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On the mathematical structure of test relativistic magnetofluiddynamics (*)

by

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ABSTRACT. — We investigate the mathematical structure of the covariant formulation of test-relativistic magnetofluiddynamics within the framework of Friedrich's quasi-linear hyperbolic systems. Explicit expressions are given for the left and right eigenvectors of the characteristic matrix and the hyperbolicity conditions are proved to be satisfied except in a special case. The reduction to a symmetric hyperbolic system is performed by using the entropy supplementary conservation law, with respect to coordinates based on a spacetime foliation.

RÉSUMÉ. — On étudie la structure mathématique de la formulation covariante de la magnétofluidodynamique relativiste test, dans le cadre des systèmes quasi-linéaires hyperboliques de Friedrichs. On détermine complètement les vecteurs propres droits et gauches de la matrice caractéristique et on vérifie les conditions d'hyperbolicité avec une exception. On réduit le système à un système symétrique en employant la loi supplémentaire de conservation de l'entropie, dans un système de coordonnées basé sur une foliation de l'espace-temps.

1. INTRODUCTION

Relativistic magnetofluiddynamics is a theory of great interest in astrophysics, cosmology and plasma physics. In particular, in astrophysics,

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relativistic magnetofluiddynamics might be important in models of pulsars [*1*], extragalactic radio-sources [*2*] and gravitational collapse [*3*]. In cosmology relativistic magnetofluiddynamics might play a significant role in theories of galaxy formation if the evidence for an intergalactic magnetic field is corroborated [*4*]. In plasma physics relativistic magnetofluiddynamical effects might be relevant in experiments on strong ionizing shock waves [*5*] and on charged relativistic particle beams [*6*].

From a mathematical viewpoint a thorough investigation of the equations of general relativistic magnetofluiddynamics (i. e., coupled with Einstein's equations) has been performed by Lichnerowicz [*6*]. In particular Lichnerowicz investigated the Cauchy problem in the framework of the Leray systems, obtaining a local existence and uniqueness theorem in a suitable Gevrey class [*6*].

In many applications in astrophysics and plasma physics one can neglect the gravitational field generated by the magnetofluid in comparison with the background gravitational field. In this case one considers only the conservation equations for the matter, neglecting the Einstein equations; the resulting theory can be called test-relativistic magnetofluiddynamics. It is interesting to study the mathematical structure of such a theory also because, being simpler than the full general relativistic one, it can be exploited more thoroughly.

In this paper we attempt at investigating the mathematical structure of test-relativistic magnetofluiddynamics with regard to the problem of hyperbolicity in the sense of Friedrichs [*7*].

In Sec. 2 we present a detailed and thorough analysis of the hyperbolicity problem for the (usual) covariant formulation of the equations of test-relativistic magnetofluiddynamics. In particular we obtain explicit and complete expressions for the right and left eigenvectors (previous results appearing in the literature [*8*] are not complete in this respect).

In Sec. 3 we treat the case when the background space-time is endowed with a space-like foliation (which occurs in many applications). Following earlier ideas of Ruggeri and Strumia [*9*] (they consider only the case of flat space-time and their treatment of the constraints does not seem to be satisfactory) we introduce the concept of main field and obtain a symmetric hyperbolic system. For such a system it is possible to apply powerful methods in order to obtain existence and uniqueness for the Cauchy problem [*10*].

2. ON THE HYPERBOLICITY OF THE COVARIANT FORMULATION

The aim of test relativistic magnetofluiddynamics is to determine the fields p (pressure), S (specific entropy, u^α (four-velocity) and b^α (related to the electromagnetic field).

The rest-mass density ρ and the total energy-density e are then obtained from the state equations

$$e = e(p, S)$$
$$\rho = \rho(p, S)$$

which are restricted by the first law of thermodynamics

(1) $$T\,ds = d(e/\rho) + p d(1/\rho),$$

where T is the absolute temperature.

Also one has the constraints

(2) $$\begin{cases} u^\alpha u_\alpha = -1 \\ u^\alpha b_\alpha = 0 \end{cases}$$

hence b^α is a space-like vector and $|b|^2 = b_\alpha b^\alpha > 0$. The unknown fields must be determined from the field equations

(3) $$\begin{cases} \nabla_\alpha(\rho u^\alpha) = 0 & \text{(conservation of mass)} \\ \nabla_\alpha T^{\alpha\beta} = 0 & \text{(conservation of energy-momentum)} \\ \nabla_\alpha \psi^{\alpha\beta} = 0 & \text{(Maxwell's equations)} \end{cases}$$

where

(4) $$\psi^{\alpha\beta} = u^\alpha b^\beta - u^\beta b^\alpha$$

and $T^{\alpha\beta}$, the energy-momentum tensor, is decomposed into

(5) $$T^{\alpha\beta} = T_f^{\alpha\beta} + T_m^{\alpha\beta}$$

with

$$T_f^{\alpha\beta} = (e+p)u^\alpha u^\beta + p g^{\alpha\beta} \qquad \text{(fluid part of } T^{\alpha\beta})$$

(6) $$T_m^{\alpha\beta} = |b|^2(u^\alpha u^\beta + g^{\alpha\beta}/2) - b^\alpha b^\beta \qquad \text{(electromagnetic part of } T^{\alpha\beta}).$$

Equations (3) are equivalent to

(7) $$(e+p+|b|^2)u^\alpha\nabla_\alpha u^\mu - b^\alpha\nabla_\alpha b^\mu + (g^{\mu\alpha} + 2u^\mu u^\alpha)b_\nu\nabla_\alpha b^\nu + \{(e+p)g^{\mu\alpha} + (e+p-e'_p|b|^2)u^\mu u^\alpha + b^\mu b^\alpha\}\nabla_\alpha p/(e+p) = 0$$

(conservation of momentum)

(8) $$u^\alpha\nabla_\alpha b^\beta - b^\alpha\nabla_\alpha u^\beta + (-e'_p b^\beta u^\alpha + u^\beta b^\alpha)\nabla_\alpha p/(e+p) = 0$$

(9) $$e'_p u^\alpha\nabla_\alpha p + (e+p)\nabla_\alpha u^\alpha = 0$$

(10) $$u^\alpha\nabla_\alpha S = 0$$

(11) $$u^\alpha u^\beta\nabla_\alpha b_\beta + \nabla_\alpha b^\alpha = 0.$$

In fact, if we call G^μ, H^β, G, K, H the LHS's of (7), (8), (9), (10), (11) respectively, we find by (1), (2), (4), (6) that the following identities hold

$$G^\mu = \nabla_\alpha T^{\alpha\mu} + (u_\beta \nabla_\alpha T^{\alpha\beta} + b_\beta \nabla_\alpha \psi^{\alpha\beta})u^\mu [1 + |b|^2(1 - e'_s/T\rho)/(e+p)]$$
$$+ b^\mu [b_\beta \nabla_\alpha T^{\alpha\beta} + (e+p+|b|^2)u_\beta \nabla_\alpha \psi^{\alpha\beta}]/(e+p) - u^\mu |b|^2 e'_s \nabla_\alpha(\rho u^\alpha)/T\rho^2$$
$$H^\beta = \nabla_\alpha \psi^{\alpha\beta} + u^\beta [b_\gamma \nabla_\alpha T^{\alpha\gamma} + (e+p+|b|^2)u_\gamma \nabla_\alpha \psi^{\alpha\gamma}]/(e+p)$$
$$+ (u_\gamma \nabla_\alpha T^{\alpha\gamma} + b_\gamma \nabla_\alpha \psi^{\alpha\gamma}) b^\beta (1 - e'_s/T\rho)/(e+p) - b^\beta e'_s \nabla_\alpha(\rho u^\alpha)/T\rho^2$$
$$G = (u_\beta \nabla_\alpha T^{\alpha\beta} + b_\beta \nabla_\alpha \psi^{\alpha\beta})(-1 + e'_s/T\rho) + (e+p)e'_s[\nabla_\alpha(\rho u^\alpha)]/T\rho^2$$
$$K = -[u_\beta \nabla_\alpha T^{\alpha\beta} + b_\beta \nabla_\alpha \psi^{\alpha\beta} + (e+p)\nabla_\alpha(\rho u^\alpha)/\rho]/T\rho$$
$$H = u_\beta \nabla_\alpha \psi^{\alpha\beta}.$$

Then (3) are verified if and only if (7), (8), (9), (10), (11) hold. We assume now these latter equations as field equations.

We can see that

$$0 = \nabla_\beta[H^\beta - u^\beta b_\mu G^\mu (e + p + |b|^2)^{-1} + b^\beta G/(e+p)] = u^\mu \nabla_\mu H + H \nabla_\mu u^\mu$$

hence, if (11) holds on a hypersurface $\mathscr{F}$ transverse to u^μ, it holds also in a neighbourhood of $\mathscr{F}$ as consequence of the other field equations.

Therefore we can take equations (7), (8), (9), (10) as the field equations for the field unknown

$$\underset{\sim}{U} = \begin{pmatrix} u^\nu \\ b^\nu \\ p \\ S \end{pmatrix}. \tag{12}$$

These equations can be written in matrix formulation

$$\mathscr{A}^{\alpha A}{}_B \nabla_\alpha U^B = 0 \qquad A, B = 0, \ldots, 9 \tag{13}$$

with

$$\underset{\approx}{\mathscr{A}}^\alpha = \begin{pmatrix} Eu^\alpha \delta^\mu{}_\nu, & -b^\alpha \delta^\mu{}_\nu + P^{\mu\alpha} b_\nu, & l^{\mu\alpha}, & 0^{\mu\alpha} \\ b^\alpha \delta^\mu{}_\nu, & -u^\alpha \delta^\mu{}_\nu, & f^{\alpha\mu}, & 0^{\mu\alpha} \\ \eta \delta^\alpha{}_\nu, & 0^\alpha{}_\nu, & e'_p u^\alpha, & 0^\alpha \\ 0^\alpha{}_\nu, & 0^\alpha{}_\nu; & 0^\alpha, & u^\alpha \end{pmatrix} \tag{14}$$

where

$$\eta = e + p, \qquad E = \eta + |b|^2, \qquad P^{\mu\alpha} = g^{\mu\alpha} + 2u^\mu u^\alpha,$$
$$l^{\mu\alpha} = [\eta g^{\mu\alpha} + (\eta - e'_p |b|^2)u^\mu u^\alpha + b^\mu b_\alpha]/\eta,$$
$$f^{\alpha\mu} = (u^\alpha b^\mu e'_p - u^\mu b^\alpha)/\eta.$$

Now we study the hyperbolicity in the sense of Friedrichs [7] of the system (13). This system will be hyperbolic in the time-direction defined by the vector field $\underset{\sim}{\xi}$, with $\xi_\alpha \xi^\alpha = -1$, if the following two conditions hold in any local chart:

i) $\det(\underset{\approx}{\mathcal{A}}^\alpha \xi_\alpha) = 0$

ii) for any ζ such that $\zeta_\alpha \xi^\alpha = 0$, $\zeta_\alpha \zeta^\alpha = 1$, the eigenvalue problem $\underset{\approx}{\mathcal{A}}^\alpha(\zeta_\alpha - \mu\xi_\alpha)\underset{\sim}{d} = 0$ has only real eigenvalues μ and ten linearly independent eigenvectors $\underset{\sim}{d}$.

In order to verify these conditions it is useful to write, for any 4-vector φ_α, the matrix $\underset{\approx}{\mathcal{A}}^\alpha \varphi_\alpha$ and calculate its determinant. We have

$$\underset{\approx}{\mathcal{A}}^\alpha \varphi_\alpha = \begin{pmatrix} \mathrm{E}a\delta^\mu{}_\nu, & m^\mu{}_\nu\,, & l^\mu\,, & 0^\mu \\ \mathrm{B}\delta^\mu{}_\nu\,, & -a\delta^\mu{}_\nu, & f^\mu\,, & 0^\mu \\ \eta\varphi_\nu\,, & 0_\nu\,, & e'_p a, & 0 \\ 0_\nu\,, & 0_\nu\,, & 0\,, & a \end{pmatrix}$$

where

$$a = u^\alpha \varphi_\alpha\,; \qquad \mathrm{B} = b^\alpha \varphi_\alpha\,;$$

$$l^\mu = \varphi^\mu + (\eta - e'_p\,|\,b\,|^2)au^\mu/\eta + \mathrm{B}b^\mu/\eta\,;$$

$$f^\mu = (ae'_p b^\mu - \mathrm{B}u^\mu)/\eta\,;$$

$$m^\mu{}_\nu = (\varphi^\mu + 2au^\mu)b_\nu - \mathrm{B}\delta^\mu{}_\nu\,.$$

Now it is easy to show that

(15) $$\det(\underset{\approx}{\mathcal{A}}^\alpha \varphi_\alpha) = \mathrm{E}a^2\mathrm{A}^2\mathrm{N}_4$$

where

(16) $$\mathrm{A} = \mathrm{E}a^2 - \mathrm{B}^2$$

(17) $$\mathrm{N}_4 = \eta(e'_p - 1)a^4 + [-(\eta + e'_p\,|\,b\,|^2)a^2 + \mathrm{B}^2]\mathrm{G}$$

with

$$\mathrm{G} = \varphi_\alpha \varphi^\alpha.$$

In the following we shall often refer to the local reference frame Σ in which

$$u^\alpha = \delta^\alpha{}_0\,; \qquad b^\alpha = \delta^\alpha{}_1\,|\,b\,|\,; \qquad \xi_3 = 0\,; \qquad \xi_2 > 0\,.$$

Moreover we assume, on physical grounds, that $e'_p - 1 > 0$. Now we prove that condition *i*) of hyperbolicity holds.

In fact for $\varphi_\alpha = \xi_\alpha$, in the frame Σ, we have

$$a = \xi_0 \neq 0$$

$$\mathrm{A} = n\xi_0^2 + |\,b\,|^2(1 + \xi_2^2) \neq 0$$

$$\mathrm{N}_4 = \mathrm{E}e'_p + \xi_2^2\,|\,b\,|^2 + \eta e'_p(\xi_1^2 + \xi_2^2) + (e'_p - 1)(\xi_1^2 + \xi_2^2)[\eta(\xi_1^2 + \xi_2^2) + \mathrm{E}] > 0\,.$$

For the eigenvalue problem, we use $\varphi_\alpha = \zeta_\alpha - \mu\xi_\alpha$ and the eigenvalues are the corresponding roots of $a = 0$; $\mathrm{A} = 0$; $\mathrm{N}_4 = 0$.

They correspond to material waves, Alfvén waves and magnetoacoustic waves, respectively.

For the right eigenvectors we use the notation

$$\underset{\sim}{d} = (d^\alpha, d^{4+\alpha}, d^8, d^9)$$

while for the left ones we use the notation

$$\underset{\approx}{s} = (s_\mu, s_{4+\mu}, s_8, s_9)\,.$$

We start investigating condition *ii*) of hyperbolicity in the case in which a material wave coincides with an Alfvén or a magnetoacoustic one.

It is easy to show that this case happens if and only if

$$\begin{vmatrix} \zeta_\alpha u^\alpha & \zeta_\alpha b^\alpha \\ \xi_\alpha u^\alpha & \xi_\alpha b^\alpha \end{vmatrix} = 0\,.$$

Then one has

$$\zeta_\alpha u^\alpha = \lambda \xi_\alpha u^\alpha\,; \qquad \zeta_\alpha b^\alpha = \lambda \xi_\alpha b^\alpha, \quad \text{for some } \lambda\,.$$

The eigenvalue corresponding to $a = 0$ is $\mu = \lambda$ and has multiplicity 8 for det $(\underset{\approx}{\mathscr{A}}^\alpha \varphi_\alpha) = 0$.

A basis for the corresponding space of right eigenvectors is given by

$$\{ (u^\alpha, 0^\alpha, 0, 0)^T ; (b^\alpha, 0^\alpha, 0, 0)^T ; (\varepsilon^\alpha{}_{\beta\gamma\delta} u^\beta b^\gamma (\zeta^\delta - \lambda \xi^\delta), 0^\alpha, 0, 0)^T ; (0^\alpha, u^\alpha, 0, 0)^T ;$$
$$(0^\alpha, b^\alpha, -|b|^2, 0)^T ; (0^\alpha, \zeta^\alpha, -b_\alpha \zeta^\alpha, 0)^T ; (0^\alpha, \varepsilon^\alpha{}_{\beta\gamma\delta} u^\beta b^\gamma \zeta^\delta, 0, 0)^T ;$$
$$(0^\alpha, 0^\alpha, 0, 1) \} \,;$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol.

A basis for the corresponding space of left eigenvectors is given by

$$\{ (u_\mu, 0_\mu, 0, 0) ; (b_\mu, 0_\mu, 0, 0) ; (\varepsilon_{\mu\beta\gamma\delta} u^\beta b^\gamma (\zeta^\delta - \lambda \xi^\delta), 0_\mu, 0, 0) ; (0_\mu, u_\mu, 0, 0) ;$$
$$(0_\mu, \varepsilon_{\mu\beta\gamma\delta} u^\beta b^\gamma \zeta^\delta, 0, 0) ; (0_\mu, b_\mu, 0, 0) ; (0_\mu, \zeta_\mu, 0, 0) ; (0_\mu, 0_\mu, 0, 1) \}.$$

This eigenvalue $\mu = \lambda$ is a root of $A = 0$ with multiplicity 2 and of N_4 with multiplicity 2. The remaining roots of N_4 arise from

$$(18) \quad (\lambda - \mu)^2 [\eta(e'_p - 1)(u^\alpha \xi_\alpha)^4 - D] + 2D\lambda(\lambda - \mu) + (1 - \lambda^2) D = 0$$

where

$$D = -(\eta + e'_p |b|^2)(u^\alpha \xi_\alpha)^2 + (b^\alpha \xi_\alpha)^2 < 0\,.$$

Now

$$1 - \lambda^2 = (\zeta_2 - \lambda \xi_2)^2 + \zeta_3^2 > 0$$

then $\mu = \lambda$ is not a solution of (18).

Moreover $\Delta/4 = D^2 - D(1 - \lambda^2)\eta(e'_p - 1)(u^\alpha \xi_\alpha)^4 > 0$. Then the remaining two roots of N_4 are real and distinct. By substituting them into

$$\begin{cases} d^\alpha = Ea^2(Bf^\alpha - al^\alpha) + Ea(B^2 - e'_p |b|^2 a^2)(\varphi^\alpha + 2au^\alpha)/\eta \\ d^{4+\alpha} = d^\alpha B/a + EAaf^\alpha \\ d^8 = Ea^2 A \\ d^9 = 0 \end{cases}$$

we obtain the corresponding two right eigenvectors, while the left ones arise from

$$\begin{cases} s_\nu = -\,B(G + 2a^2)b_\nu/\eta + a^2\eta E\phi_\nu \\ s_{4+\nu} = Ea\eta\,[(G + 2a^2)b_\nu - B\varphi_\nu] \\ s_8 = -\,EaA \\ s_9 = 0\,. \end{cases}$$

Therefore, in this case, condition *ii*) of hyperbolicity is verified.

We consider now the case

$$\begin{vmatrix} \zeta_\alpha u^\alpha & \zeta_\alpha b^\alpha \\ \xi_\alpha u^\alpha & \xi_\alpha b^\alpha \end{vmatrix} \neq 0\,,$$

i. e. the case in which no material wave coincides with an Alfvén or a magnetoacoustic one.

The real eigenvalue $\mu = (\zeta_\alpha u^\alpha)/(\xi_\alpha u^\alpha)$ corresponding to the material waves has multiplicity 2 for $\det(\underset{\approx}{\mathscr{A}}^\alpha\varphi_\alpha) = 0$. A basis for the corresponding space of right eigenvectors is given by $\{(0^\mu, \varphi^\mu, 0, 0)^T; (0^\mu, 0^\mu, 0, 1)^T\}$ while a basis for the corresponding space of left eigenvectors is given by $\{(0_\mu, -\eta\varphi_\mu, B, 0); (0_\mu, 0_\mu, 0, 1)\}$. To study the Alfvén and the magnetoacoustic waves it is useful to state the following proposition.

PROPOSITION 2.1. — « $A = 0$ has two real and distinct roots ».

In fact in the frame Σ we have $A = 0$ iff

$$\mu^2(E\xi_0^2 - |b|^2\xi_1^2) - 2\mu(E\xi_0\zeta_0 - |b|^2\xi_1\zeta_1) + E\zeta_0^2 - |b|^2\zeta_1^2 = 0$$

and
$$\Delta = 4E\,|b|^2(\zeta_0\xi_1 - \zeta_1\xi_0)^2 > 0\,. \qquad \#$$

Let us, for the sake of simplicity, consider first the case $e'_p = 1$; then (17) becomes $N_4 = -\,AG = (\mu^2 - 1)A$ and then $N_4 = 0$ has four rel and distinct roots.

Two of them are those of $A = 0$ and the remaining are $\mu_1 = -\,\mu_2 = 1$. The roots μ_3, μ_4 of $A = 0$ have multiplicity 3 for $\det(\underset{\approx}{\mathscr{A}}^\alpha\varphi_\alpha) = 0$. Six corresponding linearly independent right eigenvectors can be obtained by substituting into

$$(d^\alpha, -\,d^\alpha B/a - \eta\varphi_\gamma d^\gamma f^\alpha/a^2, -\,\eta\varphi_\gamma d^\gamma/a, 0)^T$$

the values $d^\alpha = a\eta u^\alpha - Bb^\alpha$; $d^\alpha = d_1^\alpha$; $d^\alpha = d_2^\alpha$ where d_1^α, d_2^α are two linearly independent solutions of

$$\begin{cases} d^\alpha(b_\alpha - B\varphi_\alpha/a^2) = 0 \\ d^\alpha(a\eta u_\alpha - Bb_\alpha) = 0 \end{cases}$$

for the values $\mu = \mu_3$, $\mu = \mu_4$.

Similarly, six corresponding linearly independent left eigenvectors can be obtained by substituting into $(s_\mu, -s_\mu B/a, 0, 0)$ the values $s_\mu = B\mu_\mu + ab_\mu$; $s_\mu = s^1_\mu$; $s_\mu = s^2_\mu$ where s^1_μ, s^2_μ are two linearly independent solutions of $s_\mu(\varphi^\mu + 2au^\mu) = 0$; $s_\mu(Bu^\mu + ab^\mu) = 0$ for the values $\mu = \mu_3$, $\mu = \mu_4$.

The roots μ_1, μ_2 of N_4 have multiplicity 1 for $\det(\underset{\approx}{\mathscr{A}}^\alpha \varphi_\alpha) = 0$ and the corresponding right eigenvectors can be obtained by substituting into

$$(d^\alpha, d^\alpha B/a + EaAf^\alpha, Ea^2A, 0)^T$$

the value

$$d^\alpha = Ea^2(Bf^\alpha - al^\alpha) + Ea(B^2 - e'_p |b|^2 a^2)(\varphi^\alpha + 2au^\alpha)/\eta \text{ for } \mu = 1, \quad \mu = -1.$$

Similarly the corresponding left eigenvectors can be obtained substituting into

$$(-2B\eta ab_\nu + a\eta E\varphi_\nu, E\eta(2a^2b_\nu - B\varphi_\nu), -EA, 0) \quad \text{for} \quad \mu = 1, \quad \mu = -1.$$

Before studying the eigenvalue problem for the remaining case

$$\begin{vmatrix} \zeta_\alpha u^\alpha & \zeta_\alpha b^\alpha \\ \xi_\alpha u^\alpha & \xi_\alpha b^\alpha \end{vmatrix} \neq 0 \quad \text{and} \quad e'_p > 1,$$

now we prove some lemmas.

LEMMA 2.1. — « In the frame Σ we have

$$\zeta_0^2 \leqslant (\xi_1^2 + \xi_2^2)(1 - \zeta_3^2) \tag{19}$$

and

$$\zeta_0^2 < \xi_0^2. \text{ »} \tag{20}$$

In fact, if $\xi_1 = \xi_2 = 0$, from $\zeta^\alpha \xi_\alpha = 0$, we have $\zeta^0 = 0$ and this proves (19); if $\xi_1 \neq 0$, $\xi_2 = 0$, by substituting ζ_1 from $\zeta^\alpha \xi_\alpha = 0$ into $\zeta^\alpha \zeta_\alpha = 1$ we obtain a second degree equation in the unknown ζ_2; its solutions are real if and only if (19) holds. Finally, if $\xi_2 \neq 0$, by substituting ζ_2 from $\zeta^\alpha \xi_\alpha = 0$ into $\zeta^\alpha \zeta_\alpha = 1$ we obtain a second degree equation in the unknown ζ_1; its solutions are real if and only if (19) holds.

The inequality (20) is a consequence of (19). #

LEMMA 2.2. — « In the frame Σ, $\forall c > 1$, the numbers $\mu_1(c) = p - \sqrt{q}$; $\mu_2(c) = p + \sqrt{q}$ with

$$p = \zeta_0 \xi_0 (c - 1)/[1 + \xi_0^2(c - 1)]$$
$$q = [1 + (c - 1)(\xi_0^2 - \zeta_0^2)]/[1 + \xi_0^2(c - 1)]^2$$

are real, distinct and such that

$$1 < \mu_1(c) < \zeta_0/\xi_0 < \mu_2(c) < 1. \text{»} \tag{21}$$

In fact from (20) we have $q > 0$ and then $\mu_1(c)$ and $\mu_2(c)$ are real and distinct. Moreover the function

$$(22) \qquad f(\mu)=\mu^2[1 + (c - 1)\xi_0^2] + 2\mu\xi_0\zeta_0(1 - c) - 1 + \zeta_0^2(c - 1)$$

has the coefficient of μ^2 positive and his roots are $\mu_1(c)$ and $\mu_2(c)$. But $f(\zeta_0/\xi_0) = (\zeta_0^2 - \xi_0^2)/\xi_0^2 < 0$ and then $\mu_1(c) < \zeta_0/\xi_0 < \mu_2(c)$.

Moreover $f(1)=(c-1)(\xi_0-\zeta_0)^2<0$ and $f(-1)=(c-1)(\xi_0+\zeta_0)^2>0$ and then (21) holds. #

LEMMA 2.3. — « In the frame Σ the function

$$V_\Sigma^2(\mu) = (\zeta_0 - \mu\xi_0)^2/[(\zeta_1 - \mu\xi_1)^2 + (\zeta_2 - \mu\xi_2)^2 + \zeta_3^2]$$

is such that $V_\Sigma^2[\mu_1(c)] = V_\Sigma^2[\mu_2(c)] = 1/c$ ».

In fact if we look for the values μ such that $V_\Sigma^2(\mu) = 1/c$ we find that they are the roots of the function $f(\mu)$ given by (22). These roots are $\mu_1(c)$ and $\mu_2(c)$. #

LEMMA 2.4. — « If $e'_p > 1$, N_4 and A have a common root if and only if ζ_α is one of the following 4-vectors:

$$(23) \quad \zeta_\alpha = \pm \{ \eta/|b|^2 + (u^\gamma\xi_\gamma \pm \sqrt{E}b^\gamma\xi_\gamma/|b|^2)^2]^{-\frac{1}{2}} [u_\alpha \pm \sqrt{E}b_\alpha/|b|^2 + (u^\gamma\xi_\gamma \pm \sqrt{E}b^\gamma\xi_\gamma/|b|^2)\xi_\alpha] \text{»}.$$

In fact N_4 can be written, in the frame Σ, as

$$N_4 = A^2/|b|^2 - A\,a^2[\eta-(e'_p-1)|b|^2]/|b|^2 - [(\zeta_2-\mu\xi_2)^2 + \zeta_3^2][A+a^2|b|^2(e^q{}_p-1)].$$

Then A and N_4 have a common root μ if and only if

$$\zeta_2 - \mu\xi_2 = \zeta_3 = 0\,; \qquad A(\mu) = 0\,;$$

by eliminating μ from these relations we obtain the conditions

$$\zeta_3 = 0; \qquad E(\xi_2\zeta_0 - \zeta_2\xi_0)^2 - (\zeta_1\xi_2 - \zeta_2\xi_1)^2|b|^2 = 0$$

that with $\xi_\alpha\zeta^\alpha = 0$; $\zeta_\alpha\zeta^\alpha = 1$ form a fourth degree system of four equations in the four unknowns ζ^α; its solutions are (23). We are now ready to prove the following propositions. #

PROPOSITION 2.2. — « If $e'_p > 1$ and ζ_α is not one of the 4-vectors (23), then N_4 has four real and distinct roots and there exists a basis of eigenvectors ».

In fact N_4 is a fourth degree polynomial in the unknown μ; moreover from $e'_p > 1$, $E/|b|^2 > 1$ and lemma (2.2) we have

$$(25) \qquad -1 < \mu_1(e'_p) < \zeta_0/\xi_0 < \mu_2(e'_p) < 1$$

and

$$(26) \qquad -1 < \mu_1(E/|b|^2) < \zeta_0/\xi_0 < \mu_2(E/|b|^2) < 1 .$$

Moreover, it is easy to see that

$$N_4(-1) = \eta(e'_p - 1)(\zeta_0 + \xi_0)^2 > 0$$
$$N_4[\mu_1(e'_p)] = (|b|^2/e'_p)(1 - e'_p)[(\zeta_1 - \mu_1(e'_p)\xi_1)^2 + (\zeta_2 - \mu_1(e'_p)\xi_2)^2 + \zeta_3^2][(\zeta_2 - \mu_1(e'_p)\xi_2)^2 + \zeta_3^2] \leqslant 0$$
$$N_4(\zeta_0/\xi_0) > 0$$
$$N_4[\mu_2(e'_p)] = (|b|^2/e'_p)(1 - e'_p)[(\zeta_1 - \mu_2(e'_p)\xi_1)^2 + (\zeta_2 - \mu_2(e'_p)\xi_2)^2 + \zeta_3^2][(\zeta_2 - \mu_2(e'_p)\xi_2)^2 + \zeta_3^2] \leqslant 0$$
$$N_4(1) = \eta(e'_p - 1)(\zeta_0 - \xi_0)^2 > 0$$

and then if

$$[(\zeta_2 - \mu_1(e'_p)\xi_2)^2 + \zeta_3^2][(\zeta_2 - \mu_2(e'_p)\xi_2)^2 + \zeta_3^2] \neq 0 ,$$

N_4 has four real and distinct roots alternating with the numbers in (25). If $[(\zeta_2 - \mu_1(e'_p)\xi_2)^2 + \zeta_3^2][(\zeta_2 - \mu_2(e'_p)\xi_2)^2 + \xi_3^2] = 0$, then from the hypothesis that ζ_α is distinct from each of the 4-vectors (23), from lemma (2.4), lemma (2.3) and relation (24), it follows

$$[\zeta_2 - \mu_1(E/|b|^2)\xi_2][\zeta_2 - \mu_2(E/|b|^2)\xi_2] \neq 0$$

and then

$$N_4(-1) > 0$$
$$N_4[\mu_1(E/|b|^2)] = -(\eta|b|^2/E)[(\zeta_1 - \mu_1(E/|b|^2)\xi_1)^2 + (\zeta_2 - \mu_1(E/|b|^2)\xi_2)^2][\zeta_2 - \mu_1(E/|b|^2)\xi_2]^2 < 0$$
$$N_4(\zeta_0/\xi_0) > 0$$
$$N_4[\mu_2(E/|b|^2)] = -(\eta|b|^2/E)[(\zeta_1 - \mu_2(E/|b|^2)\xi_1)^2 + (\zeta_2 - \mu_2(E/|b|^2)\xi_2)^2][\zeta_2 - \mu_2(E/|b|^2)\xi_2]^2 < 0$$
$$N_4(1) > 0$$

from which we have that N_4 has four real and distinct roots alternating with the numbers in (26).

[One of them is $\mu_1(e'_p)$ or $\mu_2(e'_p)$].

The two roots of A have multiplicity 2 for $\det(\underset{\approx}{\mathcal{A}}^\alpha \varphi_\alpha) = 0$ and four corresponding linearly independent right eigenvectors can be obtained by substituting these roots into

$$(27) \qquad \begin{pmatrix} a(Ea^2 - |b|^2G)u^\alpha - a^2Bb^\alpha + a^2|b|^2(\zeta^\alpha - \mu\xi^\alpha) \\ B(Ea^2 - |b|^2G)u^\alpha - aB^2b^\alpha + aB|b|^2(\zeta^\alpha - \mu\xi^\alpha) \\ 0 \\ 0 \end{pmatrix}$$

and

$$(28) \qquad (a\varepsilon^{\alpha}_{\beta\gamma\delta}u^{\beta}b^{\gamma}(\zeta^{\delta} - \mu\xi^{\delta}), \qquad B\varepsilon^{\alpha}{}_{\beta\gamma\delta}u^{\beta}b^{\gamma}(\zeta^{\delta} - \mu\xi^{\delta}), 0, 0)^{T}.$$

Four linearly independent left eigenvectors, corresponding to these roots of A, can be obtained by substituting them into

$$(29) \qquad (aBu_{\mu} + a^2 b_{\mu}, - B^2\mu_{\mu} - aBb_{\mu}, 0, 0)$$

and

$$(30) \qquad (a\varepsilon_{\mu\beta\gamma\delta}u^{\beta}b^{\gamma}\zeta^{\delta}, - B\varepsilon_{\mu\beta\gamma\delta}u^{\beta}b^{\gamma}\zeta^{\delta}, 0, 0).$$

Every root of N_4 has multiplicity 1 for $\det(\underset{\approx}{\mathscr{A}}^{\alpha}\varphi_{\alpha}) = 0$; by substituting it into

$$(31) \qquad (d^{\alpha}, d^{\alpha}B/a + Af^{\alpha}, aA, 0)^{T}$$

with

$$(32) \qquad d^{\alpha} = a(Bf^{\alpha} - al^{\alpha}) + (B^2 - e'_p\,|\,b\,|^2 a^2)(\varphi^{\alpha} + 2au^{\alpha})/\eta$$

and into

$$(33) \quad (- B\eta(G + 2a^2)b_{\nu} + a^2\eta E\varphi_{\nu}, Ea\eta\,[(G + 2a^2)b_{\nu} - B\varphi_{\nu}], - EaA, 0)$$

we obtain a corresponding right eigenvector and a corresponding left eigenvector, respectively. #

Another case in which the hyperbolicity condition holds is expressed in the following result.

PROPOSITION 2.3. — « If $e'_p > 1$ and ζ_{α} is one of the 4-vectors (23), but $\eta \neq (e'_p - 1)\,|\,b\,|^2$, then N_4 has four real and distinct roots and there exists a basis of eigenvectors ».

Let us consider first the case $\xi_2 = 0$ in the frame Σ.

Then from (23), we have $\zeta_3 = \zeta_2 = 0$ and from (24)

$$N_4 = (A/|\,b\,|^2)\{\,A - a^2[\eta - (e'_p - 1)\,|\,b\,|^2\,]\,\};$$

from which we see that N_4 has four real and distinct roots, those of $A = 0$ and

$$\mu_{1,2} = [(e'_p - 1)\xi_0\xi_1 \pm \sqrt{e'_p}]/[(e'_p\xi_0^2 - \xi_1^2)\xi_0].$$

The right eigenvectors corresponding to $\mu_{1,2}$ are

$$(- Bb^{\alpha}, - B\,|\,b\,|^2 u^{\alpha}, \eta\,|\,b\,|^2 a, O)^{T}$$

and the left ones are

$$(- Ea\eta u_{\nu} + \eta B(\eta - e'_p\,|\,b\,|^2)b_{\nu}/|\,b\,|^2, E\eta(ab_{\nu} + Bu_{\nu}), - Ea(E - e'_p\,|\,b\,|^2), O).$$

The roots of $A = 0$ have multiplicity 3 for $\det(\underset{\approx}{\mathscr{A}}^{\alpha}\varphi_{\alpha}) = 0$ and six corresponding linearly independent right eigenvectors are obtained by substituting these roots into

(34) $(a(\mathrm{E}-e'_p \mid b \mid^2)u^\nu - \mathrm{B}b^\nu, \mathrm{B}(\eta - e'_p \mid b \mid^2)u^\nu + a(e'_p \mid b \mid^2 - \mathrm{E})b^\nu, \eta a \mid b \mid^2, 0)^{\mathrm{T}}$

(35) $(d_1^\nu, d_1^\nu \mathrm{B}/a, 0, 0)^{\mathrm{T}}$

(36) $(d_2^\nu, d_2^\nu \mathrm{B}/a, 0, 0)^{\mathrm{T}}$

where d_1^ν and d_2^ν are two linearly independent solutions of

(37) $$d^\nu b_\nu = d^\nu u_\nu = 0\,.$$

Six linearly independent left eigenvectors corresponding to the roots of $\mathrm{A} = 0$ are obtained by substituting these roots into

(38) $(bu_\mu + ab_\mu, -\ \mathrm{E}au_\mu - bb_\mu, 0, 0)$

(39) $(s_\mu^1, -\ s_\mu^1 \mathrm{B}/a, 0, 0)$

(40) $(s_\mu^2, -\ s_\mu^2 \mathrm{B}/a, 0, 0)$

where s_μ^1 and s_μ^2 are two linearly independent solutions of

(41) $$s_\nu b^\nu = s_\nu u^\nu = 0\,.$$

Let us consider now the remaining case $\xi_2 \neq 0$ in the frame Σ.
From (23) we have $\zeta_3 = 0$ and

$$\mathrm{N}_4 = 0\,; \qquad \mathrm{A} = 0$$

which implies

$$\mu = \zeta_2/\xi_2\,.$$

Then N_4 and A have one and only one common root that is ζ_2/ξ_2 and coincides with $\mu_1(\mathrm{E}/\mid b \mid^2)$ or $\mu_2(\mathrm{E}/\mid b \mid^2)$.

Moreover from lemma (2.3) and the hypothesis $\eta \neq (e'_p - 1) \mid b \mid^2$, this common root is distinct from $\mu_1(e'_p)$ and $\mu_2(e'_p)$, whence it follows that

$$\mathrm{N}_4(-1) > 0;\ \mathrm{N}_4[\mu_1(e'_p)] < 0;\ \mathrm{N}_4(\zeta_0/\xi_0) > 0;\ \mathrm{N}_4[\mu_2(e'_p)] < 0;\ \mathrm{N}_4(1) > 0$$

and then N_4 has four real and distinct roots.

The root ζ_2/ξ_2 has multiplicity 3 for $\det(\underset{\approx}{\mathscr{A}}^\alpha \varphi_\alpha) = 0$ and three corresponding linearly independent right eigenvectors are obtained by substituting this root into (34), (35), (36), where d_1^ν, d_2^ν are two linearly independent solutions of (37). Three linearly independent left eigenvectors corresponding to ζ_2/ζ_2 are obtained by substituting this root into (38), (39), (40), where s_μ^1, s_μ^2 are two linearly independent solutions of (41). For each of the other three roots of N_4 there is a corresponding right eigenvector given by (31), (32) and left eigenvector (33). The remaining root is the solution of $\mathrm{A} = 0$ that is distinct from ζ_2/ξ_2. It has multiplicity 2 for $\det(\underset{\approx}{\mathscr{A}}^\alpha \varphi_\alpha) = 0$.

Two linearly independent right eigenvectors corresponding to it are given by (27) and (28), while two left ones are given by (29) and (30). #

The remaining case, excluded by the hypothesis in Prop. (2.2) and Prop. (2.3), is that in which $e'_p > 1$, ζ^α is one of the 4-vectors (23) and $\eta = (e'_p - 1) | b |^2$.

In this case we still find that the eigenvalues are all real, but a basis of eigenvectors does not exist.

The proof is given in the following propositions (2.4) and (2.5). They are distinguished by the value of

$$(42) \qquad \mathrm{H} = \det \begin{pmatrix} -1 & \xi_\alpha u^\alpha & \xi_\alpha b^\alpha \\ \xi_\alpha u^\alpha & -1 & 0 \\ \xi^\alpha b^\alpha & 0 & | b |^2 \end{pmatrix}.$$

PROPOSITION 2.4. — « If $e'_p > 1$, $\eta = (e'_p - 1) | b |^2$, ζ_α is one of the 4-vectors (23) and $\mathrm{H} = 0$, then both roots of A are roots of N_4, with multiplicity 2, and a basis of eigenvectors does not exist. »

In fact in the frame Σ, from (42) we have $\mathrm{H} = - | b |^2 \xi_2^2$ and then from the hypothesis $\mathrm{H} = 0$, we have $\xi_2 = 0$.

From (23) we have $\zeta_2 = \zeta_3 = 0$ and from (24) $\mathrm{N}_4 = \mathrm{A}^2/| b |^2$. Then the roots μ_1, μ_2 of A are roots of N_4 with multiplicity 2, so that their multiplicity for $\det (\mathscr{A}^\alpha \varphi_\alpha) = 0$ is 4.

But to each of them there correspond only three linearly independent right eigenvectors such us $(-\mathrm{B}b^\nu, -\mathrm{B} | b |^2 u^\nu, \eta a | b |^2, 0)^{\mathrm{T}}$; $(d_1^\nu, d_1^\nu \mathrm{B}/a, 0, 0)^{\mathrm{T}}$; $(d_2^\nu, d_2^\nu \mathrm{B}/a, 0, 0)^{\mathrm{T}}$ where d_1^ν, d_2^ν are two linearly independent solutions of $d^\nu b_\nu = d^\nu u_\nu = 0$. #

Six linearly independent left eigenvectors corresponding to μ_1 and μ_2 are obtained by substituting them into (38), (39) and (40) with $s^1{}_\mu$, $s^2{}_\mu$ independent solutions of (41).

Finally we have the following proposition.

PROPOSITION 2.5. — « If $e'_p > 1$, $\eta = (e'_p - 1) | b |^2$, ζ_α is one of the 4-vectors (23) and $\mathrm{H} \neq 0$, then N_4 and A have one and only one common root, which is a double root of N_4. Moreover a basis of eigenvectors does not exist. »

In the frame Σ from $\mathrm{H} \neq 0$ we have $\xi_2 \neq 0$ and from (23) $\zeta_3 = 0$. As consequence, (24) becomes

$$(43) \qquad \mathrm{N}_4 = \mathrm{A}^2/| b |^2 - (\zeta_2 - \mu\xi_2)^2 [\mathrm{A} + a^2 | b |^2 (e'_p - 1)].$$

From (23) we have too that ζ_2/ζ_2 is a root of A and coincides with $\mu_1(e'_p)$ or $\mu_2(e'_p)$.

From (43) we have then that ζ_2/ξ_2 is a root of N_4 too, with multiplicity 2, and that it is the only common root of A and N_4. If $\zeta_2/\xi_2 = \mu_1(e'_p)$, then from

$$\mu_1(e'_p) < \zeta_0/\xi_0 < \mu_2(e'_p) < 1$$

and

$$N_4(\zeta_0/\xi_0) > 0 ; \qquad N_4[\mu_2(e'_p)] < 0 ; \qquad N_4(1) > 0$$

we obtain that N_4 has a simple root between ζ_0/ξ_0 and $\mu_2(e'_p)$ and another simple root between $\mu_2(e'_p)$ and 1.

If $\zeta_2/\xi_2 = \mu_2(e'_p)$, then from

$$-1 < \mu_1(e'_p) < \zeta_0/\xi_0 < \mu_2(e'_p)$$

and

$$N_4(-1) > 0 ; \qquad N_4[\mu_1(e'_p)] < 0 ; \qquad N_4(\zeta_0/\xi_0) > 0$$

we obtain that N_4 has a simple root between -1 and $\mu_1(e'_p)$ and another simple root between $\mu_1(e'_p)$ and ζ_0/ξ_0.

Two linearly independent right eigenvectors corresponding to the root of A distinct from ζ_2/ξ_2 are (27) and (28), while two left ones are given by (29) and (30).

To each of the two simple roots of N_4 there corresponds a right eigenvector, given by (31), (32), and a left one, given by (33).

The remaining eigenvalue ζ_2/ξ_2 has multiplicity 4 for $\det(\underset{\approx}{\mathscr{A}}^\alpha \varphi_\alpha) = 0$, but to it there correspond only three linearly independent right eigenvectors such as (34), (35), (36) where d_1^ν, d_2^ν are two independent solutions of (37).

Three linearly independent left eigenvectors corresponding to ζ_2/ξ_2 are (38), (39),40), where s_μ^1 and s_μ^2 are two independent solutions of (41).

#

In all these considerations we have not taken into account that the field unknowns (12) are not independent, because (2) holds, and similarly that the field equations (7), (8), (9), (10), are not independent. (In fact: (7) contracted with $-u_\mu$ is equal to (8) contracted with b_β ; (8) contracted with $-u_\beta(e + p + |b|^2)$ is equal to (7) contracted with b_μ).

Therefore, after having solved the system (13) with (12) and (14), only those solutions satisfying (2) can be accepted.

In particular, this could be achieved by imposing that the constraints (2) be satisfied on a given non characteristic initial hypersurface $\mathscr{F}$. In fact, by introducing the vector

$$\underset{\sim}{D} = \begin{pmatrix} (u^\alpha u_\alpha + 1)/2 \\ u^\alpha b_\alpha \end{pmatrix}$$

the identities:

$$u_\mu G^\mu + b_\mu H^\mu = 0$$
$$u_\mu H^\mu + b_\mu G^\mu / E = 0$$

can be written in the form of a differential system

$$\underset{\approx}{C}^\alpha \nabla_\alpha \underset{\sim}{D} + \underset{\approx}{M}\underset{\sim}{D} = 0$$

with

$$\underset{\approx}{C}^{\alpha} = \begin{pmatrix} Eu^{\alpha} & -b^{\alpha} \\ -b^{\alpha} & u^{\alpha} \end{pmatrix}$$

and $\underset{\approx}{M}$ a suitable matrix.

It is immediate to check that such a system is symmetric and hyperbolic. Therefore, by applying standard results on symmetric hyperbolic systems [10], it follows that if we impose the constraints (2) on a non characteristic initial hypersurface $\mathscr{F}$, they will propagate nicely off $\mathscr{F}$.

It is physically reasonable to surmise that if one considers only independent unknowns and equations, then the resulting system would be hyperbolic.

In the next section, following ideas put forward by Boillat [11] and Ruggeri and Strumia [9] we shall introduce the main field which, altough at the cost of losing manifest covariance, will fulfill the aim of obtaining a hyperbolic system.

3. SYMMETRIZATION

Let ξ^{μ} be a time-like vector field, $\xi_{\mu}\xi^{\mu} = -1$, which is hypersurface orthogonal. This is the case in most applications. Then one can introduce, at least locally, coordinates (x^{μ}) such that

$$\xi_{\mu} = \delta^{0}_{\mu}.$$

From

$$\nabla_k \nabla_\alpha \psi^{\alpha k} = \nabla_k \nabla_i \psi^{ik} + \nabla_0 \nabla_k \psi^{0k} = \nabla_0 \nabla_k \psi^{0k}$$

we have that the equation

$$\nabla_k \psi^{0k} = 0 \tag{44}$$

holds if it is verified on a hypersurface $\mathscr{F}' : x^0 = \text{cost.}$, and moreover if $\nabla_\alpha \psi^{\alpha k} = 0$ holds.

Then we can take as independent field equations

$$\begin{cases} \nabla_\alpha T^{\alpha\beta} = 0 \\ \nabla_\alpha(\rho u^\alpha) = 0 \\ \nabla_\alpha \psi^{\alpha k} = 0 . \end{cases} \tag{45}$$

Moreover, from (1), (2), (4), (5), (6) we have

$$\nabla_\alpha(-\rho S u^\alpha) = T^{-1} u_\beta \nabla_\alpha T^{\alpha\beta} + T^{-1} b_\beta \nabla_\alpha \psi^{\alpha\beta} + (e + p - S\rho T)[\nabla_\alpha(\rho u^\alpha)]/T\rho$$

for every value of the fields.

This identity can be rewritten as

$$\nabla_\alpha h^\alpha - k \nabla_\alpha \psi^{\alpha 0} = u_\beta \nabla_\alpha T^{\alpha\beta} + u'_4 \nabla_\alpha(\rho u^\alpha) + u'_{4+k} \nabla_\alpha \psi^{\alpha k} \tag{46}$$

with

$$h^\alpha = -\rho S u^\alpha$$
$$k = -b_0 T^{-1}$$

(47)
$$\begin{cases} u'_\beta = u_\beta T^{-1} \\ u'_{4+k} = b_k T^{-1} \\ u'_4 = -S + (e+p)/T\rho . \end{cases}$$

This result could have been obtained also by applying Liu's Theorem [*12*] stating that if every suitably differentiable solution of the system (45) satisfies the constraints

$$\nabla_\alpha \psi^{0\alpha} = 0; \qquad \nabla_\alpha h^\alpha = 0,$$

then there exist quantities u'_β, u'_4, u'_{4+k}, called Lagrange multipliers, and k, such that (46) holds for every value of the fields.

The main field $\underset{\sim}{U}'$ is defined to consist of the Lagrange multipliers.

Now we show that the transformation (47) is invertible.

In fact a basic tenet of equilibrium thermodynamics [*13*] is that

$$-\mathcal{G} = TS + 1 - (e+p)/\rho$$

is a convex function (Thermodynamic stability.) Then $-\dfrac{\partial^2 \mathcal{G}}{\partial S^2} > 0$.

But, from (1), using the variables S, p, we have $-\dfrac{\partial \mathcal{G}}{\partial S} = S\,\dfrac{\partial T}{\partial S}$, from which $\dfrac{\partial T}{\partial S} > 0$.

From (47) we have

(48)
$$\begin{cases} (-u'_\beta u'^\beta)^{\frac{1}{2}} = T^{-1} \\ u'_4 = -S + (e+p)/T\rho \end{cases}$$

and then

$$\begin{vmatrix} \dfrac{\partial(-u'_\beta u'^\beta)^{\frac{1}{2}}}{\partial p} & \dfrac{\partial(-u'_\beta u'^\beta)^{\frac{1}{2}}}{\partial S} \\ \dfrac{\partial u'_4}{\partial p} & \dfrac{\partial u'_4}{\partial S} \end{vmatrix} = (\rho T^3)^{-1}\,\frac{\partial T}{\partial S} > 0$$

that assures the invertibility of (48) giving

(49)
$$\begin{cases} p = p(u'_\beta, u'_4) \\ S = S(u'_\beta, u'_4) \end{cases} .$$

The inverse of (47) are then (49) and

$$(50)\qquad \begin{cases} u_\beta = (-u'_\gamma u'^\gamma)^{-\frac{1}{2}} u'_\beta \\ b_k = (-u'_\gamma u'^\gamma)^{-\frac{1}{2}} u'_{4+k} \\ b_0 = (-u'_\gamma u'^\gamma)(u'_{4+k} u'^k)/u'_0. \end{cases}$$

Therefore we can take the components of the main field $\underset{\sim}{U}'$ as new variables. It follows that (46) must hold for every value of the new variables, too.

Defining

$$h'^\alpha = u'_\beta T^{\alpha\beta} + u'_4 \rho u^\alpha + u'_{4+k} \psi^{\alpha k} + k\psi^{0\alpha} - h^\alpha,$$

(46) can be rewritten as

$$\nabla_\alpha h'^\alpha - \psi^{0\alpha}\nabla_\alpha k = T^{\alpha\beta}\nabla_\alpha u'_\beta + \rho u^\alpha \nabla_\alpha u'_4 + \psi^{\alpha k}\nabla_\alpha u'_{4+k}$$

that must hold for every value of u'_β, u'_4, u'_{4+k}; from this statement it follows that

$$(52)\qquad \begin{cases} T^{\alpha\beta} = \dfrac{\partial h'^\alpha}{\partial u'_\beta} - \psi^{0\alpha}\dfrac{\partial k}{\partial u'_\beta} \\ \rho u^\alpha = \dfrac{\partial h'^\alpha}{u'_4} - \psi^{0\alpha}\dfrac{\partial k}{u'_4} \\ \psi^{\alpha k} = \dfrac{\partial h'^\alpha}{\partial u'_{4+k}} - \psi^{0\alpha}\dfrac{\partial k}{\partial u'_{4+k}} \end{cases}$$

which permits to write the system (45) as

$$M^\alpha{}_{AB}\nabla_\alpha U'^B = 0$$

where

$$\underset{\sim}{U}' = \begin{pmatrix} u'_\gamma \\ u'_4 \\ u'_{4+k} \end{pmatrix}$$

$$(53)\qquad M^\alpha{}_{AB} = \frac{\partial^2 h'^\alpha}{\partial U'^A \partial U'^B} - \psi^{0\alpha}\frac{\partial^2 K}{\partial U'^A \partial U'^B},$$

once (44) is used.

From the symmetry of $\underset{\approx}{M}^\alpha$ we have that the hyperbolicity condition holds if and only if $\det(\underset{\approx}{M}^\alpha \xi_\alpha) \neq 0$.

But from (53), (52) we have

$$M^\alpha{}_{AB} = \frac{\partial^2 h'^0}{\partial U'^A \partial U'^B} = \begin{pmatrix} \dfrac{\partial T^{0\beta}}{\partial U'^A} \\ \dfrac{\partial(\rho u^0)}{\partial U'^A} \\ \dfrac{\partial \psi^{0k}}{\partial U'^A} \end{pmatrix}$$

and after long and tedious calculations we find

$$\det (\underset{\approx}{M}_{\alpha}\xi_{\alpha}) = \rho^3 T^{10}(T_s)^{-1} \{ [(u_\alpha\xi^\alpha)^2(e + p + |b|^2) - (b_\alpha\xi^\alpha)^2]^2 + [(u_\alpha\xi^\alpha)^2(e+p+|b|^2)-(b_\alpha\xi^\alpha)^2][(e+p)(u_\alpha\xi^\alpha)^2+|b|^2](e'_p-1)(u_\alpha\xi^\alpha)^2\} > 0.$$

Then we may conclude that the system (45) is hyperbolic.

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