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Existence of an instanton singularity in $\phi^4{}_3$ Euclidean field theory

by

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ABSTRACT. — We improve the lower bound in the rigorous « Lipatov » analysis of the large order behavior of the perturbation expansion for $\phi^4{}_3$ Euclidean field theory [*1*]. This allows us to prove that, as was expected in this case, the Borel transform of this perturbation expansion has an « instanton » singularity on the negative real semi-axis in the Borel plane, at $t = -\mathrm{R}$, R being the radius of convergence of this Borel transform, which has been computed rigorously in [*1*]. We hope to extend in the future this study of Borel singularities to the more difficult problem of proving the existence of the first « renormalon » singularity on the positive axis in the Borel plane of $\phi^4{}_4$.

RÉSUMÉ. — Nous améliorons la borne inférieure dans l'analyse de Lipatov rigoureuse du comportement aux grands ordres du développement en perturbation pour la théorie des champs $\Phi^4{}_3$ euclidienne. Ceci nous permet de prouver que, comme on l'attend dans ce cas, la transformée de Borel de ce développement en perturbation a une singularité de type « instanton » sur le demi-axe réel négatif dans le plan de la transformée de Borel, à $t = -\mathrm{R}$, où R est le rayon de convergence de cette transformée de Borel, calculé de façon rigoureuse dans [*1*]. Nous espérons étendre dans le futur cette étude des singularités de la transformée de Borel au problème plus difficile de prouver l'existence de la première singularité de type renormalon sur l'axe positif du plan de la transformée de Borel pour la théorie $\Phi^4{}_4$.

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I. INTRODUCTION

The « Lipatov » method [2] [3] is a semi-classical Laplace expansion, which is in particular useful for analyzing large orders in perturbation theory. Applied to the (massive) ϕ^4 Euclidean field theory in the dimensions where it is renormalizable (1, 2, 3 and 4), it always gives the same kind of asymptotic behavior:

$$a_n^r = n\,!\,\mathrm{R}^{-n} n^b c(1 + 0(1/n)) \tag{1.1}$$

where $(-1)^n a_n^r$ is the n-th order of renormalized perturbation theory for the pressure or the connected Schwinger functions of the model, and R, b and c are numerical coefficients. Remark that if (1.1) is true, R is the radius of convergence of the Borel transform

$$\mathrm{B}(t) = \Sigma_n a_n^r(-t)^n/n\,! \tag{1.2}$$

of the perturbation expansion, and there is a singularity, called an « instanton » singularity [4], in the Borel plane at $t = -\mathrm{R}$. The Lipatov method gives the value of R (which is independent of the particular renormalization scheme, of the particular Schwinger function and of the set of external momenta one is considering):

$$\mathrm{R}^{-1} = e^{-\sigma+2}, \qquad \sigma = \inf \mathrm{S}(\phi) \tag{1.3}$$

$$\mathrm{S}(\phi) = (1/2)\int_{\mathbb{R}^d} [(\nabla\phi)^2(x) + \phi^2(x)]d^dx - \mathrm{Log}\int_{\mathbb{R}^d} \phi^4(x). \tag{1.4}$$

Spencer [5] and Breen [6] made this analysis rigorous at least for the leading behavior corresponding to the computation of R, respectively for ϕ^4 with a lattice regularization in any dimension and for $\phi^4{}_2$ in the continuum. This work was extended to $\phi^4{}_3$ in [1]. However an irritating detail was missing in [1]; although the radius R was proved to be given by formulae (1.3)-(1.4), the lower bound of [1] was too weak to insure that, as expected, there is indeed a singularity of B(t) at $t = -\mathrm{R}$ (i. e. the function B(t) cannot be analytically continued in a neighborhood of $t = -\mathrm{R}$). Indeed the blob graph B =  becomes divergent.

Therefore due to the minus sign of the corresponding counterterm, it is not obvious any more (as it is in $\phi^4{}_2$) that the series a_n^r is positive term by term. In the present sequel to [1], we obtain the existence of this « instanton » singularity, completing therefore the analysis of the leading asymptotic behavior of $\phi^4{}_3$ at large orders. We obtain this result by showing explicitly that some derivative of B(t) blows up when $t \to -\mathrm{R}$. Although this is not a necessary condition for an analytic function to have a singularity, it is obviously sufficient, and it seems the only easy criterion for locating singularities on the circle of convergence of a power series when there is no control over the signs of the coefficients.

Our interest in Borel singularities comes from attempts to prove the « Parisi conjecture » [7] that in the Borel plane of $\phi^4{}_4$ the closest singularity to the origin is a « renormalon » singularity sitting at $t = 2/\beta_2$ on the *positive* real axis (hence the renormalized $\phi^4{}_4$ series are not Borel summable). Such a proof would throw additional light on the « triviality » of $\phi^4{}_4$, from a point of view different from the results of [8] [9] (see [10] for a general discussion). Combining a « Lipatov bound » on the behavior of those pieces of perturbation theory which do not contain « useless » counterterms [11] with the « partly renormalized phase space expansion » with effective coupling constants of references [12] and [13], we have obtained the following result [14]:

THEOREM 1. — *The radius of convergence of the Borel transform of the renormalized $\phi^4{}_4$ series (with, for instance, the BPHZ prescription of subtraction at* 0 *external momenta), which was proved to be finite in [15], is at least $2/\beta_2$, where β_2 is the first non vanishing coefficient of the β function (with the usual conventions, $\beta_2 = 3/16\pi^2$ and $2/\beta_2 = 32\pi^2/3$ if the interaction is $g\phi^4/4\,!$).*

When $t \to 2/\beta_2$, the analysis of [7] can be transcribed in the language of [12] [13], and ultraviolet divergences appear in the Borel transform (corresponding to six-points subgraphs and similar objects). They lead to an expected behavior $B(t) \simeq (t - 2/\beta_2)^{\alpha-1}$, where α is not an integer [7]. Therefore one expects again that some derivative of B will blow up when $t \to 2/\beta_2$. We feel that we are close to a proof of this fact, although we have not succeeded yet in ruling out the very unlikely possibility that the many ultra-violet divergences at $t \to 2/\beta_2$ cancel each other completely (it seems to involve a subtle problem of linear independence of various singularities over the ring of analytic functions).

Let us return to the much easier problem treated in this paper, namely the existence of the first $\phi^4{}_3$ instanton singularity (which can be considered as a warm-up for the one above). To be as concrete as possible we shall restrict our attention to the pressure

$$p(g) = \lim_{\Lambda\to\mathbb{R}^3} (1/|\Lambda|) \operatorname{Log} Z(\Lambda) \tag{1.5}$$

of the $\phi^4{}_3$ model with coupling constant g, and we shall adopt the notations of [1] most of the time and assume some familiarity of the reader with it. Our main result is:

THEOREM 2. — *There exist an entire function $B_0(t)$, integers n_0 and α,* *and constants* $K > 0$, $A_{n_0} \geqslant 0$, *such that the* α^{th} *derivative of* $B(t) - B_0(t)$ *obeys:*

$$B^{(\alpha)}(-t) - B_0^{(\alpha)}(-t) + A_{n_0} \geqslant K\Sigma_{n=n_0,\dots,\infty} R^{-n} t^n = K(1-t/R)(t/R)^{n_0} \quad \forall t,\ 0 \leqslant t < R \qquad (1.6)$$

COROLLARY. — $\mathrm{Lim}_{t \to R_-} B^{(\alpha)}(-t) = +\infty$, *and* B *has a singularity at* $t = -R$.

The rest of this paper is devoted to the proof of this Theorem 2.

II. THE PROOF

We know that in $\phi^4{}_3$ the pressure has the asymptotic expansion

$$p(g) \sim \Sigma_{n=1,\dots,\infty} a_n^r(-g)^n \qquad (2.1)$$

where a_n^r is $1/n!$ times the sum of all connected, renormalized vacuum graphs containing n vertices, all of whom are ϕ^4 vertices. In the rest of the paper a graph means a « labeled graph » with distinguished vertices (hence a set of Wick contractions). Renormalization (indicated by the superscript r in (2.1)) prevents the occurrence of the graphs G_1, G_2, as well as (by Wick ordering) the occurence of any graph containing the « tadpole », i. e. the subgraph G_3 (see Fig. 1).

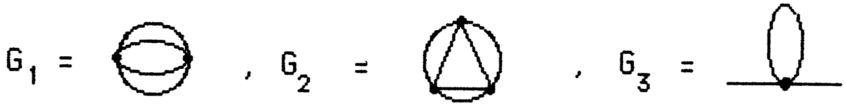

FIG. 1. — The graphs G_1, G_2 and G_3.

Hence the sum in (2.1) really starts at $n = 4$. Moreover renormalization also subtracts from every « blob » B its value at zero momentum. By translation invariance, the thermodynamic limit $\lim_{\Lambda \to \mathbb{R}^3} 1/|\Lambda| \dots$ in the definition (1.5) of $p(g)$ may be implemented at the perturbative level by holding one vertex of each vacuum graph fixed at $0 \in \mathbb{R}^3$ in position space, integrating all other vertices over all of $\mathbb{R}^3$. The Borel transform of $p(g)$ is

$$B(t) = \Sigma_n a_n^r(-t)^n/n! = \Sigma_G I_G^r(-t)^{n(G)}/[n(G)!]^2 \qquad (2.2)$$

where the sum runs over all connected vacuum graphs, I_G^r is the renormalized amplitude of G and $n(G)$ is the number of vertices of G, often called simply n in the rest of the paper. $B(t)$ is known to be analytic in $|t| < R$, R defined in (1.3) [*1*]. Similarly

$$B^{(\alpha)}(t) = \Sigma_G I_G^r(-t)^{n-\alpha}/(n-\alpha)!\,n! \qquad (2.3)$$

is analytic in $|t| < \mathrm{R}$ (we will often not write explicitly that n has to be $\geqslant \alpha$ in sums like (2.3)). The proof that, for a suitable α, $\mathrm{B}^{(\alpha)}(t)$ has a singularity at $t = -\mathrm{R}$ involves seven steps. The strategy is to relate the series $\Sigma_n a_n{}^r(-t)^n/n!$ to the series $\Sigma_n b_n(-t)^n/n!$, where

$$b_n = (1/n!) \int d\mu_{\mathrm{C}^{(n,\Lambda_n)}}(\phi) \left[\int_{\Lambda_n} d^3x : \phi^4(x) : \right]^n \tag{2.4}$$

is the coefficient of $(-g)^n$ in the perturbation expansion of the partition function (and hence contains disconnected as well as connected graphs) in a model that no longer has renormalization other than Wick ordering, but has both an order dependent volume cutoff (the compact box $\Lambda_n \subset \mathbb{R}^3$) and an order dependent ultraviolet cutoff (in the covariance $\mathrm{C}^{(n,\Lambda_n)}$ for which $d\mu_{\mathrm{C}(n,\Lambda_n)}$ is the corresponding Gaussian measure). To do so we delete all renormalized blobs (Step 1), introduce the ultraviolet cutoff (step 2), reintroduce the unrenormalized blobs (step 3), introduce the volume cutoff (step 4) and finally introduce disconnected graphs (step 5). The analysis of b_n occurs in step 6 (where b_n is bounded in terms of the infimum of a cutoff version S_n of the action S in (1.4)) and step 7 (where the infimum of S_n is related to the one of S, hence to R).

STEP 1. — In this step we show that graphs containing blobs can be ignored. This is based on two observations. Firstly, since the renormalized blob grows logarithmically in momentum space, we have for some positive C (if B′ is the graph ):

$$|\mathrm{I}_{\mathrm{B}'}{}^r(p^2)| \leqslant \mathrm{C}/(p^2+1)^{1-\varepsilon}. \tag{2.5}$$

Secondly, the number of graphs built by inserting m blobs in a blob-free graph G is small in the following sense: the number of possible ways to insert m blobs in a graph of order n is roughly $(2n)^m$ if m is small (recall that $2n$ is the number of lines of a graph with n vertices). Comparing $n!(n)^m$ with $(n+2m)!$ one gains a factor $1/n^m$, which can be used to bound the graphs with blobs by the graphs with these blobs reduced *in the full series (2.3)*, as will be done here, or in the corresponding *partial sums*, as is done in [*1*] and [*16*] (let us remark however that we did not succeed yet in doing this in a single given order of the perturbation series (since the « blob reduction » changes orders) and that therefore the conjecture that $a_n{}^r$ is positive for n large enough remains open).

To be precise, we introduce a « simplification operation » S_∞ as in [*1*] and [*16*] (but see the note added in proof of [*1*]; the operation U should be added to the original definition of S_∞ in [*16*]). It is defined in the following way. Let T be the operation which in any graph reduces every maximal chain of blobs to a single line, and U be the operation which in any graph reduces every maximal chain of tadpoles to a single line. Let $\mathrm{S} = \mathrm{U} \circ \mathrm{T}$.

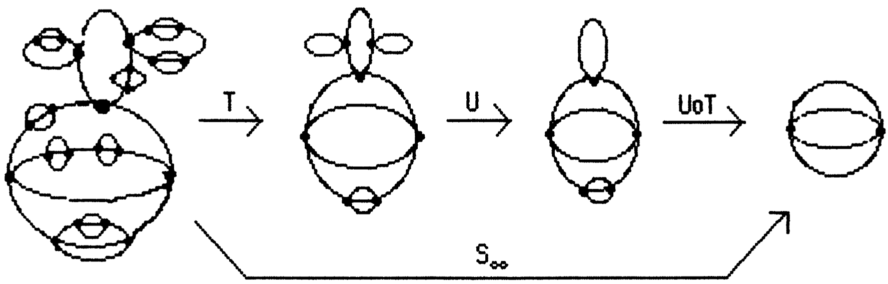

FIG. 2. — The operation S_∞.

Since any G has finitely many vertices, the sequence $S^k(G)$ must be stationary for k large enough. Then S_∞ is defined as the limit of S^k as $k \to \infty$ (see Fig. 2 for an explicit example).

The following lemma corresponds approximately to subsets of Lemmas III.4 and III.5 in [*1*]:

LEMMA 1. — Let G' be a graph with n' vertices such that $S_\infty(G') = G'$. There exists a constant K such that:

a) if $\gamma(n, G')$ is the number of graphs G, Wick ordered, with $S_\infty(G)=G'$ and $n(G) = n > n'$, one has:

$$\gamma(n, G') \leqslant K^{n-n'}[1/(n-n')!](1/n')(n!/n'!)^2 \tag{2.6}$$

b) if $S_\infty(G)=G'$ and $n(G)=n$, $n(G')=n'>3$, $|I_G^r| \leqslant K^{n-n'} I_{G'} \leqslant K^n$ (2.7)

if $S_\infty(G)=G'$ and $n(G)=n$, $n(G')=n'\leqslant 3$, $|I_G^r| \leqslant K^n$ (2.8)

Proof. — *a*) Follows from the more general combinatoric analysis in [*15*, Appendix C]. Let us sketch it here in this simpler case. With the notations of [*15*], the subgraphs of G reduced by S_∞ form a forest $\mathbf{F}$ in G such that each $F/\mathbf{F}$. $F \in \mathbf{F}$ is either a blob B or a tadpole G_3; $p = \# \mathbf{F}$ is the total number of blobs or tadpoles reduced by S_∞ (see [*15*] for the notion of the reduced graph $F/\mathbf{F}$). We organize $\mathbf{F}$ into layers $\mathbf{L}_i$, $1 \leqslant i \leqslant \lambda$

$$\mathbf{L}_1 = \{ F ; B_{\mathbf{F}}(F) = G \} ; \quad \mathbf{L}_i = \{ F ; B_{\mathbf{F}}(F) \in \mathbf{L}_{i-1} \}, \quad 2 \leqslant i \leqslant \lambda$$

where we recall that $B_{\mathbf{F}}(F)$ is the smallest graph in $\mathbf{F} \cup \{ G \}$ containing F; we put also $h_i = \# \mathbf{L}_i$. Remark that $h_i > 0$, and that the subgraphs of $\mathbf{L}_\lambda$ are the ones reduced by the first $U \circ T$ operation, the subgraphs of $\mathbf{L}_{\lambda-1}$ the ones reduced by the second $T \circ U$ operation, and so on. We observe that since the starting graph G is Wick ordered, $p < n - n'$ (since $F/\mathbf{F}$ has to be a blob for $F \in \mathbf{L}_\lambda$). To count how many G's project onto a given G′, we decide successively for each layer where to insert the corresponding blobs or bubbles. For the first layer $\mathbf{L}_1$ one has to insert on the $2n'$ original lines of G′; by a factor $(2n'+h_1-1)!/[h_1!\,(2n'-1)!] \leqslant 2^{h_1}(n'+h_1)!/(h_1!\,n'!)$

we can decide how many graphs have to be inserted on each such line of G'. Then by a factor 2^{h_1} we can decide successively for each line if the first, second, third, ... graph inserted was a blob or a tadpole. For the i-th layer, $i > 1$, the graphs have to be inserted on lines of graphs of the $i-1$ layer (hence on lines created by former insertions of blobs or tadpoles). The corresponding total number of possibilities is therefore bounded in the same way by $(3h_{i-1}+h_i-1)!/[h_i!(3h_{i-1}-1)!]2^{h_i}$, hence by $2^{3h_{i-1}+2h_i}$. Finally one should label the $n-n'$ new vertices created by the process (this can be done paying a factor $n!/n'!$). Hence since $\Sigma_i h_i = p$, we get $\gamma(n, G') \leqslant 2^{p-1}(n-n')(n!/n'!)2^{5p}(n'+h_1)!/(h_1!\,n'!)$ (The factors 2^{p-1} and $n-n'$ are for the choice of $\{ h_i \}$ with $\Sigma h_i = p$ and for the choice of p).

Using the fact that $h_1 \leqslant n - n' - 1$, one has:

$$n'(n'+h_1)!(n-n')! \leqslant 2^{n-n'}(n'+h_1+1)!(n-n'-1)! \leqslant 2^{n-n'}n!\,h_1!\,.$$

Collecting all factors give (2.6). For b) one can remark that using (2.5) with $\varepsilon < 1/2$, the tadpoles and blobs created after the first T reduction are convergent. The exponential bounds (2.7)-(2.8) are then mere consequences of the more general results of [*17*] (but in fact in this case it reduces to a trivial exercise). Remark that in (2.7) G' belongs to C, the set of all connected vacuum graphs that are completely convergent, i. e. that do not contain any divergent subgraphs. Since these graphs do not require renormalization, the superscript r has been left off $I_{G'}$. ■

LEMMA 2. — Given any α and any sufficiently large integer n_0 (depending on α) there exists an entire function $B_0(t)$ (depending on α and n_0) such that:

$$B_\alpha(t) = B^{(\alpha)}(-t) - B_0{}^{(\alpha)}(-t) \geqslant (1/2)\Sigma_{n \geqslant n_0} a_n{}^c t^{n-\alpha}/(n-\alpha)! \qquad (2.9)$$

where $a_n{}^c = (1/n!)\,\Sigma_{G \in C, n(G)=n} I_G$, the sum being over the completely convergent graphs which have n vertices, and $0 \leqslant t < R$.

Proof. — Let n_0 be an integer bigger than $\sup\{4, 2\alpha\}$. We decompose $B^{(\alpha)}(-t)$ as (with $n = n(G)$):

$$B^{(\alpha)}(-t) = \Sigma_{G, n(S_\infty(G)) < n_0} t^{n-\alpha}/[(n-\alpha)!\,n!]I_G{}^r + \Sigma_{G, n(S_\infty(G)) \geqslant n_0} t^{n-\alpha}/[(n-\alpha)!\,n!]I_G{}^r.$$

By (2.6)-(2.8), $B_0{}^{(\alpha)}(-t) = \Sigma_{G, n(S_\infty(G)) < n_0} t^{n-\alpha}/[(n-\alpha)!\,n!]I_G{}^r$ is entire in t, and:

$$B_\alpha(t) = \Sigma_{G, n(S_\infty(G)) \geqslant n_0} t^{n-\alpha}/[(n-\alpha)!\,n!]I_G{}^r \geqslant \Sigma_{n \geqslant n_0} a_n{}^c t^{n-\alpha}/(n-\alpha)! \ldots$$
$$\{ 1 - \Sigma_{k=1,\ldots,\infty}(K^{2k}/k!)[(n+k)!/n!]^2(1/n)[(n-\alpha)!\,n!]/[(n+k-\alpha)!\,(n+k)!] \} \qquad (2.10)$$

Since $n_0 > 2\alpha$, $[(n-\alpha)!\,n!]/[(n+k-\alpha)!\,(n+k)!] \leqslant 2^k[n!/(n+k)!]^2$. Hence for n_0 large enough (such that $e^{8K^2} \leqslant n_0$), the term $\{ 1 - \Sigma_k \ldots \}$ in (2.10) is larger than 1/2, and (2.9) follows from (2.10). ■

STEP 2. — In this step we introduce an ultraviolet cutoff that depends on the order n of perturbation theory we are looking at. We use a Pauli-Villars cutoff:

$$C^{(N)}=(-\Delta+1)^{-1}-(-\Delta+N)^{-1} = (N-1)(-\Delta+1)^{-1}(-\Delta+N)^{-1}. \quad (2.11)$$

Using the path integral representation

$$(-\Delta + N)^{-1}(x, y) = \int_0^\infty dte^{-Nt}\int P^t_{x,y}(d\omega) \quad (2.12)$$

where $P^t_{x,y}(d\omega)$ is the conditional Wiener measure on the set of all paths starting at x at time 0 and ending at y at time t, it is clear that:

$$0 \leqslant C^{(N)}(x, y) \leqslant (-\Delta + 1)^{-1}(x, y) \quad (2.13)$$

hence
$$a_n^c \geqslant a_n^{c,N} \quad (2.14)$$

where in $a_n^{c,N}$ amplitudes (noted I_G^N) are evaluated with the propagator $C^{(N)}$ rather than with $(p^2 + 1)^{-1}$. We have to choose N. A naive choice would be $N = n$. However in step 3 we reintroduce « bare » blobs (and associated « generalized tadpoles » containing these blobs). To end up with N equaling the order of the graph after the reintroduction of blobs, we use the following trick:

$$a_n^c = \Sigma_{k=0,\dots,\infty}2^{-(k+1)}a_n^c \geqslant \Sigma_{k=0,\dots,\infty}2^{-(k+1)}a_n^{c,n+k} \quad (2.15)$$

This leaves us with:

$$B_\alpha(t) \geqslant (1/2)\{\Sigma_{n\geqslant n_0}t^{n-\alpha}/(n-\alpha)!\,[\Sigma_{k=0,\dots,\infty}2^{-(k+1)}a_n^{c,n+k}]\} \quad (2.16)$$

STEP 3. — In this step we reintroduce the blobs but without renormalizing them (this is possible since we have now an ultraviolet cutoff). Hence we will end up with all graphs of order at least n_0 that are Wick ordered (i. e. do not contain tadpoles). In other words we end up with the Borel transform (or more precisely the α-th derivative of the part of the Borel transform of order at least n_0) for the pressure of a Wick ordered model with an order dependent ultraviolet cutoff. The corresponding result for this step is:

LEMMA 3. — There exists a constant A_{n_0} such that for $0 \leqslant t < R$:

$$\Sigma_{n\geqslant n_0}a_n^{W,n}t^{n-\alpha}/(n-\alpha)!$$
$$\leqslant 2\{\Sigma_{n\geqslant n_0}t^{n-\alpha}/(n-\alpha)!\,[\Sigma_{k=0,\dots,\infty}2^{-(k+1)}a_n^{c,n+k}]\} + 4A_{n_0} \quad (2.17)$$

where $a_n^{W,N}$ is $1/n!$ times the sum of all amplitudes (noted $I_G^{W,N}$) for connected, Wick ordered vacuum graphs G, evaluated with ultraviolet cutoff N, hence propagators $C^{(N)}$.

COROLLARY. — From (2.16) and (2.17) we have obviously:

$$B_\alpha(t) \geqslant -A_{n_0} + (1/4)\Sigma_{n\geqslant n_0} a_n{}^{W,n} t^{n-\alpha}/(n-\alpha)! \qquad (2.18)$$

Proof of Lemma 3. — We remark that in momentum space the amplitude $I_B{}^N$ of the blob with cutoff N is uniformly bounded by K' Log N, for some constant K'. We have therefore the following analogues respectively of (2.7) and (2.8) (with the same notations and conditions):

$$|I_G{}^{W,n}| \leqslant K^{n-n'}(\mathrm{Log}\, n)^{(n-n')/2} I_{G'}{}^n \leqslant K^n (\mathrm{Log}\, n)^{(n-n')/2} \qquad (2.19)$$

$$|I_G{}^{W,n}| \leqslant K^n (\mathrm{Log}\, n)^{(n-n')/2}. \qquad (2.20)$$

(Remark that we have $I_{G'}{}^n$ and not $I_{G'}{}^{n'}$ in (2.19)). We have:

$$\Sigma_{n\geqslant n_0} a_n{}^{W,n} t^{n-\alpha}/(n-\alpha)! = \Sigma_{n\geqslant n_0} a_n{}^{c,n} t^{n-\alpha}/(n-\alpha)!$$
$$+ \Sigma_{G,n(G)\geqslant n_0, S_\infty(G)\neq G} I_G{}^{W,n} t^{n-\alpha}/[(n-\alpha)!\, n!]. \qquad (2.21)$$

But by (2.19)-(2.20), $\Sigma_{G,n(G)\geqslant n_0, n(S_\infty(G))<n_0} I_G{}^{W,n} t^{n-\alpha}/[(n-\alpha)!\, n!]$ is an entire function of t and its modulus is therefore bounded uniformly for $|t| \leqslant R$ by a constant which we can call $4A_{n_0}$. Moreover, in a way similar to (2.10), by (2.19) $\Sigma_{G,n(G)\geqslant n_0, S_\infty(G)=G'\neq G \text{ and } n(G')\geqslant n_0} I_G{}^{W,n} t^{n-\alpha}/[(n-\alpha)!\, n!]$ can be bounded by

$$2\Sigma_{G,n(G)\geqslant n_0, S_\infty(G)=G} \Sigma_{k=1,\dots,\infty} 2^{-(k+1)} I_G{}^{W,n+k} (2tK^2)^k (1/k!)(1/n)[\mathrm{Log}\,(n+k)]^{k/2}$$
$$[(n+k)!/n!]^2 \{ t^{n-\alpha}/[(n-\alpha)!\, n!] \} \{ [(n-\alpha)!\, n!]/[(n+k-\alpha)!\,(n+k)!] \}$$

hence by $2\{ \Sigma_{n\geqslant n_0} t^{n-\alpha}/(n-\alpha)! [\Sigma_{k=1,\dots,\infty} 2^{-(k+1)} a_n{}^{c,n+k}] \}$ if n_0 is large enough (it suffices to require $(1/n)\{ 4K^2 \,\mathrm{Re}\, [\mathrm{Log}\,(n+k)]^{1/2}(1/k) \}^k \leqslant 1$, which can be obtained uniformly in k for n large enough). Combining this bound with (2.21) gives (2.17). ■

STEP 4. — We introduce now a volume cutoff, turning $a_n{}^{W,n}$ into $a_n{}^{W,n,\Lambda_n}$. In $a_n{}^{W,n,\Lambda_n}$, graphs have all their vertices integrated over $\Lambda_n = [-n^\beta/2, +n^\beta/2]^3$ rather than over all of $\mathbb{R}^3$ (and the value of the graph is divided by $|\Lambda_n| = n^{3\beta}$ to compensate for the lack of a fixed vertex), and the covariance $C^{(n)}$ is replaced by its analog with zero Dirichlet data on $\partial\Lambda_n$, called $C^{(n,\Lambda_n)}$. From considerations below, we need to choose β smaller than 1/6. Hence we will fix $\beta = 0.1$ in the rest of the paper. At the beginning of this step each graph has one vertex fixed and all the others integrated over all $\mathbb{R}^3$. We start by integrating the fixed vertex over Λ_n and dividing by $|\Lambda_n|$. By translation invariance, this does not change anything. Then we decrease the value of the graph (see (2.13)) by restricting the domain of integration of the remaining vertices to Λ_n. Finally we replace $C^{(n)}$ by $C^{(n,\Lambda_n)}$. This decreases still further the value of the graph since $C^{(N,\Lambda)}$ has the path integral representation:

$$C^{(N,\Lambda)}(x, y) = \int_0^t dt(e^{-t} - e^{-Nt}) \int_{\omega\cap\partial\Lambda=\phi} p^t{}_{x,y}(d\omega) \qquad (2.22)$$

(i. e. the paths ω are not allowed to cross $\partial\Lambda$). Hence

$$0 \leqslant C^{(N,\Lambda)}(x, y) \leqslant C^{(N)}(x, y). \tag{2.23}$$

Therefore at the conclusion of this step we get:

$$\begin{aligned} B_\alpha(t) &\geqslant -A_{n_0} + (1/4)\Sigma_{n\geqslant n_0} a_n^{W,n,\Lambda_n}\{t^{n-\alpha}/(n-\alpha)!\} \\ &\geqslant -A_{n_0} + (1/4)\Sigma_{n\geqslant n_0, n \text{ even}} a_n^{W,n,\Lambda_n}\{t^{n-\alpha}/(n-\alpha)!\}. \end{aligned} \tag{2.24}$$

STEP 5. — In this step we replace a_n^{W,n,Λ_n} by b_n where b_n has been defined in (2.4). Recall that in (2.4) $d\mu_C$ is the gaussian measure with mean O and covariance C, and that the Wick dots are with respect to this covariance C. Hence the only differences between a_n^{W,n,Λ_n} and b_n are that the latter contains disconnected graphs too, and that a_n^{W,n,Λ_n} contains an extra factor of $1/|\Lambda_n| = 1/n^{0.3}$.

LEMMA 4. — If n_0 is sufficiently large and $n \geqslant n_0$ is even, we have:

$$a_n^{W,n,\Lambda_n} \geqslant b_n/|\Lambda_n|\,[1 - 0(1/n^{0.2})] \geqslant (1/2)b_n/n^{0.3}.$$

Proof. — We follow the strategy of [6], eq. (3.2)-(3.3). For the duration of the proof of this lemma we fix n and write a_k for a_k^{W,n,Λ_n}, C for $C^{(n,\Lambda_n)}$ and Λ for Λ_n. We also define:

$$\begin{aligned} V(\phi) &= \int_\Lambda d^3x : \phi^4(x) : \\ d_k &= (1/k!)\int d\mu_C(\phi)V(\phi)^k \\ d_k^* &= (1/k!)\int d\mu_C(\phi)\,|V(\phi)|^k. \end{aligned}$$

Note that $d_n = b_n$. The relationship between a_k and d_k is given by

$$a_k = (1/|\Lambda|)\{d_k + \Sigma_{m=2,\dots,k}(-1)^{m-1}B(k, m)/m\}$$

where $B(k, m) = \Sigma\Pi_{i=1,\dots,m} d_{k_i}$, with the sum running over $\{\{k_i\}|\Sigma_i k_i = k, k_i \geqslant 1\}$. We shall prove below that

$$|d^*_{k-j}/d^*_k| \leqslant (C/k)^j \tag{2.25}$$

and

$$|B(k, m)| \leqslant (Cn^{0.8})^{m-1} d^*_{k-m+1} \tag{2.26}$$

with the constant C being independent of j, k, m and n. These bounds will then imply, for n even:

$$\begin{aligned} a_n &\geqslant d_n/|\Lambda|\,[1 - \Sigma_{m=2,\dots,n}(1/m)B(n, m)/d^*_n] \\ &\geqslant d_n/|\Lambda|\,[1 - \Sigma_{m=2,\dots,\infty}(Cn^{0.8})^{m-1} d^*_{n-m+1}/d^*_n] \\ &\geqslant d_n/|\Lambda|\,[1 - \Sigma_{m=2,\dots,\infty}(C^2n^{0.8}/n)^{m-1}] \\ &\geqslant d_n/|\Lambda|\,[1 - 2C^2/n^{0.2}] \end{aligned}$$

provided n is large enough.

We now prove (2.25).

$$|d^*_{k-j}/d^*_k| = k!/(k-j)!\ k^{-2j}\int|V_k(\phi/\sqrt{k})^{k-j}|\,d\mu(\phi)\left[\int|V_k(\phi/\sqrt{k})^k|\,d\mu(\phi)\right]^{-1}$$

where V_k is defined by $V(\phi)=k^2V_k(\phi/\sqrt{k})$. Were it not for the Wick ordering, V_k and V would be the same. With the help of Hölder's inequality we may derive:

$$|d^*_{k-j}/d^*_k| \leqslant k^{-j}\left[\int|V_k(\phi/\sqrt{k})^k|\,d\mu(\phi)\right]^{-j/k}.$$

To complete the proof of (2.25) it suffices to show that

$$\left[\int|V_k(\phi/\sqrt{k})^k|\,d\mu(\phi)\right]^{1/k} \geqslant C^{-1}$$

for some C independent of k and n. For $k \leqslant 32$ this may be done by explicit computation. For $k > 32$ we have for any $\psi \in C_0^\infty(\Lambda)$, using an argument that will be presented in detail in step 6 (cf. equations (2.29) to (2.31)):

$$\begin{aligned}
&\left[\int|V_k(\phi/\sqrt{k})^k|\,d\mu(\phi)\right]^{1/k}\\
&\geqslant e^{-(1/2)\langle\psi,C^{-1}\psi\rangle}\left\{\int d\mu_C(\phi)[1-(2/\sqrt{k})\langle\phi,C^{-1}\psi\rangle][V_k(\phi/\sqrt{k}+\psi)]^2\right\}^{1/2}\\
&= e^{-(1/2\langle\psi,C^{-1}\psi\rangle}\left\{\left[\int\psi^4(x)dx\right]^2(1-16/k)+\left[\int dxdy\psi^3(x)C(x,y)\psi^3(y)\right]\right.\\
&\qquad\qquad[16/k-192/k^2]\\
&+\left[\int dxdy\psi^2(x)C(x,y)^2\psi^2(y)\right][72/k^2-576/k^3]\\
&\left.+\left[\int dxdy\psi(x)C(x,y)^3\psi(y)\right][96/k^3-384/k^4]+\left[\int dxdyC(x,y)^4\right][24/k^4]\right\}^{1/2}\\
&\geqslant(1/\sqrt{2})e^{-(1/2)\langle\psi,C^{-1}\psi\rangle}\int\psi^4(x)dx,
\end{aligned}$$

since all terms dropped are positive. We shall also show in step 7 that, as $n \to \infty$, $\sup_{\psi\in C_0^\infty(\Lambda)}e^{-(1/2)\langle\psi,C^{-1}\psi\rangle}\int\psi^4(x)dx$ converges and is therefore bounded below, uniformly in n.

We prove now (2.26). By the log convexity of d^*_k, $d^*_j d^*_{k-j} \leqslant d^*_2 d^*_{k-2}$ for $2 \leqslant j \leqslant k-2$, so that:

$$\begin{aligned}
|B(k,2)| &\leqslant 2d^*_1 d^*_{k-1} + (k-3)d^*_2 d^*_{k-2}\\
&\leqslant 2d^*_1 d^*_{k-1} + c[(k-3)/(k-1)]d^*_2 d^*_{k-1}\ \text{(by (2.25))}\\
&\leqslant c.n^{0.8}d^*_{k-1}
\end{aligned}$$

for a new constant c, since $d^*_1 \leqslant (2d^*_2)^{1/2}$ and $d^*_2 \leqslant 0(n^{0.8})$ (We are using the fact that with our choice of the ultraviolet cutoff, a « linearly divergent » graph gives a divergence proportional to $n^{0.5}$). Hence by induction:

$$\begin{aligned} |B(k, m)| &= |\Sigma_{j=1,\dots,k-m+1} d_j B(k-j, m-1)| \\ &\leqslant \Sigma_{j=1,\dots,k-m+1} d^*_j (Cn^{0.8})^{m-2} d^*_{k-j-m+2} \\ &\leqslant (Cn^{0.8})^{m-1} \end{aligned}$$

as in the case of $B(k, 2)$. ■

As a consequence of Lemma 4 we have:

$$B_\alpha(t) \geqslant -A_{n_0} + (1/8)\Sigma_{n \geqslant n_0, n \text{ even}} \{ t^{n-\alpha}/(n-\alpha)! \} [1/n!n^{0.3})] \int d\mu_{C^{(n,\Lambda_n)}} \left[\int_{\Lambda_n} d^3x : \phi^4 : (x) \right]^n \quad (2.27)$$

STEP 6. — In this step we bound $\int d\mu_{C^{(n,\Lambda_n)}} \left[\int_{\Lambda_n} d^3x : \phi^4(x) : \right]^n$ in terms of

$$\sigma_n = \inf \{ S_n(\phi) \mid \phi \in C_0'(\Lambda_n) \} \quad (2.28a)$$

where $$S_n(\phi) = (1/2) \langle \phi, C^{(n,\Lambda_n)^{-1}} \phi \rangle - \operatorname{Log} \int_{\Lambda_n} \phi^4(x) d^3x . \quad (2.28b)$$

The strategy is the same as in [6], eq. (2.1)-(2.3). Let us denote:

$$V(\phi) = \int_{\Lambda_n} d^3x : \phi^4(x) : \quad (2.29)$$

$$V_n(\phi) = \int_{\Lambda_n} d^3x \{ \phi^4(x) - (6/n) C^{(n,\Lambda_n)}(x,x) \phi^2(x) + (3/n^2) [C^{(n,\Lambda_n)}(x,x)]^2 \} \quad (2.30)$$

Then for any $\psi \in C_0'(\Lambda_n)$ (we drop temporarily the (n, Λ_n) in $C^{(n,\Lambda_n)}$) :

$$\begin{aligned} \int d\mu_C V(\phi)^n &= n^{2n} \int d\mu_C [V_n(\phi/\sqrt{n})]^n \\ &= n^{2n} e^{-(n/2)\langle \psi, C^{-1}\psi \rangle} \int d\mu_C \{ e^{-2/\sqrt{n}\langle \phi, C^{-1}\psi \rangle} [V_n(\phi/\sqrt{n} + \psi)]^2 \}^{n/2} \\ &\geqslant n^{2n} e^{-(n/2)\langle \psi, C^{-1}\psi \rangle} \left\{ \int d\mu_C e^{-2/\sqrt{n}\langle \phi, C^{-1}\psi \rangle} [V_n(\phi/\sqrt{n} + \psi)]^2 \right\}^{n/2} \end{aligned}$$

by Jensen's inequality, hence:

$$\begin{aligned} &\int d\mu_C V(\phi)^n \\ &\geqslant n^{2n} e^{-(n/2)\langle \psi, C^{-1}\psi \rangle} \left\{ \int d\mu_C [1 - (2/\sqrt{n}) \langle \phi, C^{-1}\psi \rangle] [V_n(\phi/\sqrt{n} + \psi)]^2 \right\}^{n/2} \\ &\geqslant n^{2n} e^{-(n/2)\langle \psi, C^{-1}\psi \rangle} \left\{ \left[\int d^3x \psi^4(x) \right]^2 \right. \\ &\qquad \left. - (2/\sqrt{n}) \int d\mu_C \langle \phi, C^{-1}\psi \rangle [V_n \phi/\sqrt{n} + \psi)]^2 \right\}^{n/2} \quad (2.31) \end{aligned}$$

where in the last step we have dropped some positive terms.

The integral $(2/\sqrt{n})\int d\mu_C \langle \phi, C^{-1}\psi \rangle [V_n(\phi/\sqrt{n}+\psi)]^2$ may be evaluated explicitly. It is the sum of four terms:

$$(16/n)\left[\int d^3x\psi^4(x)\right]^2$$

$$(192/n^2)\int d^3xd^3y\psi^3(x)C(x,y)\psi^3(y) \leqslant 0(1)(1/n^2)\left[\int d^3x\psi^4(x)\right]^{3/2}$$

$$(576/n^3)\int d^3xd^3y\psi^2(x)C^2(x,y)\psi^2(y) \leqslant 0(1)(1/n^3)\left[\int d^3x\psi^4(x)\right]$$

$$(384/n^4)\int d^3xd^3y\psi(x)C^3(x,y)\psi(y) \leqslant 0(1)[(\mathrm{Log}\, n)/n^4]\left[\int d^3x\psi^2(x)\right]$$

$$\leqslant 0(1)[(\mathrm{Log}\, n)/n^{5/2}]\left[\int d^3x\psi^4(x)\right]^{1/2}$$

since by elementary power counting $C(x, y)$ (resp. $C^2(x, y)$, $C^3(x, y)$) is the kernel of an integral operator from L^4 to $L^{4/3}$ (resp. L^2 to L^2, L^2 to L^2), whose norm is bounded uniformly in n (resp. bounded uniformly in n, bounded by $0(1)\,\mathrm{Log}\, n$). Furthermore for any ψ obeying $S_n(\psi) \leqslant \sigma_n + 1$, we have $\int \psi^4(x)d^3x \geqslant e^{-\sigma}n^{-1} \geqslant 0(1)$, since, as we shall show in step 7, σ_n has a finite limit as $n \to \infty$. Hence for any such ψ's we may continue (2.31) into:

$$\int d\mu_C V(\phi)^n \geqslant n^{2n}e^{-(n/2)\langle\psi, C^{-1}\psi\rangle}\left[\int d^3x\psi^4(x)\right]^n [1-0(1/n)]^{n/2} \quad (2.32)$$

hence

$$\int d\mu_C V(\phi)^n \geqslant n^{2n}e^{-n\sigma_n}[1 - 0(1/n)]^{n/2}. \quad (2.33)$$

Hence (2.27) becomes

$$B_\alpha(t) \geqslant -A_{n_0} + (1/8)\Sigma_{n\geqslant n_0, n\,\mathrm{even}}\{t^{n-\alpha}/(n-\alpha)!\}(1/n!n^{0.3})n^{2n}e^{-n\sigma_n}[1-0(1/n)]^{n/2}$$
$$\geqslant -A_{n_0} + 0(1)\Sigma_{n\geqslant n_0, n\,\mathrm{even}}t^{n-\alpha}(n^n/n!)^2n^{\alpha-0.3}e^{-n\sigma_n} \quad (2.34)$$

(assuming n_0 is large enough, depending on α), hence

$$B_\alpha(t) \geqslant -A_{n_0} + 0(1)\Sigma_{n\geqslant n_0, n\,\mathrm{even}}t^{n-\alpha}n^{\alpha-1.3}e^{-n(\sigma_n-2)} \quad (2.35)$$

by Stirling's formula. To complete the proof of the theorem it suffices to choose $\alpha = 2$ and to prove that $\sigma_n = \sigma + 0(1/n)$. This is done in the next and last step.

STEP 7. — We remind the reader that

$$S(\psi) = (1/2)\int_{\mathbb{R}^3} [(\nabla\psi)^2(x) + \psi^2(x)]d^3x - \operatorname{Log}\int_{\mathbb{R}^3} \psi^4(x)d^3x$$

$$S_n(\psi) = (1/2)\langle \psi, C^{(n,\Lambda_n)^{-1}}\psi \rangle - \operatorname{Log}\int_{\Lambda_n} \psi^4(x)d^3x .$$

$$\sigma_n = \inf\{ S_n(\psi) \mid \psi \in C_0^\infty(\Lambda_n) \}$$

$$\sigma = \inf\{ S(\psi) \mid \psi \in W^{1,2}(\mathbb{R}^3) \} .$$

In this step we prove that σ and σ_n exist and that $\sigma \leqslant \sigma_n \leqslant \sigma + 0(1/n)$. For any $\psi \in C_0^\infty(\Lambda_n)$, we have $S_n(\psi) \geqslant S(\psi)$, so the existence of σ, σ_n and the inequality $\sigma \leqslant \sigma_n$ all follow simply from the Sobolev inequality $\| \psi \|_{L^4(\mathbb{R}^3)} \leqslant 0(1) \| \psi \|_{W^{1,2}(\mathbb{R}^3)}$. Furthermore following [*6*, Lemma 2.1], we may construct a function $Q_c \in W^{1,2}(\mathbb{R}^3)$ which is positive, monotone decreasing, radially symmetric and minimizes S: choose a sequence of functions $Q_i \in W^{1,2}(\mathbb{R}^3)$ such that $\sigma = \lim_{i\to\infty} S(Q_i)$; take their Schwarz symmetrization [*18*, Definition 3.3], called Q_i^{**}; since $\| Q \|_{L^p} = \| Q^{**} \|_{L^p}$, $\| \nabla Q^{**} \|_{L^2} \leqslant \| \nabla Q \|_{L^2}$ we still have $\sigma = \lim_{i\to\infty} S(Q_i^{**})$; apply [*19*, Lemma 1] to prove that $| Q_i(x) | \leqslant c \, | x |^{-1}$ for all $| x | \geqslant 1$ and all i; use the Rellich-Kondrachov theorem [*20*, page 144, part I, with $j = 0, m = 1, p = 2, q = 4$, $k = n = 3$, and Theorem 2.22 with $p = 4$] to prove that $\{ Q_i^{**} \}$ has a subsequence which converge strongly in L^4 and, by the weak compactness of the unit ball of $W^{1,2}(\mathbb{R}^3)$ [*20*, Theorem 3.5], weakly in $W^{1,2}(\mathbb{R}^3)$ to $Q_c \in W^{1,2}(\mathbb{R}^3) \cap L^4$; finally, by the weak lower semi-continuity of the norm in $W^{1,2}(\mathbb{R}^3)$, $\sigma = S(Q_c)$. We have then:

LEMMA 5. — *a*) Q_c is C^∞ and
b) Q_c and all its derivatives decay exponentially fast at infinity.

Proof. — *a*) Since Q_c minimizes S(Q) it is a weak solution of the differential equation $-\Delta Q = -Q + \lambda Q^3$, where λ is a constant (namely $4\Big/\left(\int Q^4(x)d^3x\right)$. Fix any open sphere $\Omega \subset \mathbb{R}^3$. Il we combine now the Sobolev, imbedding theorem [*20*, p. 97, case A, formula (3) with $n=3, j=i-1, m=1$, $q=6$, $p=2$], which implies

$$\psi \in W^{j,2}(\Omega) \Rightarrow [-\psi + \lambda\psi^3] \in W^{j-1,2}(\Omega) \tag{2.36}$$

with Friedrichs theorem [*21*, p. 177, with $p = j - 1$, $m = 1$], which implies

$$\{ \psi \in W^{j-1,2}(\Omega), -\Delta Q = \psi \} \Rightarrow Q \in W^{j+1,2}(\Omega) \tag{2.37}$$

we get:

$$\{ Q \in W^{j,2}(\Omega), -\Delta Q = -Q + \lambda Q^3 \} \Rightarrow Q \in W^{j+1,2}(\Omega) . \tag{2.38}$$

But $Q_c \in W^{1,2}(\Omega)$, so by induction $Q_c \in W^{j,2}(\Omega)$ for all j, and by the Sobolev imbedding theorem [*20*, p. 92, case c with $p = 2, m = 2$], $Q_c \in C^\infty(\Omega)$.

b) Q_c is radially symmetric, C^∞ and obeys the differential equation $-\Delta Q_c = -Q_c + \lambda Q_c^3$. Hence by [*22*, Lemma 7], Q_c decays exponentially at infinity. To handle the derivatives of Q_c we first observe that if $(-\Delta + 1)\psi = f$ with f a C^∞ function exponentially decaying at infinity, then any first order derivative $D\psi$ of ψ also decays exponentially. This follows from:

$$D\psi(x) = \int d^3y D_x C(x, y) f(y) \qquad (\text{with } C = (-\Delta + 1)^{-1})$$

and from the bound:

$$|D_x C(x, y)| \leqslant 0(1)[1 + |x - y|^{-2}]e^{-|x-y|}.$$

Then we observe that the n-th order derivative $D^n Q_c$ of Q_c obeys the equation $(-\Delta + 1)D^n Q_c = f$, with f being a C^∞ function built only out of derivatives of Q_c of order n and lower. Hence the result follows by induction. ■

Step 7 will be completed by one final lemma:

LEMMA 6.

$$\sigma_n \leqslant \sigma + 0(1/n). \tag{2.39}$$

Proof. — We simply construct a test function $k_n(x)Q_c(x) \in C_0^\infty(\Lambda_n)$ obeying $S_n(k_n Q_c) \leqslant S(Q_c) + 0(1/n) = \sigma + 0(1/n)$. Fix any $k \in C_0^\infty([-1/2, +1/2]^3)$ which is one on $[-1/4, +1/4]^3$ and obeys $0 \leqslant k \leqslant 1$, and define $k_n(x) = k(x/n^{0.1})$. Then, since $k_n Q_c \in C_0^\infty(\Lambda_n)$, one has (using (2.11)):

$$\begin{aligned} S_n(k_n Q_c) &= S(k_n Q_c) + [1/(2n-2)] \langle k_n Q_c, (-\Delta + 1)(-\Delta + 1)k_n Q_c \rangle \\ &\leqslant S(k_n Q_c) + 0(1/n) \end{aligned} \tag{2.40}$$

since k_n and all its derivatives are bounded uniformly in n and Q_c and all its derivatives are in L^2. Finally:

$$\begin{aligned} S(k_n Q_c) = S(Q_c - (1 - k_n)Q_c) = S(Q_c) - \langle (1 - k_n)Q_c, (-\Delta + 1)Q_c \rangle + \ldots \\ (1/2)\langle (1 - k_n)Q_c, (-\Delta + 1)(1 - k_n)Q_c \rangle - \text{Log}\left[\int (k_n Q_c)^4 dx \Big/ \int Q_c^4 dx\right] \\ \leqslant S(Q_c) + 0(1) \| (1 - k_n)Q_c \|_{L^2} - \text{Log}\left\{1 - \left[\int (1 - k_n^4)Q_c^4 dx \Big/ \int Q_c^4 dx\right]\right\} \end{aligned}$$

since $\| (-\Delta + 1)Q_c \|_{L^2} \leqslant 0(1)$, and $\| (-\Delta + 1)(1 - k_n)Q_c \|_{L^2} \leqslant 0(1)$. Hence:

$$S(k_n Q_c) \leqslant S(Q_c) + 0(1/n) - \text{Log}\{1 - 0(1/n)\} \tag{2.41}$$

since $| Q_c | \leqslant 0(1)e^{-0(1)|x|}$. We obtain:

$$S(k_n Q_c) \leqslant S(Q_c) + 0(1/n). \tag{2.42}$$

Combining (2.40) and (2.42) achieves the proof of (2.39), hence of (1.6). ■

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