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Quasi $*$-algebras valued quantized fields

by

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ABSTRACT. — Point-like fields are considered as elements of the quasi $*$-algebra $\mathscr{L}(\mathscr{D}, \mathscr{D}')$. This approach allows to recover in natural way a necessary and sufficient condition for the existence of a field $A(x)$ associated to a given Wightman field $A(f)$.

RÉSUMÉ. — Les champs localisés en un point sont considérés comme éléments de la quasi $*$-algèbre $\mathscr{L}(\mathscr{D}, \mathscr{D}')$. Cette approche permet d'obtenir de façon naturelle une condition nécessaire et suffisante pour l'existence d'un champ $A(x)$ associé à un champ de Wightman donné $A(f)$.

1. INTRODUCTION

One of the most unpleasant features in the mathematical description of quantized fields is that the point-like field $A(x)$ cannot be described as an operator over some state space. Thus it turns out that a satisfactory mathematical approach to quantum fields must make use of more singular objects. In a previous paper [*1*] one of us *et al.*, following some idea of Haag [*2*], proposed a definition of field at a point as a mapping from the Minkowski space-time M into the weak sequential completion $\tilde{C}_{\mathscr{D}}$ of the algebra of unbounded operators $C_{\mathscr{D}}$ $(= \mathscr{L}^{+}(\mathscr{D}))$. This corresponds to the heuristic approach where the field at a point is a limit of observables localized in a shrinking sequence of space-time regions.

$\tilde{C}_{\mathscr{D}}$-valued fields allowed to give a precise mathematical meaning to relation of the form

$$A(f) = \int d^4x f(x) A(x) \qquad f \in \mathscr{S}(M) \tag{1}$$

where $\mathscr{S}(M)$ is the Schwartz space of fast decreasing C^∞-functions on M.

In fact, it is proved in [*1*] that, under suitable assumptions, a Wightman field is presentable in the form (1).

In [*3*] Fredenhagen and Hertel considered a point-like field as a sesquilinear form on an appropriate pre-Hilbert space $\mathscr{D}$ which fulfils an high order energy bound.

As proposed by several authors [see reff. from *4* to *8*] we consider here the field as an element of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$, where $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is the set of all linear continuous maps from $\mathscr{D}$ endowed with a topology finer than the Hilbert-norm into its topological dual $\mathscr{D}'$. In particular we consider $\mathscr{D} = \mathscr{D}^\infty(H)$, where H is the energy operator. In this case $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is a quasi-algebra [*9*].

Starting from the notion of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$-valued field, for $\mathscr{D}$ as above, we show first that integrals of the form (1) always converge and that energy-bounds of the kind considered in [*3*] are consequences of the definition itself.

Following this approach the theorem 4 of [*1*] can be extended to a more suitable domain and completely inverted. Moreover some propositions proved in [*3*] are given here under different assumptions which appear to be in some sense, more natural.

2. MATHEMATICAL FRAMEWORK

Let us first recall some known facts about sets of operators.

Let $\mathscr{D}$ be a pre-Hilbert space and $\mathscr{H}$ its norm-completion. By $C_{\mathscr{D}}$ $(= \mathscr{L}^+(\mathscr{D}))$ we denote the set of all operators in $\mathscr{D}$ which have an adjoint in $\mathscr{D}$.

In $C_{\mathscr{D}}$ it is possible to introduce the $\mathscr{D}$-weak topology defined by the set of seminorms

$$A \rightarrow |(A\phi, \psi)| \qquad \varphi, \psi \in \mathscr{D}$$

$C_{\mathscr{D}}$ is a topological $*$-algebra, in general not complete. We call $\tilde{C}_{\mathscr{D}}$ its $\mathscr{D}$-weak sequential completion.

In recent years some partial algebraic structures, such as partial $*$-algebras [*10*], quasi $*$-algebras [*9*], have been studied by several authors. In this paper a remarkable role is playied by quasi $*$-algebras.

A quasi $*$-algebra is a pair $(\mathscr{A}, \mathscr{A}_0)$ where $\mathscr{A}$ is a linear space with involution $A \rightarrow A^+$ and $\mathscr{A}_0 \subseteq \mathscr{A}$ is a $*$-algebra such that both the left- and right-products of elements of $\mathscr{A}$ and elements of $\mathscr{A}_0$ are defined in $\mathscr{A}$. If a topology τ on $\mathscr{A}$ is given such that both the right- and left-multiplications are continuous and $\mathscr{A}_0$ is dense in $\mathscr{A}$, $\mathscr{A}$ is said to be a topological quasi *-algebra with distinguished algebra $\mathscr{A}_0$.

We are interested, in particular, to the set of operators $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ which is, for special $\mathscr{D}$, a quasi *-algebra.

Let $\mathscr{D}$ be a pre-Hilbert space, endowed with a topology t stronger than the norm-topology, $\mathscr{D}'$ its topological dual. Thus we get the familiar triplet

$$\mathscr{D} \subseteq \mathscr{H} \subseteq \mathscr{D}'$$

which is called « rigged Hilbert space ». We will consider on $\mathscr{D}'$ the strong dual topology t'.

Following Lassner [9], we denote by $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ the set of all continuous operators from $\mathscr{D}[t]$ into $\mathscr{D}'[t']$. The equation $(A^+\phi, \psi) = \overline{(A\psi, \phi)}$ $\phi, \psi \in \mathscr{D}$ defines an involution in $\mathscr{L}(\mathscr{D}, \mathscr{D}')$, which becomes so a *-invariant linear space.

If T is any self-adjoint operator in $\mathscr{H}$, the set $\mathscr{D} = \mathscr{D}^\infty(T) = \bigcap_{n>0} D(T^n)$ endowed with the t_T-topology defined by the set of seminorms

$$\phi \to \| T^n\phi \| \qquad n \in \mathbb{N}$$

provides a very simple example of the situation discussed above. In this case $\mathscr{D}[t_T]$ is a reflexive Fréchet space (and hence barrelled). For this $\mathscr{D}$, $C_{\mathscr{D}} \subseteq \mathscr{L}(\mathscr{D}, \mathscr{D}')$ and the latter is a quasi *-algebra. If we endow $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ with the $\tau_{\mathscr{D}}$-topology defined by the seminorms

$$A \to \sup_{\phi,\psi\in\mathscr{M}} |(A\phi, \psi)|, \qquad \mathscr{M} \text{ bounded in } \mathscr{D}[t_T]$$

then $C_{\mathscr{D}}$ is dense in $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ and thus $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is a topological quasi *-algebra [9].

In general, to each element $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ it corresponds a sesquilinear form on $\mathscr{D}$ which is, as is readily checked, separately continuous with respect to t. If $B \in C_{\mathscr{D}}$ then the product AB is always defined in the sense of composition of maps, whereas the product BA can be defined in the sense of forms. Neverthless both AB and BA are not, in general, elements of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$. For reflexive $\mathscr{D}$, however, we get AB, $BA \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ and $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is a quasi *-algebra with distinguished algebra $C_{\mathscr{D}}$. If, moreover, $\mathscr{D}$ is a Fréchet space (e. g. $\mathscr{D} = \mathscr{D}^\infty(T)$) the correspondence between $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ and separately continuous sesquilinear forms is an isomorphism of linear spaces, because all separately continuous sesquilinear forms are continuous.

From now on, we will consider the case where $\mathscr{D} = \mathscr{D}^\infty(T)$ for some self-adjoint operator T in $\mathscr{H}$ and prove some propositions about $\mathscr{L}(\mathscr{D}, \mathscr{D}')$.

PROPOSITION 1. — Let $\mathscr{D} = \mathscr{D}^\infty(T)[t_T]$. Then $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is sequentially complete with respect to the $\mathscr{D}$-weak topology defined by the set of semi-norms

$$A \to |(A\phi, \psi)| \qquad \phi, \psi \in \mathscr{D}.$$

Proof. — First notice that $\mathscr{D}'$ is $\sigma(\mathscr{D}', \mathscr{D})$-quasi complete since it is the dual of a barrelled space ([*11*] 23 n. 1 (3)).

Let A_n be a weak Cauchy sequence in $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ i. e.

$$((A_n - A_m)\phi, \psi) \to 0 \qquad \forall \phi, \psi \in \mathscr{D}.$$

Then the sequence $\{ A_n\phi \}$ is $\sigma(\mathscr{D}', \mathscr{D})$-Cauchy in $\mathscr{D}'$, therefore there exists an element $\Phi \in \mathscr{D}'$ such that

$$(A_n\phi, \psi) \to (\Phi, \psi) \qquad \forall \psi \in \mathscr{D}.$$

Put $\Phi = A\phi$; in this way we define an operator which maps $\mathscr{D}$ into $\mathscr{D}'$. Taking into account that the operation of taking adjoints is continuous, we can also define the operator A^+. Therefore A is continuous from $\mathscr{D}$ with the $\sigma(\mathscr{D}, \mathscr{D}')$-topology into $\mathscr{D}'$ with the $\sigma(\mathscr{D}', \mathscr{D})$ or equivalently with respect to the Mackey topologies $\tau(\mathscr{D}, \mathscr{D}')$ and $\tau(\mathscr{D}', \mathscr{D})$. Since $\mathscr{D}$ is metrizable the $\tau(\mathscr{D}, \mathscr{D}')$-topology coincides with the topology t_T ([*11*], § 21 n. 5 (3)); on the other hand, for the reflexivity of $\mathscr{D}$, the topology $\tau(\mathscr{D}', \mathscr{D})$ coincides with t'_T ([*11*] § 23 n. 3 (1)).

PROPOSITION 2. — In the hypothesis of the previous proposition, $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is isomorphic to $\tilde{C}_{\mathscr{D}}$.

Proof. — Since $\mathscr{D} = \mathscr{D}^\infty(T)$, $C_{\mathscr{D}} \subseteq \mathscr{L}(\mathscr{D}, \mathscr{D}')$ thus it is enough to prove that $C_{\mathscr{D}}$ is dense in $\mathscr{L}(\mathscr{D}, \mathscr{D}')$.

Let $T = \int_0^\infty \lambda dE(\lambda)$ be the spectral decomposition of T; put $P_n = E(n+1) - E(n)$ $n = 0, 1, \ldots$; we get thus a decomposition of $\mathscr{H}$ in mutually orthogonal subspaces $\mathscr{H}_n$. Each of the $\mathscr{H}_n$'s is contained in $\mathscr{D}$ because if $f \in \mathscr{H}$ the vector $P_n f$ is analytic for T.

On the other hand each P_n can be extended by continuity to the whole of $\mathscr{D}'$ and we get an operator $P_n : \mathscr{D}' \to \mathscr{H}_n$. $\forall k, l \in \mathbb{N}$ the operator $P_k A P_l$, with $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ is a bounded operator of $C_{\mathscr{D}}$ (see [*12*]). Let $A_{nm} = \sum_{k=0}^{n} \sum_{l=0}^{m} P_k A P_l$. We will show that $\forall A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ the sequence A_{nm}, defined in this way, converges to A.

$$|((A_{nm}-A)\phi,\psi)| = \left|\left(\left(\sum_{k=0}^{n}\sum_{l=0}^{m} P_k A P_l - A\right)\phi,\psi\right)\right| \leqslant$$
$$\leqslant \left|\left(\sum_{l=0}^{m} AP_l\phi, \sum_{k=0}^{n} P_k\psi\right) - \left(A\phi, \sum_{k=0}^{n} P_k\psi\right)\right| + \left|\left(A\phi, \sum_{k=0}^{n} P_k\psi\right) - (A\phi,\psi)\right| =$$
$$= \left|\left(\sum_{l=0}^{m} P_l\phi, \sum_{k=0}^{n} A^+P_k\psi\right) - \left(\phi, \sum_{k=0}^{n} A^+P_k\psi\right)\right| + \left|\left(A\phi, \sum_{k=0}^{n} P_k\psi\right) - (A\phi,\psi)\right| <$$
$$< \varepsilon .$$

PROPOSITION 3. — Let $\mathscr{D} = \mathscr{D}^\infty(T)$ with $T \geqslant \mathbb{1}$. If A is a sesquilinear form in $\mathscr{D}$, then $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ if, and only if, there exists a natural number k such that $T^{-k}AT^{-k}$ is defined as a bounded operator in $\mathscr{D}$.

Proof. — The operator $T^{-k}AT^{-k}$ is defined since $T^{-1}\mathscr{D} = \mathscr{D}$. Now it is enough to prove that $T^{-k}AT^{-k}$ is bounded if, and only if A is a t_T-continuous sesquilinear form. But

$$|(T^{-k}AT^{-k}\phi,\psi)| \leqslant \|\phi\| \, \|\psi\|$$

is clearly equivalent to

$$|(A\phi,\psi)| \leqslant \|T^k\phi\| \, \|T^k\psi\|$$

PROPOSITION 4. — Let $\mathscr{D} = \mathscr{D}^\infty(T)$ with $T \geqslant 1$ and $A \in C_{\mathscr{D}}$. Then there exists a natural number k such that AT^{-k} is defined as a bounded operator in $\mathscr{D}$.

Proof. — Let $A \in C_{\mathscr{D}}$; then A is $\sigma(\mathscr{D}, \mathscr{D}')$-continuous; then its graph G_A is weakly closed in $\mathscr{D} \times \mathscr{D}$. It follows that G_A is also closed in $\mathscr{D} \times \mathscr{D}$ with respect to the product topology induced on $\mathscr{D} \times \mathscr{D}$ by the t_T-topology of $\mathscr{D}$. Therefore, by the closed graph theorem, A is t_T-continuous. Then $\forall r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that

$$\|T^rA\phi\| \leqslant \|T^n\phi\| \qquad \forall\phi \in \mathscr{D} .$$

This is in particular true for $r = 0$; taking $\phi = T^{-n}\psi$ we get

$$\|AT^{-n}\psi\| \leqslant \|\psi\| \qquad \forall\psi \in \mathscr{D} .$$

3. $\mathcal{L}(\mathcal{D}, \mathcal{D}')$-VALUED FIELDS

DEFINITION 5. — An $\mathcal{L}(\mathcal{D}, \mathcal{D}')$-valued field is a mapping from the Minkowski space-time M into $\mathcal{L}(\mathcal{D}, \mathcal{D}')$

$$A : x \in M \rightarrow A(x) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$$

which satisfies the following axioms:

1) *Translation invariance:* There exists in $\mathcal{H}$ a strongly continuous representation U of the group of the translations in M such that $\forall a \in M$ $U(a)\mathcal{D} \subset \mathcal{D}$ and

$$U(a)A(x)U(-a) = A(x + a)$$

(where the product is intended in the sense discussed in § 2).

2) *Existence (and uniqueness) of a translation invariant vacuum:* There exists a unique vector $\Omega \in \mathcal{D}$ such that $\forall a \in M$

$$U(a)\Omega = \Omega$$

(Ω is unique up to a constant phase vector).

3) *Spectral condition:* The eigenvalues of the energy-momentum operator (of the theory) P^n lie in or on the plus cone.

We call Wightman field what is in general understood with these words (see, for instance, [*13*]) and suppose for this field the following axioms to be verified:

W1) Translation invariance
W2) Existence of a translation invariant vacuum
W3) Cyclicity of the vacuum vector.

From now on we choose $\mathcal{D} = \mathcal{D}^\infty(H)$ where $H = P^0$ is the energy operator and we consider in $\mathcal{D}$ the t_H-topology.

As consequence of Proposition 2, in this case, the two approaches with $\mathcal{L}(\mathcal{D}, \mathcal{D}')$-valued fields and $\tilde{C}_{\mathcal{D}}$-valued fields may be regarded as equivalent.

PROPOSITION 6. — Let $x \rightarrow A(x)$ be an $\mathcal{L}(\mathcal{D}, \mathcal{D}')$-valued field with $\mathcal{D} = \mathcal{D}^\infty(H)$, let $R = (1 + H)^{-1}$; then there exists a natural number k such that $R^k A(x) R^k$ is defined as a bounded operator in $\mathcal{D}$. (H-bound condition).

The proof follows immediately from Proposition 3 taking into account that for the spectral condition H is a self-adjoint positive operator in $\mathcal{H}$.

PROPOSITION 7. — Let $x \to \mathrm{A}(x)$ be an $\mathscr{L}(\mathscr{D}, \mathscr{D}')$-valued field with $\mathscr{D} = \mathscr{D}^\infty(\mathrm{H})$ then the integral

$$(\mathrm{A}(f)\phi, \psi) = \int d^4x f(x)(\mathrm{A}(x)\phi, \psi)$$

converges for all $\phi, \psi \in \mathscr{D}$ and defines for each $f \in \mathscr{S}(\mathrm{M})$ an operator of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$.

Proof. — Since $\mathrm{A}(x) \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ for each $\phi \in \mathscr{D}$ the form $(\mathrm{A}(x)\phi, \psi)$ is an (anti)-linear continuous form on $\mathscr{D}$. Thus there exists an integer n such that

$$| \mathrm{A}(x)\phi, \psi) | \leqslant \| \mathrm{H}^n\psi \|$$

where n may depend on $x \in \mathrm{M}$ and on $\phi \in \mathscr{D}$. But

$$| (\mathrm{A}(x)\phi, \psi) | = | (\mathrm{A}(0)\mathrm{U}(-x)\phi, \mathrm{U}(-x)\psi) | \leqslant \| \mathrm{H}^n\mathrm{U}(-x)\psi \| = \| \mathrm{H}^n\psi \|$$

therefore n does not depend on x.

$$|(\mathrm{A}(f)\phi, \psi)| = \left| \int d^4x f(x)(\mathrm{A}(x)\phi, \psi) \right| \leqslant \int d^4x |f(x)| \, |(\mathrm{A}(x)\phi, \psi)| \leqslant$$
$$\leqslant \|\mathrm{H}^n\psi\| \int d^4x |f(x)| < \infty .$$

Thus the integral exists in the sense of the weak convergence. The above inequality also shows that $\mathrm{A}(f)\phi \in \mathscr{D}'$. We will now prove that $\mathrm{A}(f) \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ i. e. that $\mathrm{A}(f)$ maps continuously $\mathscr{D}[t_\mathrm{H}]$ into $\mathscr{D}'[t'_\mathrm{H}]$. Now, since $\mathrm{A}(x) \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ it is continuous from $\mathscr{D}$ into $\mathscr{D}'$; hence for each bounded set $\mathscr{M}$ in $\mathscr{D}$ there exists $k > 0$ and $n \in \mathbb{N}$ such that

$$\sup_{\psi \in \mathscr{M}} | (\mathrm{A}(x)\phi, \psi) | \leqslant k \| \mathrm{H}^n\phi \|$$

then

$$\sup_{\psi \in \mathscr{M}} | (\mathrm{A}(f)\phi, \psi) | \leqslant \sup_{\psi \in \mathscr{M}} \int d^4x \, | f(x) | \, | (\mathrm{A}(0)\mathrm{U}(-x)\phi, \mathrm{U}(-x)\psi) | \leqslant$$
$$\leqslant \int d^4x \, | f(x) | \sup_{\psi \in \mathscr{M}} | (\mathrm{A}(0)\mathrm{U}(-x)\phi, \mathrm{U}(-x)\psi) | \leqslant k \| \mathrm{H}^n\phi \| \int d^4x \, | f(x) | .$$

Thus $\mathrm{A}(f)$ is also continuous.

At this stage of our discussion we know that the « smeared » field associated with an $\mathscr{L}(\mathscr{D}, \mathscr{D}')$-valued field, with $\mathscr{D} = \mathscr{D}^\infty(\mathrm{H})$, is also $\mathscr{L}(\mathscr{D}, \mathscr{D})$-valued. But what is usually required is that $\mathrm{A}(f) \in \mathrm{C}_\mathscr{D}$ $\forall f \in \mathscr{S}(\mathrm{M})$.

In [*3*] Fredenhagen and Hertel proposed the idea of selecting point-like fields $\mathrm{A}(x)$ in a class F satisfying the requirement that $\mathrm{R}^k\mathrm{A}(0)\mathrm{R}^k$, with $\mathrm{R} = (1 + \mathrm{H})^{-1}$, be bounded for some natural k and they proved that in this case the $\mathrm{A}(f)$'s are operators of $\mathrm{C}_\mathscr{D}$. In our approach, because of Proposition 6, if $\mathscr{D} = \mathscr{D}^\infty(\mathrm{H})$, this H-bound is a natural consequence of the definition itself. Therefore, in this case all $\mathscr{L}(\mathscr{D}, \mathscr{D}')$-valued fields are elements of the class F.

We will prove here that the H-bound condition is not only sufficient but also necessary. Actually in [3] a sort of necessary condition was proved but under stronger assumptions ([3], relation 2.6).

PROPOSITION 8. — Let $\mathscr{S}(\mathrm{M}) \ni f \to \mathrm{A}(f) \in \mathrm{C}_{\mathscr{D}}$ with $\mathscr{D} = \mathscr{D}^\infty(\mathrm{H})$ a Wightman field; then there exists a natural k (independant of f) such that $\mathrm{A}(f)\mathrm{R}^k$ is a bounded operator $\forall f \in \mathscr{S}(\mathrm{M})$.

Proof. — We procede by absurd, showing that if $\forall k > 0$ there exists an $f_k \in \mathscr{S}(\mathrm{M})$ such that $\mathrm{A}(f_k)\mathrm{R}^k$ is unbounded whereas $\mathrm{A}(f_k)\mathrm{R}^{k+1}$ is bounded, then it is possible to find a sequence $\{ a_k \}$ of real numbers and a sequence $\{ \varphi_k, \psi_k \}$ of pairs of vectors of $\mathscr{D}$ with $\| \varphi_k \| = \| \psi_k \| = 1$ such that the series

$$\sum_{k=1}^{\infty} a_k f_k$$

converges to an element $f \in \mathscr{S}(\mathrm{M})$ and such that $\forall n \in \mathbb{N}$

$$\left| \sum_{k=1}^{n} (a_k \mathrm{A}(f_k)\mathrm{R}^n \varphi_n, \psi_n) \right| > 2^n$$

and

$$| (a_{n+r}\mathrm{A}(f_{n+r})\mathrm{R}^n \varphi_n, \psi_n) | \leqslant 2^{-r}.$$

If such a sequence exists, taking into account that the map

$$f \in \mathscr{S}(\mathrm{M}) \mapsto (\mathrm{A}(f)\varphi, \psi)$$

is, $\forall \varphi, \psi \in \mathscr{D}$, a tempored distribution, one has

$$|(\mathrm{A}(f)\mathrm{R}^n\varphi_n, \psi_n)| \geqslant \left| \sum_{k=1}^{n} a_k(\mathrm{A}(f_k)\mathrm{R}^n\varphi_n, \psi_n) - \sum_{r=1}^{\infty} (a_{n+r}\mathrm{A}(f_{n+r})\mathrm{R}^n\varphi_n, \psi_n) \right| > 2^n - 1.$$

Let n_0 be the smallest number such that $\mathrm{A}(f)\mathrm{R}^{n_0}$ is bounded (Prop. 4); taking into account that $\| \mathrm{R} \| \leqslant 1$ we get for $n > n_0$

$$\begin{aligned} \| \mathrm{A}(f)\mathrm{R}^{n_0} \| &= \| \mathrm{R}^{n_0}\mathrm{A}^+(f) \| \geq \| \mathrm{R}^n\mathrm{A}^+(f) \| \geq | (\varphi_n, \mathrm{R}^n\mathrm{A}^+(f)\psi_n) | = \\ &= | (\mathrm{A}(f)\mathrm{R}^n\varphi_n, \psi_n) | \geq 2^n - 1 \end{aligned}$$

which is a contradiction.

Let us now prove the existence of the sequences $\{ a_k \}$ and $\{ \varphi_k, \psi_k \}$ as described above.

We denote with $\|\ \|_k$ the k-th seminorm defining the topology of $\mathscr{S}(\mathrm{M})$ and which may be supposed to increase with k.

To begin with, take $n = 1$.

The functional $(\mathrm{A}(f_1)\mathrm{R}\varphi, \psi)$ is unbounded then $\forall \mathrm{M} > 0$ there exist two vectors φ_1, ψ_1 with $\|\varphi_1\| = \|\psi_1\| = 1$ such that $|(\mathrm{A}(f_1)\mathrm{R}\varphi_1, \psi_1)| > \mathrm{M}$. We suppose $\mathrm{M} > 2\| f_1 \|_1 e$. We choose a number a_1^1 such that

$$2\mathrm{M}^{-1} \leqslant a_1^1 \leqslant (e\| f_1 \|_1)^{-1}$$

and a sequence $\{ a_{1+r}^1 \}_{r \geq 1}$ such that

$$a_{1+r}^1 \leqslant \min \{ (\| f_{1+r} \|_{1+r} \exp(r+1))^{-1}, 2^{-r}/|(\mathrm{A}(f_{1+r})\mathrm{R}\varphi_1, \psi_1)| \}$$

then we get

$$|(a_1^1\mathrm{A}(f_1)\mathrm{R}\varphi_1, \psi_1)| > 2$$

and

$$|(a_{1+r}^1\mathrm{A}(f_{1+r})\mathrm{R}\varphi_1, \psi_1)| \leqslant 2^{-r}.$$

We put $a_1 = a_1^1$.

Following an analogous procedure, for each n, we put

$$\mathrm{C}_n = \left\| \sum_{k=1}^{n-1} a_k \mathrm{A}(f_k)\mathrm{R}^n \right\|$$

and choose a number $\mathrm{M} > \max\limits_{j<n} \{ (2^n + \mathrm{C}_n) \| f_n \|_n \exp(n), (2^n + \mathrm{C}_n)/a_n^j \}$. The functional $(\mathrm{A}(f_n)\mathrm{R}^n\varphi, \psi)$ is unbounded; then there exist two vectors $\varphi_n, \psi_n \in \mathscr{D}$ with $\| \varphi_n \| = \| \psi_n \| = 1$ such that $|(\mathrm{A}(f_n)\mathrm{R}^n\varphi_n, \psi_n)| > \mathrm{M}$; thus we can choose a number a_n^n with

$$(2^n + \mathrm{C}_n)\mathrm{M}^{-1} \leqslant a_n^n \leqslant (\| f_n \|_n \exp(n))^{-1}$$

and a sequence a_{n+r}^n, $r \geqslant 1$ with

$$a_{n+r}^n \leqslant \min \{ (\| f_{n+r} \|_{n+r} \exp(n+r))^{-1}, 2^{-r}/|(\mathrm{A}(f_{n+r})\mathrm{R}^n\varphi_n, \psi_n)| \}$$

then we get

$$\left| \left(\sum_{k=1}^{n-1} a_k \mathrm{A}(f_k)\mathrm{R}^n\varphi_n, \psi_n \right) + (a_n^n \mathrm{A}(f_n)\mathrm{R}^n\varphi_n, \psi_n) \right| \geq$$

$$- \left| \left(\sum_{k=1}^{n-1} a_k \mathrm{A}(f_k)\mathrm{R}^n\varphi_n, \psi_n \right) \right| + a_n^n |(\mathrm{A}(f_n)\mathrm{R}^n\varphi_n, \psi_n)| > 2^n$$

and

$$|(a_{n+r}^n \mathrm{A}(f_{n+r})\mathrm{R}^n\varphi_n, \psi_n)| \leqslant 2^{-r}.$$

We put $a_n = \min \{ a_n^1, a_n^2, \dots, a_n^n \}$.

The sequence $\sum_{k=1}^{n} a_k f_k$, for the above choice of the a_k's is a Cauchy sequence in $\mathscr{S}(M)$ and then it converges to an $f \in \mathscr{S}(M)$; the other requirements are, also fulfilled and thus the theorem is proved.

PROPOSITION 9. — Let $\mathscr{D} = \mathscr{D}^\infty(H)$ and $f \to A(f)$ a hermitean (scalar) Wightman field satisfying the axioms W_1, W_2, W_3, then in order that there exists an $\mathscr{L}(\mathscr{D}, \mathscr{D}')$-valued field $x \in M \to A(x) \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ such that

$$\forall f \in \mathscr{S}(M) \qquad (A(f)\phi, \psi) = \int d^4x f(x)(A(x)\phi, \psi) \qquad \forall \phi, \psi \in \mathscr{D}$$

it is necessary and sufficient that the sesquilinear form $R^k A(0) R^k$ is bounded for some k, where $A(0) = U(-x)A(x)U(x)$.

Proof. — The sufficient part follows from the proof given in [3] taking into account that $\mathscr{L}(\mathscr{D}, \mathscr{D}')$-valued fields belong to the class F.

We prove now the necessity.

Let us call $\mathscr{D}_0$ the linear manifold of $\mathscr{D}$ which is obtained by applying to the vacuum Ω the algebra generated by the set of operators $\{ A(f) \mid f \in \mathscr{S}(M) \}$. Then, by Theorem 4 of [1] one finds a sesquilinear form $A(x)$ such that $\forall f \in \mathscr{S}(M)$

$$(A(f)\phi, \psi) = \int d^4x f(x)(A(x)\phi, \psi) \qquad \forall \phi, \psi \in \mathscr{D}_0 .$$

Now, because of Proposition 8 there exists a natural number k (independant of f) such that $R^k A(f) R^k$ is bounded $\forall f \in \mathscr{S}(M)$ and taking into account that in the proof of Theorem 4 of [1] $A(x)$ is obtained as a limit of $(A(f_n)\phi, \psi)$ where $f_n \to \delta_x$ in the topology of $\mathscr{S}'(M)$ we get, for $x = 0$ and $\phi, \psi \in \mathscr{D}_0$

$$|(A(0)\phi, \psi)| = \lim_{n \to \infty} |(A(f_n)\phi, \psi)| \leqslant \| R^{-k}\phi \| \; \| R^{-k}\psi \| .$$

Therefore the functional $R^k A(0) R^k$ is bounded on $\mathscr{D}_0$, then it can be extended to a bounded sesquilinear form all over $\mathscr{H}$. Let us call $B(0)$ the so obtained form. Let us consider the form $\hat{A}(0) = R^{-k} B(0) R^{-k}$; this is clearly a sesquilinear form on $\mathscr{D}$ which extends $A(0)$ and such that $R^k \hat{A}(0) R^k$ is bounded. For simplicity of notations we denote still $\hat{A}(0)$ by $A(0)$. The sesquilinear form $A(0)$ satisfies therefore the requirements of Proposition 3 and thus the statement is proved.

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