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STANLEY GUDDER

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Partial Hilbert Spaces and Amplitude Functions

by

Stanley GUDDER (*)

Department of Mathematics and Computer Science,
University of Denver, Denver, Colorado, 80208

ABSTRACT. — Partial Hilbert spaces are introduced as a generalization of Hilbert spaces. Motivated by quantum mechanics, a special class of partial Hilbert spaces called amplitude spaces, are defined. We characterize partial Hilbert spaces which are isomorphic to a closed subspace of an amplitude space. We also characterize amplitude spaces on the unit sphere of a Hilbert space and introduce a tensor product on amplitude spaces.

RÉSUMÉ. — On introduit des espaces de Hilbert partiels comme généralisation des espaces de Hilbert. Motivé par la Mécanique Quantique, on définit une classe particulière d'espaces de Hilbert partiels, appelés espaces d'amplitudes. On caractérise les espaces de Hilbert partiels qui sont isomorphes à un sous espace fermé d'un espace d'amplitudes. On caractérise aussi les espaces d'amplitudes sur la sphère unité d'un espace de Hilbert, et on introduit un produit tensoriel sur les espaces d'amplitudes.

1. INTRODUCTION

In this work we first introduce a generalization of Hilbert space which we call a partial Hilbert space. Roughly speaking, a partial Hilbert space

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is a set composed of a collection of Hilbert spaces which satisfy a compatibility condition on their intersections. We define a strong dual for a partial Hilbert space and show that this is again a partial Hilbert space.

We next consider a special class of partial Hilbert spaces called amplitude spaces. An amplitude space consists of the set of amplitude functions on a cover space (a set together with a covering by subsets). The definition of an amplitude function is motivated by quantum mechanics. For quantum mechanical systems, the probability of an event is frequently computed by summing (or integrating) an amplitude function over the outcomes composing the event and taking the modulus squared. This procedure is responsible for the interference effects characteristic of quantum mechanics [1] [6] [7]. We characterize partial Hilbert spaces which are isomorphic to a closed subspace of an amplitude space. Moreover, we show that the strong dual of any partial Hilbert space is isomorphic to a closed subspace of an amplitude space.

We then consider cover spaces on the unit sphere of a Hilbert space which we call Hilbertian cover spaces. Hilbertian cover spaces are characterized up to an isomorphism. Finally, we introduce tensor products on amplitude spaces.

2. PARTIAL HILBERT SPACES

Let H be a set and let: be a binary, reflexive, symmetric relation on H . For $A \subseteq H$ we define

$$A^\circ = \{f \in H : f : g \text{ for all } g \in A\}.$$

We call $A \subseteq H$ a :set if $A \subseteq A^\circ$. Thus, A is a :set if and only if $f : g$ for every $f, g \in A$. It is clear that singleton sets are :sets and hence, every $f \in H$ is contained in a :set. Moreover, by Zorn's lemma, every :set is contained in a maximal :set. Denote the collection of :sets in H by $C(H)$ and the collection of maximal :sets by $M(H)$. For a :set A we define

$$\hat{A} = \bigcap \{B \in M(H) : A \subseteq B\}.$$

LEMMA 1. — a) For any $A \subseteq H$ we have $A \subseteq A^{\circ\circ}$ and $A^{\circ\circ\circ} = A^\circ$. b) If $A \subseteq B \subseteq H$, then $B^\circ \subseteq A^\circ$. c) $A \in C(H)$ if and only if $A^{\circ\circ} \in C(H)$. d) If $A \in C(H)$, then $A \subseteq A^{\circ\circ} \subseteq A^\circ$. e) If $A \in C(H)$, then $A^\circ \in C(H)$ if and only if $A^\circ = A^{\circ\circ}$. f) $A = A^\circ$ if and only if $A \in M(H)$. g) If $A \in M(H)$, then $A = A^{\circ\circ} = A^\circ$. h) If $A \in C(H)$, then $\hat{A} = A^{\circ\circ}$.

Proof. — The proofs of a)-g) are straightforward. To prove h), let $B \in M(H)$ with $A \subseteq B$. Then by b) and g) we have, $A^{\circ\circ} \subseteq B^{\circ\circ} = B$. Hence, $A^{\circ\circ} \subseteq \hat{A}$.

Conversely, suppose $f \in \hat{A}$ and $g \in A^{\circ\circ}$. Then there exists a $B \in M(H)$ such that $A \cup \{g\} \subseteq B$. Since $f \in B$, we have $f : g$. Hence, $f \in A^{\circ\circ}$ so $\hat{A} \subseteq A^{\circ\circ}$. \square

We call (H, \cdot) a *partial Hilbert space* if it satisfies (1) every $A \in M(H)$ is a (complex) Hilbert space; (2) if $A, B \in M(H)$ and $f, g \in A \cap B$, $\alpha, \beta \in \mathbb{C}$, then $(\alpha f + \beta g)_A = (\alpha f + \beta g)_B$ and $\langle f, g \rangle_A = \langle f, g \rangle_B$. Of course, in (2) we mean by $(\alpha f + \beta g)_A$ the element $\alpha f + \beta g$ considered as a member of A and $\langle f, g \rangle_A$ is the inner product in A . If (H, \cdot) is a partial Hilbert space, we say that $A \subseteq H$ is a *closed subspace* of H if $(A, \cdot|_A)$ is a partial Hilbert space with the linear structures and inner products inherited from H . It is easy to show that $A \subseteq H$ is a closed subspace of H if every $B \in M(A)$ is a Hilbert space with linear structure and inner product inherited from H .

Let (H, \cdot) be a partial Hilbert space and let $A \in C(H)$. Then there exists a $B \in M(H)$ such that $A \subseteq B$. We define $[A]$ to be the closed linear span of A in the Hilbert space B . It is clear that $[A]$ is independent of the $B \in M(H)$ used to define it. The following lemma summarizes basic relationships for A and $[A]$.

LEMMA 2. — Let (H, \cdot) be a partial Hilbert space and let $A \in C(H)$. Then $[A]$, $A^{\circ\circ}$, \hat{A} are Hilbert spaces and $A \subseteq [A] \subseteq A^{\circ\circ} = \hat{A} \subseteq A^{\circ} = [A]^{\circ}$.

Proof. — All but the last equality are immediate consequences of Lemma 1. To show that $A^{\circ} \subseteq [A]^{\circ}$ let $f \in A^{\circ}$. Then there exists a $B \in M(H)$ such that $\{f\} \cup A \subseteq B$. Moreover, since $A \subseteq B$, $[A] \subseteq B$. It follows that $f \in [A]^{\circ}$. \square

If (H, \cdot) is a partial Hilbert space and $f \in H$, we write $\|f\| = \|f\|_A$ where $f \in A \in M(H)$ and A is arbitrary. If $f, g \in H$, $f : g$, and $\alpha, \beta \in \mathbb{C}$, we write $\alpha f + \beta g = (\alpha f + \beta g)_A$ and $\langle f, g \rangle = \langle f, g \rangle_A$ where $f, g \in A \in M(H)$ and A is arbitrary. Two partial Hilbert spaces H, K are *isomorphic* if there exists a bijection $\phi : H \rightarrow K$ satisfying (1) $f : g$ in H if and only if $\phi(f) : \phi(g)$ in K ; (2) if $f : g$ in H , $\alpha, \beta \in \mathbb{C}$, then $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$ and $\langle \phi(f), \phi(g) \rangle_K = \langle f, g \rangle_H$.

If H_0 is a Hilbert space and we define: to be the relation $H_0 \times H_0$, then clearly (H_0, \cdot) is a partial Hilbert space. Moreover, the above definitions of closed subspace and isomorphic reduce to the usual definitions on a Hilbert space. For a less trivial example, let $S_\alpha, \alpha \in \Delta$, be a collection of closed subspaces of H_0 satisfying $S_\alpha \cap S_\beta = S$, for $\alpha \neq \beta$, where $S \neq S_\alpha$ for any $\alpha \in \Delta$. Let $H = \bigcup_{\alpha \in \Delta} S_\alpha$ and for $f, g \in H$ define $f : g$ if $f, g \in S_\alpha$ for some $\alpha \in \Delta$. We then have the following result.

LEMMA 3. — With the above definitions, (H, \cdot) is a partial Hilbert space and $M(H) = \{S_\alpha : \alpha \in \Delta\}$.

Proof. — Clearly, $S_\alpha \in C(H)$ for every $\alpha \in \Delta$. To show that $S_\alpha \in M(H)$, suppose $g \in H \setminus S_\alpha$ and $g \in S_\alpha^{\circ}$. Let $f \in S_\alpha$. Since $g : f$ we must have $f, g \in S_\beta$

for some $\beta \in \Delta$. Since $g \notin S_\alpha$, $\beta \neq \alpha$. We conclude that $S_\alpha \subseteq \cup \{S_\beta : \beta \neq \alpha\}$. But then $S_\alpha \subseteq S_\alpha \cap (\cup_{\beta \neq \alpha} S_\beta) = \cup_{\beta \neq \alpha} S_\alpha \cap S_\beta = S$ so $S_\alpha = S$ which is a contradiction. Conversely, let $A \in M(H)$. Since $S \subseteq H^*$ we must have $S \subseteq A$. Since S is a proper subset of S_α for $\alpha \in \Delta$, $S \notin M(H)$. Hence, there exists an $f \in A \setminus S$. Assume $f \in S_\alpha$. Now suppose that $g \in A \setminus S$. Since $f \perp g$ we must have $f, g \in S_\gamma$ for some $\gamma \in \Delta$. If $\gamma \neq \alpha$, then $S_\alpha \cap S_\gamma = S$ which contradicts the fact that $f \in S_\alpha \cap S_\gamma$. Hence, $g \in S_\alpha$. It follows that $A \subseteq S_\alpha$ and by maximality $A = S_\alpha$. The proof is now easily completed. \square

The above example does not generalize to an arbitrary collection of closed subspaces S_α even if the S_α are required to be maximal and proper. For example, let $H_0 = \mathbb{C}^3$ and let f_1, f_2, f_3 be the standard orthonormal basis for H_0 . Define the three subspaces $S^1 = \text{span} \{f_2, f_3\}$, $S_2 = \text{span} \{f_1, f_3\}$, $S_3 = \text{span} \{f_1, f_2\}$ and let $H = S_1 \cup S_2 \cup S_3$. For $f, g \in H$, define $f \perp g$ if $f, g \in S_i$ for some $i = 1, 2, 3$. Then (H, \perp) is not a partial Hilbert space under the induced linear structure and inner product. Indeed, it is easy to show that $S = \text{span} \{f_1\} \cup \text{span} \{f_2\} \cup \text{span} \{f_3\} \in M(H)$ and yet S is not a linear space.

As another example, let X be a nonempty set and let $(X_\alpha, \Sigma_\alpha, \mu_\alpha)$ be a collection of measure spaces with $X_\alpha \subseteq X$. Let H be the set of functions $f : X \rightarrow \mathbb{C}$ such that $f \upharpoonright X_\alpha$ is measurable for all α and

$$\int_{X_\alpha} |f|^2 d\mu_\alpha = \int_{X_\beta} |f|^2 d\mu_\beta < \infty$$

for all α and β . For $f, g \in H$, define $f \perp g$ if

$$\int_{X_\alpha} f \bar{g} d\mu_\alpha = \int_{X_\beta} f \bar{g} d\mu_\beta$$

for all α and β . It can be shown using methods which are very similar to those developed in the next section that (H, \perp) is a partial Hilbert space.

We denote the topological dual of a Hilbert space H_0 by H_0^* . The dual H^* of a partial Hilbert space H is the set of functionals $F : H \rightarrow \mathbb{C}$ such that $F \upharpoonright A \in A^*$ for every $A \in M(H)$. By the Riesz lemma, if $F \in H^*$ then for any $A \in M(H)$ there exists a unique $f_F^A \in A$ such that $F(g) = \langle g, f_F^A \rangle$ for all $g \in A$. Moreover, if $B, C \in M(H)$ and $f_F^B, f_F^C \in B \cap C$, then $f_F^B = f_F^C$. We define the strong dual H^s of H to be

$$H^s = \{F \in H^* : \|F \upharpoonright A\| = \|F \upharpoonright B\| \text{ for all } A, B \in M(H)\}$$

It follows that an $F \in H^*$ is in H^s if and only if $\|f_F^A\| = \|f_F^B\|$ for all $A, B \in M(H)$. For $F, G \in H^s$ we write $F \perp G$ if $\langle f_F^A, f_G^A \rangle = \langle f_F^B, f_G^B \rangle$ for all $A, B \in M(H)$.

THEOREM 4. — If (H, \perp) is a partial Hilbert space, then (H^s, \perp) is a partial Hilbert space under the usual linear structure.

Proof. — First, it is clear that if $F, G \in H^*$, $\alpha, \beta \in \mathbb{C}$, then $\alpha F + \beta G \in H^*$ and for any $A \in M(H)$, $f_{\alpha F + \beta G}^A = \bar{\alpha} f_F^A + \bar{\beta} f_G^A$. It is also clear that if $F \in H^s$, $\alpha \in \mathbb{C}$, then $\alpha F \in H^s$. Now suppose that $F, G \in H^s$ and $F : G$. If $A, B \in M(H)$, we have

$$\begin{aligned} \|f_{F+G}^A\|^2 &= \langle f_F^A + f_G^A, f_F^A + f_G^A \rangle = \|f_F^A\|^2 + \|f_G^A\|^2 + 2\operatorname{Re} \langle f_F^A, f_G^A \rangle \\ &= \|f_F^B\|^2 + \|f_G^B\|^2 + 2\operatorname{Re} \langle f_F^B, f_G^B \rangle = \|f_{F+G}^B\|^2. \end{aligned}$$

Hence, $F + G \in H^s$. Moreover, if $\alpha, \beta \in \mathbb{C}$, then clearly $\alpha F : \beta G$ so $\alpha F + \beta G \in H^s$. For $F \in H^s$ we write $\|F\| = \|f_F^A\|$ and if $F : G$ we write $\langle F, G \rangle = \langle f_F^A, f_G^A \rangle$ where $A \in M(H)$ is arbitrary. Now let $\tilde{A} \in \tilde{M}(H^s)$. If $F, G \in \tilde{A}$, $\alpha, \beta \in \mathbb{C}$, then it is easy to show that $\alpha F + \beta G \in \tilde{A}$ so $\alpha F + \beta G \in \tilde{A}$. Also, it is clear that $\langle \cdot, \cdot \rangle$ is an inner product on \tilde{A} so \tilde{A} is an inner product space. To show \tilde{A} is complete suppose $F_n \in \tilde{A}$ is a Cauchy sequence. If $f \in H$, then $f \in A$ for some $A \in M(H)$. We then have

$$|F_n(f) - F_m(f)| = |(F_n - F_m)(f)| \leq \|F_n - F_m\| \|f\| \rightarrow 0$$

as $n, m \rightarrow \infty$. Hence, $F_n(f)$ converges. Define $F : H \rightarrow \mathbb{C}$ by $F(f) = \lim F_n(f)$. It is straightforward to show that $F \in H^s$, and for any $A \in M(H)$, $F_n|A \rightarrow F|A$ in the norm topology. It follows from the Riesz lemma that $f_{F_n}^A \rightarrow f_F^A$ for any $A \in M(H)$. Now let $G \in \tilde{A}$ and let $A, B \in M(H)$. Then

$$\langle f_F^A, f_G^A \rangle = \lim \langle f_{F_n}^A, f_G^A \rangle = \lim \langle f_{F_n}^B, f_G^B \rangle = \langle f_F^B, f_G^B \rangle.$$

Hence, $F : G$ and since $\tilde{A} \in M(H^s)$, $F \in \tilde{A}$. Thus, \tilde{A} is a Hilbert space. Condition 2) in the definition of a partial Hilbert space is clearly satisfied for H^s so the proof is complete. \square

3. AMPLITUDE SPACES

A *cover space* is a pair $X = (X, O)$ where X is a nonempty set and O is a collection of nonempty subsets such that $X = \cup O$. Cover spaces are also called pre-manuals and have been important for studies in operational statistics and the foundations of quantum mechanics [2] [3] [4] [8] [9]. A function $\mu : X \rightarrow [0, \infty) \subseteq \mathbb{R}$ is called a *positive measure* if there exists a constant $\hat{\mu} \in \mathbb{R}$ such that $\sum_{x \in E} \mu(x) = \hat{\mu}$ for all $E \in O$. If μ is a positive measure with $\hat{\mu} = 1$, we call μ a *state*. States correspond to probability measures in operational statistics. We denote the set of positive measures by $M_+(X)$ and the set of states by $\Omega(X)$. A function $f : X \rightarrow \mathbb{C}$ is called an *amplitude function* if the function $\mu_f(x) = |f(x)|^2$ is in $M_+(X)$. Amplitude functions play an important role in quantum mechanics (see Section 1 for references and more details). The *amplitude space* on X is the set $H = H(X)$ of all amplitude functions on X . For $f \in H$ we use the notation $\|f\| = \hat{\mu}_f^{1/2}$, and we define $\operatorname{supp}(f) = X \setminus \ker f$.

Let $G = \{\chi : X \rightarrow \mathbb{C} : |\chi(x)| = 1 \text{ for all } x \in X\}$. Then G is a group under pointwise multiplication and its elements correspond to « phase » transformations. If $f \in H$, then clearly $\chi f \in H$ for any $\chi \in G$. We call $f \in H$ *pure* if $\text{supp}(g) \subseteq \text{supp}(f)$ for $g \in H$ implies $g = \alpha \chi f$ for some $\alpha \in \mathbb{C}$, $\chi \in G$. The next result gives the relationship between $H(X)$ and $\Omega(X)$.

LEMMA 5. — *a*) If $r \geq 0$, $\chi \in G$, $\mu \in \Omega(X)$, then $f = r\chi\mu^{1/2} \in H(X)$. Conversely, if $f \in H(X)$, then f admits a representation $f = r\chi\mu^{1/2}$, $r \geq 0$, $\chi \in G$, $\mu \in \Omega(X)$. Moreover, if $f \neq 0$, then $r = \|f\|$, $\mu = \mu_f/\|f\|^2$, $\chi(x) = f(x)/|f(x)|$ for all $x \in \text{supp}(f)$, and χ is unique on $\text{supp}(f)$. *b*) If $0 \neq f \in H$ is pure, then $\mu_f/\|f\|^2$ is an extreme point in the convex set $\Omega(X)$.

Proof. — *a*) The first statement clearly holds. Conversely, if $f \in H(X)$, define $\chi(x) = 1$ for $x \in \ker f$ and $\chi(x) = f(x)/|f(x)|$ for $x \in \text{supp}(f)$. Then $\chi \in G$ and if $f \neq 0$ we have

$$f(x) = \|f\| \chi(x) |f(x)| / \|f\|$$

where $|f(x)|^2/\|f\|^2 = \mu \in \Omega(X)$. For uniqueness, suppose $0 \neq f \in H(X)$ has the form $f = r\chi\mu^{1/2}$, $r \geq 0$, $\chi \in G$, $\mu \in \Omega(X)$. For $E \in \mathcal{O}$ we have

$$\|f\|^2 = \sum_{x \in E} |f(x)|^2 = r^2 \hat{\mu} = r^2.$$

Hence, $r = \|f\|$, and $\mu(x) = |f(x)|^2/\|f\|^2 = \mu_f/\|f\|^2$. The result now easily follows. *b*) It is clear that $\Omega(X)$ is convex. Suppose $0 \neq f \in H$ is pure and let $\mu = \mu_f/\|f\|^2$. To show that μ is extremal, assume

$$\mu = \lambda\mu_1 + (1 - \lambda)\mu_2, \quad 0 < \lambda < 1, \quad \mu_1, \mu_2 \in \Omega(X).$$

Then $\text{supp}(\mu_1) \subseteq \text{supp}(\mu)$. Define $g(x) = \mu_1(x)^{1/2}$ for all $x \in X$. Then $g \in H$ and $\text{supp}(g) \subseteq \text{supp}(f)$. Since f is pure, $g = \alpha \chi f$ for some $\alpha \in \mathbb{C}$, $\chi \in G$. For $x \in X$, we obtain

$$\mu_1(x) = |g(x)|^2 = |\alpha|^2 |f(x)|^2 = |\alpha|^2 \mu_f(x) = |\alpha|^2 \|f\|^2 \mu(x).$$

Since $\mu_1, \mu \in \Omega(X)$, we have $\mu_1 = \mu$. Hence, $\mu_1 = \mu_2$ so μ is extremal. \square

If $f \in H(X)$, then clearly $\alpha f \in H(X)$ for all $\alpha \in \mathbb{C}$. However, if $f, g \in H(X)$ then $f + g$ need not be in $H(X)$. For example, let $X' = \{x_1, x_2, x_3, x_4\}$, $E_1 = \{x_1, x_2\}$, $E_2 = \{x_2, x_3, x_4\}$, $\mathcal{O} = \{E_1, E_2\}$. Then (X', \mathcal{O}) is a cover space. Define the functions $f, g : X' \rightarrow \mathbb{C}$ as follows

$$\begin{aligned} f(x_1) = f(x_2) = 1, & & f(x_3) = f(x_4) = 1/\sqrt{2} \\ g(x_1) = g(x_3) = 1, & & g(x_2) = \sqrt{2}, & & g(x_4) = 0. \end{aligned}$$

It is easy to check that $f, g \in H(X')$ but $f + g \notin H(X')$. We now define a relation on $H(X)$ which is necessary and sufficient for closure under linear combinations. For, $f, g \in H(X)$ we write $f : g$ if

$$\sum_{x \in E} f(x)\bar{g}(x) = \sum_{x \in F} f(x)\bar{g}(x)$$

for all $E, F \in \mathcal{O}$. It is clear that $\langle \cdot, \cdot \rangle$ is reflexive and symmetric. However, $\langle \cdot, \cdot \rangle$ need not be transitive. For instance, in the above example $f : 0, 0 : g$ but f does not $: g$. If $f : g$, we write $\langle f, g \rangle = \sum_{x \in E} f(x)\bar{g}(x)$, $E \in \mathcal{O}$. Notice that $\langle f, f \rangle = \|f\|^2$.

LEMMA 6. — If $f, g \in H(X)$, then $f : g$ if and only if $f + g, f + ig \in H(X)$.

Proof. — For any $E \in \mathcal{O}$ we have

$$\sum_{x \in E} [f(x) + g(x)][\bar{f}(x) + \bar{g}(x)] = \|f\|^2 + \|g\|^2 + 2\operatorname{Re} \sum_{x \in E} f(x)\bar{g}(x) \quad (1)$$

Hence, if $f : g$ then $f + g \in H(X)$. Moreover, $f : ig$ so $f + ig \in H(X)$. Conversely, if $f + g \in H(X)$, then from (1) we have

$$\operatorname{Re} \sum_{x \in E} f(x)\bar{g}(x) = \operatorname{Re} \sum_{x \in F} f(x)\bar{g}(x)$$

for any $E, F \in \mathcal{O}$. If in addition, $f + ig \in H(X)$, then

$$\sum_{x \in E} [f(x) + ig(x)][\bar{f}(x) - \bar{ig}(x)] = \|f\|^2 + \|g\|^2 + 2\operatorname{Im} \sum_{x \in E} f(x)\bar{g}(x).$$

Hence

$$\operatorname{Im} \sum_{x \in E} f(x)\bar{g}(x) = \operatorname{Im} \sum_{x \in F} f(x)\bar{g}(x)$$

for any $E, F \in \mathcal{O}$. It follows that $f : g$. \square

For $E \in \mathcal{O}$, we define the Hilbert space

$$l^2(E) = \{f : E \rightarrow \mathbb{C} : \sum_{x \in E} |f(x)|^2 < \infty\}$$

where the inner product is given by $\langle f, g \rangle_E = \sum_{x \in E} f(x)\bar{g}(x)$. Notice that $f \in H(X)$ if and only if $f|E \in l^2(E)$ for all $E \in \mathcal{O}$ and $\|f|E\|_E = \|f|F\|_F$ for all $E, F \in \mathcal{O}$.

THEOREM 7. — If (X, \mathcal{O}) is a cover space, then $(H(X), \langle \cdot, \cdot \rangle)$ is a partial Hilbert space.

Proof. — Suppose $A \in M[H(X)]$. If $f \in A, \alpha \in \mathbb{C}$, then $\alpha f : g$ for all $g \in A$. Since A is maximal, $\alpha f \in A$. If $f, g \in A$, then by Lemma 6, $f + g \in H(X)$. Also, it is clear that $(f + g) : h$ for all $h \in A$. Again by maximality $f + g \in A$. Hence, A is a linear space. It is straightforward to show that $\langle \cdot, \cdot \rangle$ is an inner product on A . For completeness, let $f_n \in A$ be a Cauchy sequence. Let $x_0 \in X$ and suppose $x_0 \in E \in \mathcal{O}$. Then

$$|f_n(x_0) - f_m(x_0)|^2 \leq \sum_{x \in E} |f_n(x) - f_m(x)|^2 = \|f_n - f_m\|.$$

Hence, $f_n(x_0)$ is Cauchy in \mathbb{C} so $f_n(x_0)$ converges. Define $f : X \rightarrow \mathbb{C}$ by $f(x) = \lim f_n(x)$. A standard argument shows that $f|E \in l^2(E)$ for all $E \in \mathcal{O}$ and that $\|f|E\|_E = \lim \|f_n|E\|_E$. But since $\|f_n|E\|_E = \|f_n|F\|_F$ for all $E, F \in \mathcal{O}$ we have $\|f|E\|_E = \|f|F\|_F$ for all $E, F \in \mathcal{O}$. Hence $f \in H(X)$. If $g \in A$ and $E, F \in \mathcal{O}$, then a standard argument gives

$$\begin{aligned} \langle f|E, g|E \rangle_E &= \lim \langle f_n|E, g|E \rangle_E = \lim \langle f_n|F, g|F \rangle_F \\ &= \langle f|F, g|F \rangle_F \end{aligned}$$

Hence, $f : g$ and by maximality, $f \in A$. Condition (2) for a partial Hilbert space clearly holds so the proof is complete. \square

A set $A \subseteq H(X)$ is *unital* if for any $x \in X$ there exists a $0 \neq f \in A$ such that $|f(x)| = \|f\|$.

THEOREM 8. — If $A \in C[H(X)]$ is unital, then $[A]$ is the unique maximal :set containing A and

$$A \subseteq [A] = A^{\ddot{}} = \hat{A} = A^{\dot{}} = [A]^{\dot{}}.$$

Proof. — Let $B \in M[H(X)]$ with $A \subseteq B$. Then clearly $[A] \subseteq B$. Let $f \in B$ and suppose $f \perp A$. If $x \in X$, then $x \in E$ for some $E \in O$. Since A is unital, there exists a $g \in A$ such that $g(x) \neq 0$ and $g(y) = 0$ for all $y \in E \setminus \{x\}$. Then

$$0 = \sum_{z \in E} f(z)\bar{g}(z) = f(x)\bar{g}(x).$$

Hence, $f(x) = 0$. It follows that $f = 0$ and hence, $B = [A]$. The result now follows from Lemmas 1 and 2. \square

COROLLARY 9. — Let $A \in C[H(X)]$ be unital. a) If A is a linear space, then its closure \bar{A} is the unique maximal :set containing A . b) If A is a Hilbert space, then $A \in M[H(X)]$.

A set of functions $Y \subseteq \mathbb{C}^S$ is *separating* for a set S if $x \neq y \in S$ implies that there exists an $f \in Y$ such that $f(x) \neq f(y)$. We denote the power set of a set S by $P(S)$. The next result characterizes closed subspaces of $H(X)$ up to an isomorphism.

THEOREM 10. — A partial Hilbert space H is isomorphic to a closed subspace of $H(X)$ for some cover space (X, O) if and only if there exists a $Y \subseteq P(H^*)$ satisfying (1) $\cup Y$ is separating for H ; (2) if $f : g$ in H , then $\sum_{F \in E} F(f)\bar{F}(g) = \langle f, g \rangle$ for every $E \in Y$; (3) if

$$\sum_{F \in E_1} F(f)\bar{F}(g) = \sum_{F \in E_2} F(f)\bar{F}(g)$$

for every $E_1, E_2 \in Y$, then $f : g$.

Proof. — Suppose $\phi : H \rightarrow K$ is an isomorphism, where K is a closed subspace of $H(X)$ for some cover space (X, O) . For $x \in X$, let $F_x : K \rightarrow \mathbb{C}$ be defined by $F_x(f) = f(x)$. We first show that $F_x \in K^*$. Let $A \in M(K)$. It is clear that $F_x | A$ is a linear functional. To show that $F_x | A$ is continuous, suppose $f_n \in A$ and $f_n \rightarrow f \in A$ in the norm topology of A . If $x \in E \in O$, we have

$$\begin{aligned} |F_x(f_n) - F_x(f)|^2 &= |f_n(x) - f(x)|^2 \\ &\leq \sum_{y \in E} |f_n(y) - f(y)|^2 = \|f_n - f\|^2 \rightarrow 0. \end{aligned}$$

Hence, $F_x | A \in A^*$. For $F \in K^*$, it is easy to show that $F \circ \phi \in H^*$. Let

$$Y = \{ \{F_x \circ \phi : x \in E\} : E \in O \} \subseteq P(H^*).$$

To show that $\cup Y$ is separating for H , let $f \neq g \in H$. Then $\phi(f) \neq \phi(g) \in K$ so there exists an $x \in X$ such that $\phi(f)(x) \neq \phi(g)(x)$. Now $F_x \circ \phi \in UY$ and

$$(F_x \circ \phi)f = \phi(f)(x) \neq \phi(g)(x) = (F_x \circ \phi)g.$$

To show that (2) holds, let $f : g$ in H , $E \in O$, and

$$E' = \{F_x \circ \phi : x \in E\} \in Y.$$

Then $\phi(f) : \phi(g)$ in K and

$$\sum_{F \in E'} F(f)F(g) = \sum_{x \in E} \phi(f)(x)\phi(g)(x) = \langle \phi(f), \phi(g) \rangle_K = \langle f, g \rangle_H.$$

Finally, suppose the hypothesis in (3) holds for $f, g \in H$. For $E_1, E_2 \in O$ we have $E'_1 = \{F_x \circ \phi : x \in E_1\} \in Y$, $E'_2 = \{F_x \circ \phi : x \in E_2\} \in Y$ and

$$\begin{aligned} \sum_{x \in E_1} \phi(f)(x)\bar{\phi}(g)(x) &= \sum_{F \in E'_1} F(f)\bar{F}(g) = \sum_{F \in E'_2} F(f)\bar{F}(g) \\ &= \sum_{x \in E_2} \phi(f)(x)\bar{\phi}(g)(x). \end{aligned}$$

Hence, $\phi(f) : \phi(g)$ in K and since ϕ is an isomorphism, we have $f : g$.

Conversely, suppose there exists a $Y \subseteq P(H^*)$ satisfying (1), (2) and (3). If $X = UY$ and $O = Y$, then (X, O) is a cover space. For $f \in H$, define $\phi(f) : X \rightarrow \mathbb{C}$ by $\phi(f)(x) = x(f)$. To show that $\phi(f) \in H(X)$, since $f : f$, applying Condition (2) we have for every $E \in Y$

$$\sum_{x \in E} |\phi(f)(x)|^2 = \sum_{x \in E} |x(f)|^2 = \|f\|^2.$$

It follows that ϕ is a map from H into $H(X)$. To show that ϕ is injective, suppose $f \neq g \in H$. Since X is separating for H , there exists an $x \in X$ such that $x(f) \neq x(g)$. Hence, $\phi(f)(x) \neq \phi(g)(x)$ so $\phi(f) \neq \phi(g)$. Suppose $f : g$ in H . If $E_1, E_2 \in O$, then applying Condition (2) gives

$$\sum_{x \in E_1} \phi(f)(x)\bar{\phi}(g)(x) = \sum_{x \in E_1} x(f)\bar{x}(g) = \langle f, g \rangle = \sum_{x \in E_2} \phi(f)(x)\bar{\phi}(g)(x).$$

Hence, $\phi(f) : \phi(g)$ and $\langle \phi(f), \phi(g) \rangle_{H(X)} = \langle f, g \rangle_H$. Moreover, it is clear that $\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$ for any $\alpha, \beta \in \mathbb{C}$. Also, it easily follows from (3) that if $\phi(f) : \phi(g)$, then $f : g$. We now show that $\phi(H)$ is a closed subspace of $H(X)$. Let $A \in M[\phi(H)]$. It is clear that $\phi^{-1}(A) \in C(H)$. Suppose $f \in H$ and $f \in \phi^{-1}(A)$. Then $\phi(f) \in A$ and since A is maximal in $\phi(H)$, $\phi(f) \in A$. Hence, $f \in \phi^{-1}(A)$ so $\phi^{-1}(A) \in M(H)$ and $\phi^{-1}(A)$ is a Hilbert space. Since $\|\phi(f)\| = \|f\|$ for all $f \in H$, it follows that A is a Hilbert space in $\phi(H)$. We conclude that $\phi(H)$ is a closed subspace of $H(X)$ and that H and $\phi(H)$ are isomorphic. \square

If (X, O) is a cover space and $x, y \in E \in O$ with $x \neq y$ we write $x \perp y$. If $A \subseteq E \in O$ we call A an *event* and denote the set of events by \tilde{O} . For $B \subseteq X$ we use the notation $B^\perp = \{x \in X : x \perp y \text{ for all } y \in B\}$. We call X *coherent* if $A, B \in \tilde{O}$ and $A \subseteq B^\perp$ implies $A \cup B \in \tilde{O}$. We call X *irredundant* if every $E \in O$ is maximal in O . Coherent, irredundant, cover spaces have been studied extensively [2] [3] [4] [8] [9].

THEOREM 11. — If H is a partial Hilbert space, then H^s is isomorphic to a closed subspace of $H(X)$ for some coherent, irredundant cover space (X, O) .

Proof. — For each $A \in M(H)$, let E_A be the collection of all orthonormal bases for the Hilbert space A . Let (X, O) be the cover space in which $O = \{ E : E \in E_A, A \in M(H) \}$ and $X = \cup O$. For $F \in H^s$, define $\phi(F) = F|X$. To show that $\phi(F) \in H(X)$, let $E \in O$. Then $E \in E_A$ for some $A \in M(X)$ and we have

$$\sum_{x \in E} |\phi(F)(x)|^2 = \sum_{x \in E} |F(x)|^2 = \sum_{x \in E} |\langle x, f_F^A \rangle|^2 = \|f_F^A\|^2 = \|F\|^2$$

Hence, $\phi : H^s \rightarrow H(X)$. To show that ϕ is injective, suppose $\phi(F) = \phi(G)$ so $F|X = G|X$. For $f \in H$, we have $f \in A \in M(H)$ for some A . If $f = 0$, then clearly $F(f) = G(f)$. Otherwise, $f/\|f\| \in E$ for some $E \in E_A$. Hence,

$$F(f) = \|f\| F(f/\|f\|) = \|f\| G(f/\|f\|) = G(f).$$

Thus, $F = G$ and ϕ is injective. For $F, G \in H^s, A \in M(H)$ we have for any $E \in E_A$

$$\begin{aligned} \langle f_F^A, f_G^A \rangle &= \sum_{x \in E} \langle f_F^A, x \rangle \langle x, f_G^A \rangle \\ &= \sum_{x \in E} G(x) \overline{F(x)} = \sum_{x \in E} \phi(G)(x) \overline{\phi(F)(x)}. \end{aligned}$$

It follows that $F : G$ in H^s if and only if $\phi(F) : \phi(G)$ in $H(X)$. Moreover, if $F : G$ in H^s we have $\langle G, F \rangle = \langle \phi(G), \phi(F) \rangle$. It is also clear that if $F : G$ in H^s and $\alpha, \beta \in \mathbb{C}$, then $\phi(\alpha F + \beta G) = \alpha \phi(F) + \beta \phi(G)$. It is straightforward to show that $\phi(H)$ is a closed subspace of $H(X)$. To show that X is irredundant, suppose $E_1, E_2 \in O$ with $E_1 \subseteq E_2$. If $E_1, E_2 \in E_A$ for some $A \in M(H)$, then clearly $E_1 = E_2$. Otherwise, $E_1 \in E_A, E_2 \in E_B$ for $A, B \in M(H)$. Applying Lemmas 1 and 2, we have

$$B = B^\cdot = [E_2]^\cdot = E_2^\cdot \subseteq E_1^\cdot = [E_1]^\cdot = A^\cdot = A.$$

Hence, $A = B$ and again $E_1 = E_2$. To show that X is coherent, let $A, B \in \tilde{O}$ with $A \subseteq B^\perp$. It follows that $f : g$ for every $f \in A, g \in B$. Hence, there exists a $C \in M(H)$ such that $A \cup B \subseteq C$. We can extend $A \cup B$ to an orthonormal basis for C . Thus, $A \cup B \subseteq E \in E_C$ so $A \cup B \in \tilde{O}$. \square

4. HILBERTIAN COVER SPACES

Let K be a complex Hilbert space with unit sphere $S(K)$ and let $O(K)$ be the collection of all maximal orthogonal sets in $S(K)$. We call $S(K) = (S(K), O(K))$ a *Hilbertian* cover space. For $v \in K$, define the function $f_v : S(K) \rightarrow \mathbb{C}$ by $f_v(x) = \langle v, x \rangle$. It is easy to show that $f_v \in H(S(K))$ and $\|f_v\| = \|v\|$. Moreover, $f_v : f_u$ for any $v, u \in K$ and $\langle f_v, f_u \rangle = \langle v, u \rangle$.

We denote the set of pure elements in $H(S(K))$ by $H_p(S(K))$. The next result characterizes $H(S(K))$ and $H_p(S(K))$ when $\dim K \geq 3$.

THEOREM 12. — Let $S(K)$ be a Hilbertian cover space with $\dim K \geq 3$. Then $f \in H(S(K))$ if and only if there exists a unique positive trace class operator T_f on K such that $f(x) = \chi(x) \langle T_f x, x \rangle^{1/2}$ where $\chi \in G$ is unique on $\text{supp}(f)$. Also, $f \in H_p(S(K))$ if and only if $f = \chi f_v$ where $v \in K$ is unique up to a multiple of modulus one and $\chi \in G$ is unique on $\text{supp}(f)$.

Proof. — If $f \in H(S(K))$, then it follows from the proof of Gleason's theorem [6] that there exists a unique, positive, trace class operator T_f such that $|f(x)|^2 = \langle T_f x, x \rangle$ for all $x \in S(K)$. For $x \in \text{supp}(f)$, let

$$\chi(x) = f(x)/|f(x)|$$

and let $\chi(x) = 1$ on $\ker f$. Then $\chi \in G$ and

$$f(x) = \chi(x) |f(x)| = \chi(x) \langle T_f x, x \rangle^{1/2}$$

for all $x \in S(K)$. For uniqueness, suppose $f(x) = \chi_1(x) \langle T x, x \rangle^{1/2}$ for a positive, trace class operator T and a $\chi_1 \in G$. Then

$$\langle T x, x \rangle = |f(x)|^2 = \langle T_f x, x \rangle$$

for all $x \in S(K)$. Hence, $T = T_f$ and $\chi_1 = \chi$ on $\text{supp}(f)$.

Conversely, suppose $f : S(K) \rightarrow \mathbb{C}$ has the form $f(x) = \chi(x) \langle T x, x \rangle^{1/2}$ for a positive, trace class operator T and a $\chi \in G$. Then for any $E \in O(K)$ we have

$$\sum_{x \in E} |f(x)|^2 = \sum_{x \in E} \langle T x, x \rangle = \text{tr}(T).$$

Hence, $f \in H(S(K))$.

Now let $f \in H_p(S(K))$. Then from the above, $f(x) = \chi_0(x) \langle T x, x \rangle^{1/2}$ for a positive, trace class operator T and a $\chi_0 \in G$. If $T = 0$, then $f(x) = \langle 0, x \rangle$ for all $x \in S(K)$. Otherwise, by Lemma 5(b)

$$\mu_f / \|f\|^2 = \langle T x, x \rangle / \text{tr}(T)$$

is an extremal state. It follows that $T = \lambda P_u$ is a one-dimensional projection onto some $u \in S(K)$ and $\lambda > 0$. Letting $v = \lambda^{1/2} u$ we have

$$f(x) = \chi_0(x) \langle \lambda P_u x, x \rangle^{1/2} = \chi_0(x) |\langle v, x \rangle|$$

for all $x \in S(K)$. Define $\chi(x) = 1$ if $\langle v, x \rangle = 0$ and

$$\chi(x) = \chi_0(x) |\langle v, x \rangle| / \langle v, x \rangle$$

if $\langle v, x \rangle \neq 0$. Then $\chi \in G$ and $f(x) = \chi(x) \langle v, x \rangle$ for all $x \in S(K)$. For uniqueness, suppose $f(x) = \chi_1(x) \langle v_1, x \rangle$ for some $v_1 \in K$, $\chi_1 \in G$. Then

$|\langle v_1, x \rangle| = |\langle v, x \rangle|$ for all $x \in S(K)$. Then $v_1 \perp x$ if and only if $v \perp x$. Hence, there exists an $\alpha \in \mathbb{C}$ such that $v_1 = \alpha v$. Moreover,

$$|\alpha| |\langle v, x \rangle| = |\langle v, x \rangle|.$$

Thus, if $v \neq 0$, $|\alpha| = 1$.

Conversely, it is clear that any function of the form $f = \chi f_v$ is in $H_p(S(K))$. \square

The next result gives a large class of maximal \cdot sets in $H(S(K))$. For a fixed $\chi \in G$, define $H_\chi(S(K)) = \{\chi f_v : v \in K\}$.

THEOREM 13. — *a) $H_\chi(S(K))$ is a unital \cdot set. b) $H_\chi(S(K))$ is a Hilbert space and the map $\phi : H_\chi(S(K)) \rightarrow K$ given by $\phi(\chi f_v) = v$ is an isomorphism. c) $H_\chi(S(K)) \in M[H(S(K))]$.*

Proof. — *a)* If $x \in S(K)$, then $|\chi f_x(x)| = \|f_x\|$. Hence, $H_\chi(S(K))$ is unital. Moreover, it is clear that $H_\chi(S(K))$ is a \cdot set. *b)* The map ϕ is well-defined since $\chi f_v = \chi f_u$ implies $v = u$. It is clear that $H_\chi(S(K))$ is a linear space and that $\alpha \chi f_v = \chi f_{\alpha v}$, $\chi f_v + \chi f_u = \chi f_{v+u}$ for all $\alpha \in \mathbb{C}$, $v, u \in K$. Moreover, ϕ is surjective. To show ϕ is injective, suppose $\chi f_v \neq \chi f_u$. Then there is an $x \in S(K)$ with $\chi(x) \langle v, x \rangle \neq \chi(x) \langle u, x \rangle$ so $v \neq u$. To show that ϕ is an isomorphism, we have for any $E \in O(S(K))$, $v, u \in K$,

$$\begin{aligned} \langle v, u \rangle &= \sum_{x \in E} \langle v, x \rangle \langle x, u \rangle = \sum_{x \in E} \chi(x) \langle v, x \rangle \overline{\chi(x) \langle u, x \rangle} \\ &= \langle \chi f_v, \chi f_u \rangle. \end{aligned}$$

The result now follows. *c)* This follows from *a)*, *b)* and Corollary 9 *b)*. \square

We conjecture that every unital maximal \cdot set in $H(S(K))$ has the form $H_\chi(S(K))$ for some $\chi \in G$.

Two cover spaces (X, O) and (X', O') are *isomorphic* if there exists a bijection $\phi : X \rightarrow X'$ such that the map $\overline{\phi}(E) = \{\phi(x) : x \in E\}$ is surjective from O to O' . It follows that $\overline{\phi} : O \rightarrow O'$ is bijective. Notice that if (X, O) is a cover space and $\phi : X \rightarrow X'$ is bijective, then $(X', \overline{\phi}(O))$ is a cover space which is isomorphic to (X, O) . A cover space (X', O') is a *subcover space* of a cover space (X, O) if $O' \subseteq O$. For a cover space (X, O) , a set $A \subseteq H(X)$ is *strongly separating* if for $x \neq y \in X$ there exists an $f \in A$ such that $f(x) = \|f\| \neq f(y)$. Notice that a strongly separating set is unital.

A subcover space of a Hilbertian cover space is called a *Hilbertian subcover space*. The next theorem characterizes Hilbertian subcover spaces up to an isomorphism. This theorem generalizes a result in [6].

THEOREM 14. — *A cover space (X, O) is isomorphic to a Hilbertian subcover space if and only if $H(X)$ contains a strongly separating \cdot set.*

Proof. — Suppose (X, O) is isomorphic to a Hilbertian subcover space (X', O') , $X' \subseteq S(K)$, under an isomorphism $\phi : X \rightarrow X'$. For each $v \in X'$,

define $g_v : X \rightarrow \mathbb{C}$ by $g_v(x) = \langle v, \phi(x) \rangle$, and let $A = \{g_v : v \in X'\}$. It is clear that $A \subseteq H(X)$. To show that $A \in C[H(X)]$, suppose that $g_v, g_u \in A$, and $E, F \in O$. Then since $\overline{\phi}(E), \overline{\phi}(F) \in O'$ we have

$$\begin{aligned} \sum_{x \in E} g_u(x) \overline{g_v}(x) &= \sum_{x \in E} \langle v, \phi(x) \rangle \langle \phi(x), u \rangle = \langle v, u \rangle \\ &= \sum_{x \in F} \langle v, \phi(x) \rangle \langle \phi(x), u \rangle = \sum_{x \in F} g_v(x) \overline{g_u}(x). \end{aligned}$$

To show that A is strongly separating, suppose $x \neq y \in X$. Then $\phi(x) \neq \phi(y)$ and

$$g_{\phi(x)}(x) = \|\phi(x)\|^2 = 1 = \|g_{\phi(x)}\|.$$

Now suppose that $g_{\phi(x)}(y) = 1$. Then

$$\|\phi(x)\| \|\phi(y)\| = g_{\phi(x)}(y) = |\langle \phi(x), \phi(y) \rangle|.$$

Since we have equality in Schwarz's inequality, there exists an $\alpha \in \mathbb{C}$ such that $\phi(x) = \alpha \phi(y)$. But then

$$1 = g_{\phi(x)}(y) = \langle \phi(x), \phi(y) \rangle = \alpha \langle \phi(y), \phi(y) \rangle = \alpha.$$

Hence, $\phi(x) = \phi(y)$ which is a contradiction

Conversely, suppose that $H(X)$ contains a strongly separating set A . Since A is unital, it follows from Theorem 8 that $K = A^2$ is a Hilbert space. For $x \in X$, we show that there exists a unique $g \in K$ such that $g(x) = \|g\| = 1$. Since A is strongly separating, there exists a $0 \neq f \in A$ such that $f(x) = \|f\|$. Then if $g = f/\|f\|$, we have $g \in K$ and $g(x) = \|g\| = 1$. Now suppose $g, h \in K$ with

$$g(x) = h(x) = \|g\| = \|h\| = 1.$$

If $y \perp x$, then it follows that $g(y) = h(y) = 0$. Therefore, if $x \in E \in O$, we have

$$\langle g, h \rangle = \sum_{y \in E} g(y) \overline{h}(y) = g(x) \overline{h}(x) = \|g\| \|h\|.$$

We conclude that there exists an $\alpha \in \mathbb{C}$ such that $g = \alpha h$. Hence, $1 = \alpha \|h\|^2 = \alpha$ so $g = h$. Define $\phi : X \rightarrow S(K)$ by $\phi(x) = g_x$ where $g_x(x) = \|g_x\| = 1$. To show that ϕ is injective, suppose that $x, y \in X$ with $x \neq y$. Since A is strongly separating, there exists $0 \neq f \in A$ such that $f(x) = \|f\| \neq f(y)$. Then $g_x = f/\|f\|$ and $g_x(y) \neq 1$. Hence,

$$\phi(x) = g_x \neq g_y = \phi(y).$$

Now consider the cover space $(\phi(X), \overline{\phi}(O))$. To complete the proof, it suffices to show that $\overline{\phi}(E)$ is an orthonormal basis in K for each $E \in O$. For $x \neq y \in E$, we have

$$\langle g_x, g_y \rangle = \sum_{z \in E} g_x(z) \overline{g_y}(z) = \sum_{z \in E} \delta_{xz} \delta_{yz} = 0$$

so $\phi(x) \perp \phi(y)$. Now let $f \in K$ and suppose $f \perp \overline{\phi}(E)$. If $x \in E$, then $f \perp g_x$. Hence,

$$0 = \langle f, g_x \rangle = \sum_{y \in E} f(y) \overline{g_x}(y) = f(x).$$

It follows that $f(x) = 0$ for all $x \in E$, so

$$\|f\|^2 = \sum_{x \in E} |f(x)|^2 = 0.$$

Hence, $f = 0$ and $\overline{\phi}(E)$ is an orthonormal basis. \square

We have actually proven a stronger result than that given in Theorem 14.

COROLLARY 15. — If (X, O) admits a strongly separating set $A \subseteq H(X)$, then there exists an isomorphism ϕ from (X, O) to a subcover space of the Hilbertian cover space $(S(A), O(A))$. Moreover, for each $f \in A$ we have $f(x) = \langle f, \phi(x) \rangle$ for all $x \in X$.

5. TENSOR PRODUCTS

An important problem in operational statistics is to find an appropriate definition for a tensor product which generalizes the Hilbert space tensor product [5] [10]. In this section we present an approach to this problem.

Let $X_1 = (X_1, O_1)$ and $X_2 = (X_2, O_2)$ be cover spaces. We define $X_1 \times X_2 = (X_1 \times X_2, O_1 \times O_2)$ where $X_1 \times X_2$ is the Cartesian product and

$$O_1 \times O_2 = \{ E \subseteq X_1 \times X_2 : E = E_1 \times E_2, E_1 \in O_1, E_2 \in O_2 \}.$$

It is clear that $X_1 \times X_2$ is a cover space. For $f_1 \in H(X_1)$, $f_2 \in H(X_2)$ define

$$f_1 \times f_2 : X_1 \times X_2 \rightarrow \mathbb{C} \text{ by } f_1 \times f_2(x_1, x_2) = f_1(x_1) f_2(x_2).$$

Then $f_1 \times f_2 \in H(X_1 \times X_2)$. Indeed, let $E, F \in O_1 \times O_2$ where $E = E_1 \times E_2$, $F = F_1 \times F_2$. Then

$$\begin{aligned} \sum_{(x_1, x_2) \in E} |f_1 \times f_2(x_1, x_2)|^2 &= \sum_{x_1 \in E_1} |f_1(x_1)|^2 \sum_{x_2 \in E_2} |f_2(x_2)|^2 \\ &= \sum_{x_1 \in F_1} |f_1(x_1)|^2 \sum_{x_2 \in F_2} |f_2(x_2)|^2 \\ &= \sum_{(x_1, x_2) \in F} |f_1 \times f_2(x_1, x_2)|^2 \end{aligned}$$

We also see from the above that $\|f_1 \times f_2\| = \|f_1\| \|f_2\|$. For $A_1 \subseteq H(X_1)$, $A_2 \subseteq H(X_2)$, define

$$A_1 \times A_2 = \{ f_1 \times f_2, f_1 \in A_1, f_2 \in A_2 \} \subseteq H(X_1 \times X_2)$$

LEMMA 16. — If $A_1 \in C[H(X_1)]$, $A_2 \in C[H(X_2)]$, then

$$A_1 \times A_2 \in C[H(X_1 \times X_2)].$$

Proof. — Let $f_1 \times f_2, g_1 \times g_2 \in A_1 \times A_2$, $E = E_1 \times E_2$, $F = F_1 \times F_2 \in O_1 \times O_2$. Then

$$\begin{aligned} \sum_{(x_1, x_2) \in E} f_1 \times f_2(x_1, x_2) \overline{g_1 \times g_2(x_1, x_2)} &= \sum_{x_1 \in E_1} f_1(x_1) \overline{g_1(x_1)} \sum_{x_2 \in E_2} f_2(x_2) \overline{g_2(x_2)} \\ &= \sum_{x_1 \in F_1} f_1(x_1) \overline{g_1(x_1)} \sum_{x_2 \in F_2} f_2(x_2) \overline{g_2(x_2)} \\ &= \sum_{(x_1, x_2) \in F} f_1 \times f_2(x_1, x_2) \overline{g_1 \times g_2(x_1, x_2)} \quad \square \end{aligned}$$

If $A_1 \in C[H(X_1)]$, $A_2 \in C[H(X_2)]$, we define $A_1 \hat{\otimes} A_2 = [A_1 \times A_2]$. As we shall see, $A_1 \hat{\otimes} A_2$ can be thought of as a tensor product of A_1 and A_2 . The main point here is the following. Although it is not at all obvious how to define the tensor product of two cover spaces, there is a natural definition for the tensor product of sets in their amplitude spaces. If H_1 and H_2 are Hilbert spaces, we denote their tensor product by $H_1 \otimes H_2$.

THEOREM 17. — If $A_1 \in C[H(X_1)]$, $A_2 \in C[H(X_2)]$ are Hilbert spaces, then $A_1 \otimes A_2$ and $A_1 \hat{\otimes} A_2$ are isomorphic and the map $f_1 \otimes f_2 \rightarrow f_1 \times f_2$ extends to a unique isomorphism.

Proof. — Let $\psi_\alpha, \phi_\beta, \alpha \in A, \beta \in B$ be orthonormal bases for A_1 and A_2 , respectively. It is well known that $M = \{\psi_\alpha \otimes \phi_\beta : \alpha \in A, \beta \in B\}$ is an orthonormal basis for $A_1 \otimes A_2$. If $E_1 \in O_1, E_2 \in O_2$, then we have for any $\alpha, \alpha' \in A, \beta, \beta' \in B$

$$\begin{aligned} \langle \psi_\alpha \times \phi_\beta, \psi_{\alpha'} \times \phi_{\beta'} \rangle &= \sum_{(x_1, x_2) \in E_1 \times E_2} \psi_\alpha(x_1) \phi_\beta(x_2) \bar{\psi}_{\alpha'}(x_1) \bar{\phi}_{\beta'}(x_2) \\ &= \sum_{x_1 \in E_1} \psi_\alpha(x_1) \bar{\psi}_{\alpha'}(x_1) \sum_{x_2 \in E_2} \phi_\beta(x_2) \bar{\phi}_{\beta'}(x_2) \\ &= \langle \psi_\alpha, \psi_{\alpha'} \rangle \langle \phi_\beta, \phi_{\beta'} \rangle = \delta_{\alpha\alpha'}, \delta_{\beta\beta'}. \end{aligned}$$

It now easily follows that $\tilde{M} = \{\psi_\alpha \times \phi_\beta : \alpha \in A, \beta \in B\}$ is an orthonormal basis for $\text{span } A_1 \times A_2$ and hence for $A_1 \hat{\otimes} A_2$. Define the map $T : \text{span } M \rightarrow \text{span } \tilde{M}$ by

$$T(\sum c_{\alpha\beta} \psi_\alpha \otimes \phi_\beta) = \sum c_{\alpha\beta} \psi_\alpha \times \phi_\beta.$$

Then it is clear that T is a linear map and $\langle Tf, Tg \rangle = \langle f, g \rangle$ for all $f, g \in \text{span } M$. It follows that T has a unique extension to an isomorphism \bar{T} from $A_1 \otimes A_2$ to $A_1 \hat{\otimes} A_2$. Moreover, it is easy to show that $\bar{T}(f_1 \otimes f_2) = f_1 \times f_2$ for all $f_i \in A_i, i = 1, 2$. \square

COROLLARY 18. — If $A_i \in M[H(X_i)], i = 1, 2$, then the map

$$f_1 \otimes f_2 \rightarrow f_1 \times f_2$$

extends to a unique isomorphism from $A_1 \otimes A_2$ to $A_1 \hat{\otimes} A_2$.

The next result shows that if $A_i \in C[H(X_i)], i = 1, 2$, are unital, then $A_1 \hat{\otimes} A_2$ is defined strictly in terms of the algebraic relation. In this way the « tensor product » $A_1 \hat{\otimes} A_2$ is intrinsically defined.

THEOREM 19. — If $A_i \in C[H(X_i)], i = 1, 2$, are unital, then $A_1 \hat{\otimes} A_2$ is the unique maximal set containing $A_1 \times A_2$. Moreover,

$$A_1 \hat{\otimes} A_2 = (A_1 \times A_2)^{\ddot{}} = (A_1 \times A_2)^{\dot{}}.$$

Proof. — Applying Theorem 8, it suffices to show that $A_1 \times A_2$ is unital in $H(X_1 \times X_2)$. If $x_i \in X_i, i = 1, 2$, then there exist $f_i \in A_i, i = 1, 2$, such

that $f_i(x_i) \neq 0$, $f_i(y_i) = 0$ for all $y_i \perp x_i$, $i = 1, 2$. Then $f_1 \times f_2 \in A_1 \times A_2$, and $f_1 \times f_2(x_1, x_2) \neq 0$ and $f_1 \times f_2(y_1, y_2) = 0$ for all

$$(y_1, y_2) \perp (x_1, x_2). \quad \square$$

REFERENCES

- [1] R. FEYNMAN and A. HIBBS, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.
- [2] D. FOULIS and C. RANDALL, Operational Statistics I. Basic Concepts, *J. Math. Phys.*, t. **13**, 1972, p. 1167-1175.
- [3] D. FOULIS and C. RANDALL, *The empirical logic approach to the physical sciences*, in: *Foundations of Quantum Mechanics and Ordered Linear Spaces* (Hartkampfer and Neumann, Eds.), Springer Verlag, Heidleberg, 1974, p. 230-249.
- [4] D. FOULIS and C. RANDALL, *What are quantum logics and what ought they to be?* in: *Current Issues in Quantum Logic*, Beltrametti and Van Fraassen, Eds., Plenum Press, New York, 1981, p. 35-52.
- [5] D. FOULIS and C. RANDALL, *Empirical logic and tensor products*, in: *Interpretations and Foundations of Quantum Theory*, H. Neumann, Ed., B. I. Wissenschaftsverlog, Bibliographisches Institut, Mannheim, 1981, p. 9-20.
- [6] S. GUDDER, Hilbertian interpretations of manuals, *Proc. Amer. Math. Soc.*, t. **85**, 1982, 251-255.
- [7] S. GUDDER, Amplitude phase-space model for quantum mechanics, *Intern. J. Theor. Phys.*, t. **24**, 1985, p. 343-353.
- [8] C. RANDALL and D. FOULIS, Operational statistics II. Manuals of operations and their logics, *J. Math. Phys.*, t. **14**, 1972, p. 1472-1480.
- [9] C. RANDALL and D. FOULIS, *The Operational approach to quantum mechanics*, in: *The Logico-Algebraic Approach to Quantum Mechanics III. Physical Theory as Logical-Operational Structures*, C. Hooker, Ed., Reidel, Dordrecht, 1978, p. 167-201.
- [10] C. RANDALL and D. FOULIS, *Operational statistics and tensor products*, in: *Interpretations and Foundations of Quantum Theory*, H. Neumann, Ed., B. I. Wissenschaftsverlag, Bibliographisches Institut, Mannheim, 1981, p. 21-28.

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