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Transforms associated to square integrable group representations II: examples

by

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ABSTRACT. — We give examples of integral transforms defined through square integrable group representations, described in the first paper of this series. We focus on the Weyl and « $ax + b$ » groups and study their relevance in physics and applied mathematics.

RÉSUMÉ. — On donne des exemples de transformations intégrales définies au moyens de représentations de carré intégrable de groupes, décrites dans le premier article de cette série. On se concentre sur le groupe de Weyl et le groupe « $ax + b$ » et on examine leur pertinence en Physique et en Mathématiques appliquées.

1. INTRODUCTION

In a preceding paper [*1*], we discussed orthogonality relations for square integrable group representations, and claimed that they allowed a unified discussion of many situations of interest in quantum mechanics

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and in signal analysis. The main purpose of this paper is to illustrate this claim.

We first recall the statement of orthogonality relations, in a form almost identical to the one given in [*1*].

Let G be a locally compact group, with left Haar measure $d\mu(x)$ and right Haar measure $d\mu_R(x)$.

Let U:

$$\gamma \to U(\gamma) \qquad (\gamma \in G)$$

be a continuous unitary representation in a Hilbert space $\mathcal{H}(U)$.

In the present paper, we assume U to be irreducible. We shall see however, in the comments and remarks at the end of this paper, that the irreducibility assumption is unnaturally restrictive. It will be relaxed in work in preparation.

We say that U is *square integrable* if there exists in $\mathcal{H}(U)$ at least one non-zero vector φ such that the matrix element $(U(\gamma)\varphi, \varphi)$,—a complex-valued function on G—is square integrable with respect to the measure $d\mu(\gamma)$. It follows then that the function $(U(\gamma)\varphi, \varphi)$ is also square integrable with respect to the right-invariant measure $d\mu_R(\gamma)$. Any such vector will be called an *admissible analyzing wavelet*.

If U is square integrable, then the set A(U) of admissible analyzing wavelets is dense in $\mathcal{H}(U)$.

Now choose any $\varphi \in A(U)$, and an arbitrary $\psi \in \mathcal{H}(U)$, which need not be admissible.

Then,

i) The matrix element $(U(\gamma)\varphi, \psi)$ is square integrable with respect to $d\mu(\gamma)$, i. e. it belongs to $L^2(G, d\mu(\gamma))$. (It does not necessarily belong to $L^2(G, d\mu_R(\gamma))$; there is no difficulty in deriving the appropriate $L^2(G, d\mu_R(\gamma))$-statements).

We next consider two admissible analyzing wavelets φ_1, φ_2, and two arbitrary vectors in $\mathcal{H}(U)$, namely ψ_1, and ψ_2. The orthogonality relations are a statement about the inner product of $(U(\gamma)\varphi_1, \psi_1)$ and of $(U(\gamma)\varphi_2, \psi_2)$, in $L^2(G, d\mu(\gamma))$, namely;

ii) There exists in A(U) a positive quadratic form

$$q(\varphi_1, \varphi_2) = (\varphi_1, \varphi_2)_a \qquad (\varphi_1, \varphi_2) \in A(U)$$

such that A(U) is a Hilbert space with respect to the scalar product $(\ ,\ )_a$, and such that

$$\int \overline{(U(\gamma)\varphi_1, \psi_1)}(U(\gamma)\varphi_2, \psi_2)d\mu(\gamma) = (\varphi_2, \varphi_1)_a(\psi_1, \psi_2).$$

In this series of papers, we shall be mainly concerned with the case $\varphi_1 = \varphi_2$. If we define then the number

$$C_\varphi = \frac{(\varphi, \varphi)_a}{(\varphi, \varphi)}$$

we obtain the following statement:

For fixed admissible φ, the correspondence between the vector $\psi \in \mathscr{H}(U)$ and the function

$$(L_\varphi \psi)(\gamma) = \frac{1}{\sqrt{C_\varphi}} (U(\gamma)\varphi, \psi)$$

in $L^2(G, d\mu(\gamma))$, is an isometry from $\mathscr{H}(U)$ into $L^2(G, d\mu(\gamma))$.

The function $L_\varphi \psi$ is called the *left transform* of ψ with respect to G, U and φ.

It is sometimes also convenient to introduce the right transform

$$(R_\varphi \psi)(\gamma) = (L_\varphi \psi)(\gamma^{-1}).$$

Examples will be found in all the sections that follow.

The range of L_φ is in general a proper closed subspace of $L^2(G, d\mu(\gamma))$. The usefulness of the transforms L_φ is largely a consequence of the fact that there exists a simple general characterization of this range.

Consider on G the function

$$P_\varphi(\gamma) = \frac{1}{C_\varphi} (U(\gamma)\varphi, \varphi) \qquad (\varphi \in A(U), \quad \gamma \in G).$$

Consider on $L^2(G, d\mu(\gamma)$ the integral operator P_φ given by the kernel

$$P_\varphi(\gamma, \gamma') = P_\varphi(\gamma'^{-1}\gamma)$$

(a convolution operator on the group). Then P_φ is the orthogonal projection operator on the range of L_φ. It acts as the identity on any function $\psi(\gamma)$ in the range of L_φ, and sends to zero any function orthogonal to that range. The range of L_φ is a Hilbert space with reproducing kernel $P_\varphi(\gamma, \gamma')$.

The above results can be written very transparently with the help of Dirac notation, in which $\langle \varphi |$ denotes the linear functional corresponding to a vector $\varphi = | \varphi \rangle$ in a Hilbert space (To avoid possible misunderstandings, we mention that we deal here in bona fide Hilbert space vectors, and that a label on a vector, say $| \gamma \rangle$ does not imply that $| \gamma \rangle$ is the eigenvector of some operator). If we choose any admissible analyzing wavelet φ and write $| \gamma ; \varphi \rangle$ (or $| \gamma \rangle$ for shortness) to denote

$$| \gamma \rangle = \frac{1}{\sqrt{C_\varphi}} U(\gamma) | \varphi \rangle,$$

then the basic isometry relation takes the familiar form

$$\int_G |\gamma\rangle \, d\mu(\gamma) \langle \gamma | = \mathbb{1}$$

(identity operator in $\mathscr{H}(U)$). The transform L_φ associates, to the vector $\psi \in \mathscr{H}(U)$, the function $\langle \gamma | \psi \rangle$ on G.

The reproducing equation becomes evident: it states that

$$\langle \gamma | \psi \rangle = \int \langle \gamma | \gamma' \rangle \, d\mu(\gamma') \langle \gamma' | \psi \rangle$$

where

$$\langle \gamma | \gamma' \rangle = \frac{1}{C_\varphi}(U(\gamma)\varphi, U(\gamma')\varphi) = \frac{1}{C_\varphi}(U(\gamma'^{-1}\gamma)\varphi, \varphi).$$

There exists a vast literature dealing with square integrable representations of various classes of groups, and we shall not attempt here even a superficial survey. Our main purpose in this paper is the study of some very particular cases which fall under the general scheme just described, and which are of independent interest.

We shall start with the Weyl-Heisenberg group and some closely related objects, and then discuss transforms associated with the « $ax + b$ »-group (affine group in one dimension).

Notations: some families of operators.

Many calculations in this paper can be made trivially with the help of straightforward operator identities which we give here for the reader's convenience.

Consider first a space $L^2(\mathbb{R})$ (line with Lebesgue measure). Introduce the following operators:

i) T^x: shift by x

$$(T^x\psi)(x') = \psi(x - x') \qquad x \in \mathbb{R}.$$

ii) E^b: multiplication by exponential

$$(E^b\psi)(x') = e^{ibx'}\psi(x') \qquad b \in \mathbb{R}.$$

iii) D^y: Unitary dilation operator:

$$(D^y\psi)(t) = |y|^{-1/2}\psi\left(\frac{t}{y}\right) \qquad y \in \mathbb{R}, \quad y \neq 0.$$

iv) Fourier transform:

$$(F\psi)(\omega) = \frac{1}{\sqrt{2\pi}}\int e^{-i\omega t}\psi(t)dt.$$

Inverse Fourier transform:

$$(\mathrm{F}^{-1}\chi)(t) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t}\chi(\omega)d\omega .$$

All these operators are unitary in $L^2(\mathbb{R})$. They satisfy

$$\begin{aligned} \mathrm{T}^x\mathrm{T}^{x'} &= \mathrm{T}^{x+x'} && (x, x' \in \mathbb{R}) \\ \mathrm{E}^b\mathrm{E}^{b'} &= \mathrm{E}^{b+b'} && (b, b' \in \mathbb{R}) \\ \mathrm{D}^y\mathrm{D}^{y'} &= \mathrm{D}^{yy'} && (y, y' \in \mathbb{R}, y \neq 0, y' \neq 0). \end{aligned}$$

Furthermore

$$\begin{aligned} \mathrm{E}^b\mathrm{T}^x &= e^{ibx}\mathrm{T}^x\mathrm{E}^b & \mathrm{T}^x\mathrm{E}^b &= e^{-ibx}\mathrm{E}^b\mathrm{T}^x \\ \mathrm{F}\mathrm{T}^x &= \mathrm{E}^{-x}\mathrm{F} & \mathrm{F}\mathrm{E}^b &= \mathrm{T}^b\mathrm{F} \\ \mathrm{F}^{-1}\mathrm{T}^x &= \mathrm{E}^x\mathrm{F}^{-1} & \mathrm{F}^{-1}\mathrm{E}^b &= \mathrm{T}^{-b}\mathrm{F}^{-1} \\ \mathrm{D}^y\mathrm{E}^b &= \mathrm{E}^{b/y}\mathrm{D}^y & \mathrm{E}^b\mathrm{D}^y &= \mathrm{D}^y\mathrm{E}^{by} \\ \mathrm{D}^y\mathrm{T}^x &= \mathrm{T}^{xy}\mathrm{D}^y & \mathrm{T}^x\mathrm{D}^y &= \mathrm{D}^y\mathrm{T}^{x/y} \\ \mathrm{F}\mathrm{D}^y &= \mathrm{D}^{1/y}\mathrm{F} & \mathrm{F}^{-1}\mathrm{D}^y &= \mathrm{D}^{1/y}\mathrm{F}^{-1} \end{aligned}$$

(identity) $\quad \mathrm{F}^4 = \mathbb{1}$

(parity operator) $\quad \mathrm{F}^2 = \mathrm{D}^{(-1)}$.

2. THE WEYL-HEISENBERG GROUP

a) The group.

The Weyl-Heisenberg group appears naturally whenever one considers simultaneously displacements (shifts) in a space $\mathbb{R}^n$ and in the space of Fourier conjugate variables. In order to motivate the definition below, consider, in the Hilbert space $L^2(\mathbb{R}^n, d^nx)$ the unitary displacement operators

$$\mathrm{T}^b \qquad (\mathrm{T}^b\psi)(x) = \psi(x-b) \qquad (b \in \mathbb{R}^n).$$

If we define F (the Fourier transform) as the unitary map from $L^2(\mathbb{R}^n, d^nx)$ to $L^2(\mathbb{R}^n, d^np)$ defined by

$$(\mathrm{F}\psi)(p) = \hat{\psi}(p) = (2\pi)^{-n/2}\int e^{-ipx}\psi(x)d^nx$$

we notice that

$$\mathrm{F}^{-1}\mathrm{T}^b\mathrm{F} = \mathrm{E}^b$$

where E^b is the (unitary) operator of multiplication by an exponential

$$(\mathrm{E}^b\psi)(x) = e^{ibx}\psi(x).$$

Consider now the set of operators generated in the sense of operator (multiplication) by the T^s ($s \in \mathbb{R}^n$) and the E^b ($b \in \mathbb{R}^n$). One has

$$T^sE^b = e^{-ibs}E^bT^s$$

and it is convenient, (but not essential) to introduce the two-parameter family of operators

$$W(s, b) = e^{i\frac{sb}{2}}T^sE^b = e^{-i\frac{sb}{2}}E^bT^s .$$

The operators $W(s, b)$ satisfy

$$W(s, b)W(s', b') = e^{\frac{i}{2}(bs'-sb')}W(s + s', b + b') .$$

We are now ready to define the Weyl-Heisenberg group:

Consider the set $G_{HW}^{(n)}$ of triplets $\gamma = \{ s, b, \tau \}$ with $s \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $\tau \in \mathbb{C}$, $|\tau| = 1$. Define

$$\gamma\gamma' = \{ s + s', b + b', \tau\tau' e^{\frac{i}{2}(bs'-sb')} \}$$
$$\gamma^{-1} = \{ -s, -b, \tau^{-1} \} .$$

With these operations, $G_{HW}^{(n)}$ becomes a group, the Weyl-Heisenberg group.

The discussion preceding this definition gives immediately a unitary representation of $G_{HW}^{(n)}$ in $L^2(\mathbb{R}^n, d^nx)$:

$$U(s, b, \tau) = \tau W(s, b) .$$

It is not difficult to see that U is irreducible.

The group $G_{HW}^{(n)}$ is unimodular. A suitable (right and left) invariant measure is

$$d\mu(\gamma) = d^nsd^nbd\varphi$$

with $\tau = e^{i\varphi}$.

We shall see that the representation U is square integrable.

Remark. — The Weyl-Heisenberg group with dilations.

Let, as above, $s \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, $\tau \in \mathbb{C}$, $|\tau| = 1$, $\gamma = \{ s, b, \tau \}$. If a is any non-zero real number, $a \in \mathbb{R}$, $a \neq 0$, consider the pair $\{ a, \gamma \}$:

$$\{ a, \gamma \} = \{ a, s, b, \tau \} .$$

The set of such pairs can be made into a group by defining

$$\{ a, s, b, \tau \} \{ a', s', b', \tau' \} = \left\{ aa', s + as', b + \frac{1}{a}b', \tau\tau' \exp\left(\frac{i}{2}\left(abs' - \frac{1}{a}sb'\right)\right)\right\}$$
$$\{ a, s, b, \tau \}^{-1} = \left\{ a^{-1}, -\frac{1}{a}s, -ab, \tau^{-1} \right\} .$$

(This is a semidirect product construction).

This group will be exploited in another paper.

b) The transform L_φ: arbitrary φ.

For the sake of simplicity, we consider now the case $n = 1$; the generalization to arbitrary n is trivial in the case of the Weyl-Heisenberg group.

The results of the introduction give now the following statements. Let φ, (the analyzing wavelet) be an arbitrary vector in $L^2(\mathbb{R})$. For every $b \in \mathbb{R}$, $s \in \mathbb{R}$, consider the vector $\varphi^{(s,b)} \in L^2(\mathbb{R})$ defined as

$$\varphi^{(s,b)} = \frac{2}{\sqrt{C_\varphi}} W(s, b)\varphi$$

where

$$C_\varphi = 2\pi \| \varphi \|^2 .$$

To every $\psi \in L^2(\mathbb{R})$ associate the function

$$(L_\varphi \psi)(s, b) = (\varphi^{(s,b)}, \psi) .$$

Then the transformation L_φ is isometric from $L^2(\mathbb{R}, dx)$ into $L^2(\mathbb{R}^2, dsdb)$. The range of L_φ is characterized as follows: consider the function

$$\Phi^{(s,b)}(s', b') = (\varphi^{(s',b')}, \varphi^{(s,b)}) = (F\varphi^{(s',b')}, F\varphi^{(s,b)})$$
$$= \frac{1}{C_\varphi} e^{\frac{i}{2}(b's' - bs)} \int e^{i(b-b')x} \overline{\varphi}(x - s')\varphi(x - s)dx .$$

Then a function $f(s, b)$ belongs to the range of L_φ if and only if it satisfies

$$f(s, b) = \int \overline{\Phi}^{(s,b)}(s', b') f(s', b') ds' db' .$$

If F denotes the Fourier transform as in (1), one has

$$(L_\varphi \psi)(s, b) = (L_{F\varphi} F\psi)(b, - s) .$$

If D(a) is the unitary dilation operator, one has

$$(L_\varphi \psi)(s, b) = (L_{D(a)\varphi} D(a)\psi)\left(as, \frac{1}{a} b \right).$$

If the displacement operator T^{s_0} and the multiplication operator E^{b_0} are defined as in p. 8, then

$$(L_\varphi T^{s_0} \psi)(s, b) = e^{-ibs_0/2}(L_\varphi \psi)(s - s_0, b)$$
$$(L_\varphi E^{b_0} \psi)(s, b) = e^{ibs_0/2}(L_\varphi \psi)(s, b - b_0) .$$

The adjoint of L_φ (which is the inverse of L_φ on the range of L_φ and zero on the orthogonal complement) is given by

$$\psi(x) = \iint \varphi^{(s,b)}(x) L_\varphi(\psi)(s, b) ds db$$

with appropriate definition of convergence.

We mention finaly that the L_φ-transform allows an easy characterization of « band-limited » (or « Paley-Wiener ») functions.

Assume that ψ is such that the support of the Fourier transform of ψ is contained in a finite interval Δ. Then in addition to the reproducing equation, $L_\varphi \psi$ satisfies a more stringent reproducing equation, determined by the kernel $\Phi_\Delta^{(s,b)}(s', b') = (F\varphi^{(s',b')}, F\varphi^{(s,b)})_\Delta$ where the subscript Δ means that the inner product is defined by integration over the interval Δ.

c) The transform L_φ: gaussian analyzing wavelet.

Choose a fixed $\sigma > 0$, and define the gaussian analyzing wavelet

$$\varphi_\sigma(x) = \frac{\sigma^{-1/2}}{\sqrt{2}\pi^{3/4}} e^{-x^2/2\sigma^2}.$$

The normalization has been chosen so that $\| \varphi_\sigma \| = \dfrac{1}{\sqrt{2\pi}}$ for all σ.

The transform $(L_{\varphi_\sigma}\psi)(s, b)$ can then be written conveniently in terms of the complex variable

$$z = \frac{1}{\sqrt{2}} \frac{s}{\sigma} - \frac{i}{\sqrt{2}} b\sigma .$$

The reproducing kernel $(\varphi_\sigma^{(s,b)}, \varphi_\sigma^{(s',b')})$ can be explicitly calculated. It is

$$\begin{aligned} P(z, z') &= \frac{1}{2\pi} e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2} e^{\bar{z}' z} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}|z - z'|^2} e^{i \operatorname{Im}(\bar{z}' z)} . \end{aligned}$$

The reproducing equation shows then that the range of L_{φ_σ} consists exactly of functions that are *i)* square integrable with respect to $d(\operatorname{Re} z) d(\operatorname{Im} z)$ and *ii)* of the form

$$e^{-\frac{1}{2}|z|^2} f(z)$$

where f is entire analytic.

This is the canonical coherent states representation. The function f belongs to Bargmann's space of analytic functions.

3. THE « $ax + b$ » GROUP

The « $ax + b$ » group is the group of affine transformations of the real axis

$$x \in \mathbb{R} \rightarrow ax + b .$$

It can be defined as the set of pairs $\{ a, b \}$, with $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a \neq 0$, with the group law

$$\{ a, b \} \{ a', b' \} = \{ aa', b + ab' \} .$$

The « $ax + b$ »-group is not unimodular. Its left Haar measure is

$$d\mu(a, b) = \frac{dadb}{a^2}$$

and the right Haar measure is

$$d\mu_R(a, b) = \frac{dadb}{|a|} .$$

3.1. Wavelet transforms on the line: general admissible φ.

We now consider the case where G is the « $ax + b$ »-group and where U is the irreducible representation

$$U(b, a) = T^b D^a \qquad (b \in \mathbb{R}, a \in \mathbb{R}, a \neq 0)$$

acting in the Hilbert space $L^2(\mathbb{R}, dt)$. The left transform defined by $U(b, a)$ has arisen independently in mathematical analysis and in the study of signals. Following Y. Meyer, we call it a wavelet transform.

In this section we first discuss some features of the wavelet transform that hold for an arbitrary (admissible) analyzing wavelet, and then compute it more explicitly for some special analyzing wavelets.

It is important to notice right from the start that the matrix element

$$(U(a, b)\varphi, \psi) = (T^b D^a \varphi, \psi) \qquad (\psi \in L^2(\mathbb{R}, dt), \varphi \in L^2(\mathbb{R}, dt))$$

can also be written as an expression involving the Fourier transforms $\hat{\varphi} = F\varphi$ and $\hat{\psi} = F\psi$. The commutation relations of Sec. 1 give

$$(T^b D^a \varphi, \psi) = (E^{-b} D^{1/a} \hat{\varphi}, \hat{\psi}) .$$

a) *The admissibility condition.*

The expression for the left Haar measure of the « $ax + b$ »-group shows that a function $\varphi \in L^2(\mathbb{R}, dt)$ is admissible if and only if one has

$$\iint |(T^b D^a \varphi, \varphi)|^2 \frac{dadb}{a^2} < \infty .$$

With the help of Sect. 2, one can calculate

$$\iint |(T^b D^a \varphi, \varphi)|^2 \frac{da\,db}{a^2} = \|\varphi\|^2 2\pi \int \frac{|\hat{\varphi}(\omega)|^2}{|\omega|} d\omega .$$

This means that a function $\varphi \in L^2(\mathbb{R}, dt)$ is admissible if and only if its Fourier transform belongs to $L^2\left(\mathbb{R}, \frac{d\omega}{|\omega|}\right)$; one sees that the number C_φ is now

$$C_\varphi = 2\pi \int \frac{|\hat{\varphi}(\omega)|^2}{|\omega|} d\omega .$$

If t is interpreted as time, then the admissibility condition says essentially that the analyzing wavelet $\varphi(t)$ has no zero-frequency component. Indeed, regularity conditions on φ together with the assumption $\hat{\varphi}(0) \neq 0$, imply the divergence of the integral C_φ.

If, with slight loss of precision, we write the admissibility condition as $\hat{\varphi}(0) = 0$, we see that it implies $\int \varphi(t)dt = 0$; an integrable admissible analyzing wavelet has zero mean. Consequently if the wavelet transform is extended from $L^2(\mathbb{R}, dt)$ to a larger space containing functions that are constant over $\mathbb{R}$, the wavelet transform of all such functions vanishes. Stronger statements hold if $\hat{\varphi}(\omega)$ has zeroes of higher order at $\omega = 0$.

The general definition of $L_\varphi \psi$ can now be written as

$$(L_\varphi \psi)(b, a) = \frac{1}{\sqrt{C_\varphi}} (T^b D^a \varphi, \psi) = \frac{1}{\sqrt{C_\varphi}} (E^{-b} D^{1/a} \hat{\varphi}, \hat{\psi}) .$$

b) *Explicit expression for the transform.*

If the operators and the inner products are written out explicitly, one finds

$$(L_\varphi \psi)(b, a) = \frac{|a|^{-1/2}}{\sqrt{C_\varphi}} \int \overline{\varphi}\left(\frac{b - t}{a}\right) \psi(t)dt \tag{A}$$

$$= \frac{|a|^{1/2}}{\sqrt{C_\varphi}} \int e^{ib\omega} \overline{\hat{\varphi}}(a\omega) \hat{\psi}(\omega) d\omega .$$

These equations can of course be the starting point of a discussion of the wavelet transform, shortcircuiting the generalized nonsense.

c) *Reality and symmetry properties.*

In contrast to Fourier transform and to transforms based on the Weyl-Heisenberg group, the wavelet transform need not involve integrals over rapidly oscillating functions. This can be seen from the first equation (A), and is one of the main features that singles out wavelet transforms among more general left transforms, and makes them computationally and theoretically appealing.

If φ and ψ are real, then $L_\varphi \psi$ is real. There is no loss in generality in assuming that ψ is real. There is also no loss of generality in assuming that $\hat{\varphi}$ is real, since the admissibility condition is valid, separately, for the real and imaginary part of $\hat{\varphi}$.

Under the two assumptions above one has

$$(L_\varphi \psi)(b, -a) = \overline{(L_\varphi \psi)(b, a)}$$

and one can restrict one's attention to the half-plane $a > 0$.

At this stage, we do not discuss decompositions involving symmetrization or support properties, since such decompositions may affect the smoothness of the functions concerned.

d) *Isometry and inversion.*

The general isometry equation becomes now

$$\int |\psi(t)|^2 dt = \iint |(L_\varphi \psi)(b, a)|^2 \frac{dadb}{a^2}$$

which gives also

$$\int \overline{\psi_1(t)} \psi_2(t) dt = \iint \overline{L_\varphi \psi_1(b, a)} L_\varphi \psi_2(b, a) \frac{dadb}{a^2} .$$

The transform L_φ is inverted, on its range, by its adjoint, i. e.

$$\psi(t) = \frac{1}{\sqrt{C_\varphi}} \iint (L_\varphi \psi)(b, a) |a|^{-1/2} \varphi\left(\frac{t-b}{a}\right) \frac{dadb}{a^2}$$

with suitable definition of integral.

e) *Characterization of the range of* L_φ.

The function $P_\varphi(\gamma)$ of Sec. 1 is now

$$\begin{aligned} P_\varphi(b, a) &= \frac{1}{C_\varphi} |a|^{-1/2} \int \overline{\varphi}\left(\frac{b-t}{a}\right) \varphi(t) dt \\ &= \frac{1}{C_\varphi} |a|^{1/2} \int \overline{\hat{\varphi}}(\omega a) e^{ib\omega} \hat{\varphi}(\omega) d\omega \end{aligned}$$

and the characterization of functions $\psi(a, b)$ that belong to the range of L_φ is

$$\psi(a, b) = \iint P_\varphi\left(\frac{b - b'}{a}, \frac{a}{a'}\right)\psi(b', a')\frac{db'da'}{a'^2}.$$

This characterization is important in practice, since it underlies interpolation procedures adapted to the range of L_φ.

f) *Wavelet transform of shifted functions.*

Consider the shifted « signal » $T^x\psi$. The definition shows then that

$$(L_\varphi T^x\psi)(b, a) = (L_\varphi\psi)(b - x, a).$$

If the function ψ is displaced, its transform undergoes the same displacement along the b-axis of the cut b, a-plane.

g) $L_\varphi\psi$ *as a « time-frequency » representation; localization.*

The transform (L_φ) is non-local, in the sense that the integral $(L_\varphi\psi)$ involves all values of ψ. If however the analyzing wavelet is suitably « concentrated », then the value of $L_\varphi\psi$ at a given point of the cut (b, a)-plane depends effectively only on the values of ψ and $\hat{\psi}$ in appropriate domains. Assume that φ is in some sense negligible outside a compact set $I(\varphi)$, (« effective support of φ »), and that $\hat{\varphi}$ is in some sense negligible outside a compact set $J(\hat{\varphi})$; (« effective support of $\hat{\varphi}$ »). Then the integral $(L_\varphi\psi)$ can be effectively replaced by an integral over the set $b - aI(\varphi)$. The arguments t of ψ that effectively influence a given value $(L_\varphi\psi)(b, a)$ lie in the « neighbourhood » $b - t \in aI(\varphi)$. This neighbourhood increases linearly with a. If a is large, then $(L_\varphi\psi)(b, a)$ is a very smoothed-out picture of ψ, showing only large-scale features of ψ that lie in the set $b - aI(\varphi)$.

The integral in ω can be effectively replaced by an integral over $\frac{1}{a}J(\hat{\varphi})$. The Fourier components $\hat{\varphi}(\omega)$ that effectively influence $(L_\varphi\psi)(b, a)$ at a given point b, a, correspond to values of ω that satisfy

$$\omega \in \frac{1}{a}J(\hat{\varphi}).$$

It is telling to use a logarithmic scale for ω and to write

$$\log(\omega) - \log\left(\frac{1}{a}\right) \in \log I(\hat{\varphi}).$$

The relevant Fourier components of ψ lie in a fixed « logarithmic neighbourhood » of $\log\left(\frac{1}{a}\right)$. (The logarithmic distance between frequencies corresponds of course to musical intervals; for this reason wavelet transforms were called « cycle octave transforms ».)

If a is large, then $(L_\varphi\psi)(a, b)$ catches low frequencies of ψ, in concordance with the picture in t-space.

The above discussion has been voluntarily kept imprecise. The compact sets $I(\varphi)$ and $J(\hat{\varphi})$ cannot be true supports of φ and $\hat{\varphi}$ and the definition of effective support will depend on the problem on hand.

A natural rough measure of the (lack of) concentration of is the « uncertainty parameter »

$$\eta = (\Delta t)^2(\Delta\omega)^2$$

where

$$(\Delta t)^2 = \frac{1}{\|\varphi\|^2}\int (t - t_0)^2 |\varphi(t)|^2 dt$$

$$t_0 = \frac{1}{\|\varphi\|^2}\int t |\varphi(t)|^2 dt$$

$$(\Delta\omega)^2 = \frac{1}{\|\hat{\varphi}\|^2}\int (\omega - \omega_0)^2 |\hat{\varphi}(\omega)|^2 d\omega$$

$$\omega_0 = \frac{1}{\|\hat{\varphi}\|^2}\int \omega |\hat{\varphi}(\omega)|^2 d\omega .$$

It is well known that

$$\eta \geqslant \frac{1}{4}$$

for every φ.

h) *Wavelet transform on the line: special analyzing wavelets.*

i) Linear combinations of gaussians. — A gaussian is not admissible, even after shifts and multiplication by exponentials, since its Fourier transform does not vanish at zero. It is however easy to construct linear combinations of gaussians that are admissible. The corresponding reproducing kernels can be explicitly calculated. Some details can be found in [*3*].

ii) Derivative of gaussians. — The derivative of any function in the Schwartz space $\mathcal{S}(\mathbb{R})$ is an admissible analyzing wavelet, as can be seen by a look at the Fourier transform of $\frac{d}{dt}g$.

Consider in particular the case where g is a gaussian:

$$g(t) = \exp\left(-\frac{t^2}{2}\right)$$

and choose as analyzing wavelet the negative second derivative of g

$$\varphi(t) = -\frac{d^2}{dt^2}g(t) = (1 - t^2)g(t).$$

The wavelet transform with respect to this φ has a very simple intuitive interpretation. One has, with $\partial = \frac{d}{dt}$

$$\begin{aligned}(\mathrm{L}_\varphi\psi)(b, a) &= \frac{1}{\sqrt{\mathrm{C}_\varphi}}(\mathrm{T}^b\mathrm{D}^a\partial^2 g, \psi) \\ &= -\frac{a^2}{\sqrt{\mathrm{C}_\varphi}}(\mathrm{T}^b\mathrm{D}^a g, \partial^2\psi) \\ &= -\frac{a^{3/2}}{\sqrt{\mathrm{C}_\varphi}}\int g\left(\frac{t-b}{a}\right)\psi''(t)dt\end{aligned}$$

For fixed a, the function $(\mathrm{L}_\varphi\psi)(b, \omega)$ is a gaussian average of the negative second derivative of ψ, and is thus really a picture of the negative second derivative of ψ, on a scale given by a. (Compare the less specific discussion of section g).

If the function ψ has an isolated discontinuity at, say $t = t_0$, then the function $\mathrm{L}_\varphi\psi(b, a)$ (a constant and sufficiently small) will try to imitate the derivative of a Dirac δ-function in a neighbourhood of $b = t_0$. The imitation will be closer, and the variation of $\mathrm{L}_\varphi\psi(b, a)$ more rapid, as a tends to zero.

These facts have been used, (without the formalism of wavelet transforms) in models of vision (edge detection) (We are grateful to R. Balian who brought this to our attention.)

The methods described here allow the construction of appropriate extrapolation procedures in the parameter a.

iii) Logarithmic gaussian. — Let $\alpha > 0$. Consider the function $\hat{g}_\alpha(\omega)$ defined as

$$\begin{aligned}\hat{g}_\alpha(\omega) &= \exp\left(-\frac{\alpha}{2}\ln^2(\omega)\right) \qquad && \omega > 0 \\ &= 0 && \omega < 0.\end{aligned}$$

The function $\hat{g}_\alpha(\omega)$ belongs to $\mathscr{S}(\mathbb{R})$. It has a zero of infinite order at $\omega = 0$.

Consider now $g_\alpha = \mathrm{F}^{-1}g_\alpha$ and choose it as analyzing wavelet. The resulting L_φ-transform L_α has the following basic property: If $\partial = \frac{d}{dt}$ then

$$(\mathrm{L}_\alpha\partial^n\psi)(b, a) = i^n e^{n(n+1)/2\alpha}a^{-n}(\mathrm{L}_\alpha\psi)(b, e^{n/\alpha}a)$$

which will be exploited in a forthcoming paper.

3.2. Wavelet transform on the half-line: affine coherent states.

i) The group.

In this section we restrict our attention to the subgroup of affine transformations with positive dilation parameter $a > 0$. The group law and the two Haar measures are the same as for the full group described at the beginning of this chapter. It is well known [*6*] [*7*], that this group has two faithful unequivalent irreducible unitary representations (up to unitary equivalence). Both are square integrable in the sense of (I) and we can apply the method of (I).

We will often use in this section the two following realisations of the two representations of the « $ax + b$ » group ($a > 0$) $U_\pm$ defined on $L^2(\mathbb{R}^+, x^{n-1}dx)$ for $n \geqslant 1$, by

$$(U_\pm(a, b)\psi)(x) = a^{n/2}e^{\pm ibx}\psi(ax)$$

and their Fourier transforms defined by

$$(V_\pm(a, b)\tilde{\psi})(k) = a^{n/2}\tilde{\psi}\left(\frac{k \pm b}{a}\right).$$

An easy computation gives the fact that the (unbounded) operator defined in [*1*] by (3.1) associated to $U_\pm$ is given by

$$(C\psi)(x) = x^{-1/2}\psi(x)$$

defined on the domain

$$\mathcal{D} = \left\{ \psi \in L^2(\mathbb{R}^+, x^{n-1}dx), \int |\psi(x)|^2 x^{n-2}dx < +\infty \right\}.$$

This means that, for ψ_1 and ψ_2 in $\mathcal{D}$ and every φ_1 and φ_2 in $L^2(\mathbb{R}^+, x^{n-1}dx)$, we have

$$\int (\varphi_1, U_\pm(a, b)\psi_1)(U_\pm(a, b)\psi_2, \varphi_2)\frac{dadb}{a^2} = (\varphi_1, \varphi_2)(C\psi_2, C\psi_1)$$

and allows to define

$$(L^\pm_{\varphi_0}\psi)(a, b) = \frac{1}{\sqrt{C_{\varphi_0^\pm}}}a^{n/2}\int_0^{+\infty} e^{\mp ibx}\overline{\varphi_0(ax)}\psi(x)x^{n-1}dx$$

and

$$(R^\pm_{\varphi_0}\psi)(a, b) = \frac{1}{\sqrt{C_{\varphi_0^\pm}}}a^{n/2}\int_0^{+\infty} \overline{\varphi_0(x)}e^{\pm ibx}\psi(ax)x^{n-1}dx$$

as isometric transform between $L^2(\mathbb{R}^+, x^{n-1}dx)$ and $L^2\left(\mathbb{R}^+ \times \mathbb{R}, \frac{dadb}{a^2}\right)$.

The reproducing kernels of the range of the transforms (see I(V)) is given by the function

$$P^{\pm}_{\varphi_0}(a, b) = \frac{1}{C_{\varphi_0^{\pm}}} a^{n/2} \int e^{+ibx}\overline{\varphi_0(ax)}\varphi_0(x)x^{n-1}dx .$$

The following paragraph is devoted to a family of functions φ_0 which gives a characterization of the range of $L^{-}_{\varphi_0}$ in terms of analyticity in the variable $z = b + ia$ (affine coherent states).

ii) Affine coherent states.

In this section we consider the representation U_- and the family of « analyzing wavelets » given by the functions

$$\varphi^l(x) = x^{l-1}e^{-x} \qquad l \geq 1 .$$

This family of analyzing wavelets gives rise to the following transforms

$$(L^l\psi)(a, b) = \frac{2^{l+\frac{n}{2}-1}}{\sqrt{2\pi\Gamma(2l + 2n)}} a^{l+\frac{n}{2}-1} \int_0^{+\infty} x^{l-1}e^{ibx-ax}\psi(x)x^{n-1}dx ,$$

$L^l\psi = f$ is clearly the product of an analytic function of the variable $z = b + ia$ by $a^{l+\frac{n}{2}-1}$. The general theory exposed in (I) and recalled in the introduction gives that

$$\int | f(a, b) |^2 \frac{dadb}{a^2} = \| \psi \|^2 .$$

It turns out [5] that the range of the transform L^l is exactly the set of functions which are the product of $a^{l+\frac{n}{2}-1}$ by an analytic function, square integrable with respect to the two-dimensional measure $\frac{dadb}{a^2}$ (Note the analogy with Bargmann space.)

Let us call H_l the range of L^l i. e.

$$H_l = \left\{ f(a, b) = a^{\frac{n}{2}+l-1}g(z), \quad z = b + ia, \quad a > 0, \quad g \text{ analytic with } \int | f(a, b) |^2 \frac{dadb}{a^2} < + \infty \right\}.$$

H_l can be considered as the state space of quantum mechanics of a particle moving on the half-line, or of a particle with radial hamiltonian.

The family of spaces H_l has been used in [5] and has many properties of the Bargmann space described in section (2):

— they solve the time-independent Schrödinger equation for the hydrogen atom in an explicit way [5]

— they provide a natural setting for large N (or l) limits [*5*]
— they allow us to introduce Wigner functions for the radial Schrödinger equation [*5*].

Remark. — If we consider the « $ax + b$ » group as a subgroup of SL(2, R) via the parametrization

$$\begin{pmatrix} \sqrt{a} & \dfrac{b}{\sqrt{a}} \\ 0 & \dfrac{1}{\sqrt{a}} \end{pmatrix}$$

we can consider the following realisation of the two representations of « $ax + b$ » W_+, W_-, obtained by restricting the metaplectic representation

$$(W_\pm \psi)(x) = a^{n/4} e^{\pm ib\frac{x^2}{2}} \psi(a^{1/2}x)$$

with the following choice of « analyzing wavelets »

$$\psi^l(x) = x^l e^{-x^2} \qquad l > 0 .$$

They give rise to affine coherent states very well adapted to the radial harmonic oscillator ([*5*]).

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