligned}|(v(t)-v(s), v(t))| \leqq& \int_s^t \|\nabla v(\tau)\|_2 d\tau \cdot \|\nabla v(t)\|_2 \\ &+ \mathrm{C}\int_s^t \tau^{\gamma-2}\|\mathrm{R}^{1/2}|v(\tau)|^2\|_2^2 d\tau \\ &+ \mathrm{C}\int_s^t \tau^{\gamma-2}\|\mathrm{R}^{1/2}|v(\tau)|^2\|_2 d\tau \cdot \|\mathrm{R}^{1/2}|v(t)|^2\|_2\end{aligned} \tag{3.12}$$

for $0 < s \leqq t < +\infty$. Here we have used the following inequality:

$$|(f(|v|^2)v(\tau), v(t))| \leqq \int_{\mathbb{R}^n} (\mathrm{R}\,|v(\tau)|^2)\,|v(\tau)|\,|v(t)|\,dx$$
$$\leqq \frac{1}{2}\int_{\mathbb{R}^n} (\mathrm{R}\,|v(\tau)|^2)(|v(\tau)|^2 + |v(t)|^2)dx$$
$$= \frac{1}{2}\|\mathrm{R}^{1/2}|v(\tau)|^2\|_2^2 + \frac{1}{2}\|\mathrm{R}^{1/2}|v(\tau)|^2\|_2 \cdot \|\mathrm{R}^{1/2}|v(t)|^2\|_2 . \quad (3.13)$$

If $1 < \gamma \leqq 2$, by (3.8) we have

$$|(v(t) - v(s), v(t))| \leqq \mathrm{C}(t^{\gamma-1} - s^{\gamma/2}\cdot t^{(\gamma/2)-1}) + \mathrm{C}(t^{\gamma-1} - s^{\gamma-1}), \qquad 0 < s \leqq t < 1 . \quad (3.14)$$

Let $s \to +0$ in (3.14) and use (3.11) to obtain

$$|(v(t) - v_+(0), v(t))| \leqq \mathrm{C}t^{\gamma-1}, \qquad t \in (0, 1], \quad (3.15)$$

where $\mathrm{C} = \mathrm{C}(n, \gamma, \|v(1)\|_{\Sigma^{1,1}})$. For $2 < \gamma < \mathrm{Min}(4, n)$, we obtain (3.15) with $t^{\gamma-1}$ replaced by t in the same way. Therefore,

$$\|v(t) - v_+(0)\|_2^2 = (v(t) - v_+(0), v(t)) - (v(t) - v_+(0), v_+(0)) \leqq \mathrm{C}t^{\min(\gamma-1,1)} + |(v(t) - v_+(0), v_+(0))| \to 0$$

as $t \to +0$. This completes the proof of (1).

2) We next prove (2). We assume that for some non-trivial solution $v(t)$ in $\mathrm{C}((0, +\infty); \Sigma^{1,1})$ of (3.2) there exists a $v_+(0) \in \mathrm{L}^2$ satisfying (3.3) and we deduce the contradiction. $v(t)$ can be represented as follows:

$$v(t) = \mathrm{U}(t - r)v(r) - i\lambda \int_r^t \tau^{2-\gamma}\mathrm{U}(t - \tau)f(|v(\tau)|^2)v(\tau)d\tau \quad (3.16)$$

for $0 < t, r < +\infty$. (3.3) and (3.8) give us

$$\mathrm{R}^{1/2}|v_+(0)|^2 \in \mathrm{L}^2, \quad (3.17)$$

$$\mathrm{R}^{1/2}|v(t)|^2 \to \mathrm{R}^{1/2}|v_+(0)|^2 \quad \text{weakly in } \mathrm{L}^2 \quad (t \to +0), \quad (3.18)$$

since for any $\psi \in \mathscr{S}(\mathbb{R}^n)$

$$|(\mathrm{R}^{1/2}|v(t)|^2 - \mathrm{R}^{1/2}|v_+(0)|^2, \psi)| = |(|v(t)|^2 - |v_+(0)|^2, \mathrm{R}^{1/2}\psi)|$$
$$\leqq (\|v(t)\|_2 + \|v_+(0)\|_2)\|v(t) - v_+(0)\|_2 \|\mathrm{R}^{1/2}\psi\|_\infty$$
$$\leqq (\|v(t)\|_2 + \|v_+(0)\|_2)\|v(t) - v_+(0)\|_2 \times (\|\psi\|_{(2n+\varepsilon)/(n-\gamma+\varepsilon)} + \|\psi\|_{(2n-\varepsilon)/(n-\gamma+\varepsilon)}), \quad (3.19)$$

where ε is a sufficiently small positive number.

In addition, (3.3) and (3.5) give us

$$\|v_+(0)\|_2 \neq 0 . \quad (3.20)$$

Therefore, we can choose a $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$- \operatorname{Re} i(f(|v_+(0)|^2)v_+(0), \psi) > 0 .$$

We consider the inner product between (3.16) and $U(t)\psi$ and take the real part. This leads us to

$$\operatorname{Re}(U(-t)v(t) - U(-r)v(r), \psi) = \lambda \int_r^t \tau^{\gamma-2} \{ - \operatorname{Re} i(U(-\tau)f(|v(\tau)|^2)v(\tau), \psi) \} \, d\tau . \quad (3.21)$$

Now we show that

$$i(U(-\tau)f(|v(\tau)|^2)v(\tau), \psi) \to i(f(|v_+(0)|^2)v_+(0), \psi) \quad (\tau \to +0) \quad (3.22)$$

For that purpose, we consider

$$\begin{aligned} & i(U(-\tau)f(|v(\tau)|^2)v(\tau), \psi) - i(f(|v_+(0)|^2)v_+(0), \psi) \\ & = i(f(|v(\tau)|^2)v(\tau), U(\tau)\psi - \psi) + i(\{ f(|v(\tau)|^2) - f(|v_+(0)|^2) \} v(\tau), \psi) \\ & + i(f(|v_+(0)|^2)(v(\tau) - v_+(0)), \psi) . \end{aligned} \quad (3.23)$$

By K_1, K_2 and K_3 we denote the first term, the second term and the third term at the R. H. S. of (3.23), respectively. By (3.8) and Lemma 2.4 we have

$$\begin{aligned} |K_1| & \leqq |(R^{1/2}|v(\tau)|^2, R^{1/2}(\overline{v}(\tau)(U(\tau)\psi - \psi)))| \\ & \leqq \| R^{1/2}|v(\tau)|^2 \|_2 \| R^{1/2}(\overline{v}(\tau)(U(\tau)\psi - \psi)) \|_2 \\ & \leqq C \| \overline{v}(\tau)(U(\tau)\psi - \psi) \|_{2n/(3n-\gamma)} \\ & \leqq C \| v(\tau) \|_2 \| U(\tau)\psi - \psi \|_{2n/(2n-\gamma)} \to 0 \quad (\tau \to +0) . \end{aligned} \quad (3.24)$$

Since we have by (3.3) and Lemma 2.4

$$\begin{aligned} \| R^{1/2}(\overline{v}(\tau)\psi) - R^{1/2}(\overline{v}_+(0)\psi) \|_2 & \leqq C \| (\overline{v}(\tau) - \overline{v}_+(0))\psi \|_{2n/(2n-\gamma)} \\ & \leqq C \| v(\tau) - v_+(0) \|_2 \| \psi \|_{2n/(n-\gamma)} \to 0 \quad (\tau \to +0), \end{aligned}$$

we obtain by (3.18)

$$K_2 = (R^{1/2}|v(\tau)|^2 - R^{1/2}|v_+(0)|^2, R^{1/2}(\overline{v}(\tau)\psi)) \to 0 \quad (\tau \to +0) . \quad (3.25)$$

In the same way as (3.24) we have

$$K_3 \to 0 \qquad (\tau \to +0) . \quad (3.26)$$

(3.24)-(3.26) show (3.22).

It follows from (3.22) that there exists $0 < \delta < 1$ such that

$$\begin{aligned} - \operatorname{Re} i(U(-\tau)f(|v(\tau)|^2)v(\tau), \psi) & > - \operatorname{Re} \frac{i}{2}(f(|v_+(0)|^2)v_+(0), \psi) \\ & > 0, \qquad 0 < \tau < \delta . \end{aligned} \quad (3.27)$$

Therefore, by (3.21) and (3.27) we have

$$\begin{aligned} &\mathrm{Re}\,(\mathrm{U}(-t)v(t) - \mathrm{U}(-r)v(r), \psi) \\ &= -\lambda \int_r^t \mathrm{Re}\, i(\mathrm{U}(-\tau)f(|v(\tau)|^2)v(\tau), \psi)\tau^{\gamma-2}d\tau \\ &\geqq -\frac{\lambda}{2}\mathrm{Re}\, i(f(|v_+(0)|^2)v_+(0), \psi)\int_r^t \tau^{\gamma-2}d\tau, \qquad 0 < r \leqq t < \delta. \qquad (3.28) \end{aligned}$$

Since $\gamma - 2 \leqq -1$ for $0 < \gamma \leqq 1$, the R. H. S. of (3.28) tends to $+\infty$ as $r \to +0$. This contradicts the boundedness in L^2 of $v(t)$. Q. E. D.

4. SCATTERING THEORY FOR SMALL DATA IN $\Sigma^{l,m}$

In this section and the next section we let $q=4n/(2n-\gamma)$, $q'=4n/(2n+\gamma)$, $r = 2n/(n-\gamma)$ and $r' = 2n/(n+\gamma)$, unless specified otherwise. In this section we will give the global existence theorem for the Cauchy problem (1.1)-(1.2) for small data which yields the scattering theory for small data. For convenience we introduce the following Banach spaces $B_1^{l,m}$ and $B_2^{l,m}$ by

$$B_1^{l,m} = \{ \mathrm{U}(-t)u(t) \in C(\mathbb{R}; \Sigma^{l,m}); |||u|||_{l,m,1} = \sup_{t\in\mathbb{R}} \|\mathrm{U}(-t)u(t)\|_{\Sigma^{l,m}} < \infty \},$$

$$\begin{aligned} B_2^{l,m} = \{ &\mathrm{U}(-t)u(t) \in C(\mathbb{R}; \Sigma^{l,m}), D^\alpha u(t) \in L^{8/\gamma}(\mathbb{R}; L^q), \\ &\mathrm{U}(t)x^\beta \mathrm{U}(-t)u(t) \in L^{8/\gamma}(\mathbb{R}; L^q), \ |\alpha| \leqq l, \ |\beta| \leqq m; \\ &|||u|||_{l,m,2} = |||u|||_{l,m,1} + \sum_{|\alpha|\leqq l}\left(\int \|D^\alpha u(t)\|_q^{8/\gamma}dt\right)^{\gamma/8} \\ &\qquad + \sum_{|\beta|\leqq m}\left(\int \|\mathrm{U}(t)x^\beta \mathrm{U}(-t)u(t)\|_q^{8/\gamma}dt\right)^{\gamma/8} < \infty \Big\}, \end{aligned}$$

where $l, m \in \mathbb{N}$, and the closed balls $B_{1,\rho}^{l,m}$ and $B_{2,\rho}^{l,m}$ by

$$B_{1,\rho}^{l,m} = \{ u \in B_1^{l,m}; |||u|||_{l,m,1} \leqq \rho \},$$
$$B_{2,\rho}^{l,m} = \{ u \in B_2^{l,m}; |||u|||_{l,m,2} \leqq \rho \}.$$

THEOREM 4.1. — Let $\lambda \in \mathbb{R}$. There exists an $\varepsilon > 0$ depending only on n, γ and λ such that if $\phi \in \Sigma_\varepsilon^{l,m} = \{ \psi \in \Sigma^{l,m}; \|\psi\|_{\Sigma^{1,1}} \leqq \varepsilon \}$, then the following results hold:

1) Suppose that $1 < \gamma \leqq (4/3)$. Then there exists a unique solution u of (1.1)-(1.2) such that

$$u \in B_1^{l,m}.$$

2) Suppose that $(4/3) < \gamma < \mathrm{Min}(4, n)$. Then there exists a unique solution u of (1.1)-(1.2) such that

$$u \in \mathrm{B}_2^{l,m}.$$

Proof. — We may assume $t > 0$. We consider the following linear Schrödinger equations

$$iv_t + \Delta v = \lambda f(|w|^2)w, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{4.1}$$

$$v(0, x) = \phi(x), \qquad x \in \mathbb{R}^n. \tag{4.2}$$

We define the operator S formally $v = \mathrm{S}w$.

1) We prove (1). We first construct the solution of (1.1)-(1.2) in $\mathrm{B}_1^{1,1}$ by the contraction mapping principle. We have by Lemma 2.2, (2.7), (2.8) and the integral equation corresponding to (4.1)-(4.2)

$$\sum_{|\alpha| \leqq 1} \| \mathrm{D}^\alpha v(t) \|_2 \leqq \sum_{|\alpha| \leqq 1} \| \mathrm{D}^\alpha \phi \|_2 + \mathrm{C} \int_0^t g(s) \cdot \sum_{|\alpha| \leqq 1} \| \mathrm{D}^\alpha w(s) \|_2 ds, \tag{4.3}$$

$$\sum_{|\beta| \leqq 1} \| \mathrm{J}^\beta v(t) \|_2 \leqq \sum_{|\beta| \leqq 1} \| x^\beta \phi \|_2 + \mathrm{C} \int_0^t g(s) \cdot \sum_{|\beta| \leqq 1} \| \mathrm{J}^\beta w(s) \|_2 ds, \tag{4.4}$$

where $g(s) = \| w(s) \|_r^2 + \| w(s) \|_{r+\varepsilon_1}^2 + \| w(s) \|_{r-\varepsilon_1}^2$, and $\varepsilon_1 > 0$ is sufficiently small. Here we have used the fact that D^α and J^β commute with $i\partial/\partial t + \Delta$ (see [*12*] [*14*] [*15*] and [*24*]). Using Lemma 2.1, we have for $\rho > 0$ and $w \in \mathrm{B}_{1,\rho}^{1,1}$

$$\begin{aligned} \| w(t) \|_r &\leqq \mathrm{C} \| w(t) \|_2^{1-(\gamma/2)} \cdot \sum_{|\beta|=1} \| \mathrm{J}^\beta w(t) \|_2^{\gamma/2} t^{-\gamma/2} \\ &\leqq \mathrm{C}\rho t^{-\gamma/2}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \| w(t) \|_r &\leqq \mathrm{C} \| w(t) \|_2^{1-(\gamma/2)} \cdot \sum_{|\alpha|=1} \| \mathrm{D}^\alpha w(t) \|_2^{\gamma/2} \\ &\leqq \mathrm{C}\rho, \end{aligned} \tag{4.6}$$

where ρ is a small positive constant to be determined later. Therefore we obtain from (4.5) and (4.6)

$$\| w(t) \|_r \leqq \mathrm{C}\rho(1 + t)^{-\gamma/2}. \tag{4.7}$$

In the same way as in the proof of (4.7) we have

$$\| w(t) \|_{r+\varepsilon_1} \leqq \mathrm{C}\rho(1 + t)^{-\left(\gamma + \frac{n}{2}\varepsilon_2\right)/(2+\varepsilon_2)}, \tag{4.8}$$

and

$$\| w(t) \|_{r-\varepsilon_1} \leqq \mathrm{C}\rho(1 + t)^{-\left(\gamma - \frac{n}{2}\varepsilon_2\right)/(2-\varepsilon_2)}, \tag{4.9}$$

where $\varepsilon_2 = \varepsilon_1\left(1 - \frac{\gamma}{n}\right)$. (4.3), (4.4) and (4.7)-(4.9) show

$$\sum_{|\alpha|\leq 1} \| D^\alpha v(t) \|_2 \leqq \sum_{|\alpha|\leq 1} \| D^\alpha \phi \|_2 + C\rho^3 \int_0^t (1+s)^{-\left(\gamma - \frac{n}{2}\varepsilon_2\right)/\left(1-\frac{1}{2}\varepsilon_2\right)} ds$$

$$\leqq \sum_{|\alpha|\leq 1} \| D^\alpha \phi \|_2 + C\rho^3 \leqq \varepsilon + C\rho^3 , \qquad (4.10)$$

and

$$\sum_{|\beta|\leq 1} \| J^\beta v(t) \|_2 \leqq \sum_{|\beta|\leq 1} \| x^\beta \phi \|_2 + C\rho^3 \leqq \varepsilon + C\rho^3 . \qquad (4.11)$$

Now we let $w_1, w_2 \in B^{1,1}_{1,\rho}$ and $v_1 = Sw_1$, $v_2 = Sw_2$.

We put $V = v_1 - v_2$ and $W = w_1 - w_2$. Then V satisfies with zero initial condition:

$$\begin{aligned} iV_t + \Delta V &= \lambda(f(|w_1|^2)w_1 - f(|w_2|^2)w_2) \\ &= \lambda f(|w_1|^2)W + \lambda(f(w_1 \cdot \overline{W}) + f(W \cdot \overline{w}_2))w_2 . \end{aligned} \qquad (4.12)$$

In the same way as in the proof of (4.10)-(4.11) we have by (4.12)

$$\sum_{|\beta|\leq 1} \| D^\alpha V(t) \|_2 \leqq C\rho^2 ||| W |||_{1,1,1} , \qquad (4.13)$$

and

$$\sum_{|\beta|\leq 1} \| J^\beta V(t) \|_2 \leqq C\rho^2 ||| W |||_{1,1,1} . \qquad (4.14)$$

From (4.10), (4.11), (4.13) and (4.14) it follows that

$$||| Sw |||_{1,1,1} \leqq C\varepsilon + C\rho^3 , \qquad (4.15)$$

and

$$||| Sw_1 - Sw_2 |||_{1,1,1} \leqq C\rho^2 ||| w_1 - w_2 |||_{1,1,1} . \qquad (4.16)$$

Now we choose ε and ρ so that

$$C\varepsilon \leqq \rho/2 , \qquad C\rho^3 \leqq \rho/2 \qquad \text{and} \qquad C\rho^2 \leqq 1/2 .$$

Then (4.15) and (4.16) imply that S is a contraction mapping from $B^{1,1}_{1,\rho}$ to itself. This implies that there exists a unique solution $u(t)$ of (1.1)-(1.2) such that $u \in B^{1,1}_{1,\rho}$. We next prove $u(t) \in B^{l,m}_1$. The calculations below are rather formal, but they can be easily justified by the regularizing technique.

In the same way as (4.10) and (4.11) we easily obtain by (1.1)-(1.2)

$$\sum_{|\alpha|\leq l} \| D^\alpha u(t) \|_2 \leqq \sum_{|\alpha|\leq l} \| D^\alpha \phi \|_2 + C\rho^2 \int_0^t (1+s)^{-\left(\gamma - \frac{n}{2}\varepsilon_2\right)/\left(1-\frac{1}{2}\varepsilon_2\right)} \cdot \sum_{|\alpha|\leq l} \| D^\alpha u(s) \|_2 ds, \quad (4.17)$$

and

$$\sum_{|\beta|\leq m} \| J^\beta u(t) \|_2 \leqq \sum_{|\beta|\leq m} \| x^\beta \phi \|_2 + C\rho^2 \int_0^t (1+s)^{-\left(\gamma - \frac{n}{2}\varepsilon_2\right)/\left(1-\frac{1}{2}\varepsilon_2\right)} \sum_{|\beta|\leq m} \| J^\beta u(s) \|_2 ds. \quad (4.18)$$

By (4.17), (4.18) and Gronwall's inequality we have

$$\sum_{|\alpha|\leq l} \| D^\alpha u(t) \|_2, \sum_{|\beta|\leq m} \| J^\beta u(t) \|_2 \leqq \| \phi \|_{\Sigma^{l,m}} \exp C \int_0^\infty (1+s)^{-\left(\gamma - \frac{n}{2}\varepsilon_2\right)/\left(1-\frac{1}{2}\varepsilon_2\right)} ds.$$

This completes the proof of (1).

2) We next prove (2). In the same way as in Part (1) we first construct the solution of (1.1)-(1.2) in $B_2^{1,1}$ by the contraction mapping principle. (2.5), (2.6), the integral equation corresponding to (4.1)-(4.2) and Lemma 2.2 yield

$$\sum_{|\alpha|\leq 1} \| D^\alpha v(t) \|_q \leqq \sum_{|\alpha|\leq 1} \| U(t) D^\alpha \phi \|_q + C \int_0^t (t-s)^{-\gamma/4} \| w(s) \|_q^2 \cdot \sum_{|\alpha|\leq 1} \| D^\alpha w(s) \|_q ds, \quad (4.19)$$

$$\sum_{|\beta|\leq 1} \| J^\beta v(t) \|_q \leqq \sum_{|\beta|\leq 1} \| U(t) x^\beta \phi \|_q + C \int_0^t (t-s)^{-\gamma/4} \| w(s) \|_q^2 \cdot \sum_{|\beta|\leq 1} \| J^\beta w(s) \|_q ds. \quad (4.20)$$

For $\rho > 0$ and $w \in B_{2,\rho}^{1,1}$ we have by the same argument as (4.7)

$$\| w(s) \|_q \leqq C\rho(1+t)^{-\gamma/4}, \quad (4.21)$$

where ρ is a small positive constant to be determined later. We apply (4.21), Lemmas 2.3-2.4 and Hölder's inequality to (4.19) to obtain

$$\begin{aligned}
&\sum_{|\alpha|\leqq 1}\left(\int \|D^\alpha v(t)\|_q^{8/\gamma}dt\right)^{\gamma/8} \\
&\leqq C\sum_{|\alpha|\leqq 1}\|D^\alpha\phi\|_2 + C\rho^2\left(\int (1+|s|)^{-4\gamma/(8-\gamma)}\cdot\sum_{|\alpha|\leqq 1}\|D^\alpha w(s)\|_q^{8/(8-\gamma)}ds\right)^{(8-\gamma)/8} \\
&\leqq C\sum_{|\alpha|\leqq 1}\|D^\alpha\phi\|_2 + C\rho^2\left(\int (1+|s|)^{-4\gamma/(8-2\gamma)}ds\right)^{(8-2\gamma)/8} \\
&\times\sum_{|\alpha|\leqq 1}\left(\int \|D^\alpha w(s)\|_q^{8/\gamma}ds\right)^{\gamma/8} \\
&\leqq C\sum_{|\alpha|\leqq 1}\|D^\alpha\phi\|_2 + C\rho^3 \\
&\leqq C\varepsilon + C\rho^3 \qquad \text{for} \qquad \gamma > 4/3. \qquad (4.22)
\end{aligned}$$

Similary we have by (4.20) and (4.21)

$$\begin{aligned}
\sum_{|\beta|\leqq 1}\left(\int \|J^\beta v(t)\|_q^{8/\gamma}dt\right)^{\gamma/8} &\leqq C\sum_{|\beta|\leqq 1}\|x^\beta\phi\|_2 + C\rho^3 \\
&\leqq C\varepsilon + C\rho^3 \qquad \text{for} \qquad \gamma > 4/3. \qquad (4.23)
\end{aligned}$$

By the integral equation corresponding to (4.1)-(4.2) we have

$$\begin{aligned}
\sum_{|\alpha|\leqq 1}\|D^\alpha v(t)\|_2^2 &\leqq C\sum_{|\alpha|\leqq 1}\|D^\alpha\phi\|_2^2 + C\sum_{|\alpha|\leqq 1}\left\|\int_0^t U(t-s)D^\alpha f(|w|^2)w(s)ds\right\|_2^2 \\
&\leqq C\sum_{|\alpha|\leqq 1}\|D^\alpha\phi\|_2^2 \\
&+C\sum_{|\alpha|\leqq 1}\left(\int_0^t U(t-s)D^\alpha f(|w|^2)w(s)ds, \int_0^t U(t-\tau)D^\alpha f(|w|^2)w(\tau)d\tau\right). \qquad (4.24)
\end{aligned}$$

Using Lemma 2.2 and (2.5) we can estimate the second term of the R. H. S. of (4.24) as follows

$$\begin{aligned}
&C\sum_{|\alpha|\leqq 1}\int_0^t \|D^\alpha f(|w|^2)w(s)\|_{q'}\cdot\int_0^t |s-\tau|^{-\gamma/4}\|D^\alpha f(|w|^2)w(\tau)\|_{q'}dsd\tau \\
&\leqq C\sum_{|\alpha|\leqq 1}\int_0^t \|w(s)\|_q^2\|D^\alpha w(s)\|_q \times \int_0^t |s-\tau|^{-\gamma/4}\|w(\tau)\|_q^2\|D^\alpha w(\tau)\|_q dsd\tau. \qquad (4.25)
\end{aligned}$$

Applying (4.21), Hölder's inequality and Lemma 2.4, we see that (4.25) is dominated by

$$C\rho^4 \cdot \sum_{|\alpha|\leq 1} \int_0^t (1+s)^{-\gamma/2} \| D^\alpha w(s) \|_q \times \int_0^t |s-\tau|^{-\gamma/4}(1+\tau)^{-\gamma/2} \| D^\alpha w(\tau) \|_q ds d\tau$$

$$\leqq C\rho^4 \sum_{|\alpha|\leq 1} \left(\int_0^t (1+s)^{-4\gamma/(8-\gamma)} \| D^\alpha w(s) \|_q^{8/(8-\gamma)} ds \right)^{(8-\gamma)/8}$$

$$\times \left(\int_0^t \left| \int_0^t |s-\tau|^{-\gamma/4}(1+\tau)^{-\gamma/2} \| D^\alpha w(\tau) \|_q d\tau \right|^{8/\gamma} ds \right)^{\gamma/8}$$

$$\leqq C\rho^4 \left(\int (1+|s|)^{-4\gamma/(8-2\gamma)} ds \right)^{(8-2\gamma)/4} \times \sum_{|\alpha|\leq 1} \left(\int \| D^\alpha w(s) \|_q^{8/\gamma} ds \right)^{\gamma/4}$$

$$\leqq C\rho^6 \quad \text{for} \quad \gamma > 4/3. \tag{4.26}$$

Hence we have by (4.25) and (4.26)

$$\sum_{|\alpha|\leq 1} \| D^\alpha v(t) \|_2 \leqq C \sum_{|\alpha|\leq 1} \| D^\alpha \phi \|_2 + C\rho^3$$
$$\leqq C\varepsilon + C\rho^3. \tag{4.27}$$

We have by (4.23) and the same argument as in the proof of (4.27)

$$\sum_{|\beta|\leq 1} \| J^\beta v(t) \|_2 \leqq C\varepsilon + C\rho^3. \tag{4.28}$$

Let W and V be defined as in (4.12). In the same way as in the proof of (4.22)-(4.23) and (4.27)-(4.28) we have

$$\sum_{|\alpha|\leq 1} \left(\int \| D^\alpha V(t) \|_q^{8/\gamma} dt \right)^{\gamma/8} \leqq C\rho^2 ||| W |||_{1,1,2}, \tag{4.29}$$

$$\sum_{|\beta|\leq 1} \left(\int \| J^\beta V(t) \|_q^{8/\gamma} dt \right)^{\gamma/8} \leqq C\rho^2 ||| W |||_{1,1,2}, \tag{4.30}$$

$$\sum_{|\alpha|\leq 1} \| D^\alpha V(t) \|_2 \leqq C\rho^2 ||| W |||_{1,1,2}, \tag{4.31}$$

and

$$\sum_{|\beta|\leq 1} \| J^\beta V(t) \|_2 \leqq C\rho^2 ||| W |||_{1,1,2}. \tag{4.32}$$

We have by (4.22), (4.23) and (4.27)-(4.32)

$$||| Sw |||_{1,1,2} \leqq C\varepsilon + C\rho^3 , \tag{4.33}$$

and

$$||| Sw_1 - Sw_2 |||_{1,1,2} \leqq C\rho^2 ||| w_1 - w_2 |||_{1,1,2} . \tag{4.34}$$

If we choose ε and ρ so that

$$C\varepsilon \leqq \rho/2 , \qquad C\rho^3 \leqq \rho/2 \qquad \text{and} \qquad C\rho^2 \leqq 1/2 ,$$

(4.33) and (4.34) show that S is a contraction mapping from $B^{1,1}_{2,\rho}$ to itself. This implies that there exists a unique solution $u(t)$ of (1.1)-(1.2) such that $u \in B^{1,1}_{2,\rho}$. In the same way as in the proof of (4.22)-(4.23) and (4.27)-(4.28) we easily have

$$\sum_{|\alpha| \leqq l} \left(\int || D^\alpha u(t) ||_q^{8/\gamma} dt \right)^{\gamma/8} \leqq C \sum_{|\alpha| \leqq l} || D^\alpha \phi ||_2 < \infty , \tag{4.35}$$

$$\sum_{|\beta| \leqq m} \left(\int || J^\beta u(t) ||_q^{8/\gamma} dt \right)^{\gamma/8} \leqq C \sum_{|\beta| \leqq m} || x^\beta \phi ||_2 < \infty , \tag{4.36}$$

$$\sum_{|\alpha| \leqq l} || D^\alpha u(t) ||_2 \leqq C \sum_{|\alpha| \leqq l} || D^\alpha \phi ||_2 < \infty , \tag{4.37}$$

and

$$\sum_{|\beta| \leqq m} || J^\beta u(t) ||_2 \leqq C \sum_{|\beta| \leqq m} || x^\beta \phi ||_2 < \infty . \tag{4.38}$$

(4.35)-(4.38) imply $u(t) \in B^{l,m}_2$. Q. E. D.

REMARK 4.1. — From the proof of Theorem 4.1 and the uniqueness of solutions in $B^{1,1}_1$ of (1.1)-(1.2) we can easily see that if the initial datum ϕ is in $\Sigma^{l,m}$ ($l, m \in \mathbb{N}$), then the solution $u(t)$ in $B^{1,1}_1$ of (1.1)-(1.2) belongs to $B^{l,m}_2$ for $(4/3) < \gamma < \mathrm{Min}(4, n)$ and to $B^{l,m}_1$ for $1 < \gamma \leqq (4/3)$.

In the same way as in the proof of Theorem 4.1 we have the following results.

THEOREM 4.2. — Suppose that $1 < \gamma < \mathrm{Min}(4, n)$, and $l, m \in \mathbb{N}$. There exists an $\varepsilon > 0$ depending only on n, γ and λ such that the following results hold valid;

1-*a*) For any $u_+ \in \Sigma^{l,m}_\varepsilon$ there exists a unique $\phi \in \Sigma^{l,m}$ such that

$$|| u_+ - U(-t) ||_{\Sigma^{l,m}} \to 0 \qquad (t \to +\infty) , \tag{4.39}$$

where $u(t)$ is the solution of (1.1)-(1.2) with $U(-t)u(t)$ in $C(\mathbb{R}; \Sigma^{l,m})$.

1-*b*) For any $u_- \in \Sigma^{l,m}_\varepsilon$ the same result as above holds valid with $+\infty$ replaced by $-\infty$ in (4.39).

2) For any $\phi \in \Sigma_\varepsilon^{l,m}$ there exist unique $u_\pm \in \Sigma^{l,m}$ such that the solution $u(t)$ of (1.1)-(1.2) with $U(-t)u(t)$ in $C(\mathbb{R}; \Sigma^{l,m})$ satisfies

$$\| u_\pm - U(-t)u(t) \|_{\Sigma^{l,m}} \to 0 \qquad (t \to \pm \infty).$$

Proof. — We consider the following integral equation:

$$u(t) = U(t)u_+ - i\lambda \int_t^{+\infty} U(t-s) f(|u|^2) u(s) ds. \tag{4.40}$$

(4.40) is the integral version of the initial value problem of (1.1) with the initial data given at $+\infty$. In the same way as in the proof of Theorem 4.1 we can prove that there exists an $\varepsilon > 0$ depending only on n, γ and λ such that for any $u_+ \in \Sigma_\varepsilon^{l,m}$ (4.40) has a unique solution $u(t)$ satisfying (4.39) in $B_1^{l,m}$, if $1 < \gamma < (4/3)$ and in $B_2^{l,m}$, if $(4/3) < \gamma < \mathrm{Min}(4, n)$. Then, we put

$$\phi = u(0) = u_+ - i\lambda \int_0^{+\infty} U(-s) f(|u|^2) u(s) ds \in \Sigma^{l,m}. \tag{4.41}$$

This completes the proof of 1-*a*). 1-*b*) and 2) can be proved in the same way. Q. E. D.

Remark 4.2. — Theorem 4.2 and the proof of Theorem 4.1 imply that sufficiently small $\varepsilon > 0$ the wave operators $W_\pm : u_\pm \to \phi$ and $W_\pm^{-1} : \phi \to u_\pm$ are well defined as mappings from $\Sigma_\varepsilon^{l,m}$ into $\Sigma_{2\varepsilon}^{l,m}$ and are one-one and continuous from $\Sigma_\varepsilon^{l,m}$ into $\Sigma_{2\varepsilon}^{l,m}$. Accordingly, for sufficiently small $\varepsilon > 0$ the scattering operator $S = W_+^{-1} \cdot W_-$ is well defined as a mapping from $\Sigma_\varepsilon^{l,m}$ into $\Sigma_{4\varepsilon}^{l,m}$ and is one-one and continuous from $\Sigma_\varepsilon^{l,m}$ into $\Sigma_{4\varepsilon}^{l,m}$.

COROLLARY 4.1. — Suppose that the assumptions of Theorem 4.1 hold valid with $l, m \geqq [n/2] + 1$, where $[n/2]$ denotes the largest integer smaller than or equal to $n/2$. Then the unique solution $u(t)$ of (1.1)-(1.2) constructed in Theorem 4.1 satisfies

$$\| u(t) \|_\infty \leqq C(1 + |t|)^{-n/2}. \tag{4.42}$$

Proof. — By Lemma 2.1 we have for any $u \in B_1^{l,m}$

$$\begin{aligned} \| u(t) \|_\infty &\leqq C \| u(t) \|_2^{1-a} \sum_{|\beta| = [n/2]+1} \| J^\beta u(t) \|_2^a \, |t|^{-([n/2]+1)a} \\ &\leqq C |t|^{-([n/2]+1)a}, \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} \| u(t) \|_\infty &\leqq C \| u(t) \|_2^{1-a} \sum_{|\alpha| = [n/2]+1} \| D^\alpha u(t) \|_2^a \\ &\leqq C, \end{aligned} \tag{4.44}$$

where a satisfies $([n/2] + 1)a = n/2$. Hence (4.42) follows from (4.43) and (4.44) immediately. Q. E. D.

5. SCATTERING THEORY FOR ARBITRARY DATA IN $\Sigma^{l,m}$

In this section we will prove the existence of the wave operators defined in $\Sigma^{l,m}$ and asymptotic completeness for $(4/3) < \gamma < \mathrm{Min}(4, n)$ and $\lambda > 0$. Our proof is based on the conservation laws of L^2-norm and of the energy, and the pseudoconformal conservation law. We first give the global existence theorem for the Cauchy problem (1.1)-(1.2) for arbitrary data in $\Sigma^{l,m}$.

THEOREM 5.1. — Suppose that $\lambda > 0$, $l, m \in \mathbb{N}$ and $(4/3) < \gamma < \mathrm{Min}(4, n)$. Then, for any $\phi \in \Sigma^{l,m}$ there exists a unique solution u of (1.1)-(1.2) such that

$$u \in B_2^{l,m}.$$

Proof. — By [6]-[8] there exists a unique solution $u(t)$ of (1.1)-(1.2) satisfying

$$U(-t)u(t) \in C(\mathbb{R}; \Sigma^{1,1})$$

(see, e. g., [7, § 2 and § 3]). By Remark 4.1 it is sufficient to prove that $\| U(-t)u(t) \|_{\Sigma^{1,1}}$ is uniformly bounded for any t in $\mathbb{R}$. For any $t \in \mathbb{R}$, Ginibre-Velo [7] showed that $u(t)$ satisfies

$$\| u(t) \|_2 = \| \phi \|_2, \tag{5.1}$$

$$\sum_{|\alpha|=1} \| D^\alpha u(t) \|_2^2 + P(u(t)) = \sum_{|\alpha|=1} \| D^\alpha \phi \|_2^2 + P(\phi), \tag{5.2}$$

$$\sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 + 4t^2 P(u(t)) = \sum_{|\beta|=1} \| x^\beta \phi \|_2^2 + 4(2-\gamma) \int_0^t sP(u(s))ds. \tag{5.3}$$

We have by (2.3), Lemma 2.1 and (5.1)-(5.2)

$$\sum_{|\alpha| \leqq 1} \| D^\alpha u(t) \|_2 \leqq C\Big(\sum_{|\alpha| \leqq 1} \| D^\alpha \phi \|_2 \Big), \qquad t \in \mathbb{R}. \tag{5.4}$$

Next we prove

$$\sum_{|\beta| \leqq 1} \| J^\beta u(t) \|_2 \leqq C(\| \phi \|_{\Sigma^{1,1}}), \qquad t \in \mathbb{R}. \tag{5.5}$$

Since it is clear that (5.5) holds valid in the case of $2 \leqq \gamma < \mathrm{Min}(4, n)$, we only give the proof in the case of $(4/3) < \gamma < 2$. We only consider the case of $t > 0$. Differentiating (5.3) with respect to t, we obtain

$$\frac{d}{dt}\Big\{ \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 + 4t^2 P(u(t)) \Big\} = 4(2-\gamma)tP(u(t)), \quad t > 0. \tag{5.6}$$

Multiplying (5.6) by $t^{\gamma-2}$, we have

$$\frac{d}{dt}\left\{ t^{\gamma-2} \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 + 4t^\gamma P(u(t)) \right\}$$
$$= (\gamma - 2)t^{\gamma-3} \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 \leqq 0, \qquad t > 0. \quad (5.7)$$

Integrating (5.7) with respect to t, we get

$$t^{\gamma-2} \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 + 4t^\gamma P(u(t)) \leqq \sum_{|\beta|=1} \| J^\beta u(1) \|_2^2 + 4P(u(1)), \quad t \geqq 1. \quad (5.8)$$

By (2.3), (5.1)-(5.3) and Lemma 2.1, the R. H. S. of (5.8) is dominated by $C(\| \phi \|_{\Sigma^{1,1}})$. Hence we have

$$P(u(t)) \leqq C(\| \phi \|_{\Sigma^{1,1}})(1 + t)^{-\gamma}, \qquad t > 0, \qquad (5.9)$$

$$\sum_{|\beta|=1} \| J^\beta u(t) \|_2 \leqq C(\| \phi \|_{\Sigma^{1,1}})(1 + t)^{1-(\gamma/2)}, \quad t > 0. \qquad (5.10)$$

We consider the following integral equation

$$u(t) = U(t)\phi - i\lambda \int_0^t U(t - s) f(| u |^2) u(s) ds. \qquad (5.11)$$

This is the integral version of the initial value problem (1.1)-(1.2). Taking the L^q norm and using Lemma 2.2, Lemma 2.6, Lemma 2.1 and (5.9), we have

$$\| u(t) \|_q \leqq C \| U(t)\phi \|_q + C \int_0^t (t - s)^{-\gamma/4} \| f(| u |^2) u(s) \|_{q'} ds$$
$$\leqq C(\| \phi \|_{\Sigma^{1,1}})(1 + t)^{-\gamma/4} + C \int_0^t (t - s)^{-\gamma/4} P(u(s))^{1/2} \| u(s) \|_q ds$$
$$\leqq C(\| \phi \|_{\Sigma^{1,1}})(1 + t)^{-\gamma/4}$$
$$+ C(\| \phi \|_{\Sigma^{1,1}}) \int_0^t (t - s)^{-\gamma/4}(1 + s)^{-\gamma/2} \| u(s) \|_q ds, \quad t > 0. \qquad (5.12)$$

By virtue of Lemma 2.4 we obtain

$$\| u(t) \|_q \leqq C(\| \phi \|_{\Sigma^{1,1}})(1 + t)^{-\gamma/4} \quad \text{for} \quad (4/3) < \gamma < 4. \qquad (5.13)$$

Using Lemma 2.1, (5.9)-(5.10) and (5.13), we have

$$\| u(t) \|_r \leqq C \| u(t) \|_q^{(4-2\gamma)/(4-\gamma)} \times \sum_{|\beta|=1} \| J^\beta u(t) \|_2^{\gamma/(4-\gamma)} t^{-\gamma/(4-\gamma)}$$
$$\leqq C(\| \phi \|_{\Sigma^{1,1}})(1+t)^{-\gamma(4-2\gamma)/4(4-\gamma)} \times (1+t)^{(1-(\gamma/2))\gamma/(4-\gamma)} t^{-\gamma/(4-\gamma)}$$
$$\leqq C(\| \phi \|_{\Sigma^{1,1}}) t^{-\gamma/(4-\gamma)}, \quad t>0, \qquad (5.14)$$

and we obtain by (5.4)

$$\| u(t) \|_r \leqq C(\| \phi \|_{1,2}) . \tag{5.15}$$

(5.14) and (5.15) imply

$$\| u(t) \|_r \leqq C(\| \phi \|_{\Sigma^{1,1}})(1 + t)^{-\gamma/(4-\gamma)} . \tag{5.16}$$

Since the operator J_β commutes with $i\partial/\partial t + \Delta$, we have from (1.1)-(1.2)

$$i(J^\beta u)_t + \Delta(J^\beta u) = \lambda J^\beta(f(|u|^2)u) , \tag{5.17}$$

$$J^\beta u(0, x) = x^\beta \phi . \tag{5.18}$$

Multiplying (5.17) by $\overline{J^\beta u}$, integrating with respect to x and taking the imaginary part, we get

$$\frac{1}{2} \cdot \frac{d}{dt} \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 = \operatorname{Im} \lambda \sum_{|\beta|=1} (J^\beta(f(|u|^2)u(t)), J^\beta u(t)) . \tag{5.19}$$

We apply Hölder's inequality, Lemma 2.4 and (5.16) to the R. H. S. of (5.19) to obtain

$$\begin{aligned} \frac{1}{2} \cdot \frac{d}{dt} \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 &\leqq C \| u(t) \|_r^2 \cdot \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 \\ &\leqq C(\| \phi \|_{\Sigma^{1,1}})(1 + t)^{-2\gamma/(4-\gamma)} \cdot \sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 . \end{aligned} \tag{5.20}$$

Since $2\gamma/(4 - \gamma) > 1$ for $\gamma > 4/3$, (5.20) and Gronwall's inequality yield

$$\sum_{|\beta|=1} \| J^\beta u(t) \|_2^2 \leqq C(\| \phi \|_{\Sigma^{1,1}}), \qquad t > 0 . \tag{5.21}$$

We also obtain (5.21) for $t < 0$ in the same way. Therefore, Theorem 5.1 follows from (5.4) and (5.21). Q. E. D.

THEOREM 5.2. — Suppose that $\lambda > 0$, $(4/3) < \gamma < \operatorname{Min}(4, n)$, $l, m \in \mathbb{N}$.

1-*a*) For any $u_+ \in \Sigma^{l,m}$ there exists a unique $\phi \in \Sigma^{l,m}$ such that

$$\| u_+ - U(-t)u(t) \|_{\Sigma^{l,m}} \to 0 \quad (t \to +\infty), \tag{5.22}$$

where $u(t)$ is the solution of (1.1)-(1.2) with $U(-t)u(t)$ in $C(\mathbb{R}; \Sigma^{l,m})$.

1-*b*) For any $u_- \in \Sigma^{l,m}$ the same result as above holds valid with $+\infty$ replaced by $-\infty$ in (5.22).

2) For any $\phi \in \Sigma^{l,m}$ there exist unique $u_\pm \in \Sigma^{l,m}$ such that the solution $u(t)$ of (1.1)-(1.2) with $U(-t)u(t)$ in $C(\mathbb{R}; \Sigma^{l,m})$ satisfies

$$\| u_\pm - U(-t)u(t) \|_{\Sigma^{l,m}} \to 0 \quad (t \to \pm\infty) .$$

Proof. — The theorem follows from the same argument as in the proof of Theorem 4.2 and the fact $\| U(-t)u(t) \|_{\Sigma^{l,m}}$ is uniformly bounded as a function of t, which is shown in Theorem 5.1. Q. E. D.

COROLLARY 5.1. — Under the assumptions of Theorem 5.2, the wave operators and the scattering operator constructed in Theorem 5.2 are homeomorphisms from $\Sigma^{l,m}$ to $\Sigma^{l,m}$.

Proof. — 1-*a*) and 1-*b*) of Theorem 5.2 implies the wave operators $W_\pm : u_\pm \to \phi$ are well defined in $\Sigma^{l,m}$.

(2) of Theorem 5.2 implies that $\text{Range}(W_+) = \text{Range}(W_-) = \Sigma^{l,m}$. Therefore $W_\pm$ are bijections from $\Sigma^{l,m}$ onto $\Sigma^{l,m}$. Accordingly, the scattering operator $S = W_+^{-1} W_-$ is well defined in $\Sigma^{l,m}$ and a bijection from $\Sigma^{l,m}$ onto $\Sigma^{l,m}$. The continuity properties of $W_\pm$, S, $W_\pm^{-1}$, S^{-1} are proved by the fact that the nonlinear term $f(|u|^2)u$ is infinitely differentiable with respect to u and $\bar{u}$. Q. E. D.

COROLLARY 5.2. — Suppose that the assumptions of Theorem 5.1 hold valid with $l, m \geqq [n/2] + 1$. Then the unique solution $u(t)$ of (1.1)-(1.2) constructed in Theorem 5.1 satisfies

$$\| u(t) \|_\infty \leqq C(1 + |t|)^{-n/2} .$$

Proof. — Corollary 5.2 is proved in the same way as in the proof of Corollary 4.1. Q. E. D.

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Note added in Proof. — After this paper was completed, Ginibre [*25*] pointed out to the authors that if the nonlinear function $f(|u|^2)$ in (1.1) satisfies the following three assumptions:

(1) $$|(\psi, f(|v_1|^2)v_1 - f(|v_2|^2)v_2)| \leqq K(\| v_1 \|_2 + \| v_2 \|_2)\{ \| \psi \|_\infty + \| \psi \|_2 \} \| v_1 - v_2 \|_2$$

for any $v_1, v_2 \in L^2$ and $\psi \in L^2 \cap L^\infty$, where $K(s)$ is a nonnegative increasing function defined on $[0, \infty)$,

(2) for some $\delta \geqq 1$ $$f(|Cv|^2)(x) = |t|^{\delta-2} f(|v|^2)\left(\frac{x}{t}\right), \quad v \in L^2, \quad t \neq 0,$$

where the transform C is defined in (1.8),

(3) $$f(|v|^2)v = 0 \text{ is equivalent to } v = 0,$$

then the same result as Theorem 3.1 (2), that is, the non-existence of scattering states holds for all non-trivial solutions in $C(\mathbb{R}; L^2)$ of (1.1) satisfying the L^2 norm conservation law. Under suitable assumptions on f, for any $\phi \in L^2$ we have a solution $u(t)$ in $C(\mathbb{R}; L^2)$ of (1.1)-(1.2) satisfying the L^2 norm conservation law (see, e. g., [26] and [27]). Under the assumption (2) the inverse transform of (1.8) translates a solution $u(t)$ in $C(\mathbb{R}; L^2)$ of (1.1) into a solution $v(t) = (C^{-1}u)(t)$ in $C(\mathbb{R}\setminus\{0\}; L^2)$ of the new equation:

$$iv_t = -\Delta v + \lambda |t|^{-\delta} f(|v|^2)v, \quad t \in \mathbb{R}\setminus\{0\}, \quad x \in \mathbb{R}^n, \qquad (*)$$

and the transform (1.8) conserves the L^2 norm. The assumptions (1)-(3) ensure that the proof of Theorem 3.1 (2) can be directly applied to the solution in $C(\mathbb{R}\setminus\{0\}; L^2)$ of (*) with the L^2 norm conservation law. The assumptions (1)-(3) cover the following cases.

a) $f(|u|^2) = |u|^{p-1}$ for $1 < p \leqq 1 + \dfrac{2}{n}$ $(n \geqq 2)$ and $1 < p \leqq 2$ $(n = 1)$.

b) $f(|u|^2) = |x|^{-\gamma} * |u|^2$ for $0 < \gamma \leqq 1$ $(n \geqq 3)$ and $0 < \gamma < 1$ $(n = 2)$.

REFERENCES

[1] J. E. Barab, Nonexistence of asymptotic free solutions for a nonlinear Schrödinger equation. *J. Math. Phys.*, t. **25**, 1984, p. 3270-3273.

[2] P. Brenner, On space-time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations. *Math. Z.*, t. **186**, 1984, p. 383-391.

[3] P. Brenner, On scattering and everywhere defined scattering operators for nonlinear Klein-Gordon equations. *J. Differential Equations*, t. **56**, 1985, p. 310-344.

[4] J. M. Chadam and R. T. Glassey, Global existence of solutions to the Cauchy problem for time dependent Hartree equations. *J. Math. Phys.*, t. **16**, 1975, p. 1122-1130.

[5] A. Friedman, *Partial Differential Equations*. Holt-Rinehart and Winston, New York, 1969.

[6] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equation I, II. *J. Funct., Anal*, t. **32**, 1979, p. 1-32, 33-71 ; III, *Ann. Inst. Henri Poincaré, Physique Théorique*, t. **28**, 1978, p. 287-316.

[7] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations with non local interactin. *Math. Z.*, t. **170**, 1980, p. 109-136.

[8] J. Ginibre and G. Velo, Sur une équation de Schrödinger non linéaire avec interaction non locale, in *Nonlinear partial differential equations and their applications*. Collège de France, Séminaire, vol. II, Pitman, Boston, 1981.

[9] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, *J. Math. pures et appl.*, t. **64**, 1985, p. 363-401.

[10] R. T. Glassey, Asymptotic behavior of solutions to a certain nonlinear-Hartree equations, *Comm. Math. Phys.*, t. **53**, 1977, p. 9-18.

[11] N. Hayashi and M. Tsutsumi, *L^∞-decay of classical solutions for nonlinear Schrödinger equations*, preprint.

[12] N. Hayashi, K. Nakamitsu and M. Tsutsumi, On solutions of the initial value problem for the nonlinear Schrödinger equations, to appear *in J. Funct. Anal.*

[13] N. HAYASHI and Y. TSUTSUMI, *Remarks on the scattering problem for nonlinear Schrödinger equations*, preprint.

[14] W. HUNZIKER, On the space-time behavior of Schrödinger wavefunctions. *J. Math. Phys.*, t. **7**, 1965, p. 300-304.

[15] A. JENSEN, Commutator methods and a smoothing property of the Schrödinger evolution group. *Math. Z.*, t. **191**, 1986, p. 53-59.

[16] E. M. STEIN, Singular Integral and Differentiability Properties of Functions, Princeton Univ. Press. *Princeton Math.* Series 30, 1970.

[17] W. A. STRAUSS, Nonlinear invariant wave equations, in *Invariant Wave Equations* (Erice, 1977), *Lecture Notes in Physics*, t. **78**, Springer-Verlag, Berlin-Heidelberg-New York, 1978, p. 197-249.

[18] W. A. STRAUSS, Nonlinear scattering theory at low energy. *J. Funct. Anal.*, t. **41**, 1981, p. 110-133.

[19] W. A. STRAUSS, Nonlinear Scattering theory at low energy: Sequel. *J. Funct. Anal.*, t. **43**, p. 281-293,

[20] R. S. STRICHARTZ, Restrictions of Fourier Transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, t. **44**, 1977, p. 705-714.

[21] Y. TSUTSUMI, Global existence and asymptotic behavior of solutions for nonlinear Schrödinger equations, *Doctor Thesis*, Univ. of Tokyo, 1985.

[22] Y. TSUTSUMI, Scattering problem for nonlinear Schrödinger equations. *Ann. Inst. Henri Poincaré, Physique Théorique*, t. **43**, 1985, p. 321-347.

[23] Y. TSUTSUMI and K. YAJIMA, The asymptotic behavior of nonlinear Schrödinger equations, *Bull. (New Series). Amer. Math. Soc.*, t. **11**, 1984, p. 186-188.

[24] K. YAJIMA, The surfboard Schrödinger equations. *Comm. Math. Phys.*, t. **96**, 1984, p. 349-360.

[25] J. GINIBRE, *Private communication.*

[26] T. KATO, *On nonlinear Schrödinger equations*, preprint, University of California, Berkeley, 1986.

[27] Y. TSUTSUMI, L^2-solutions for nonlinear Schrödinger equations and nonlinear groups, to appear *in Funkcialaj Ekvacioj.*

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