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Small random perturbations of infinite dimensional dynamical systems and nucleation theory

by

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ABSTRACT. — We consider the stochastic differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - V'(u) + \varepsilon\alpha;\ u(x,\ 0) = u_0(x);\ u(0,\ t) = u(L,\ t) = 0,\ x \in [0,\ L],\ t > 0$$

where α is the standard space-time white noise and V is a double well non symmetric potential. The equation without the white noise term ($\varepsilon = 0$) exhibits several equilibria two of which are stable. We study, in the double limit zero noise and thermodynamic limit ($\varepsilon \to 0$, $L \to \infty$), the large fluctuations and compute the transition probability between the two stable equilibria (tunnelling). In this way we extend previous results of Faris and Jona-Lasianio [*1*] who considered the symmetric problem in a fixed interval [0, L]. The unique stationary measure associated to the stochastic process described by our equation is strictly related to the Gibbs measure for a ferromagnetic spin system subject to a Kac interaction. Our double limit corresponds to the one considered by Lebowitz and Penrose in their rigorous version of the mean field theory of the first order phase transitions. The tunnelling between the two (non equivalent) equilibrium configura-

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tions is interpreted as the decay from the metastable to the stable state. Our results are in qualitative agreement with the usual nucleation theory.

RÉSUMÉ. — On considère l'équation différentielle stochastique

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-V'(u)+\varepsilon\alpha\,;\ u(x,0)=u_0(x)\,;\ u(0,t)=u(L,t)=0,\ x\in[0,L],\ t>0,$$

où α est le bruit blanc standard dans l'espace temps et V un potentiel à double puits non symétrique. L'équation sans le terme de bruit blanc ($\varepsilon = 0$) possède plusieurs états d'équilibre dont deux sont stables. Dans la double limite du bruit nul et du volume infini ($\varepsilon \to 0$, $L \to \infty$), on étudie les grandes fluctuations et on calcule la probabilité de transition entre les deux états d'équilibre stables (effet tunnel). On étend ainsi des résultats antérieurs de Faris et Jona-Lasinio *[1]* qui ont considéré le problème symétrique dans un intervalle fixe $[0, L]$. L'unique mesure stationnaire associée au processus stochastique décrit par notre équation est reliée de façon stricte à la mesure de Gibbs d'un système de spins ferromagnétiques avec une interaction de Kac. Notre double limite correspond à celle considérée par Lebowitz et Penrose dans leur version rigoureuse de la théorie du champ moyen pour les transitions de phase du premier ordre. L'effet tunnel entre les deux configurations d'équilibre (non équivalentes) est interprété comme le passage de l'état métastable à l'état stable. Nos résultats sont en accord qualitatif avec la théorie usuelle de la nucléation.

SECTION I

INTRODUCTION

In this paper we study the large deviations for an infinite dimensional dynamical system subject to small random perturbations.

The model that we consider is (formally) defined by the following stochastic differential equation.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-V'(u)+\varepsilon\alpha \qquad 1.1$$

where the field u depends on the one-dimensional space variable $x \in [0, L]$ and on the time t; $u(x, t)$ satisfies Dirichlet boundary conditions:

$$u(0, t) = u(L, t) = 0$$

α is the standard space time white noise, i. e. α is a Gaussian random field with zero mean and covariance:

$$\mathbb{E}(\alpha(x, t), \alpha(x', t')) = \delta(x - x')\delta(t - t') \qquad 1.2$$

Obviously equations 1.1 and 1.2 do not make sense in their present form and we refer to Section 2 where precise definitions will be given.

Eq. 1.1 describes a stochastic diffusion process whose trajectories are made by continuous functions of time taking values in some space of functions (profiles) of the space variable x; it is obtained by adding a white noise forcing term with intensity ε to a deterministic non linear heat equation which is of gradient type: in fact eq. 1.1 can be written, for $\varepsilon = 0$, as:

$$\frac{\partial u}{\partial t} = - \frac{\delta S(u)}{\delta u} \qquad 1.3$$

where

$$S(u) = \int_0^L \left[\frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + V(u) \right] dx \qquad 1.4$$

The function $V(u)$ is called potential; the functional $S(u)$ is called stationary action.

The typical example of the kind of potentials that we shall consider is of the form $V(u) = \overline{V}_{a,h}(u) = V_h(u + a)$ where $V_h(u) = V_0(u) + hu$ and $V_0(u)$ is an even two well shaped function. The graph of $V(u)$ is given in fig. 1.

The simplest physical interpretation of eq. 1.1 is that of an elastic string moving in a very viscous noisy environment, submitted to a non symmetric ($h \neq 0$) anharmonic potential and satisfying non zero boundary conditions ($a \neq 0$).

Our model is very similar to the one considered in [1] by Faris and Jona-Lasinio and, for the moment, the only difference is that our potential $V(u)$ is non symmetric.

In [1], henceforth referred to as F.-J., the authors study the large deviations giving explicit estimates to the probability of some tunelling events; they show that for fixed L, in the limit $\varepsilon \to 0$ the analysis developed by Ventzell and Freidlin for the finite dimensional case is still valid; but, as it was remarked in [1], the case where the interval $[0, L]$ is replaced by the full real line is not a trivial extension of the previous one and, probably, it requires completely different considerations.

In the framework of the above mentioned physical interpretation, the infinite volume (thermodynamic) limit should describe the motion of an infinitely extended string but the interest of this limit can better be understood if we consider another physical application of our equation, namely the so called stochastic quantization. This method has been recently proposed by Parisi and Wu [2] to study quantum field theory: the aim is to obtain the probability distribution associated to a Euclidian quantum field theory in d dimensions as the limiting (stationary) distribution of a diffusion process taking values on fields on a d-dimensional space.

The stationary measure for the stochastic process described by our equation, whose formal density is given by $\exp[-S(u)/\varepsilon^2]$, is associated to the quantum mechanical description of an anharmonic oscillator. This description, of course, becomes complete only in the limit $L \to \infty$. In this paper we assume that $\varepsilon = \varepsilon(L)$ and study a joint thermodynamic and zero noise limit.

We find that for $\varepsilon = \exp[-\exp(cL)]$, $c > 0$, the basic features of the F.-J. analysis still hold.

From a mathematical point of view, this provides some informations about the « true » infinite volume limit, giving an estimate (upper bound) for the dependence on L that still preserves the finite length and then the finite dimensional picture.

From a physical point of view the relevance of the above mentioned joint limit becomes clear in the framework of a third possible interpretation of our equation 1.1 which constitutes the main motivation of the present paper. It comes from the theory of metastability in statistical mechanics.

Let us say a few words about the phenomenon of metastability in the case of ferromagnetic phase transitions at low temperature. If we add a small positive magnetic field to a negatively spontaneously magnetized spin system, then, in some particular experimental situations, the system, instead of undergoing the right phase transition towards a stable state with positive magnetization, still persists for a long (macroscopic) time in an apparently equilibrium situation (called « metastable equilibrium ») with negative magnetization, untill an external perturbation or a spontaneous large fluctuation, which « nucleates » the new phase, starts an ireversible process that leads the system to the correct stable equilibrium.

For more details on metastability and on its possible rigorous theory we refer to the review papers [*3*] [*4*].

In a recent paper [*5*] a « pathwise approach » to metastability has been proposed in a general context of stochastic dynamics. The Contact Process, as well as the simple Curie Weiss mean field model, were studied from this point of view. The dynamics of this last model is very similar to the one described by an Itô *one dimensional* stochastic differential equation where the drift is given by the derivative of a double wells potential like our $V(u)$.

Physically, u represents the magnetization, $V(u)$ the canonical free energy, h the external magnetic field (see [*5*] for more details).

In another paper [*6*] some generalizations to finite dimensional cases were considered where, in order to perform the « pathwise approach » program proposed in [*5*], a wide use of the fundamental works of Ventzell-Freidlin was made [*7*] [*8*].

We notice that the Curie Weiss theory in spite of its extreme simplicity (only one variable, the total magnetization, is introduced to describe the state of the system; absence of any spatial structure) is completely unacceptable from the point of view of statistical mechanics: in fact, since in this theory the range of the interaction among the spins equals the volume of the container, we get that the fundamental property of the free energy that ensures the thermodynamic stability, namely the convexity, is violated. And, even worse, it is this unphysical feature of the free energy that provides the potential barrier (or better, the activation energy) necessary to produce the metastable behaviour.

In the framework of equilibrium statistical mechanics a rigorous and acceptable version of the mean field theory is now well known. It is based on the introduction of the so called « Kac potentials » [*9*]. In the following we shall describe how the Gibbs equilibrium measure of a simple one dimensional model « à la Kac » can be related to the unique stationary measure of the process described by eq. 1.1, giving rise to a dynamical theory where all the above mentioned unpleasant and unphysical features are absent.

The spatial structure of the model will play an important role: it will be necessary to garantee the correctness of the theory but, at the same time, it will produce a remarkable complication with respect to the naive Curie-Weiss model.

Consider a finite interval $\Lambda = [1, N]$ on the lattice $\mathbb{Z}$, with $N = ML$, M, L positive integers.

A spin σ_α ranging from $-\infty$ to $+\infty$ is sitting on each site α of Λ. We suppose given an *a priori* independent measure on each site: $\rho(\sigma_\alpha)d\sigma_\alpha$ with the properties:

a) $\rho(\sigma_\alpha)$ is even

b) $$\gamma_\rho(t) = \ln \int e^{t\sigma}\rho(\sigma)d\sigma < +\infty \qquad \forall t \in \mathbb{R}$$

and $$\frac{d^3}{dt^3}\gamma_\rho(t) \leqslant 0 \qquad \forall t > 0$$

c) $$\int \exp\,[t\sigma^2]\rho(\sigma)d\sigma < +\infty \qquad \forall t \in \mathbb{R}$$

Let us remark that these conditions are the same as in [*10*] where generalized Curie-Weiss was studied from a rigorous point of view.

The volume Λ is subdivided into L disjoint blocks $A_1, \ldots, A_L$ of length $|A_i| = 1/\gamma \equiv M$, the Hamiltonian is given by:

$$H = -\gamma\sum_{i=1}^{L}\Big[J_0\sum_{\substack{\alpha\in A_i\\ \alpha'\in A_i}}\sigma_\alpha\sigma_{\alpha'} + J_1\sum_{\substack{\alpha\in A_i\\ \alpha'\in A_{i+1}}}\sigma_\alpha\sigma_{\alpha'}\Big] - h\sum_{i=1}^{L}\sum_{\alpha\in A_i}\sigma_\alpha \qquad 1.5$$

with $A_{L+1} = A_1$

where J_0, J_1 and h are positive and in equation 1.5 we have chosen for the sake of simplicity periodic boundary conditions on Λ. The Hamiltonian H describes a (not-strictly translationally invariant) interaction of range $M = 1/\gamma$ and strengh γ.

Let us introduce the quantity

$$Z_N^{\Delta}(m, \beta) = \int \prod_{\alpha=1}^{N} \rho(\sigma_\alpha) d\sigma_\alpha \exp\left[-\beta H_0(\underline{\sigma}_N) \right] \chi_\Delta\left(\frac{1}{N} \sum_{\alpha=1}^{N} \sigma_\alpha - m \right) \quad 1.6$$

where

$$\chi_\Delta(x) = \begin{cases} 1 & \text{if } |x| \leqslant \Delta \\ 0 & \text{if } |x| > \Delta \end{cases} \quad 1.7$$

H_0 is Hamiltonian 1.5 with $h = 0$ and $N = L/\gamma$.

We define, in general, the canonical free energy for a continuous spin magnetic system in the following way

$$\psi(m, \beta) = \frac{1}{\beta} \lim_{\Delta \to 0} \lim_{N \to \infty} \psi_{N,\Delta}(m, \beta) \quad 1.8$$

where

$$\psi_{N,\Delta}(m, \beta) = -\frac{1}{N} \ln Z_N^{\Delta}(m, \beta)$$

Now consider the canonical free energy of our model for fixed L, and call it $\psi_{\gamma,L,\Delta}(m, \beta)$. In theorem A.1 of Appendix A we show that, in the so called Van der Waals [9] limit ($\gamma \to 0$, $L \to \infty$) followed by the limit $\Delta \to 0$, the function $\psi_{\gamma,L,\Delta}(m, \beta)$ converges to a convex function of m that turns out to be the convex envelop of the corresponding Curie-Weiss free energy ($L = 1$). Moreover, we show that the limits $\gamma \to 0$, $L \to \infty$ can be interchanged and also taken simultaneously with arbitrary relative velocity: namely, we can take $\gamma = \gamma(L)$ with an arbitrary dependance $\gamma(L)$, provided $\gamma(L) \to 0$ as $L \to \infty$.

In this way we extend to our case the well known result obtained by Lebowitz and Penrose for translationally invariant Kac potentials [9].

Now for L and γ fixed we can introduce the variables

$$m_i = \gamma \sum_{\alpha \in A_i} \sigma_\alpha$$

(average magnetization in the block A_i).

It follows from Appendix A that, for γ small, the Gibbs measure associated to our model is well approximated, for $h \neq 0$, by the following distribution

$$P(m_1, \ldots, m_L) \sim \exp\left[-\frac{1}{\gamma} F(m_1, \ldots, m_L) \right] \quad 1.9$$

where

$$F(m_1, \ldots, m_L) = \sum_{i=1}^{L} [V_h(m_i) + \beta J_1(m_i - m_{i+1})^2] \quad \text{with} \quad m_1 = m_{L+1}$$

and

$$V_h(m) = -\beta(J_0 + J_1)m^2 - hm + \inf_t \left\{ tm - \ln \int \exp [t\sigma]\rho(\sigma)d\sigma \right\} \qquad 1.10$$

In eq. 1.9 we have assumed periodic boundary conditions but the modification to describe arbitrary boundary conditions is obvious. In particular for $m_0 = m_{L+1} = a$, by translating all the spin variables by $-a$ we get:

$$\begin{aligned} F(m_1, \ldots, m_L) = \sum_{i=1}^{L-1} [V_h(m_i + a) + \beta J_1(m_i - m_{i+1})^2] + \\ + V_h(m_1 + a) + V_h(m_2 + a) + \beta J_1(m_1^2 + m_L^2) \end{aligned} \qquad 1.11$$

Looking at the expression in eq. 1.9, one immediately realizes that the stationary measure with formal density $\exp[-S(u)/\varepsilon^2]$ is in fact, a continuous version of $\exp[-1/\gamma F(m_1, \ldots, m_L)]$ provided $\gamma = \varepsilon^2$: it suffices to substitute the sum with the integral and the finite difference with the derivative.

We remark that, as far as we know, no limit exists in which $F(\underline{m_L})$ reduces to $S(u)$; furthermore, the dynamics described by eq. 1.1 involves only global variables and it is not even clear how to relate a discrete version of eq. 1.1 to a Glauber-like microscopic dynamics involving the single spin variables σ_α. But it is reasonable to think that the discrete and the continuous model share most of the physically relevant features, so that eq. 1.1 provides a qualitatively good description for the time evolution of the magnetization profile of a ferromagnetic Kac's system.

Let us now briefly outline the time evolution of a system described by eq. 1.1 to illustrate how nicely it fits with the phenomenological nucleation theory. As we shall see in the next section, when the random noise is absent ($\varepsilon = 0$) there exists only two stable equilibria corresponding to two non equivalent minima of the stationary action $S(u)$. Apart from the effect of the boundary conditions, these configurations are spatially homogeneous ($u(x) \sim$ const.). For one of them (the local minimum) the constant value of the magnetization is antiparallel to the external magnetic field h (metastable state), whereas for the other (the absolute minimum of S) it is parallel (stable state). If at time $t = 0$ we are in the metastable state and switch on the random noise keeping ε very small, the magnetization profile starts oscillating near the local minimum untill some very large fluctuation leads the system near to the absolute minimum. We evaluate the probability of the occurence of the tunnelling in a given large interval of time: this quantity is related to the mean transition time.

We get that the most probable mechanism of tunnelling is given by a path going « up » against the gradient of $S(u)$ and passing near a particular saddle point of S.

The spatial structure of this saddle point is, in our case ($h \neq 0$), characterized by a very pecular profile (see fig. 2): the magnetization has the value typical of the metastable state all over L, except for a finite region (practically independent from L) where the magnetization takes an opposite value.

This region can be interpreted as the critical droplet and its formation turns out to be necessary to allow the growth of the new stable phase all over the volume. The position of this critical droplet is affected by the choice of the boundary conditions; in other words what we describe is not the homogeneous nucleation but, rather, the nucleation in presence of « defects ».

In section 2 we shall clarify this point; now we only say that, meanwhile from a static point of view all the relative velocities in the joint limit $\gamma \to 0$, $L \to \infty$ are equivalent (in the sense that they all produce the same limiting convex free energy), from the dynamical point of view we expect very different kind of behaviours and it is only when the strength of the noise goes to zero sufficiently rapidly, with respect to the volume, that we are able to avoid the phenomena of the true infinite volume situation. In this last case, the large local fluctuation can take place everywhere in the space and even a global description of the profile looses any meaning.

The organization of the paper will be the following:

In section 2 we give the basic definitions and properly define our stochastic process via an integral equation. We further classify the critical points of $S(u)$ and state our main result in Theorem 2.1.

In section 3 we prove the theorem. The strategy is obvious: since for any fixed L the situation is similar to the one already considered in F.-J. what we have to do is to trace back the L dependence in all the F.-J. estimates and adapt some arguments to our non symmetric case.

It turns out that the main new estimate, that we need, concerns the time needed by the deterministic flow, described by eq. 1.1 when $\varepsilon = 0$, to reach the neighborhood of a stable configuration starting from the vicinities of the saddle point.

Section 4 is devoted to this estimate: we shall find an upper bound to these times containing an explicit L-dependence.

Appendix A is devoted to the analysis of the previously defined statistical mechanical model; appendix B contains the proof of some results concerning the critical points of $S(u)$.

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SECTION 2

DEFINITIONS AND RESULTS

Let us start by saying that in this paper, our exposition will not be self-contained: we shall rephrase several definitions and results of F.-J., when we need their adaptation to our case, but we shall, often, quote the F.-J. paper without reporting the arguments of proof.

In the sequel we shall use the following notations: $|||\ \ |||_p$ will denote the L^p norm of a function of the space and time variables; given a function f, we set:

$$|||\, f\, |||_p = \left(\int_0^T dt \int_0^L dx\, |\, f(x, t)\, |^p \right)^{1/p}$$

Moreover

$$|||\, f\, |||_\infty = \sup_{x \in [0,L],\, t \in [0,T]} |\, f(x, t)\, |$$

We shall denote by $||\ \ ||_p$ the L^p norm of a function of the space variable:

given $$u : [0, L] \to \mathbb{R}$$

we set $$||\, u\, ||_p = \left(\int_0^L dx\, |\, u(x)\, |^p \right)^{1/p}$$

and

$$||\, u\, ||_\infty = \sup_{x \in [0,L]} |\, u(x)\, |$$

Equation (1.1) that has been introduced at a formal level in the introduction can be restated as an integral equation:

$$u = -\, GV'(u) + \varepsilon W + g u_0 \qquad 2.1$$

Where

(1) g is the integral operator that solves the initial value problem for the heat equation with Dirichlet boundary conditions (D. b. c.) on $[0, L]$.

The Kernel of g has the following expression

$$g(x, y, t) = \frac{2}{L} \sum_{n=1}^{\infty} \exp\left[-\frac{n^2\pi^2}{L^2} t \right] \sin \frac{n\pi x}{L} \sin \frac{n\pi y}{L} \qquad 2.2$$

Notice that for any positive t the operator g maps the space $C(0, L)$ of continuous functions on $[0, L]$ into the space $C_D(0, L)$ of continuous functions on $[0, L]$ with (D. b. c.) on $0, L$.

(2) $u_0 \in C_D(0, L)$ is the initial datum

(3) G is the operator that solves the inhomogeneous heat equation with zero initial condition.

$$G : C[[0, L] \times [0, T]] \to C_D[[0, L] \times [0, T]]$$

is given by: $(Gf)(x, t) = \int_0^t ds \int_0^L dy g(x, y, t - s) f(y, s)$.

(4) W, formally given by: $W = G\alpha$, α = space-time standard white noise, is the Gaussian process with covariance

$$\mathbb{E}(W(x, t)W(x', t')) = \int_0^T ds \int_0^L dy G(x, t, y, s)G(x', t', y, s)$$

with $G(x, t, y, s) = g(x, y, t - s) \cdot \mathbb{1}_{[0,\infty]}(t - s)$.

One gets:

$$\begin{aligned} \mathbb{E}(W(x, t)W(x', t')) &= \Gamma(x, t, x', t') \\ &= \frac{1}{2} \sum_n \frac{1}{\mu_n} [\exp(-|t-t'|\mu_n) - \exp(-(t+t')\mu_n)] \cdot \phi_n(x)\phi_n(x') \end{aligned} \quad 2.3$$

with

$$\mu_n = \frac{n^2\pi^2}{L^2}, \quad \phi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{\pi n x}{L}\right)$$

For more details, see F.-J.

We assume the following properties of the potential V. (See Fig. 1.)

I) $$V \in C^\infty(\mathbb{R})$$

II) $$\lim_{|m| \to \infty} V'(m)/m^3 = \lambda > 0$$

III) There exists only three hyperbolic critical points $m_{\pm,0}$ such that

$$V(m_+) < V(m_-) < V(m_0),\ V'(m_+) = V'(m_-) = V'(m_0) = 0,$$
$$V''(m_-) < 0,\ V''(m_+) < 0,\ V''(m_0) > 0.$$

IV) $\exists \overline{m} : m_0 \leqslant \overline{m} < m_+$ such that

$$V'''(m) > 0 \quad \text{for} \quad m > \overline{m}, \quad V'''(m) < 0 \quad \text{for} \quad m < \overline{m}, \quad V'''(\overline{m}) = 0.$$

In Appendix A we show that, provided a further assumption is made on $\rho(\sigma_\alpha)$ defined in introduction (see point II of lemma A.2), a particular example can be found in the framework of mean field theory of ferromagnetism. In this case one gets:

$$V(m) \equiv \overline{V}_{a,h}(m) = V_h(m + a)$$

where:

$$V_h(m) = V_0(m) + hm$$

and $V_0(m)$ is an even function of m satisfying I) II) and a symmetric version of III). Namely:

$$m_+ = -m_-, \quad m_0 = 0.$$

Property IV) is a consequence of the hypothesis *b*) of section 1 and is satisfied for $\overline{m} = 0$. Fig. 1.

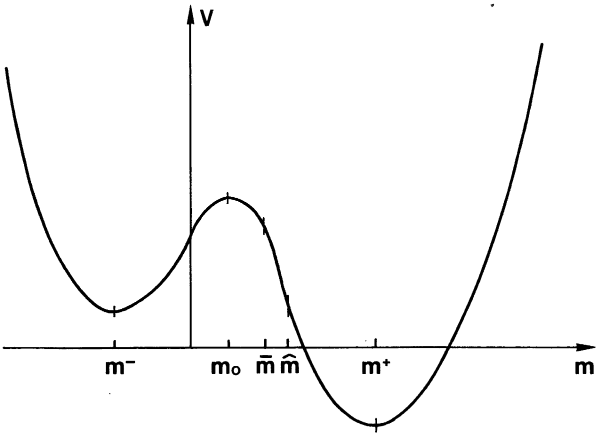

FIG. 1.

The following properties of the Gaussian process W are proved in F.-J.:

1) The random functions $W(x, t)$ are Hölder continuous with exponent $< 1/4$ with probability one.

2) They satisfy the boundary conditions $W(0, t) = W(L, t) = 0$ and $W(x, 0) = 0$ with probability one.

Thus for any $u_0 \in C_D(0, L)$ and any realization $W \in C_D^0[[0, L] \times [0, T]]$ (the space of continuous functions on $[0, L] \times [0, T]$ that satisfy D. b. c. on 0 and L and zero initial condition) we try to find a unique solution to eq. 2.1.

Since the space $C_D^0[[0, L] \times [0, T]]$ is of full measure with respect to the Gaussian process W, in this way we correctly define the non-linear (non Gaussian) random process u.

The following proposition summarizes the properties of the solution of eq. 2.1.

PROPOSITION 2.1.

I. $\forall u_0 \in C_D(0, L)$, $\forall T$, $\forall W \in C_D^0[[0, L] \times [0, T]]$ there exists a unique solution u of eq. 2.1 in $C_D^{u_0}[[0, L] \times [0, T]]$ (the space of continuous real functions on $[0, L] \times [0, T]$ with D. b. c. and u_0 initial condition).

II. The solution u depends continuously in the uniform norm on

$$Z = \varepsilon W + u_0$$

and we have

$$|||\, u_2 - u_1 \,|||_{\gamma} \leqslant \exp\,[K(T, L).T]\, |||\, Z_2 - Z_1 \,|||_{\infty} \qquad 2.4$$

for any pair u_1, u_2 of solutions corresponding to Z_1, Z_2.

The constant K(T, L) is given by

$$K(T, L) = \text{const}\ L^{9/2}T^{5/4} \qquad 2.5$$

DEFINITION 2.2. — The previous proposition allows us to define the continuous one to one map $\phi_{u_0} : \varepsilon W \to u = \phi_{u_0}(\varepsilon W)$ where u is the unique solution to eq. 2.1.

For the proof we refer to F.-J. Here we only prove the L^4 *a priori* estimate which is the analog of Lemma 5.4 of F.-J. The result is contained in the following:

PROPOSITION 2.3. — Let u be a solution of eq. 2.1:

$$u = -\, GV^1(u) + Z$$

Then $\exists$ constants $\alpha > 0$, $\beta > 0$ independent from L and T such that either

$$|||\, u - Z \,|||_4 \leqslant 1/\alpha\, |||\, Z \,|||_4$$

or

$$|||\, u - Z \,|||_4 \leqslant 1/\beta(LT)^{1/4}$$

From this Proposition and from Lemmas 5.5, 5.6, 5.8 and Proposition 5.7 of F.-J., it is easy to get the expression given by eq. 2.5.

Proof of Proposition 2.3. — Let $q = u - Z$.

We denote by $(\cdot, \cdot)$ the scalar product in $L^2([0, L] \times [0, T])$. It follows from condition II) above on the potential that $\exists M > 0$: if $|\, m \,| > M$ then $V'(m) > \frac{\lambda}{2} m^3$.

We have

$$\langle V'(u), u - Z \rangle \geqslant \left\langle \left(V'(u) - \frac{\lambda}{2} u^3 \right) 1\!\!1_{|u| < M}, u - Z \right\rangle + \frac{\lambda}{2} \langle (q + Z)^3, q \rangle \geqslant$$

$$\geqslant -c_1 L.T - c_2 |||Z|||_4 (LT)^{3/4} + \lambda/2(|||q|||_4^4 - 3|||Z|||_4 |||q|||_4^3 - 3|||q|||_4 |||Z|||_4^3) \qquad 2.6$$

where

$$c_1 = \sup_{|m| \leqslant M} \left\{ \left| V'(m) - \frac{\lambda}{2} m^3 \right| |\, m \,| \right\}; \quad c_2 = \sup_{|m| < M} \left| V'(m) - \frac{\lambda}{2} m^3 \right|$$

We shall prove that the statement of the Lemma is true with α, β such that

$$3(\alpha + \alpha^3) < \lambda/8 \quad \text{and} \quad c_1\beta^4 + c_2\beta^3 < \lambda/8$$

For, if $|||\, Z \,|||_4 < \alpha \,|||\, q \,|||_4$ and $(LT)^{1/4} < \beta \,|||\, q \,|||_4$ we get from eq. 2.6 that $\langle V(u), u - Z \rangle > 0$.

In this way we get a contradiction with Lemma 5.3 of F.-J. and so Proposition 2.3 is proved. ■

Now we give some results concerning the equilibrium solutions of the deterministic time evolution given by eq. 2.1 when $\varepsilon = 0$. These stationary solutions are the critical points of the stationary action:

$$S(u) = \int_0^L \left[\frac{1}{2} \left(\frac{du}{dx} \right)^2 + V(u) \right] dx \qquad 2.7$$

$S(u)$ is conventionally $= +\infty$ if u does not satisfy D. b. c., if it is not absolutely continuous or if the integral is not convergent.

The critical points of S are the solutions of the Newton's equation

$$\frac{d^2u}{dx^2} = V'(u) \quad \text{with the conditions} \quad u(0) = u(L) = 0 \qquad 2.8$$

In Appendix B we shall prove the following:

PROPOSITION 2.4. — Call $\hat{m}$ the real number such that $V(\hat{m}) = V(m_-)$, $m_- < \hat{m} < m_+$ (see Fig. 1).

We distinguish three cases:

1) $\hat{m} \leqslant 0$
2) $\hat{m} > 0 > m_-$
3) $m_- \geqslant 0$.

In the case 1 there exists a unique critical point u^* which is the absolute minimum of $S(u)$.

In the case 2 the critical points exhibit an oscillating behaviour. Calling I the number of positive oscillations and J that of the negative ones we have:

i) $|I - J| \leqslant 1$
ii) $\exists K_1 > 0$ such that $I + J \leqslant K_1 L$
iii) $\exists L_0 \in \mathbb{N}$, $K_2 > 0$ such that $\forall L > L_0$ to any $I < K_2 L$ is associated a unique pair of critical points

$$u_{II,\varepsilon_{II}} \quad \text{with} \quad \varepsilon_{II} = \text{sign} \left. \frac{du_{II}}{dx} \right|_{x=0} = \pm 1$$

and to any pair I, J with $|I - J| = 1$, $I \leqslant K_2 L$ is associated a unique critical point

$$u_{IJ,\varepsilon_{IJ}} \quad \text{with} \quad \varepsilon_{IJ} = \begin{cases} -1 & \text{if} \quad I = J + 1 \\ +1 & \text{if} \quad I = J - 1 \end{cases}$$

Moreover: u_{01} is the absolute minimum, u_{10} is the only other (local) minimum and all the other critical points are unstable.

$u_{11,+}$ and $u_{11,-}$ are saddle points and have only one instability direction. Finally

$$\forall I, J : I + J \leqslant K_1 L$$

$$S(u_{IJ,\varepsilon_{IJ}}) > S(u_{11,-}) = S(u_{11,+})$$

$$0 < \Delta S(L) \equiv S(u_{11,\pm}) - S(u_{10})$$

$$\lim_{L\to\infty} \Delta S(L) = \Delta S > 0$$

$$\lim_{L\to\infty} (S(u_{01}) - S(u_{10}))/L = [V(m^+) - V(m^-)] .$$

In the case 3 for L sufficiently large, we have exactly three critical points u_a, u_b, u_c where u_a is a local minimum, u_b is a saddle point with a unique direction of instability, u_c is the absolute minimum and:

$$\lim_{L\to\infty} S(u_b) - S(u_a) > 0$$

$$\lim_{L\to\infty} \frac{S(u_c) - S(u_a)}{L} < 0$$

Proof. — This proposition summarizes the most relevant points of Appendix B. Namely Proposition B.4, Lemma B.5 and B.6, Lemma B.7, Lemma B.8 and Proposition B.9.

In fig. 2 we represent, for large L, the critical points in the cases 1), 3) and the first critical points in the case 2.

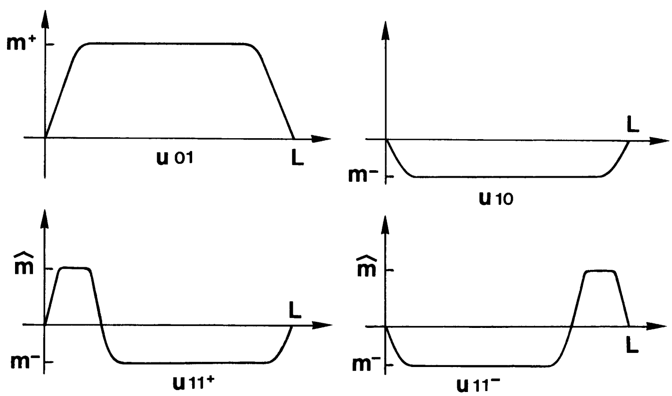

FIG. 2 *a*. — The first four critical points of case 2 (see Proposition 2.2).

From now on, we shall consider only the case 2): $\hat{m} > 0 > m_-$. The other two cases will be discussed in Remark 2.2 after the statement of Theorem 2.1.

We call $\mathbb{B}(u_{01})$ and $\mathbb{B}(u_{10})$ the basins of attraction of u_{01} and u_{10}, respectively, with respect to the deterministic gradient flow described by eq. 1.1 when $\varepsilon = 0$.

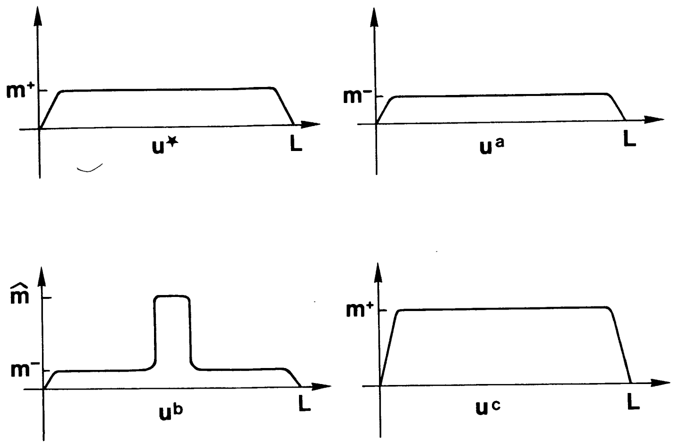

FIG. 2 *b*. — The critical points of case 1 and 3 (see Proposition 2.2).

For any sufficiently large L, we consider a neighbourhood Y of u_{01} in the uniform topology satisfying:

1) Y is contained in a uniform ball centered at u_{01} of radius $r < \frac{| m_+ - | m_- ||}{2}$.

2) Y contains a uniform ball centered at u_{01} of radius $\theta(\mathrm{L}) = \exp\left[-\frac{\mu}{2}\mathrm{L} \right]$

where μ is a suitable positive constant (see point II), III) of Th. 4.1).

3) $\mathrm{Y} \subset \mathbb{B}(u_{01})$.

We define the « tunnelling event »

$$\mathrm{A}^{\mathrm{T}}_{u_0} = \{ u \in \mathrm{C}^{u_0}_{\mathrm{D}}([0, \mathrm{L}] \times [0, \mathrm{T}]) : u(.,\mathrm{T}) \in \mathrm{Y} \}$$

We define also

$$\overline{\mathrm{A}}^{\mathrm{T}}_{u_0} = \{ u \in \mathrm{C}^{u_0}_{\mathrm{D}}([0, \mathrm{L}] \times [0, \mathrm{T}]) : u(.,\mathrm{T}) \in \overline{\mathrm{Y}} \}$$

where $\overline{\mathrm{Y}}$ is the closure of Y in the uniform topology.

The Sobolev H_1 norm of a function $f : [0, \mathrm{L}] \to \mathrm{R}$ is defined as

$$\| f \|_{\mathrm{H}_1} = \left(\| f \|_2^2 + \left\| \frac{df}{dx} \right\|_2^2 \right)^{1/2}$$

Now we are able to state our main result concerning the probability of tunnelling.

THEOREM 2.1. — $\exists \mathrm{L}_0 : \forall \mathrm{L} > \mathrm{L}_0, \forall \zeta > 0$ there exists a neighbourhood N in the Sobolev H_1 topology, whose radius depends only on ζ; there exists $\mathrm{T}_0(\mathrm{L}, \zeta)$, $\varepsilon_0(\mathrm{L}, \mathrm{T}, \zeta)$ such that

$$\forall \mathrm{T} > \mathrm{T}_0, \quad \forall \varepsilon < \varepsilon_0, \quad \forall u_0 \in \mathrm{N} \cap \mathbb{B}(u_{10})$$

such that $\frac{d^2u}{dx^2} \in C_D(0, L)$ we have:

$$\exp\left[-\frac{2\Delta S(L)+\zeta}{\varepsilon^2}\right] \leqslant \mathbb{P}(A^T_{u_0}) \leqslant \mathbb{P}(\overline{A^T_{u_0}}) \leqslant \exp\left[-\frac{2\Delta S(L)-\zeta}{\varepsilon^2}\right] \quad 2.9$$

Moreover it is possible to choose

$$T_0 = \exp\,[c_1 L] \quad 2.10$$

$$\varepsilon_0(L, T, \zeta) = \frac{c_2}{L^3}\exp\,[-(L)^{9/2}T^{9/4}c_3] \quad 2.11$$

where c_1, c_2, $c_3 > 0$ depend only on ζ.

Remark 2.1. — It comes out from the proof of the above theorem given in Section 3 that we can prove the lower bound in 2.9 even in the case when the initial configuration u_0 is only supposed to belong to a neighbourhood $\tilde{N}$ of u_{10} in the uniform topology with $S(u_0)$ bounded from above by some constant c.

Remark 2.2. — It will be clear from the following Sections that a result, completely analogous to theorem 2.1, is still true in the case $0 < m_-$ if we substitute, respectively, u_{10}, $u_{11,\pm}$ and u_{01} with u_a, u_b and u_c; on the other hand in the case $\hat{m} \leqslant 0$ we cannot have tunnelling phenomenon at all (absence of metastability).

We recall that, in the magnetic interpretation these three cases correspond to different boundary conditions (see eq. 1.11). The case $0 \geqslant \hat{m}$ corresponds to boundary conditions that favour the metastable phase with negative magnetization and so the fact that the configuration u_b (saddle point) contains a critical drop of positive magnetization, placed in the center of the interval [0, L], can be interpreted in the following way: the boundary conditions repell the positively magnetized phase and so the most likely position of the critical drop is as far as possible from the boundary.

The case $\hat{m} > 0 > m_-$ corresponds to boundary conditions that attract the positively magnetized phase and so the most likely position where the critical drop is formed, during the tunnelling, turns out to be near the boundary.

In the case $\hat{m} \leqslant 0$ the boundary conditions are so favourable to the positively magnetized phase that we don't have any other stable critical point.

If we notice that $\hat{m}$ is the magnetization inside the critical droplet (in the limit $L \to \infty$), it is easy to understand why a change in the sign of $\hat{m}$ implies a qualitative change in the behaviour of the system.

In fact for $\hat{m} \leqslant 0$, it is the boundary itself that behaves like a critical or overcritical droplet, driving the system directly to the stable state with positive magnetization.

SECTION 3

PROOF OF THEOREM 2.1

We first evaluate a lower bound to the probability of tunnelling $\mathbb{P}(u \in A_{u_0}^T)$ following exactly the arguments of F.-J. Since $A_{u_0}^T$ is open $\forall f \in A_{u_0}^T$, $\exists \delta > 0$ such that

$$\mathbb{P}(u \in A_{u_0}^T) \geqslant \mathbb{P}(||| f - u |||_\infty < \delta) \tag{3.1}$$

From Proposition 2.1 and definition 2.2, we get:

$$\mathbb{P}(||| u - f |||_\infty < \delta) \geqslant \mathbb{P}(||| \varepsilon W - (\phi_{u_0})^{-1}(f) |||_\infty < \delta \exp[-K(L,T)T]) \tag{3.2}$$

Now we define the Gaussian action functional $I_0(f)$ as

$$I_0(f) = \frac{1}{2} ||| G^{-1} f |||_2^2$$

and we set, conventionally, $I_0(f) = +\infty$ if f does not satisfy the D. b. c. or the condition $f = 0$ at $t = 0$.

The expression of $I_0(f)$ is formally given by

$$I_0(f) = \frac{1}{2} \int_0^T dt \int_0^L dx \left(\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} \right)^2$$

From Proposition 3.1 of F.-J. (see eq. 3.6 of F.-J.) we have

$$\mathbb{P}(||| \varepsilon W - \phi_{u_0}^{-1}(f) |||_\infty < \delta \exp[-K(L,T)T]) \leqslant \\ \leqslant \mathbb{P}(||| \varepsilon W |||_\infty < \delta \exp[-K(L,T)T]) . \exp\left[-\frac{I(f)}{\varepsilon^2} \right] \tag{3.3}$$

where

$$I(f) = I_0[\phi_{u_0}^{-1}(f)] .$$

The formal expression of $I(f)$ is given by

$$I(f) = \frac{1}{2} \int_0^T dt \int_0^L \left(\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} + V'(f) \right)^2 dx \tag{3.4}$$

From the proof of Proposition 3.3 below it easily follows that

$$\mathbb{P}(||| W |||_\infty < \bar{\delta}/\varepsilon) > 1/2 \quad \text{if} \quad \bar{\delta}/\varepsilon > \text{const } L^2$$

so that if $\varepsilon < \frac{\delta}{L^3} \exp[-K(L,T)T]$ it will certainly be true that

$$\mathbb{P}(||| u - f |||_\infty < \delta) \geqslant \frac{1}{2} \exp[-I(f)/\varepsilon^2] \tag{3.5}$$

Then the proof of the first part of eq. 2.9 is reduced to find $\forall\zeta > 0$ a function in $A^T_{u_0}$ such that

$$I(f) < 2\Delta S(L) + \zeta \qquad 3.6$$

provided T is larger than some $T_0(L, \zeta)$.

To show 3.6 we construct four paths and then we piece them together to form f. The first one is constructed by linear interpolation from u_0 to a point u_1 near u_{10} (see lemma 9.2 of F.-J.). The third path is again a linear interpolation from a point u_2 to a point u_3 both near u_{11}, we will show that the contribution to $I(f)$ coming from these two paths is small. The second path is from u_1 to u_2 and it follows the gradient flow with reversed velocity. Finally the fourth path is along the gradient flow from u_3 to the vicinities of u_{01}. The contribution to $I(f)$ of this fourth path is zero and so the main contribution will come from the second path.

We shall show that we can choose the points u_1, u_2 and u_3 with the following conditions

I) $\| u_i \|_2$, $\| S'(u_i) \|_2$, $i = 1, 2, 3$ are bounded by $\hat{C}L$ where $\hat{C}$ is a positive constant and $S'(u) = -\dfrac{d^2u}{dx^2} + V'(u)$.

II) The gradient flow starting at u_3 reaches the neighbourhood of u_{01} in a finite time $\overline{T}_2$.

The gradient flow with reversed velocity starting at u_1 reaches u_2 in a time $\overline{T}_1$:

III) $\overline{T}_1 + \overline{T}_2 \equiv \overline{T} < \exp(\overline{C}_1 L)$ for some positive constant $\overline{C}_1$.

We first show that in any uniform neighbourhood of the saddle point $u_{11} = u_{11,\pm}$ one can find points u_2 and u_3 belonging respectively to the basin of attraction of u_{10} and u_{01}.

Consider the case when the potential V is symmetric (i. e. $V(u) = V(-u)$). In the case considered in Appendix A this means $a = 0$ and $h = 0$.

Since u_{11} is the minimal saddle point it is certainly possible to find, in any neighbourhood of u_{11} points in the basin of attraction of one stable equilibrium; because of the symmetry we find points in $\mathbb{B}(u_{10})$ and also points in $\mathbb{B}(u_{01})$.

We want to show that the same happens in the non symmetric case. In general we consider a one parameter family $V_\mu(u)$ where μ can represents either h or a and $V_0(u)$ is symmetric.

Call $u^\mu_{11,+}$ one of the two minimal saddle points of $S(u)$ (we can suppose $|\mu| < |\mu_0|$ in such a way that the two wells-shape of $V_\mu(u)$ is preserved and Proposition 2.4 still holds).

Consider the second differential of $S(u)$

$$S''(u) = -\frac{\partial^2}{\partial x^2} + V''(u) \qquad 3.7$$

with $u = u^{\mu}_{11,+}$ and D. b. c. and call ψ_μ the eigenfunction of this operator associated to the only negative eigenvalue.

The following proposition is true:

PROPOSITION 3.1. — There exists $\delta > 0$ such that for any $\bar{\mu} : |\bar{\mu}| < |\mu_0|$ and for any L there exists $\eta > 0$ such that if we can assume that $u^{\bar{\mu}}_2 = u^{\bar{\mu}}_{11,+} + \delta\psi_{\bar{\mu}} \in \mathbb{B}(u^{\bar{\mu}}_{01})$ and $u^{\bar{\mu}}_3 = u^{\bar{\mu}}_{11,+} - \delta\psi_{\bar{\mu}} \in \mathbb{B}(u^{\bar{\mu}}_{10})$ then the same is true for $\mu = \bar{\mu} + \Delta\mu$ provided $|\Delta\mu| < \eta$.

Proof. — It is very easy to see that the functions: $\mathbb{R} \to C^1_D(0, L)$ given by $u^{\mu}_{11,+}$, u^{μ}_{01}, u^{μ}_{10} are continuous.

To see this property it is sufficient to look at the equation of Newton's type satisfied by the critical points of the action $S_\mu(u)$ and remark that the energy $E^{\mu}_{IJ} = \frac{1}{2}\left(\frac{du^{\mu}_{IJ}}{dx}\right)^2 + V_\mu(u^{\mu}_{IJ})$ is a continuous function of μ (see eq. B9, B12).

Now by definition of ψ_μ, if δ is sufficiently small we have

$$S_{\bar{\mu}}(u^{\bar{\mu}}_2) < S_{\bar{\mu}}(u^{\bar{\mu}}_{11,+}) \qquad 3.8$$

By the continuity of $S_\mu(u^{\mu}_{11,+})$ as a function of μ (that immediately follows from the previously discussed continuity of $u^{\mu}_{10,+}$ as a function of μ) we have that if η is sufficiently small

$$S_{\bar{\mu}}(u^{\bar{\mu}}_2) < S_{\bar{\mu}+\Delta\mu}(u^{\bar{\mu}+\Delta\mu}_{11,+}) \qquad 3.9$$

Now $u^{\bar{\mu}}_2$ can be written as:

$$u^{\bar{\mu}}_2 = \delta[c_1\psi_{\bar{\mu}+\Delta\mu} + \Phi] + u^{\bar{\mu}}_{11,+} \qquad 3.10$$

when Φ is orthogonal to $\psi_{\bar{\mu}+\Delta\mu}$ in $L^2([0, L])$ and necessarily $c_1 \neq 0$.

We define for $\tau : 0 \leqslant \tau \leqslant 1$

$$u_\tau = c_1\psi_{\bar{\mu}+\Delta\mu} + \tau\Phi \qquad 3.11$$

then

$$S(u_\tau) - S(u^{\bar{\mu}+\Delta\mu}_{11,+}) < \delta^2[c_1^2(\psi_{\bar{\mu}+\Delta\mu}, S''_{u^{\bar{\mu}+\Delta\mu}_{11}}\psi_{\bar{\mu}+\Delta\mu}) + 0(\delta)] \qquad 3.12$$

for δ sufficiently small. This implies that all the points of the segment defined by eq. 3.11 considered as starting points of the gradient flow corresponding to $\mu = \bar{\mu} + \Delta\mu$ are all approaching when $t \to \infty$ either $u^{\bar{\mu}+\Delta\mu}_{10}$ or $u^{\bar{\mu}+\Delta\mu}_{01}$.

In particular this means that

$$u^{\bar{\mu}}_2 \quad \text{and} \quad u^{\bar{\mu}+\Delta\mu}_2 = u^{\bar{\mu}+\Delta\mu}_{11,+} + \delta\psi_{\bar{\mu}+\Delta\mu}$$

belong to the same basin of attraction (for the gradient flow corresponding to $\mu = \bar{\mu} + \Delta\mu$).

Now since we know, by hypothesis, that $u^{\bar{\mu}}_2$ belongs to the basin of attraction, say $\mathbb{B}_{\bar{\mu}}(u^{\bar{\mu}}_{10})$, of $u^{\bar{\mu}}_{10}$ with respect to the gradient flow corresponding to $\mu = \bar{\mu}$, to conclude the proof we have only to show that $u^{\bar{\mu}}_2 \in \mathbb{B}_{\bar{\mu}+\Delta\mu}(u^{\bar{\mu}+\Delta\mu}_{10})$ namely $u^{\bar{\mu}}_2$ belongs to the basin of attraction of $u^{\bar{\mu}+\Delta\mu}_{01}$ with respect to the gradient flow corresponding to $\bar{\mu} + \Delta\mu$. In fact a completely analogous argument can be applied to $u_3^{\bar{\mu}}$. To see that $u_2^{\bar{\mu}} \in \mathbb{B}_{\bar{\mu}+\Delta\mu}(u_{10}^{\bar{\mu}+\Delta\mu})$ consider a time T so large that the evoluted $u_T^{\bar{\mu}}$ at time T along the gradient flow corresponding to $\mu = \bar{\mu}$ and starting at $u_2^{\bar{\mu}}$ is such that

$$\| u_T^{\bar{\mu}} - u_{10}^{\bar{\mu}} \|_\infty < \bar{\delta}/2$$

Now it is easy to show along the same lines of the proof of Proposition 2.1 a continuity property of the dynamics (at fixed times) with respect to μ that ensures that if $u_T^{\bar{\mu}+\Delta\mu}$ is the evoluted at time T of $u_2^{\bar{\mu}}$ with the dynamics corresponding to $\mu = \bar{\mu} + \Delta\mu$, then:

$$\| u_T^{\bar{\mu}} - u_T^{\bar{\mu}+\Delta\mu} \|_\infty \underset{\Delta\mu\to 0}{\longrightarrow} 0 \tag{3.13}$$

From the continuity in μ of the critical points and from 3.13 we get, $\forall\bar{\delta}$

$$\| u_T^{\bar{\mu}+\Delta\mu} - u_{10}^{\bar{\mu}+\Delta\mu} \|_\infty < \bar{\delta} \tag{3.14}$$

for $\Delta\mu$ sufficiently small.

By choosing $\bar{\delta}$ sufficiently small since $u_{10}^{\bar{\mu}+\Delta\mu}$ is a local minimum of S we conclude

$$u_T^{\bar{\mu}+\Delta\mu} \in \mathbb{B}_{\bar{\mu}+\Delta\mu}(u_{10}^{\mu+\Delta\mu})$$

and so

$$u_2^{\bar{\mu}} \in \mathbb{B}_{\bar{\mu}+\Delta\mu}(u_{10}^{\bar{\mu}+\Delta\mu}) \quad \blacksquare$$

Now by iterating a sufficiently large (possibly depending on L) number of times the argument of Proposition 3.1 since the value of δ can be chosen uniformely in μ for $|\mu| < |\mu_0|$ we get point II) above.

To show I) it is sufficient to remark that, in our construction u_2 and u_3 are bounded by a constant (of course u_1 satisfies the same property) and that $\psi_\mu(x)$, being a solution of a Schrödinger equation with bounded potential, has a Sobolev H_2-norm bounded by a constant.

Point III) is more difficult to prove.

Its proof will be the subject of the following section 4.

Now the contribution to $I(f)$ coming from the path against the flow (from u_1, to u_2) is bounded by $2\Delta S(L)$ (see the proof of theorem 9.1 of F.-J.). The contribution to $I(f)$ coming from the linear interpolation between u_0 and u_1 and between u_2 and u_3 is bounded by ζ provided

$$\begin{aligned} \| u_0 - u_1 \|_2 &< \text{const } \zeta/L \\ \| u_3 - u_2 \|_2 &< \text{const } \zeta/L \end{aligned} \tag{3.15}$$

as it follows from Lemma 9.2 of F.-J. This concluded the proof of eq. 3.6. $\blacksquare$

To get an upper bound to the probability of tunnelling, we have to consider all the possible mechanisms of tunnelling. The proof will not follow the topological arguments used in F.-J. but it will be based on the concept of basin of attraction.

The crucial property of the gradient flow that we use is the fact that the basins of attraction $\mathbb{B}(u_{10})$, $\mathbb{B}(u_{01})$ are disjoint and open in the uniform topology (Theorem 8.4 of F.-J.). We start proving a preliminary result:

PROPOSITION 3.2. — If $u(.,t)$ is regular, namely $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ are in $C_D([0,L] \times [0,T])$ then $\exists \delta$ such that $\forall \zeta > 0$ there exists an H_1 neighbourhood N of u_{10} such that if $u_0 \in N$ the inequality

$$I(u) \leqslant 2\Delta S - \zeta \tag{3.16}$$

implies that

$$\operatorname{dist}(u, \overline{A}^{T}_{u_0}) \geqslant \delta.$$

Proof. — For, consider a neighbourhood $\overline{Y}' \supset \overline{Y}$, $\overline{Y}' \subset \mathbb{B}(u_{01})$ such that $\operatorname{dist}(\partial\overline{Y}, \partial\overline{Y}') > 2\delta$.

If a regular u belongs to $\overline{A}'_{u_0}$:

$$\overline{A}'_{u_0} = \{ u \in C^{u_0}_D([0,L] \times [0,T]) : u(.,T) \in \overline{Y}') \tag{3.17}$$

then $I(u) \geqslant 2\Delta S(L) - \zeta$.

In fact if $u(.,T) \in \mathbb{B}(u_{01})$ necessarily an instant $T' < T$ exists such that $S(u(.,T)) \geqslant S(u_{11})$, otherwise the trajectory $u(.,t)$ would always belong to the basin of attraction either of u_{10} or of u_{01} and this is impossible since these basins are open.

If u is regular it follows (see eq. 10.4 of F.-J.) that

$$I(u) \geqslant 2(S(u(.,T')) - S(u(.,0))) \tag{3.18}$$

and since for any ζ there exists an H_1-neighbourhood N of u_{10} such that for any $u(.,0)$ in N we have

$$| S(.,0) - S(u_{10}) | < \zeta/2$$

then

$$I(u) > 2\Delta S(L) - \zeta/2 \tag{3.19}$$

We conclude that if

$$I(u) \leqslant 2\Delta S(L) - \zeta/2$$

then $u(.,T) \notin \overline{Y}'$ and so $\operatorname{dist}(u, A_{u_0}) > \delta$.

We know by theorem 6.9 of F.-J. that if u is such that $I(u) < +\infty$ and $u(.,0)$ is regular then $\exists$ a sequence u_n of regular functions such that

i) $u_n \to u$ uniformly

ii) $I(u_n) \to I(u)$.

It is easy to extend the previous results, valid for regular u. It is sufficient to consider the approximating sequence $u_n \to u$, for n sufficiently large:

$$| I(u_n) - I(u) | < \zeta/2.$$

Now, since $I(u_n) < 2\Delta S(L) - \zeta/2$, $u_n(.,T) \notin \overline{Y}'$ and so $\operatorname{dist}(u_n, \overline{A}_{u_0}) > 2\delta$ and since $\| u - u_n \|_\infty \underset{n\to\infty}{\to} 0$ we get $\operatorname{dist}(u, \overline{A}_{u_0}) > \delta$. ■

If we define $I^\alpha = \{ u : I(u) \leqslant \alpha \}$ we can say:

$$\mathbb{P}(u \in \bar{A}_{u_0}) \leqslant \mathbb{P}(\text{dist}\,(u, I^{2\Delta S[L)-\zeta}) \geqslant \delta) \qquad 3.18$$

Now, using the continuous map ϕ_{u_0} we get, like in eq. 3.2:

$$\mathbb{P}(u \in \bar{A}_{u_0}) \leqslant \mathbb{P}(\text{dist}\,(\varepsilon W, I_0^{2\Delta S(L)-\zeta}) \geqslant \delta \exp[-K(LT)T]) \qquad 3.19$$

To bound the r. h. s. of eq. 3.19, we need some probability estimate on the gaussian process W that are contained in the following:

PROPOSITION 3.3. — $\forall S > 0$, $\forall \delta > 0$, $\zeta > 0$

$$\mathbb{P}(\text{dist}\,(\varepsilon W, I_0^S) > \delta) < \exp\left[-\frac{S-\zeta}{\varepsilon^2}\right] \qquad 3.20$$

if

$$\varepsilon < \varepsilon_0(\zeta, \delta, L) = \sqrt{\frac{2\zeta}{\left(\ln\frac{2S}{\zeta}\right)}\frac{\delta^8}{L^{16}.32.S^4}} \qquad 3.21$$

Proof. — We follow the proof of Proposition 3.2 of F.-J. making the estimates explicitely.

We first solve the eigenvalue problem associated to the covariance operator Γ.

We can write

$$\Gamma^{-1} = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right) \qquad 3.22$$

with the condition that any f in the domain of Γ^{-1} must satisfy $f = 0$ for $t = 0$, $x = 0$ and $x = L$,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)f = 0; \qquad t = T, \quad x = 0, \quad x = L. \qquad 3.23$$

It is easily seen that the eigenvalue problem

$$\Gamma^{-1} f_{nm} = \gamma_{nm}^{-1} f_{nm} \qquad 3.24$$

is solved by

$$f_{nm}(x, t) = \sin\frac{n\pi x}{L}\sin b_{n,m} t$$

$$\gamma_{nm}^{-1} = b_{nm}^2 + \left(\frac{n\pi}{L}\right)^4 \quad \begin{matrix} n = 1, 2, \ldots \\ m = 1, 2, \ldots \end{matrix} \qquad 3.25$$

with b_{nm}, satisfying

$$-\frac{L^2}{n^2\pi^2} b_{nm} = \text{tg}\,(b_{nm}T) \qquad 3.26$$

from eq. 3.26 we get that $\forall T$, L, n

$$b_{nm} \in \left[\frac{2m-1}{2}\pi, m\pi\right] \qquad 3.27$$

we have

$$W = \sum_{n,m} (W, f_{nm}) f_{nm}$$

where the coefficients (W, f_{nm}) are independent Gaussian random variables with variance γ_{nm}.

We get

$$\Gamma(x, t, x, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \gamma_{nm} \mid f_{nm}(x, t) \mid^2 \qquad 3.29$$

We set

$$W_N = \sum_{n=1}^{N} \sum_{m=1}^{N} (W, f_{nm}) f_{nm} \quad \text{and} \quad \tilde{W}_N = W - W_N \qquad 3.30$$

For the variance $\tilde{\Gamma}_N$ of $\tilde{W}_N$ we have

$$\tilde{\Gamma}_N(x, t, x, t) = \sum_{n=1}^{N} \sum_{m=N+1}^{\infty} \gamma_{nm} \mid f_{nm} \mid^2 + \sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} \gamma_{nm} \mid f_{nm} \mid^2 \leqslant$$
$$\leqslant \sum_{n=1}^{N} \sum_{m=N+1}^{\infty} \frac{L^4}{m^2 + \pi n^4} + \sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} \frac{L^4}{m^2 + \pi n^4}. \qquad 3.31$$

It is not difficult to estimate the double sums in the last side of eq. 3.31 by using the corresponding double integrals.

The result is

$$\tilde{\Gamma}_N(x, t, x, t) \leqslant L^4/N^{1/2} \quad \text{for} \quad N > 10. \qquad 3.32$$

Now we have

$$\mathbb{P}(\text{dist}(\varepsilon W, I_0^S) > \delta) \leqslant \mathbb{P}(\| \varepsilon \tilde{W}_N \|_\infty \geqslant \delta) + \mathbb{P}(\varepsilon W_N \notin I_0^S) \qquad 3.33$$

By eq. 3.13 of F.-J. we get

$$\mathbb{P}(\| \varepsilon \tilde{W}_N \| \geqslant \delta) \leqslant \text{const} \exp[-S/\varepsilon^2] \qquad 3.34$$

provided

$$L^4/N^{1/2} < \delta^2/2S \qquad 3.35$$

and by eq. 3.15 of F.-J. we get

$$\mathbb{P}(\varepsilon W_N \notin I_0^S) \leqslant \exp\left(-\frac{S - \zeta/2}{\varepsilon^2}\right) \cdot \left(\sqrt{\frac{2S}{\zeta}}\right)^{N^2} \qquad 3.36$$

If $\left(\sqrt{\frac{2S}{\zeta}}\right)^{N^2} < \exp[\zeta/\varepsilon^2]$ namely if

$$\varepsilon^2 < \frac{2\zeta}{N^2} \frac{1}{\ln(2S/\zeta)}$$

we get the desired result and this is just condition 3.21 if we choose:

$$N = \left(\frac{L^4 2S}{\delta^2}\right)^2 \sqrt{2} \qquad \blacksquare$$

By eq. 3.19 and Proposition 3.3 we reduce the proof of the theorem to the point III which will follow from theorem 4.1 below. ■

SECTION 4

TIME ESTIMATE FOR THE GRADIENT FLOW

In this section we give an upper bound, with an explicit L-dependence, to the time needed by the gradient flow to reach a given neighbourhood of u_{01} starting from a neighbourhood of $u_{11,+}$ and to the time needed by the flow with reversed velocity to reach a neighbourhood of $u_{11,+}$ starting from a neighbourhood of u_{10}. In particular we shall prove point III of the previous section. We only consider here the first case, the second one being completely analogous. Moreover, nothing changes of course with $u_{11,-}$ instead of $u_{11,+}$. We want to prove the following:

THEOREM 4.1. — Let $u(x, t)$ be a solution of:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + W'(u), \quad t \geqslant 0, \quad x \in [0, L] \qquad 4.1$$

$$u(0, t) = u(L, t) = 0 \quad \forall t \geqslant 0, \quad u(x, 0) = u_0(x), \quad x \in [0, L]$$

with $W(u) = -V(u)$.

i) u_0 belongs to the unstable manifold of $u_{11,+}$ and to the basin of attraction $\mathbb{B}(u_{01})$ of u_{01}.

ii) $\| u_0 - u_{11,+} \|_2 = [\theta(L)]^{1/3} \equiv \exp[-\mu L/2]$, $\mu \geqslant \mu_0 > 0$.

iii) $S(u_0) < S(u_{11,+})$

(such u_0's exist as we have seen in section 3).

Then $\exists L_0$, such that $\forall L > L_0$

$$T_{\theta[L]} \equiv \inf\{ t > 0 : u(.,t) \cap B_{\theta(L)}(u_{01}) \neq \emptyset \} \leqslant \\ \leqslant \exp[\alpha L] . [S(u_{11,+}) - S(u_{01})]/[K\theta^2(L)] \qquad 4.2$$

where K, μ_0 and α are suitable positive constants and

$$B_\theta(u) = \{ v : \| u - v \|_\infty \leqslant \theta \}.$$

Let us first explain the basic idea of the proof:
it is not difficult to see that if u satisfies eq. 4.1.

$$\frac{d}{dt} S(u(.,t)) = - \int_0^L \left[\frac{\partial^2 u}{\partial x^2} + W'(u) \right]^2 dx \leqslant 0 \qquad 4.3$$

in particular $\frac{d}{dt} S(u^*) = 0$ iff u^* is a critical point of $S(u)$.

We show that $\frac{d}{dt} S(u(.,t))$ cannot be too small unless $u(.,t)$ is near some critical point.

More precisely, we shall prove that under the hypotheses of Theorem 4.1 the inequality

$$\left| \frac{d}{dt} S(u(.,t)) \right| \leqslant K\theta^2 \exp[-2\alpha L] \qquad 4.4$$

implies, for α sufficiently large:

$$\| u(.,t) - u^* \|_{c^1} \leqslant \theta$$

where u^* is one of the critical points $u_{11,+}, u_{11,-}, u_{01}, u_{10}$ and

$$\| u \|_{c^1} = \sup_{x \in [0,L]} \left(| u(x) | + \left| \frac{du}{dx} \right| \right).$$

Let $B_\theta^1(u) = \{ v : \| u - v \|_{c^1} < \theta \}$. If we can exclude *a priori* that

$$u(.,t) \in B_\theta^1(u_{11,+}) \cup B_\theta^1(u_{11,-}) \cup B_\theta^1(u_{01}) \cup B_\theta^1(u_{10}) \quad \forall t \in [0, T_\theta]$$

then necessarely

$$\left| \frac{dS}{dt}(u(.,t)) \right| > K\theta^2 \exp[-\alpha L] \qquad \forall t \in [0, T_\theta]$$

and so inequality 4.2 follows.

Before stating the main preliminary result of this section, namely Proposition 4.2, let us introduce another way to look at our equation 4.1.

First of all, it is known (see for instance theorem 3.5.2 of reference 12) that if u_0 belongs to the Sobolev space H_1 then $\forall t > 0$ the same is true for $\frac{\partial u(.,t)}{\partial t}$. In particular $\frac{\partial u}{\partial t}$ is an absolutely continuous function of x in $[0, L]$. The above hypothesis is verified in our case because of *iii*).

Now, for any $t > 0$, the solution of eq. 4.1 must satisfy the inomogeneous boundary problem:

$$\frac{d^2 u}{dx^2} + W'(u(x)) = g_t(x) \qquad 4.5$$
$$u(0) = u(L) = 0$$

where $g_t(x)$ is nothing but the continuous function $\frac{\partial u(x, t)}{\partial t}$. It is natural to expect, when $g_t(x)$ is small, the solutions of eq. 4.5 to be near to the solutions of the corresponding equation with $g = 0$. But since eq. 4.5 corresponds to a boundary value and not to an initial value problem the comparison between the two equations is not immediate. Our result is contained in the following:

PROPOSITION 4.2. — There exists positive constants L_0, α_0 such that for $L > L_0$, $\alpha > \alpha_0$ if $g(x)$ is a real continuous function on $[0, L]$ with

$$\int_0^L |g(x)|^2 dx \leqslant K\theta^2 \exp[-2\alpha L] \tag{4.6}$$

then the solutions of the boundary problem with forced term

$$\frac{d^2u}{dx^2} + W^1(u(x)) = g(x) \qquad 0 \leqslant x \leqslant L; \quad u(0) = u(L) = 0 \tag{4.7}$$

that satisfy $S(u) < S(u_{11})$ are such that

$$\min_{u^* \in \{u_{11,+}, u_{11,-}, u_{01}, u_{10}\}} \| u - u^* \|_{c^1} < \theta \tag{4.8}$$

The proof of Proposition 4.2 will be postponed at the end of this section.

Proof of theorem 4.1. — From the previous discussion it is clear that, to get the theorem, it is sufficient to prove that if L is large enough the solution of equation 4.1 cannot enter in the balls

$$B^1_{\theta(L)}(u_{11,-}),\ B^1_{\theta(L)}(u_{11,+}) \text{ and } B^1_{\theta(L)}(u_{10}) \quad \text{for} \quad t \in [0, T_{\theta(L)}].$$

By Proposition B.8 of Appendix B it is possible to choose a μ_0 such that if λ_0 is the minimal eigenvalue (negative) of $S''(u_{11,\pm})$ and $\bar{\lambda}_0$ is the minimal eigenvalue (positive) of $S''(u_{10})$ then $\exp[-\mu_0 L/2] < \min(|\lambda_0|, \bar{\lambda}_0)$.

It is immediate to see that, for some constant K:

$$\inf_{u \in B^1_\rho(u_{11,\pm})} \{ S(u) \} > S(u_{11,\pm}) - KL\rho$$

Moreover, since $\theta = \exp[-3/2\mu L]$ with $\mu > \mu_0$, $S(u_0) - S(u_{11,-}) < -|\lambda_0|\theta^2/2$ for L sufficiently large and $S(u_{11,-}) = S(u_{11,+})$ we get

$$\inf_{u \in B^1_\theta(u_{11,\pm})} S(u) > S(u_0)$$

and since S is decreasing along the gradient flow $B^1_{\theta(L)}(u_{11,\pm})$ cannot be reached.

On the other hand, for some constant K

$$\sup_{u \in B^1_\theta(u_{10})} S(u) \leqslant S(u_{10}) + L\theta K$$

and

$$\sup_{u \in B^1_\theta(u_{10})} \| u - u_{10} \|_2 \leqslant \theta L^{1/2}$$

moreover

$$\inf_{u:\|u-u_{10}\|=\sigma} S(u) \geqslant \bar{\lambda}_0 \sigma^2 + 0(\sigma^3) + S(u_{10})$$

therefore it is easy to check that if $\sigma > \exp\left[-\mu \frac{L}{2} \right]$ then

$$\inf_{u:\|u-u_{10}\|_2=\sigma} S(u) > \sup_{u \in B^1_\theta(u_{10})} S(u)$$

for L sufficiently large.

If we start from any point of $B^1_\theta(u_{10})$ the gradient flow will never reach the set $\{ u : \| u - u_{10} \|_2 = \sigma \}$; consequently since, for σ sufficiently small, in the interior of the ball $\{ u : \| u - u_{10} \|_2 \leqslant \sigma \}$ there are no critical point other than u_{10}, we conclude that, for L sufficiently large:

$$B^1_\theta(u_{10}) \subset \mathbb{B}(u_{10})$$

and so by hypothesis *i)* $B^1_\theta(u_{10})$ cannot be reached by the solution of eq. 4.1 for $t < T_{\theta(L)}$. ■

In the sequel we shall give some lemmas from which Proposition 4.2 will be deduced.

We start by noticing that, like in the case of the critical points of $S(u)$ treated in Appendix B, we can see eq. 4.7 as Newton's equation of motion for a particle of mass 1 with potential energy $W(u)$ and a forcing term $g(x)$.

In this interpretation x is a time variable and Dirichlet boundary conditions say that the particle returns to the origin at time L.

From our hypothesis on the potential it follows that W is strictly decreasing on $]m_-, m_0[$ and strictly increasing on $]m_0, m_+[$ so that we can define the local inverse functions of W

$$\begin{aligned} W_-^{-1} &: [W(m_-), W(m_0)] \to [m_-, m_0] \\ W_+^{-1} &: [W(m_0), W(m_+)] \to [m_0, m_+] \end{aligned} \qquad 4.9$$

LEMMA 4.3. — Let u be a solution of eq. 4.7 that satisfies: $S(u) < S(u_{11})$ if

$$E(x) = \frac{1}{2}\left(\frac{du(x)}{dx}\right)^2 + W(u(x)) \qquad 4.10$$

and

$$\int_0^L | g(x) |^2 dx \leqslant \delta$$

then, there exists a constant β such that

$$\sup_{(x,y)\in[0,L]} | E(x) - E(y) | \leqslant \sqrt{\delta L \beta}$$

Proof. — By differentating $E(x)$ and using eq. 4.7 we get

$$\frac{dE}{dx} = g(x)\frac{du}{dx}$$

therefore by Schwarz inequality

$$\sup_{(x,y)\in[0,L]} |E(x) - E(y)| \leqslant \left(\int_0^L |g(x)|^2 dx\right)^{1/2} \left(\int_0^L \left|\frac{du}{dx}\right|^2 dx\right)^{1/2}.$$

Since

$$S(u) = \int_0^L \left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 - W(u)\right] dx < S(u_{11})$$

and W is bounded above, there exists a constant $\beta > 0$ such that

$$\int_0^L \left(\frac{du}{dx}\right)^2 dx \leqslant \beta L$$

from which we get the result. ■

It is proved in Appendix B that the only possible solutions of eq. 4.7 with $g \equiv 0$ correspond to some particular values E_{IJ} of the energy $E = \frac{1}{2}\left(\frac{du}{dx}\right)^2 + W(u)$ which, in that case, is a conserved quantity, (the integers I and J count the number of extrema of u).

Now from the previous Lemma we know that the solutions of eq. 4.7 with $g \not\equiv 0$ but small in the L^2 norm are such that $\forall x \in [0, L]$, $E(x)$ is in a narrow band in the plane u, $W(u)$ of width $\Delta E \leqslant \sqrt{\delta L \beta}$.

Since the typical distances between the energy levels E_{IJ} is $\exp(-\text{const } L)$, we choose $\delta = \frac{\theta^4}{\beta L} \exp[-2\alpha L]$; conditions on α will be specified later.

Consider now the initial value problem

$$\begin{aligned} &\frac{d^2u}{dx^2} + W'(u) = g(x) \\ &u(0) = 0 \\ &u'(0) = \varepsilon[2(E_0 - W(0))]^{1/2} \end{aligned} \qquad 4.12$$

with $\varepsilon = \pm 1$ and $\int_0^L |g(x)|^2 dx \leqslant \frac{\theta^4}{\beta L} \exp[-2\alpha L]$.

Given a solution $u(x)$ of the inhomogeneous boundary problem 4.7 we can find an E_0 and an $\varepsilon = \pm 1$ such that $u'(0) = \varepsilon[2(E_0 - W(0))]^{1/2}$. Therefore this solution of eq. 4.7 solves also the inhomogeneous initial value problem 4.12. Let us now sketch the basic ideas for the proof of Proposition 4.2.

We shall prove that, under the hypotheses of Proposition 4.2, the possible values of E_0 must be very near E_{11}, E_{10}, or E_{01} so that by continuous dependence on initial conditions we get that the possible solutions of eq. 4.7 have to be very near to one of the critical points $u_{11,+}$, $u_{11,-}$, u_{10} or u_{01} in the C_1 topology. To get the above restrictions on E_0 we first exclude the number of oscillations of the forced motion described by eq. 4.12 to be too high. Then we show that for α sufficiently large the times needed to come back to the origin by the forced motion and by the conservative motion with $g = 0$ and same initial condition are very near for any E_0 far enough from $W(m_-)$ and $W(m_+)$. Consequently we can replace the non conservative return times with the conservative ones within a small error. Finally by evaluating the variation of the conservative return times with the energy we can conclude that even for the non conservative forced motion it is impossible to satisfy D. b. c. unless the center of the corresponding band of energy lies very near some level E_{IJ}. The case when E_0 is near $W(m_-)$ or $W(m_+)$ is treated separately and it is shown that the return time is so large that even with only one oscillation it is impossible to satisfy D. b. c. Let Γ_i, $i = 1, 2, 3, 4$, be the intervals

$$\begin{aligned}
\Gamma_1 &= (W(0) - \Delta E, W(m_-) - \exp[-\beta L]) \\
\Gamma_2 &= (W(m_-) - \exp[-\beta L], W(m_-) + \exp[-\beta L]) \\
\Gamma_3 &= (W(m_-) + \exp[-\beta L], W(m_+) - \exp[-\beta L]) \\
\Gamma_4 &= (W(m_+) - \exp[-\beta L], W(m_+) + \exp[-\beta L])
\end{aligned} \qquad 4.13$$

where $\Delta E = \exp[-\alpha L]$.

LEMMA 4.3. — There exists an L_0 such that if $L \geqslant L_0$ and $E_0 \in \Gamma_1 \cup \Gamma_3$ then, if u_ε is a solution of equation, 4.12 which satisfies also D. b. c. $u_\varepsilon(0) = u_\varepsilon(L) = 0$ there exist two integers I, J such that

$$\begin{aligned}
S(u_\varepsilon) \geqslant\ & 2I\sqrt{2}\int_{W_-^{-1}(E_0-\Delta E)}^{0} [E_0 - \Delta E - W(u)]^{1/2} du + \\
&+ 2J\sqrt{2}\int_{0}^{W_+^{-1}(E_0-\Delta E)} [E_0 - \Delta E - W(u)]^{1/2} du - L(E_0 + \Delta E)
\end{aligned} \qquad 4.14$$

Proof. — We consider first the case $E \in \Gamma_1$.

Let u_ε be a solution of 4.12; since $\dfrac{du_\varepsilon}{dx}$ is continuous and

$$\left(\frac{du_\varepsilon}{dx}\right)^2 = 2[E(x) - W(u_\varepsilon(x))]$$

$\dfrac{du_\varepsilon}{dx}$ can change sign only if $E(x) - W(u_\varepsilon(x)) = 0$. Therefore u_ε starts at $x = 0$ with a given sign of $\dfrac{du_\varepsilon}{dx}$ which is unchanged as long as $u_\varepsilon(x) \in [0, W_\varepsilon^{-1}(E_0 - \Delta E)[$.

If $u_\varepsilon(x) \in [W_\varepsilon^{-1}(E_0 - \Delta E), W_\varepsilon^{-1}(E_0 + \Delta E)]$ with $E(x) - W(u_\varepsilon(x)) \geqslant 0$, $\frac{du_\varepsilon}{dx}$ can change sign in an *a priori* complicated way but if for some x^*, $u(x^*) \in]W_-^{-1}(E_0 - \Delta E), W_+^{-1}(E_0 - \Delta E)[$, $\frac{du}{dx}$ will not change sign until $u(x) \in [W_{-\varepsilon}^{-1}(E_0 - \Delta E), W_{-\varepsilon}^{-1}(E_0 + \Delta E)]$.

Using

$$S(u_\varepsilon) = \int_0^L \left\{ \frac{1}{2} \left(\frac{du_\varepsilon}{dx} \right)^2 - W(u) \right\} dx \tag{4.15}$$

and eq. 4.10, we get

$$S(u_\varepsilon) \geqslant \int_0^L \left(\frac{du_\varepsilon}{dx} \right)^2 du - (E_0 + \Delta E)L \tag{4.16}$$

On each subinterval of $[0, L]$ where the sign of $\frac{du}{dx}$ is unchanged we can perform the change of variable $x = x(u)$ in the integral on the right hand side of 4.16.

Using

$$\left| \frac{du_\varepsilon}{dx} \right| = \sqrt{2[E(x) - W(u_\varepsilon(x))]} \geqslant \sqrt{2}\sqrt{E_0 - \Delta E - W(u_\varepsilon)}$$

it is not difficult to see that if I(resp. J) is the number of subintervals of $[0, L]$ where u_ε is negative (resp. positive), then by forgetting the contribution in $\int_0^L \left(\frac{du}{dx} \right)^2 dx$ which comes from the x's such that

$$u_\varepsilon(x) \in [W_\varepsilon^{-1}(E_0 - \Delta E), W_\varepsilon^{-1}(E_0 + \Delta E)]$$

for $\varepsilon = +1$ and $\varepsilon = -1$, we get

$$\begin{aligned} \int_0^L \left(\frac{du_\varepsilon}{dx} \right)^2 dx \geqslant 2I\sqrt{2} \int_{W_-^{-1}[E_0 - \Delta E)}^0 [E_0 - \Delta E - W(u)]^{1/2} du \, + \\ + 2J\sqrt{2} \int_0^{W_+^{-1}(E_0 - \Delta E)} [E_0 - \Delta E - W(u)]^{1/2} du \end{aligned} \tag{4.17}$$

which leads to 4.14.

The case $E \in \Gamma_3$ is much simpler since $I = 0$ and $J = 1$ is obviously the only possibility compatible with the return to the origin at time L.

Let us define

$$S_-(E) = 2\sqrt{2} \int_{W_-^{-1}(E)}^0 [E - W(u)]^{1/2} du \tag{4.18}$$

and

$$S_+(E) = 2\sqrt{2} \int_0^{W_+^{-1}(E)} [E - W(u)]^{1/2} du \tag{4.19}$$

LEMMA 4.4. — Under the hypothesis of lemma 4.3, for any η: $0 < \eta < W(m_-) - W(0)$ if $S(u_\varepsilon) < S(u_{11})$ then, for L large enough

$$E_0 > W(m_-) - \eta.$$

Proof. — It follows from equation B.38 of Appendix B that

$$S(u_{11}) = S_+(E_{11}) + S_-(E_{11}) - E_{11}L.$$

For any $\eta : 0 < \eta < W(m_-) - W(0)$, if u_ε satisfies equation 4.12, D. b. c. and $E_0 < W(m_-) - \eta$ we get:

$$S(u_\varepsilon) - S(u_{11}) \geqslant [\eta - W(m_-) - E_{11})]L - S_-(E_{11}) - S_+(E_{11})$$

Since we know from eq. B.22 of Appendix B that

$$W(m^-) - E_{11} < \exp[-\text{const}\, L]$$

and furthermore $S_-(E_{11}) + S_+(E_{11})$ is bounded by a constant uniformely in L, we conclude that for L sufficiently large, the inequality $S(u_\varepsilon) < S(u_{11})$ implies: $E_0 > W(m_-) - \eta$.

Now we choose $\eta = \bar{\eta}$ in such a way that in the set $\Delta\bar{\eta} \equiv [W(m_-) - \eta, W(m_-)]$ the quantity $T_+(E) + T_-(E)$ defined in eq. B.11 of Appendix B is strictly increasing with E (this is always possible as it is shown in Appendix B) and we set $\bar{\Gamma}_1 = \Gamma_1 \cap \Delta\bar{\eta}$.

LEMMA 4.5. — Under the hypotheses of Lemma 4.3 if L is large enough and $S(u_\varepsilon) < S(u_{11})$ then the only possibilities for the pair (I, J) are (1, 1), (1, 0) and (0, 1).

Proof. — We have only to consider the case $E_0 \in \bar{\Gamma}_1$ since, as we have seen, for $E_0 \in \Gamma_3$ we have (I, J) = (0, 1) and so the result is proved and for $E_0 \in \Gamma_1 \backslash \bar{\Gamma}_1 : S(u_\varepsilon) > S(u_{11})$ in contradiction with the hypothesis.

Now let I and J be strictly bigger than 1; from the inequality 4.14 we get:

$$\begin{aligned} S(u_\varepsilon) - S(u_{11}) \geqslant & (I-1)S_-(E_0 - \Delta E) + (J-1)S_+(E_0 - \Delta E) \\ & + S_+(E_0 - \Delta E) - S_+(E_{11}) + S_-(E_0 - \Delta E) - S_-(E_{11}) \\ & + [E_{11} - (E_0 + \Delta E)]L \end{aligned} \quad 4.20$$

First of all, from eq. 4.18, 4.19 and B.9 we have

$$\frac{d}{dE} S_\pm(E) = T_\pm(E) > 0.$$

Now we distinguish two cases. If $E_0 - \Delta E > E_{11}$ from eq. B.26 we get $|E_{11} - (E_0 - \Delta E)| < \exp[-\text{const}\, L]$ and so for L sufficiently large, since $S_\pm(E_0 - \Delta E) - S_\pm(E_{11}) \geqslant 0$ and

$$\begin{aligned} (I-1)S_-(E_0 - \Delta E) + (J-1)S_+(E_0 - \Delta E) \geqslant \\ \geqslant \max\{ S_-(W(m_-) - \eta), S_+(W(m_-) - \eta) \} > 0 \end{aligned}$$

we get a contradiction with the hypothesis $S(u_\varepsilon) < S(u_{11})$ by choosing L sufficiently large.

If $E_0 - \Delta E < E_{11}$ (but $E_0 > W(m_-) - \bar{\eta}$ so that $T(E) = T_+(E) + T_-(E)$ is increasing in E), we have

$$|S_+(E_0-\Delta E)+S_-(E_0-\Delta E)-S_+(E_{11})-S_-(E_{11})| \leqslant$$
$$\leqslant |E_{11}-(E_0-\Delta E)| \sup_{E\in[E_0-\Delta E,E_{11}]} (T_+(E)+T_-(E)) \leqslant |E_{11}-(E_0-\Delta E)| L$$

and so

$$S(u_\varepsilon) - S(u_{11}) \geqslant S_+(W(m_-) - \eta) + S_-(W(m_-) - \eta) - 2\Delta EL$$

from which for L sufficiently large we get again a contradiction. ■

For a given ε, if u_ε is the solution of 4.12 let us now define the non conservative times

$$\tilde{T}_\varepsilon^{(1)} = \inf(x > 0 \mid u_\varepsilon(x) = W_\varepsilon^{-1}(E_0 - \Delta E))$$
$$\tilde{T}_\varepsilon^{(2)} = \inf\left(x > \tilde{T}_\varepsilon^{(1)} \mid u_\varepsilon(x) = W_\varepsilon^{-1}(E_0 - \Delta E), \varepsilon\frac{du_\varepsilon}{dx} < 0\right)$$
$$\tilde{T}_\varepsilon^{(3)} = \inf(x > \tilde{T}_\varepsilon^{(2)} \mid u_\varepsilon(x) = 0)$$

if $E_0 \in \Gamma_1$ we can also define

$$\tilde{T}_\varepsilon^{(4)} = \inf(x > \tilde{T}_\varepsilon^{(3)} \mid u_\varepsilon(x) = W_{-\varepsilon}^{-1}(E_0 - \Delta E))$$
$$\tilde{T}_\varepsilon^{(5)} = \inf\left(x > \tilde{T}_\varepsilon^{(4)} \mid u_\varepsilon(x) = W_{-\varepsilon}^{-1}(E_0 - \Delta E), \varepsilon\frac{du_\varepsilon}{dx} > 0\right)$$
$$\tilde{T}_\varepsilon^{(6)} = \inf(x > \tilde{T}_\varepsilon^{(6)} \mid u_\varepsilon(x) = 0)$$

call $\tilde{T}_\varepsilon$ the sum of the previous times: $\tilde{T}_\varepsilon = \sum_{i=1}^{6} \tilde{T}_\varepsilon^{(i)}$.

LEMMA 4.6. — If $\Delta E = \theta^2 \exp[-\alpha L]$ then, there exists a constant K such that for any $\varepsilon = \pm 1$

if $E_0 \in \overline{\Gamma}_1 \cup \Gamma_3$:

$$|\tilde{T}_\varepsilon^{(1)} + \tilde{T}_\varepsilon^{(2)} + \tilde{T}_\varepsilon^{(3)} - T_\varepsilon(E_0)| \leqslant \theta K \exp\left[-\left(\frac{\alpha}{2} - 2\beta - \frac{\hat{\hat{K}}}{2}\right)L\right] \quad 4.23$$

if $E_0 \in \overline{\Gamma}_1$ we also have:

$$|\tilde{T}_\varepsilon - T(E_0)| < 2K\theta \exp\left[-\left(\frac{\alpha}{2} - 2\beta - \frac{\hat{\hat{K}}}{2}\right)L\right] \quad 4.24$$

where

$$\hat{\hat{K}} = \sup_{|x| \leqslant \max(|m_-|, m_+)} |W''(x)|.$$

Proof. — We consider first the case $E \in \overline{\Gamma}_1$.

If $\varepsilon = -1$ one has

$$\tilde{T}_{-}^{(1)} + \tilde{T}_{-}^{(3)} \geqslant T_{-}(E_0 + \Delta E) - \int_{W_{-}^{-1}(E_0+\Delta E)}^{W_{-}^{-1}(E_0-\Delta E)} (E_0 + \Delta E - W(u))^{-1/2} du \quad 4.25$$

Calling $I(E_0, \Delta E)$ the previous integral, by a simple change of variable we get

$$I(E_0, \Delta E) \leqslant \sqrt{\Delta E}(-W'_{-}(W_{-}^{-1}(E_0 + \Delta E)))^{-1} \quad 4.26$$

By Lagrange theorem one has

$$|W'_{-}(W_{-}^{-1}(E_0 + \Delta E))| \geqslant \text{const} \exp[-2\beta L] \quad 4.27$$

therefore if $2\beta < \alpha/2$, by 4.26 and 4.27, using that $T_{-}(E)$ is increasing with E in $\overline{\Gamma}_1$ we get

$$\tilde{T}_{-}^{(1)} + \tilde{T}_{-}^{(2)} + \tilde{T}_{-}^{(3)} \geqslant T_{-}(E_0) - \text{const}\, \theta \exp\left[-\left(\frac{\alpha}{2} - 2\beta\right)L\right] \quad 4.28$$

Now from lemmas B.2 and B.3 we get

$$\tilde{T}_{+}^{(1)} + \tilde{T}_{+}^{(2)} + \tilde{T}_{+}^{(3)} \geqslant T_{+}(E_0) - \text{const}\, \theta \exp\left[-\left(\frac{\alpha}{2} - 2\beta\right)L\right] \quad 4.29$$

and

$$\tilde{T}_{\varepsilon} \geqslant T(E_0) - \text{const}\, \theta \exp\left[-\left(\frac{\alpha}{2} - 2\beta\right)L\right] \quad 4.30$$

On the other hand, by lemmas B.2 and B.3 of Appendix B one has

$$\tilde{T}_{\varepsilon}^{(1)} + \tilde{T}_{\varepsilon}^{(3)} \leqslant T_{\varepsilon}(E - \Delta E) \leqslant T_{\varepsilon}(E_0) + \text{const}\, \theta^2 \exp[-\alpha L] \quad 4.31$$

$$\tilde{T}_{\varepsilon} - \tilde{T}_{\varepsilon}^{(2)} - \tilde{T}_{\varepsilon}^{(5)} \leqslant (T_{+} + T_{-})(E_0 - \Delta E) \leqslant T_{+} + T_{-})(E_0)\,. \quad 4.32$$

We give an upper bound on $\tilde{T}_{\varepsilon}^{(2)}$ and $\tilde{T}_{\varepsilon}^{(5)}$: we consider only $\tilde{T}_{-}^{(2)}$, the other cases are similar.

Let us first consider the two vectors

$$U(x) = \left(u(x), \frac{du}{dx}\right)$$
$$\tilde{U}(x) = \left(\tilde{u}(x), \frac{d\tilde{u}}{dx}\right) \quad 4.33$$

where u is the solution of the initial value problem

$$\frac{d^2u}{dx^2} = -W'(u) + g(x + \tilde{T}_{-}^{1})$$
$$u(\tilde{T}_{-}^{(1)}) = W_{-}^{-1}(E_0 - \Delta E) \quad 4.34$$
$$\frac{du}{dx}(\tilde{T}_{-}^{(1)}) = -\sqrt{2(E(\tilde{T}_{-}^{(1)}) - W(E_0 - \Delta E)}$$

$\tilde{u}$ is the solution of the same differential equation with $g \equiv 0$ and same initial conditions. If we call

$$\begin{aligned} \mathrm{F(U)} &= \left(\left.\frac{du}{dx}\right|_{x=y}, -\mathrm{W}'(u(y)) \right) \\ \mathrm{G}(y) &= (0, g(y + \tilde{\mathrm{T}}^{(1)}_{-})) \end{aligned} \tag{4.35}$$

and U and $\tilde{\mathrm{U}}$ satisfy the following integral equations:

$$\begin{aligned} \mathrm{U}(x) &= \mathrm{U}_0(x) + \int_0^x [\mathrm{F}(\mathrm{U}(y)) + \mathrm{G}(y)]dy \\ \tilde{\mathrm{U}}(x) &= \mathrm{U}_0(x) + \int_0^x \mathrm{F}(\tilde{\mathrm{U}}(y))dy \end{aligned} \tag{4.36}$$

For $x = (x_1, x_2) \in \mathbb{R}^2$ let $\| x \| = | x_1 | + | x_2 |$.

We get

$$\| \mathrm{U}(x) - \mathrm{Y}(x) \| \leqslant \hat{\hat{\mathrm{K}}} \int_{\tilde{\mathrm{T}}^{(1)}_{-}}^{x} \| \mathrm{U}(y) - \mathrm{Y}(y) \| \, dy + \int_{\tilde{\mathrm{T}}^{(1)}_{-}}^{x} \| \mathrm{G}(y) \| \, dy \tag{4.37}$$

with

$$\hat{\hat{\mathrm{K}}} = \sup_{x \leqslant \max(|m^-|, m_+)} | \mathrm{W}''(x) | .$$

By Schwarz inequality, the last integral is less than $\sqrt{\mathrm{L}\delta}$ and so from Gronwall's inequality we get

$$\begin{aligned} \sup_{\tilde{\mathrm{T}}^{1}_{-} \leqslant x \leqslant \mathrm{X}} \| \mathrm{U}(x) - \tilde{\mathrm{U}}(x) \| &\leqslant \sqrt{\mathrm{L}\delta} \exp [\hat{\hat{\mathrm{K}}}(\mathrm{X} - \tilde{\mathrm{T}}^{(1)}_{-})] = \\ &= \mathrm{const}\, \theta^2 \exp [- \alpha \mathrm{L} + \hat{\hat{\mathrm{K}}}(\mathrm{X} - \tilde{\mathrm{T}}^{(1)}_{-})] \end{aligned} \tag{4.38}$$

We choose $\alpha > \hat{\hat{\mathrm{K}}}$.

Now if L is large enough let us define

$$x_1 : \inf(x \mid x > \tilde{\mathrm{T}}^{(1)}_{-}, 0 > \tilde{u}(x) > \mathrm{W}_{-}^{-1}(\mathrm{E}_0 - \Delta\mathrm{E}) + 2\sqrt{\mathrm{L}\delta} \exp [\hat{\hat{\mathrm{K}}}\mathrm{L}]) \tag{4.39}$$

then from 4.38 we get

$$u(x_1) > \mathrm{W}_{-}^{-1}(\mathrm{E}_0 - \Delta\mathrm{E}) + \sqrt{\mathrm{L}\delta} \exp [\hat{\hat{\mathrm{K}}}\mathrm{L}]$$

this implies $\tilde{\mathrm{T}}^{(2)}_{-} < x_1$.

Since $\tilde{u}$ is a conservative motion we have

$$\begin{aligned} x_1 \leqslant (\mathrm{const}) &\int_{\mathrm{W}_{-}^{-1}(\mathrm{E}_0 + \Delta\mathrm{E})}^{\mathrm{W}_{-}^{-1}(\mathrm{E}_0 - \Delta\mathrm{E})} [\mathrm{E}_0 + \Delta\mathrm{E} - \mathrm{W}(u)]^{-1/2} du + \\ &+ \int_{\mathrm{W}_{-}^{-1}(\mathrm{E}_0 - \Delta\mathrm{E})}^{\mathrm{W}_{-}^{-1}(\mathrm{E}_0 - \Delta\mathrm{E}) + 2\sqrt{\mathrm{L}\delta} \exp [\hat{\hat{\mathrm{K}}}\mathrm{L}]} [\mathrm{E}_0 - \Delta\mathrm{E} - \mathrm{W}(u)]^{-1/2} du \end{aligned} \tag{4.40}$$

As before, one gets

$$x_1 \leqslant \text{const}\,\theta \exp\left[-\left(\frac{\alpha}{2} - \frac{\hat{\hat{K}}}{2} - 2\beta\right)L\right]$$

therefore from 4.31, 4.32 and 4.41 we get the result.

LEMMA 4.7. — If L is large enough, there exists a constant $\tilde{K}$ such that if $E_0 \in \tilde{\Gamma}_1 \cup \Gamma_3$
but

$$\text{Inf}(|E_0 - E_{11}|, |E_0 - E_{10}|, |E_0 - E_{01}| \geqslant \theta \exp[-\alpha L] \qquad 4.42$$

where γ is such that

$$\tilde{K} < \gamma < \frac{\alpha}{2} - 2\beta - \frac{\hat{\hat{K}}}{2} \qquad 4.43$$

then the solution u_ε of the initial value problem 4.12 which satisfies $S(u_\varepsilon) < S(u_{11})$ cannot satisfy D. b. c.

Proof. — Let $\tilde{\Gamma}_1$, $\tilde{\Gamma}_3$ be the subsets of $\overline{\Gamma}_1$, Γ_3 where 4.42 is true;

$$\tilde{\Gamma}_1 = \tilde{\Gamma}_1^{(1)} \cup \tilde{\Gamma}_1^{(2)} \cup \tilde{\Gamma}_1^{(3)}, \qquad \tilde{\Gamma}_3 = \tilde{\Gamma}_3^{(1)} \cup \tilde{\Gamma}_3^{(2)}$$

with:

$$\tilde{\Gamma}_1^{(1)} = \{ E \in \overline{\Gamma}_1 \mid E \leqslant E_{11} - \theta \exp[-\gamma L] \}$$
$$\tilde{\Gamma}_1^{(2)} = \{ E \in \overline{\Gamma}_1 \mid E_{11} + \theta \exp[-\gamma L] \leqslant E \leqslant E_{10} - \theta \exp[-\gamma L] \}$$
$$\tilde{\Gamma}_1^{(3)} = \{ E \in \overline{\Gamma}_1 \mid E_{10} + \theta \exp[-\gamma L] \leqslant E \leqslant W(m_-) - \exp[-\beta L] \}$$
$$\tilde{\Gamma}_3^{(1)} = \{ E \in \Gamma_3 \mid W(m_-) + \exp[-\beta L] \leqslant E \leqslant E_a - \theta \exp[-\gamma L] \}$$
$$\tilde{\Gamma}_3^{(2)} = \{ E \in \Gamma_3 \mid E_{01} + \theta \exp[-\gamma L] \leqslant E \leqslant W(m^+) - \exp[-\beta L] \}$$

From eq. B.27, for L sufficiently large we have:

$$\inf(|E_{10} - W(m_-)|, |E_{11} - E_{10}|, |E_{01} - W(m_+)|) > \frac{1}{L}\exp[-\tilde{K}L]$$

for some constant ($\tilde{K}$). Therefore from the first inequality in 4.43 the $\tilde{\Gamma}_1^{(i)}$'s are disjoint.

If $E_0 \in \tilde{\Gamma}_1^{(1)}$, by lemma B.2 and B.3 we get

$$T(E_0) < T(E_{11} - \theta \exp[-\gamma L]) \leqslant L - K_1 \theta \exp[-\gamma L]$$

for some constant K_1 if L is large enough.

Therefore by lemma 4.6 if $\gamma < \alpha/2 - 2\beta - \hat{\hat{K}}/2$ for L large enough we get $\tilde{T}_\varepsilon < L$. Since $\tilde{T}_\varepsilon^{(1)} + \tilde{T}_\varepsilon^{(2)} + \tilde{T}_\varepsilon^{(3)} < \tilde{T}_\varepsilon$, u_ε cannot satisfy D. b. c. at $x = 0$ and $x = L$ with $(I, J) = (1, 1)$, $(0, 1)$ or $(1, 0)$ and since by lemma 4.4 all other values for (I, J) are excluded this concludes the proof in the case $E \in \tilde{\Gamma}_1^{(1)}$.

By similar argument, if $\gamma < \alpha/2 - 2\beta - \hat{\hat{K}}/2$ and L is large enough, one gets:

for $E_0 \in \tilde{\Gamma}_1^{(2)}$

$$\tilde{T}_\varepsilon > L, \quad \tilde{T}_\varepsilon^{(1)} + \tilde{T}_\varepsilon^{(2)} + \tilde{T}_\varepsilon^{(3)} < L$$

for $E_0 \in \tilde{\Gamma}_1^{(3)}$

$$\tilde{T}_-^{(1)} + \tilde{T}_-^{(2)} + \tilde{T}_-^{(3)} > L \quad \text{and} \quad \tilde{T}_+^{(1)} + \tilde{T}_+^{(2)} + \tilde{T}_+^{(3)} < L$$

for $E_0 \in \tilde{\Gamma}_3^{(1)}$

$$\tilde{T}_+^{(1)} + \tilde{T}_+^{(2)} + \tilde{T}_+^{(3)} < L$$

for $E_0 \in \Gamma_3^{(2)}$

$$\tilde{T}_+^{(1)} + \tilde{T}_+^{(2)} + \tilde{T}_+^{(3)} > L$$

and the lemma is proved. ■

LEMMA 4.8. — There exists a constant $\tilde{\tilde{K}}$ such that if L is large enough, $\beta > \tilde{\tilde{K}}$, $\alpha/2 - 2\beta - \tilde{\tilde{K}}/2 > 0$ and $E_0 \in \Gamma_2 \cup \Gamma_4$, then the solution of 4.12 cannot satisfy Dirichlet boundary conditions.

Proof. — For $E_0 \in \Gamma_2$.

If $\varepsilon = -1$ it is not difficult to check that if $\alpha > \beta$ then

$$\tilde{T}_-^{(1)} \geqslant \sqrt{2} \int_{W^{-1}(W(m_-) - \exp[-\beta L])}^{0} [W(m_-) + \exp[-\beta L] - W(u)]^{-1/2} du \quad 4.47$$

as in the proof of lemma B.1, we get

$$\tilde{T}_-^{(1)} \geqslant K_1 \beta_1 L \qquad \text{for some constant } K_1.$$

If $\varepsilon = +1$ and $\alpha/2 - 2\beta - \hat{\hat{K}}/2 > 0$ we get

$$\tilde{T}_+^{(1)} + \tilde{T}_+^{(2)} + \tilde{T}_+^{(3)} \leqslant K_2 \qquad \text{for some constant } K_2.$$

For $E_0 \in \Gamma_4$ we have

$$\tilde{T}_+^{(1)} \geqslant K_1' \beta L \qquad \text{for some constant } K_1'.$$

By choosing $\tilde{K} = \max\left(\frac{1}{K_1}, \frac{1}{K_1'}\right)$ the lemma is proved. ■

Proof of Proposition 4.2. — In order to satisfy all the previous conditions we choose $\beta = \alpha/8$, $\gamma = \alpha/8 + \tilde{K}/2 - \hat{\hat{K}}/4$ with $\alpha > \max(4\tilde{\tilde{K}}, 4\tilde{\tilde{K}} + 2\hat{\hat{K}})$.

It is clear from lemma 4.7 and 4.8 that, if we consider any real value of E_0 such that

$$\min(|E_0 - E_{11}|, |E_0 - E_{10}|, |E_0 - E_{01}|) > \theta \exp[-\gamma L] \quad 4.48$$

then the corresponding solution of the initial value problem 4.12 with $S(u_\varepsilon) < S(u_{11})$ cannot satisfy D. b. c. on $[0, L]$.

We define now, as in the proof of lemma 4.6 two vectors:

$$U_\varepsilon = \left(u_\varepsilon, \frac{du_\varepsilon}{dx}\right)$$
$$U_{I,J,\varepsilon_{IJ}} = \left(u_{IJ,\varepsilon_{IJ}}, \frac{du_{IJ,\varepsilon_{IJ}}}{dx}\right) \qquad 4.49$$

where u_ε is the solution of equation 4.12 and $u_{IJ,\varepsilon_{IJ}}$ is the solution of the analogous conservative equation with $g \equiv 0$ and initial conditions:

$$u_{IJ,\varepsilon_{IJ}}(0) = 0$$
$$\frac{du_{IJ,\varepsilon_{IJ}}}{dx} = \varepsilon_{IJ}\sqrt{2(E_{IJ} - W(0))}.$$

If E_0 and ε satisfy one of the following conditions:

$$\begin{array}{lll} i) & |E_0 - E_{11}| \leqslant \theta \exp[-\gamma L] & \varepsilon\varepsilon_{IJ} = +1 \\ ii) & |E_0 - E_{10}| \leqslant \theta \exp[-\gamma L] & \varepsilon\varepsilon_{10} = -1 \\ iii) & |E_0 - E_{01}| \leqslant \theta \exp[-\gamma L] & \varepsilon = \varepsilon_{01} = +1 \end{array} \qquad 4.50$$

from Gronwall's inequality we get:

$$\sup_{0\leqslant x\leqslant L} \|U_\varepsilon(x) - U_{IJ,\varepsilon_{IJ}}(x)\| \leqslant \text{const}(\max(\theta \exp[-\gamma L], \theta \exp[-\alpha L])). \exp[\hat{\hat{K}} L] \qquad 4.51$$

where $\hat{\hat{K}}$ is defined in lemma 4.6.

Therefore, since $\gamma < \alpha$, the right hand side of 4.51 does not exceed $\theta \exp[-(\gamma - \hat{\hat{K}})L]$ and, choosing $\alpha_0 = \frac{5}{4}\tilde{\tilde{K}}$, we end the proof of proposition 4.2. ■

APPENDIX A

In this appendix we study the system introduced in section 1 and described by the Hamiltonian 1.5.

The finite volume Gibbs equilibrium measure is

$$P_N(d\underline{\sigma}_N) = P(d\sigma_1 \dots d\sigma_N) = \exp[-\beta H(\underline{\sigma}_N)] \prod_{x=1}^{N} \rho(\sigma_x)d\sigma_x / Z_N(\beta) \qquad A.1$$

where

$$Z_N(\beta) = \int \prod_{\alpha=1}^{N} \rho(\sigma_\alpha)d\sigma_\alpha \exp[-\beta H(\underline{\sigma}_N). \qquad A.2$$

If we introduce the variables

$$m_i = \frac{1}{M}\sum_{\alpha\in A_i}\sigma_\alpha = \gamma\sum_{\alpha\in A_i}\sigma_\alpha \qquad i = 1, 2, \dots, L \qquad A.3$$

the Gibbs equilibrium measure expressed in terms of the block magnetizations becomes

$$P(d\underline{m}_L) = P(dm_1, \dots, dm_L) = \exp[-\beta/\gamma E(\underline{m}_L) \prod_{i=1}^{L} R_\gamma(dm_i)/Z_{\gamma,L}(\beta) \qquad A.4$$

where

$$E(\underline{m}_L) = -\sum_{i=1}^{L}[(J_0 + J_1)m_i^2 - J_1(m_i - m_{i+1})^2 + hm_i] \qquad A.5$$

and $R_\gamma(dm)$ is the measure induced on $\mathbb{R}$ by the application $(\sigma_\alpha)_{\alpha\in A_i} \to \sum_{x\in A_i}\sigma_\alpha$ i. e.:

$$\int R_\gamma(dm_i)\chi_\Delta(m_i - m) = \int \prod_{x\in A_i}\rho(d\sigma_x)\chi_\Delta\Big(\gamma\sum_{x\in A_i}\sigma_x - m\Big) \qquad A.6$$

with

$$\chi_\Delta = \begin{cases} 1 & \text{if } |x| \leqslant \Delta \\ 0 & \text{if } |x| > \Delta \end{cases}$$

and

$$Z_{\gamma,L}(\beta) \equiv Z_N(\beta) = \int \prod_{i=1}^{L} R_\gamma(dm_i)\exp[-\beta/\gamma E(\underline{m}_L)] \qquad A.7$$

THEOREM A.1. — For the partition function $Z^{\Delta}_{\gamma,L}(m)$ defined by eq. 1.6 with H_0 given by eq. 1.5 when $h = 0$ and ρ satisfying conditions *a*), *b*), *c*) of section 1:

$$-\lim_{\Delta\to 0}\lim_{\substack{\gamma\to 0\\ L\to\infty}} \frac{\gamma}{L}\ln Z^{\Delta}_{\gamma,L}(m) = \text{convex envelope}(V_0(m)) \qquad A.8$$

where $V_0(m)$ is defined by

$$V_0(m) = -\tilde{J}_0\beta m^2 + I(m) \qquad A.9$$

with

$$I(m) = \inf_{t \in \mathbb{R}} \left\{ tm - \ln \int e^{t\sigma} \rho(\sigma) d\sigma \right\} \qquad \text{A.10}$$
$$\tilde{J}_0 = J_0 + J_1 .$$

Proof. — The strategy of the proof is that of properly bounding $\frac{\gamma}{L} \ln Z^{\Delta}_{\gamma,L}(m)$ and then show that the lower and the upper bound tend to the same value in the limit considered in eq. A.8. These bounds will require estimate of the Curie-Weiss free energy

$$\lim_{\gamma \to 0} \gamma \ln \int R_\gamma(dm) \exp [(\beta hm + \beta \tilde{J}_0 m^2)/\gamma] = - \beta F_{C.W.}(\beta, h) \qquad \text{A.11}$$

by Theorem B.1 of [*10*] we have:

$$- \beta F_{C.W.}(\beta, h) = \sup_{m \in \mathbb{R}} [\beta hm + \beta \tilde{J}_0 m^2 - I(m)] \equiv \sup_m [- V_h(m)] \qquad \text{A.12}$$

where $I(m)$ is defined in eq. A.10.

The quantity in square bracket in A.12 is well known.

In fact (see for instance [*10*]) $\exists \beta_c$ such that

if $\beta \leqslant \beta_c$ $V_0(m)$ is a convex function of m that attains its minimum at $m = 0$;

if $\beta > \beta_c$ $V_0(m)$ has three extrema and two of them ($\pm m^*(\beta)$) are equal minima;

if $\beta > \beta_c$ and $h \neq 0$ $V_h(m)$ has at most three extrema but only one of them $m^*(h, \beta)$ is the absolute minimum.

Let us consider the $\beta > \beta_c$ first.

a) $$| m_0 | < m^*(\beta) = m^*$$

calling $$\alpha = \frac{m_0 + m^*}{2m^*}$$

since $$\chi_\Delta\left(\frac{1}{L}\sum_{i=1}^{L} m_i - m_0\right) \geqslant \prod_{i=1}^{\alpha L} \chi_\Delta(m_i - m^*) \prod_{i=\alpha+1}^{L} \chi_\Delta(m_i + m^*)$$

we get

$$\frac{\gamma}{L} \ln Z^{\Delta}_{\gamma,L}(m_0) \geqslant - \beta J_1 \Delta^2 - \frac{(2m^* + \Delta)^2}{L} + \gamma\alpha \ln \int R_\gamma(dm) \exp [\beta \tilde{J}_0 m^2/\gamma] \chi_\Delta(m - m^*) + $$
$$+ \gamma\alpha \ln \int R_\gamma(dm) \exp [\beta \tilde{J}_0 m^2/\gamma] \chi_\Delta(m + m^*) \qquad \text{A.13}$$

on the other hand we have:

$$\frac{\gamma}{L} \ln Z^{\Delta}_{\gamma,L}(m_0) \leqslant \gamma \ln \int R_\gamma(dm) \exp \left[\frac{\beta \tilde{J}_0 m^2}{\gamma} \right] \qquad \text{A.14}$$

since the interaction energy among blocks is always positive.

If we remark that A.12 is a consequence of large deviation theory it is easy to convince ourself that

$$\lim_{\gamma \to 0} \gamma \ln \int R_\gamma(dm) \exp [\beta \tilde{J}_0 m^2/\gamma] \chi_\Delta(m \mp m^*) = \lim_{\gamma \to 0} \gamma \ln \int R_\gamma(dm) \exp [\beta \tilde{J}_0 m^2/\gamma] = V_0(m^*) \qquad \text{A.15}$$

therefore

$$- \lim_{\Delta \to 0} \lim_{\substack{\gamma \to 0 \\ L \to \infty}} \frac{\gamma}{L} \ln Z^{\Delta}_{\gamma,L}(m_0) = V_0(m^*) \qquad \forall m_0 : | m_0 | \leqslant m^* \qquad \text{A.16}$$

and therefore, since m^* is the value of m for which $V_0(m)$ is minimum we have also

$$V_0(m^*) = \text{convex envelope}\,(V_0(m)) \qquad \forall m : |m| \leqslant m^*.$$

b) $$|m_0| > m^*(\beta)$$

since
$$\chi_\Delta\left(\frac{1}{L}\sum_{i=1}^{L} m_i - m_0\right) \geqslant \prod_{i=1}^{L} \chi_\Delta(m_i - m_0) \qquad \text{A.17}$$

we get
$$\frac{\gamma}{L}\ln Z^\Delta_{\gamma,L}(m_0) \geqslant -\beta J_1 \Delta^2 + \gamma \ln \int \chi_\Delta(m' - m_0)\exp\left[\frac{\beta \tilde{J}_0}{\gamma} m'^2\right].R_\gamma(dm') \qquad \text{A.18}$$

Let us introduce the « translated » measure

$$P_{t_0}(dm) = \frac{\exp[t_0 m/\gamma]R_\gamma(dm)\exp[\beta\tilde{J}_0 m^2/\gamma]}{\int \exp[t_0 m'/\gamma]R_\gamma(dm')\exp[\beta\tilde{J}_0 m'^2/\gamma]} \qquad \text{A.19}$$

where t_0 is such that

$$m_0 = \frac{\int m' R_\gamma(dm')\exp[\beta\tilde{J}_0 m'^2/\gamma + t_0 m'/\gamma]}{\int R_\gamma(dm')\exp[\beta\tilde{J}_0 m'^2/\gamma + t_0 m'/\gamma]} \qquad \text{A.20}$$

We remark that such a t_0 exists always by convexity argument.
Therefore since

$$\gamma \ln \int \chi_\Delta(m-m_0)R_\gamma(dm)\exp[\beta\tilde{J}_0 m^2/\gamma] =$$
$$= \gamma \ln \int P_{t_0}(dm)\chi_\Delta(m-m^0)\exp[-t_0 m/\gamma] + \gamma \ln \int \exp[t_0 m/\gamma]R_\gamma(dm)\exp[\beta\tilde{J}_0 m^2/\gamma] \qquad \text{A.21}$$

we get

$$\frac{\gamma}{L}\ln Z^\Delta_{\gamma,L}(m_0) \geqslant -4\beta J_1 \Delta^2 - t_0 m_0 - 0(\Delta) + 0(\gamma) + \sup_m \{ t_0 m + \beta\tilde{J}_0 m^2 - I(m) \} \qquad \text{A.22}$$

where we have used A.12 and the fact that

$$\lim_{\gamma\to 0} \ln \int P_{t_0}(dm)\chi_\Delta(m - m_0) = \text{const.}$$

which follows from A.20 and from the analogous of A.15 with $h \neq 0$. Using A.20 to compute the sup { ... } in A.22 we get

$$\frac{\gamma}{L}\ln Z^\Delta_{\gamma,L} \geqslant -\beta\tilde{J}_0 m_0^2 - I(m_0) - 0(\Delta) - \text{const}\,\gamma \qquad \text{A.23}$$

Now
$$\frac{\gamma}{L}\ln Z^\Delta_{\gamma,L}(m_0) \leqslant \frac{\gamma}{L}\ln \int \prod_{i=1}^{L}(R_\gamma(dm_i)\exp[\beta\tilde{J}_0 m^2/\gamma])\chi_\Delta\left(\frac{1}{L}\sum_{i=1}^{L} m_i - m_0\right) \qquad \text{A.24}$$

By Tchebytchev exponential inequality we can get

$$\frac{\int \prod_{i=1}^{L}\left(\prod_{\alpha\in A_i}\rho(\sigma_\alpha)d\sigma_\alpha \exp\left[\beta\gamma\tilde{J}_0\left(\sum_{\alpha\in A_i}\sigma_\alpha\right)^2\right]\right)\chi_\Delta\left(\frac{1}{L}\sum_{\alpha=1}^{L\gamma}\sigma_\alpha > m_0 - \Delta\right)}{\int \prod_{i=1}^{L}\left(\prod_{\alpha\in A_i}\rho(\alpha_\alpha)d\sigma_\alpha \exp\left[\beta\gamma\tilde{J}_0\left(\sum_{\alpha\in A_i}\sigma_\alpha\right)^2\right]\right)} \leqslant$$

$$\leqslant \exp\left[-\frac{L}{\gamma}\left\{\sup_t\left[t(m_0-\Delta)-\gamma\ln\left(\frac{\int R_\gamma(dm)\exp\left[(\beta\tilde{J}_0 m^2 + tm)/\gamma\right]}{\int R_\gamma(dm)\exp\left[\beta\tilde{J}_0 m^2\right]}\right)\right]\right\}\right] \qquad \text{A.25}$$

therefore

$$\frac{\gamma}{L}\ln Z^\Delta_{\gamma,L}(m_0) \leqslant -\sup_t\left\{t(m_0-\Delta)-\gamma\ln\int R_\gamma(dm)\exp\left[(\beta\tilde{J}_0 m^2 + tm)\gamma\right]\right\} = \qquad \text{A.26}$$

$$= \beta\tilde{J}_0 m_0^2 - I(m_0) + 0(\Delta) + 0(\gamma).$$

From A.23 and A.26 we get

$$\lim_{\Delta\to 0}\lim_{\substack{\gamma\to 0\\ L\to\gamma}}\frac{\gamma}{L}\ln Z^\Delta_{\gamma,L}(m_0) = \beta\tilde{J}_0 m_0^2 - I(m_0) \qquad \text{A.27}$$

since if $|m_0| > m^*$ the right hand side of A.27 is convex (see [*10*]) this concludes the proof for $\beta > \beta_c$. For the case $\beta < \beta_c$ the proof goes along the same lines of part *b*.

Our previous results are consistent with the probability distribution that for γ and L finite reads:

$$P(d\underline{m}_L) = \frac{\exp\left[-F(\underline{m}_L)/\gamma\right]d\underline{m}}{Z_{\gamma,L}(\beta, h)} \qquad \text{A.28}$$

with

$$F(\underline{m}_L) = \sum_{i=1}^{L}\left\{V_h(m_i) + \beta J_1(m_i - m_{i+1})^2\right\} \qquad \text{A.29}$$

where

$$V_h(m) = -(\tilde{J}_0\beta m^2 + \beta h m) + I(m). \qquad \text{A.30}$$

We conclude this appendix by giving some properties of the function $V_h(m)$ that we have assumed for the continuous version of this model namely condition I) II) III) and IV) of section 2.

LEMMA A.2. — Under the hypothesis *a*) *b*) *c*) for the measure $\rho(\sigma_\alpha)$ the following properties hold for the function $V_h(m)$ given by eq. A.30.

I. $\forall\beta > 0$ and h finite $V_h(m)\in C'(-\infty, +\infty)$;

II. if we further assume $\rho(\sigma)\propto\exp[-a\sigma^4]$ we have also that $\forall\beta > 0$ and h finite $V_h'(m) \underset{m\to\infty}{\to} m^3$;

III. $\exists\beta_c$ such that $\forall\beta > \beta_c$ and h sufficiently small $V_h^1(m) = 0$ has three solutions m_+, m_-, m_0 where m_+ and m_- correspond to minima and m_0 to a maximum;

IV. sign $V'''(m)$ = sign (m).

Proof. — From A.30 and A.10 it follows that

$$\frac{dV_h}{dm} = -2\tilde{J}_0\beta m - h + t^* \qquad \text{A.31}$$

$$\frac{d^2V_h}{dm^2} = -2\tilde{J}_0\beta + \frac{dt^*}{dm} \qquad \text{A.32}$$

$$\frac{d^K V_h}{dm^K} = \frac{d^{K-1}}{dm^{K-1}}t^*, \qquad K > 2 \qquad \text{A.33}$$

where

$$t^* : m = \frac{\int \sigma\rho(\sigma)d\sigma \exp[t^*\sigma]}{\int \rho(\sigma)d\sigma \exp[t^*\sigma]} \equiv \langle \sigma \rangle_{t^*} \qquad \text{A.34}$$

from A.34 we also get

$$\frac{dm}{dt^*} = \langle \sigma^2 \rangle_{t^*} - (\langle \sigma \rangle_{t^*})^2 > 0 \qquad \text{A.35}$$

$$\frac{d^2m}{dt^*} = \langle \sigma^3 \rangle_{t^*} - 3\langle \sigma^2 \rangle_{t^*}\langle \sigma \rangle_{t^*} + 2\langle \sigma \rangle_{t^*}^3 = \begin{cases} > 0 & t^* < 0 \\ < 0 & t^* > 0 \end{cases} \qquad \text{A.36}$$

where the strict inequality in A.35 comes from the non singular nature of $\rho(\sigma)$ and the strict inequality in A.36 is a consequence of the properties *b*) and *c*) of $\rho(\sigma)$ (see section 1).

Furthermore, assuming that $\rho(\sigma) \propto \exp[-a\sigma^4]$ by explicit calculation, it is possible to get:

$$m = \frac{\int \exp[-a\sigma^4]\exp[t\sigma]\sigma d\sigma}{\int \exp[-a\sigma^4]\exp[t\sigma]d\sigma} \underset{t\to\infty}{\rightarrow} \frac{t^{1/3}}{(4a)^{1/3}} \qquad \text{A.37}$$

In conclusion:

Property I follows from A.33 and A.35.
Property II from A.31 and A.37.
Property IV from A.34, A.35, A.36.

To obtain property III we remark that the eq.

$$\frac{dV_h(m)}{dm} = 0 \qquad \text{A.39}$$

coupled with A.31 is the selfconsistency eq. of the Curie-Weiss model and, as it is well known, (see for instance [*10*]) $\exists \beta_c : \forall \beta > \beta_c$ and h sufficiently small property III holds.

APPENDIX B

In this appendix, we will prove some results concerning the equilibrium solutions of the deterministic time evolution given by eq. 2.1 when $\varepsilon = 0$.

These stationary solutions are the critical points of the stationary action

$$S(u) = \int_0^L \left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 - W(u)\right]dx \qquad B.1$$

with D. b. c. on $[0, L]$, where $W(u) = -V(u)$ i. e. the solutions of the boundary value problem:

$$\frac{d^2u}{dx^2} = -W'(u) \qquad B.2$$

$$u(0) = u(L) = 0. \qquad B.3$$

If we interpret x as a time variable, eq. B.2 becomes the Newton's equation for a particle of mass one moving in the potential $+W(u)$ and the solution of eq. B.2 satisfying the conditions B.3 are the trajectories that pass through the origin at times 0 and L.

When u is a solution of eq. B.2, the energy

$$E = \frac{1}{2}\left(\frac{du}{dx}\right)^2 + W(u) \qquad B.4$$

is a conserved quantity and in our case, (boundary conditions B.3) we will have $W(m_+) > E > W(0)$.

Fig. B.1 shows the three possibilities that can arise and that are discussed in Proposition 2.2.

We will start with some preliminary considerations and Lemmas. We will consider first the case 2 of fig. B.1 and the results for case 1 and 3 will be briefly summarized at the end of this section.

Given a real number $E \in [W(0), W(m_-)]$, under our hypothesis on W (see fig. B.1), there exists two periodic solutions of the initial value problem

$$\frac{d^2u}{dx^2} + W'(u) = 0 \qquad B.5$$

$$u(0) = 0; \quad \left.\frac{du}{dx}\right|_{x=0} = \varepsilon[2(E - W(0))]^{1/2} \qquad B.6$$

with $\varepsilon = +1$ and $\varepsilon = -1$, each corresponding to a given value of ε.

On the other hand, for $E \in \,]W(m^-), W(m^+)[$ only the solution of the initial value problem B.5 and B.6 with $\varepsilon = +1$, returns to the origin in a finite time.

Under our hypothesis (see fig. B.1), W is strictly decreasing on (m_-, m_0) and strictly increasing on (m_0, m_+), therefore we can define local inverse of W

$$W_-^{-1} : [W(m_-, W(m_0)] \to [m_-, m_0] \qquad B.7$$

$$W_+^{-1} : [W(m_0), W(m_+)] \to [m_0, m_+] \qquad B.8$$

given $E \in [W(0), W(m^+)]$, if

$$W_-^{-1}(E) \leqslant x < y \leqslant W_+^{-1}(E)$$

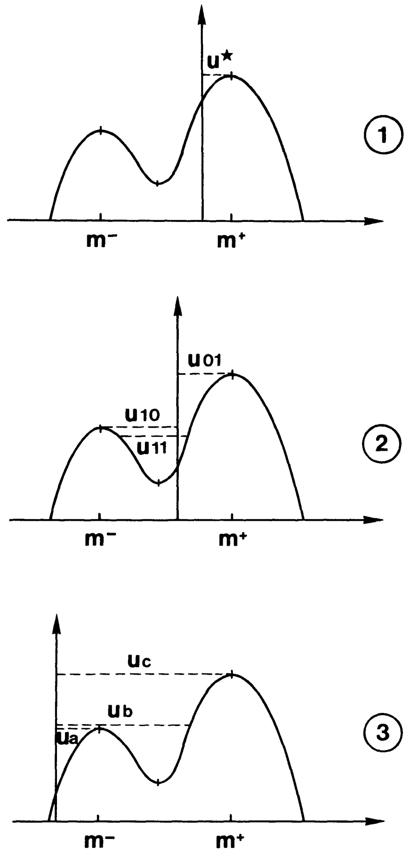

FIG. B.1.

we define

$$\tau_{[x,y]}(\mathrm{E}) = \frac{1}{\sqrt{2}} \int_x^y [\mathrm{E} - \mathrm{W}(u)]^{-1/2} du \qquad \mathrm{B.9}$$

that is the time for a particle which satisfies eq. B.2 and has energy E to go from x to y.

Consider for sake of definitness the case $\mathrm{E} - \mathrm{W}(\overline{m}) \geqslant 0$ and call:

$$\begin{aligned} \tau_-(\mathrm{E}) &= \tau_{[\mathrm{W}_-^{-1}(\mathrm{E}), m_0]} \\ \tau_0(\mathrm{E}) &= \tau_{[m_0, 0]} \\ \tau_{\overline{m}}(\mathrm{E}) &= \tau_{[0, \overline{m}]} \\ \tau_+(\mathrm{E}) &= \tau_{[\overline{m}, \mathrm{W}_+^{-1}(\mathrm{E})]} \end{aligned} \qquad \mathrm{B.10}$$

The period of a periodic solution of eq. B.2 will be

$$\mathrm{T} = 2(\mathrm{T}_-(\mathrm{E}) + \mathrm{T}_+(\mathrm{E})) \qquad \mathrm{B.11}$$

where

$$\mathrm{T}_- = \tau_- + \tau_0 ; \quad \mathrm{T}_+ = \tau_{\overline{m}} + \tau_+$$

(the modifications required for the case $E - W(\overline{m}) < 0$ or more generally for the case $\overline{m} \leqslant 0$ are obvious and they imply only a different way of splitting the times T_+ and T_-.

Given a real number L, that in our problem can be arbitrary large, two integer I, J with $|I - J| \leqslant 1$ and $\varepsilon_{IJ} \in \{-1, +1\}$ such that

$$\varepsilon_{IJ} = -1 \quad \text{if} \quad I = J + 1; \qquad \varepsilon_{IJ} = +1 \quad \text{if} \quad J = I + 1$$

but $\varepsilon_{IJ} = +1$ or -1, then, if E_{IJ} is such that

$$L = 2IT_-(E_{IJ}) + 2JT_+(E_{IJ}) \qquad \text{B.12}$$

the solution $u_{IJ,\varepsilon_{IJ}}$ of the initial value problem B.5, B.6 with $E = E_{IJ}$ and $\varepsilon = \varepsilon_{IJ}$ is a solution of the boundary value problem B.2, B.3.

On the other hand for a given L, if u is a solution of the boundary value problem B.2, B.3, it is clear that the value of $\left.\frac{du}{dx}\right|_{x=0}$ gives a pair (ε, E) such that B.6 is satisfied, D. b. c. gives the two integers I, J such that B.12 is satisfied. To discuss the solutions of equation B.10 we will give first some properties of the various times τ, which are crucial to our analysis.

LEMMA B.1. — It is possible to find a constant $\delta > 0$ such that if $W(m_-) - E > \delta$ then

$$\tau_-(E) \geqslant -K_1(\delta)\ln(W(m_-) - E) \qquad \text{B.13}$$

$$\tau_-(E) \leqslant -K_2(\delta)\ln(W(m_-) - E) \qquad \text{B.14}$$

$$\tau_0(E) + \tau_{m_0}(E) + \tau_+(E) \leqslant K_3(\delta) \qquad \text{B.15}$$

for some constants $K_i = K_i(\delta)$, $i = 1, 2, 3$. For $W(m_+) - E < \delta$ the analogous of B.13 and B.14 holds.

Proof. — We study $\tau_-(E)$, the proof for $\tau_+(E)$ is similar. Let us first choose $\delta > 0$ such that for $|W(m_-) - E| < \delta$ it also follows that $\tilde{m} - W_-^{-1}(E) > \delta$ where $\tilde{m}$ is the only inflection point in the interval (m_-, m_0), then

$$\begin{aligned}\tau_-(E, \delta) \equiv \frac{1}{\sqrt{2}}\int_{W_-^{-1}(E)}^{W_-^{-1}(E)+\delta/2} [E - W(u)]^{-1/2}du \leqslant \tau_-(E) \leqslant \\ \leqslant \tau_-(E, \delta) + \frac{1}{2}(m_+ - m_-)(E - W(W_-^{-1}(E) + \delta/2))\end{aligned} \qquad \text{B.16}$$

Using Taylor formula and also property IV of $V(m)$ that controls the sign of W''', if we notice that $\overline{m} > m_0$, we get

$$\begin{aligned}\tau_-(E, \delta) \geqslant \frac{1}{\sqrt{-W''(m_-)}}\frac{1}{\sqrt{2}}\int_0^{\delta/2} [y(W_-^{-1}(E) - m_-) + y^2/2]^{-1/2}dy = \\ = \frac{1}{\sqrt{-W''(m_-)}}\ln\frac{\sqrt{\delta} + \sqrt{4(W_-^{-1}(E) - m_- + \delta)}}{\sqrt{4(W_-^{-1}(E) - m_-)}}\end{aligned} \qquad \text{B.17}$$

and also

$$(W(m_-) - E)/\alpha \geqslant \frac{E - W(m^-)}{W''(W_-^{-1}(E))} \geqslant \frac{(W_-^{-1}(E) - m_-)^2}{2} \qquad \text{B.18}$$

for some $\alpha > 0$ for $W_-^{-1}(E) + \delta/2 < \tilde{m}$ and B.13 follows.

The arguments to get B.14 are very similar.

To get B.15 it is sufficient to notice that for $|W(m^-) - E| < \delta$ the r. h. s. of eq. B.9, when the integration intervals are those prescribed by the definitions of τ_0, $\tau_{\overline{m}}$ and τ_+ in eq. B.10, is always finite and bounded uniformely on E.

The following lemma is an adaptation of a result of Loud [*11*] (see also Chafee and Infante [*12*]).

LEMMA B.2. — $\tau_{\pm}(E)$ is differentiable on its domain $(W(0), W(m_{\pm}))$ and

$$\frac{d\tau_{\pm}(E)}{dE} \geqslant 0 \tag{B.19}$$

Moreover there exist two constants K_1, K_2 such that

$$\frac{d\tau_{\pm}(E)}{dE} \geqslant K_1 \tau_{\pm}(E) \tag{B.20}$$

$$\frac{d\tau_{\pm}(E)}{dE} \leqslant K_2 \tau_{\pm}(E)/[W(m_{\pm}) - E] \tag{B.21}$$

if $W(m_{\pm}) - E$ is small enough.

Proof. — We consider $\tau_-(E)$, the other cases are similar. If we perform the change of variable $W(u) = [E - W(-m_0)y^2 + W(-m_0)]$ in the integral expression of $\tau_-(E)$ we get

$$\tau_-(E) = \frac{1}{\sqrt{2}} \int_{-1}^{0} \frac{(E - W(m_0))^{1/2} y dy}{W'(W_-^{-1}((E - W(m_0))y^2 + W(m_0)))} \tag{B.22}$$

Differentiating with respect to E, setting $u = W_-^{-1}((E - W(m_0))y^2 + W(m_0))$ we get

$$\frac{d\tau_-(E)}{dE} = \frac{1}{2\sqrt{2}}(E - W(m_0))^{-1/2} \int_{-1}^{0} \frac{y dy}{W'(u)(1 - y^2)^{1/2}} \frac{[(W'(u))^2 - 2W''(u)(W(u) - W(m_0))]}{(W'(u))^2} \tag{B.23}$$

Calling $G(u)$ the term in square bracket in B.23, it is not difficult to see that

$$G(m_0) = [W'(m_0)]^2 > 0.$$

Since, by property IV, $G'(u) < 0$ we get $G(u) > G(m_0) > 0$ from which we deduce B.19. Moreover we also get

$$\frac{d\tau_-(E)}{dE} \geqslant \frac{(W'(m_0))^2}{2\sqrt{2}(E - W(m_0))^{1/2}} \int_{-1}^{0} \frac{y dy}{(1 - y^2)^{1/2}[W'(u)]^3} \tag{B.24}$$

from which, by Lagrange theorem, we get B.20. Using property IV and Lagrange theorem in B.23 it is not difficult to prove B.21.

LEMMA B.3. — If $E - W(\overline{m}) \geqslant \rho > 0$ there exists a constant $C(\rho)$ such that

$$0 \geqslant \frac{d\tau_0}{dE} \geqslant -C(\rho) \tag{B.25}$$

$$0 \geqslant \frac{d\tau_{\overline{m}}}{dE} \geqslant -C(\rho) \tag{B.26}$$

Proof. — We consider only $\tau_{\overline{m}}$, the proof for τ_0 is similar.
From the definition of $\tau_{\overline{m}}$ we get

$$\frac{d\tau_{\overline{m}}}{dE} = -\frac{1}{2\sqrt{2}} \int_{0}^{\overline{m}} [E - W(u)]^{-3/2} du \tag{B.27}$$

and therefore since $E - W(\overline{m}) \geqslant \rho > 0$ B.26 follows.

The following proposition is the main result of this section.

PROPOSITION B.4. — There exists K_1 such that only for $I < K_1L$ the equation B.12 has solution for any L.

There exists a constant K_2 such that if $I < K_2L$, $J < K_2L$ then if L is large enough, equation B.12 has only one solution E_{IJ}, moreover

$$E_{II} > E_{I,I+1} > E_{I+1,I} > E_{I+1,I+1} \tag{B.28}$$

If L/I is large enough then

$$\text{for } I \geqslant 1 \quad W(m_-) - \text{const}\exp\left[-\frac{L}{I}K_3\right] \leqslant E_{IJ} \leqslant W(m_-) + \text{const}\exp\left[-\frac{L}{I}K_4\right] \tag{B.29}$$

$$W(m_+) - \text{const}\exp\left[-\frac{L}{I}K_3\right] \leqslant E_{01} \leqslant W(m_+) + \text{const}\exp\left[-\frac{L}{I}K_4\right] \tag{B.30}$$

for some constants K_3 and K_4.

If ΔE_I is the difference between two consecutive terms in B.28 then if L/I is large enough

$$|\Delta E_I| \geqslant \text{const } 1/L \exp\left[-\frac{L}{I}K_5\right] \tag{B.31}$$

$$|\Delta E_I| \leqslant \text{const}\exp\left[-\frac{L}{I}K_6\right] \tag{B.32}$$

For a given L, large enough, if E_{IJ} is a solution with $I > K_2L$ then I, J cannot belong to any interval $[E_{\tilde{I}\tilde{J}}, E_{10}]$.

Proof. — Under our hypothesis ($W''(m_0) \neq 0$). There exists a constant K_1, independant from L such that

$$\inf_{W(m_0) < E < W(m_-)} \{ T(E) \} \geqslant 1/K_1 > 0$$

from which we get the first assertion.

From lemma B.1, B.2, B.3 there exists $E_* < W(m^-)$ such that if $E \subset [E_*, W(m_-)]$ then $T_-(E)$ is strictly increasing, moreover there exists an $E^* \in [W(0), E_*]$:

$$T_-(E^*) = \max_{E\in[W(0),E_*]} T_-(E) \tag{B.34}$$

Similarly there exists $\tilde{E}_*$ such that, if $E \in [\tilde{E}_*, W(m_+)]$, $T_+(E)$ is strictly increasing and $\tilde{E}^* > \tilde{E}_*$ such that

$$T_+(\tilde{E}^*) = \max_{E\in(W(0),\tilde{E}_*)} T_+(E) \tag{B.35}$$

Therefore if $L \geqslant \max(2T_+(\tilde{E}^*), 2T_-(E^*))$, there exists one solution of B.12 for IJ = (10), (01), (11). Let $K_2 < (T(E^*))^{-1}$ then if $1 \leqslant I, J \leqslant K_2L$ and L large enough we get

$$2IT_+(E) + 2JT_-(E) \leqslant K_2L(T_+(E) + T_-(E))$$

since for $E = E^*$ one gets $2IT_+(E^*) + 2JT_-(E^*) < L$ and $2IT_+(E) + 2JT_-(E)$ is strictly increasing if $E > E^*$ there exists only one E_{IJ} such that B.12 is true.

Using the fact that $T_+ + T_-$ is increasing if $E > E^*$ and that

$$2IT_-(E_{II}) + 2IT_+(E_{II}) = 2IT_-(E_{I,I+1}) + 2(I+1)T_+(E_{I,I+1})$$

we get also $E_{II} > E_{I,I+1}$ and similarly all other inequality given in B.28. B.29, B.30 are a direct consequence of equations B.13, B.14, B.15 in Lemma B.1.

B.32 is a direct consequence of B.29.

We prove B.31 in the case $\Delta E_I = E_{II} - E_{I,I+1}$. The other cases are proved with easy modification of the same argument.

Using B.12 for two pairs (I, I), (I, I + 1) one gets

$$2T_+(E_{I,I+1}) < 2I\left(\int_{E_{I,I+1}}^{E_{II}} \frac{d\tau_-}{dE}(\tilde{E})d\tilde{E}\right) + 2(I+1)\int_{E_{I,I+1}}^{E_{II}} \frac{d\tau_+}{dE}(\tilde{E})d\tilde{E} \tag{B.36}$$

From B.20 and B.29 we have, if L/I is large enough, that

$$2T_+(E_{I,I+2}) \leqslant 2I \frac{(\Delta E_I)K_1(\tau_-(E_{I,I+2})}{W(m_-) - E_{II}} + 2IC_3 \exp\left[-\frac{L}{I}K\right] \qquad B.37$$

for some constant K_1, C_3, K.

Since $2I\tau_-(E_{I,I+2}) < L$ and $T_+(E_{I,I+2}) \geqslant C$ for some constant C we get B.31.

To prove the last statement of the proposition it is sufficient to prove that if I, J > K_2L then $E_{IJ} \notin [E_{\tilde{I}\tilde{J}}, E_{\tilde{I}'\tilde{J}'}]$ for $\tilde{I}$, $\tilde{J}$, $\tilde{I}'$, $\tilde{J}' \leqslant K_2L$ with the properties that no other solutions of B.12 belongs to the interval.

Assume $E_{IJ} \in [E_{\tilde{I}\tilde{J}}, E_{\tilde{I}'\tilde{J}'}]$, if $E_{IJ} = E_{\tilde{I}\tilde{J}}$ since $T_+ + T_-$ is a bijection, I must be equal to $\tilde{I}$ which is absurd. By the same argument we can assume $E_{\tilde{I}\tilde{J}} < E_{IJ} < E_{\tilde{I}'\tilde{J}'}$ therefore if L is sufficiently large

$$IT_-(E_{\tilde{I}\tilde{J}}) + JT_+(E_{\tilde{I}\tilde{J}}) < IT_-(E_{IJ}) + JT_+(E_{IJ}) = L$$

which is absurd since

$$\tilde{I}T_-(E_{\tilde{I}\tilde{J}}) + \tilde{J}T_+(E_{\tilde{I}\tilde{J}}) = L \quad \text{with} \quad \tilde{I} < I \text{ and } \tilde{J} < J.$$ ■

Evaluation of the action

We want to evaluate the value of the action S(*u*) associated to the stationary solution $u_{IJ,\varepsilon_{IJ}}$

$$S(u) = \int_0^L \left(\frac{1}{2}\left(\frac{du}{dx}\right)^2 - W(u)\right)dx$$

that is the critical values of this functional.

Using energy conservation we get

$$S(u_{IJ,\varepsilon_{IJ}}) = \int_0^L \left(\frac{du}{dx}\right)^2 dx - E_{IJ}L$$

and a simple change of variable leads to

$$S(u_{IJ,\varepsilon_{IJ}}) = 2IS_-(E_{IJ}) + 2JS_+(E_{IJ}) - E_{IJ}L \qquad B.38$$

where

$$S_-(E) = \sqrt{2}\int_{W_-^{-1}(E)}^{0} du(E - W(u))^{1/2}$$

$$S_+(E) = \sqrt{2}\int_0^{W_+^{-1}(E)} du(E - W(u))^{1/2}$$

It is not difficult to see that $S_\pm(E)$ are increasing functions of E and that $\dfrac{dS_\pm(E)}{dE} = T_\pm(E)$.

LEMMA B.5. — There exists a L_0 such that $\forall L > 0$, $\forall (I, J) : \begin{matrix} I + J > 2 \\ |I - J| \leqslant 1 \end{matrix}$

$$S(u_{IJ,-}) = S(u_{IJ,+}) > S(u_{11,+}) = S(u_{11,-}) \qquad B.39$$

Proof. — By definition we have

$$\begin{aligned} S(u_{IJ}) - S(u_{11}) = {} & (I - 1)S_-(E_{IJ}) + (J - 1)S_+(E_{IJ}) + S_+(E_{IJ}) - S_+(E_{11}) + \\ & + S_-(E_{IJ}) - S_-(E_{11}) + (E_{11} - E_{IJ})L \end{aligned} \qquad B.40$$

In the case $E_{11} - E_{IJ} > \alpha/L^{1-\varepsilon}$ where α is a constant independent of L, the proof is straightforward since $S_+(E)$ and $S_-(E)$ are positive and for L going to infinity are uniformely bounded, meanwhile the last term in B.40 goes to infinity.

Let us discuss the case when $E_{IJ} \underset{L\to\infty}{\to} W(m_-)$. By Lagrange theorem we get

$$| S_+(u_{IJ}) + S_-(u_{IJ}) - S_+(u_{11}) - S_-(u_{11}) | \leqslant | E_{11} - E_{IJ} | \sup_{E\in[E_{IJ},E_{11}]} | T_+(E) + T_-(E) | \qquad B.41$$

If we remark that $L = T_+(E_{11}) + T_-(E_{11})$, then by Proposition B.4, if L is large enough, we get that the r. h. s. of B.41 does not exceed $(E_{11} - E_{IJ})L$ and therefore the left hand side of B.40 will be larger than $(I - 1)S_-(E_{IJ}) + (J - 1)S_+(E_{IJ})$ a quantity which is strictly positive for any L and $I + J > 2$.

LEMMA B.6.

$$\lim_{L\to\infty} \Delta S(L) = 2S_+(W(m^-))$$

$$\lim_{L\to\infty} (S(u_{01}) - S(u_{10}))/L = - [W(m^+) - W(m^-)]$$

Proof. — Using equation B.38 we get

$$\Delta S(L) = 2S_-(E_{11}) + 2S_+(E_{11}) - 2S_-(E_{10}) + (E_{10} - E_{11})L$$

By Proposition B.4, eq. B.29: $E_{10} - E_{11} \leqslant \text{const} \exp\left[-\frac{L}{I}K^4\right]$ therefore the two last terms in the r. h. s. side of the previous equation go to zero and E_{11} and $E_{10} \underset{L\to\infty}{\to} W(m^-)$.
On the other hand, by equation B.38 we get also

$$\frac{S(u_{01}) - S(u_{10})}{L} = \frac{2S_+(E_{01}) - 2S_-(E_{10})}{L} - (E_{01} - E_{10})$$

By definition of S_- and S_+ the first term in the right hand side of the previous equation goes to zero when L goes to infinity and, using equation B.29-B.30, we get the result.

Stability properties of the critical points

The stability properties of a solution u^* of the boundary value problem

$$-\frac{d^2u}{dx^2} - W'(u) = 0 \qquad B.42$$
$$u(0) = u(L) = 0$$

follow from the dimensionality of the subspace $\mathbb{D}(u^*)$ of $L^2(0, L)$ spanned by the eigenvectors corresponding to the negative eigenvalues of the self adjoint operator:

$$S''(u^*) = -\frac{d^2}{dx^2} - W''(u^*) \qquad B.43$$

with D. b. c. on [0, L]. In this part, we will give the exact dimension of $\mathbb{D}(u_{IJ,\varepsilon_{IJ}})$, moreover we will give also lower bounds on the first eigenvalues of $S''(u_{10})$ and $S''(u_{01})$ which are positive and upper bound on the first eigenvalue of $S''(u_{11,\pm})$ which is negative. These bounds depend on L and are needed in section 3. Let us remark that the exact dimensionality of $\mathbb{D}(u_{IJ,\varepsilon_{IJ}})$ was given in J.-F. [*1*], Laetsch [*14*], Berestycki [*15*] by using convexity hypothesis, namely $-\frac{W'(u)}{u} \leqslant W''(u)$ that are not true in our case. The constructive approach of Chafee, Infante [*12*] together with the fact that L can be arbitrarily large allow us to solve these problems.

PROPOSITION B.7. — If L is large enough, then for any critical point corresponding to I, J such that $I, J \leqslant K_2L$ where K_2 is defined in Proposition B.4 one has:

$$\dim \mathbb{D}(u_{IJ,\varepsilon_{IJ}}) = I + J - 1.$$

Remark. — The restriction on I in Proposition B.7 is for technical reasons, let us remark this in section 3, we need only results for $0 \leqslant I \leqslant 1$.

Proof of Proposition B.7. — Let us first consider $u^* = u_{10}$. Let $u(x, v_0)$ be the solution of the initial value problem

$$-\frac{d^2 f}{dx^2} - W'(f) = 0$$

$$f(0) = 0 \qquad \text{B.44}$$

$$f'(0) = v_0 < 0 \quad \text{where} \quad E_{10} = \frac{1}{2}v_0^2 + W(0)$$

It is not difficult to see that $Z_1(x) = \frac{\partial u}{\partial v_0}(x, v_0)$ is the solution of the second order linear equation

$$-\frac{d^2 g}{dx^2} - W''(u_{10})g = 0 \qquad \text{B.45}$$

with initial condition $g(0) = 0$, $g'(0) = 1$.

On the other hand, if we consider $u_{10}(x)$ as the restriction to $[0, L]$ of a periodic solution of B.5, B.6, it is not difficult to see that

$$Z_2(x) = \frac{du_{10}(x)}{dx} \qquad 0 < x < L + 2T_+(E_{10})$$

is also a solution of B.45. Moreover, it is easy to check that

$$Z_2(x) < 0 \quad \text{if} \quad 0 < x < L/2$$

$$Z_2(L/2) = 0$$

$$Z_2(x) > 0 \quad \text{if} \quad L/2 < x < L + 2T_+(E_{10}), \quad Z_2(L + 2T_+(E_{10})) = 0$$

Since Z_1 and Z_2 are two solutions of the same equation B.45 and $Z_1 \neq \lambda Z_2$ for $\forall \lambda \in \mathbb{R}$, by the theorem on separation of zeroes [*16*] there must exist only one x^* in $]L/2, L+T_+(E_{10})[$ such that $Z_1(x^*) = 0$, moreover since $Z_1(0) = 0$, $Z_1'(0) = 1$ we have $Z_1(x) > 0$ if $0 < x < x^*$. Assume that $x^* > L$ then $Z_1(x)$ is the eigenvector corresponding to the eigenvalue 0 of the self adjoint operator

$$L_D(0, x^*) = -\frac{d^2}{dx^2} - W''(u_{10})$$

with Dirichlet boundary condition on $[0, x^*]$.

By the theorem on oscillation properties of eigenvector [*16*] this is the first eigenvector, from the quadratic form inequality

$$L_D(0, x^*) \leqslant S''(u_{10}) \qquad \text{B.46}$$

it follows [*17*] that the first eigenvalue of $S''(u_{10})$ is strictly positive and therefore $\dim \mathbb{D}(u_{10}) = 0$.

Now to prove that $x^* > L$ it is sufficient, since $Z_1(x) > 0$ if $0 < x < x^*$, to show that $Z_1(L) > 0$ uniformly with respect to L.

If

$$2\tau_-(E_{10}) + \tau_0(E_{10}) < x < 2T_-(E_{10})$$

$$x = 2T_-(E_{10}) + \int_0^{u[x,v_0)} (E - W(u))^{-1/2} du \qquad \text{B.47}$$

therefore, since $\frac{dx}{dv_0} = 0$ we get

$$Z_1(x) = v_0\left[\frac{1}{2}\int_0^{u(x,v_0)}[E_{10} - W(u)]^{-3/2}du - 2\frac{dT}{dE}(E_{10})\right] \qquad B.48$$

Let us remark that in B.48, $u(x, v_0) = u_{10}(x) < 0$, $v_0 < 0$ therefore $Z_1(x) > 0$, since by construction $u_{10}(L) = 0$ we get

$$\lim_{x\to L} Z_1(x) = -2v_0\frac{dT_-(E_{10})}{dE}$$

by lemma B.2, B.3 and Proposition B.4 $\frac{dT_-}{dE}(E_{10}) \geqslant KT_-(E_{10}) = KL > 0$ therefore using B.20 and $E_{10} = \frac{1}{2}v_0^2 + W(0)$, there exist two constants L_0, $\tilde{K}$ such that if $L > L_0$, $Z_1(L) \geqslant \tilde{K}L$.

The case $u^* = u_0$ is similar with the following modification: we define $\tilde{Z}_2 = \frac{d\tilde{u}_{01}}{dx}$ where $\tilde{u}_{01}(x)$ is the solution of $-\frac{d^2f}{dx^2} - \tilde{W}'(f) = 0$ with initial condition $f(0) = 0$; $f'(0) = \pm [2(E_{01} - W(0))]^{1/2}$ where $\tilde{W} = W$ on $[m_-, +\infty]$ but on $]-\infty, m_-[$ $\tilde{W}$ is equal to a C^∞ modification of W such that the newtonian motion is periodic and the time interval with $\tilde{u}_{01}(x)$ negative, is bounded uniformaly with respect to L. $\tilde{Z}_1$ is defined by B.44 but with $v_0 > 0$ and such that $E_{01} = 1/2v_0^2 + W(0)$. We consider the case $u^* = u_{11,-}$.

Let $\check{Z}_2$ be $\frac{du_{11,-}}{dx}$, it is not difficult to check that $\check{Z}_2(T_-(E_{11})) = \check{Z}_2(2T_-(E_{11}) + T_+(E_{11})) = 0$ and $\check{Z}_2(x) > 0$ if $T_-(E_{11}) < x < 2T_-(E_{11}) + T_+(E_{11})$.

Therefore, since $\check{Z}_2(x)$ is solution of $-\frac{d^2}{dx^2} + - W''(u_{11,-}) = 0$; zero is the first eigenvalue and $\check{Z}_2(x)$ is the first eigenvector corresponding to the operator

$$-\frac{d^2}{dx^2} - W''(u_{11,-}) = L_D(T_-(E_{11}), 2T_-(E_{11}) + T_+(E_{11})) \qquad B.49$$

with D. b. c. on $[T_-(E_{11}), 2T_-(E_{11}) + T_+(E_{11})]$.

It follows from the quadratic form inequality

$$S''(u_{11,-}) \leqslant L_D(T_-(E), 2T_-(E_{11}) + T_+(E_{11})) \qquad B.50$$

that the first eigenvalue of $S''(u_{11,-})$ is strictly negative.

Let $\check{Z}_1(x) = \frac{\partial u}{\partial v_0}(x, v_0)$ where $u(x, v_0)$ is the solution of B.44, but here $v_0 < 0$ and such that $E_{11} = \frac{1}{2}v_0^2 + W(0)$. In this case $\check{Z}_1(x)$ is solution of the differential equation

$$-\frac{d^2}{dx^2}f - W''(u_{11,-})f = 0$$

with initial condition $f(0) = 0$, $f'(0) = 1$. Using the theorem on separation of zeroes, since $\check{Z}_2(x)$, extended to $[0, 3T_-(E_{11}) + 2T_+(E_{11})]$ satisfies the same differential equation, there exists one and only one zero of $\check{Z}_1$, say x^*, between $T_-(E_{11}), 2T_-(E_{11}) + T_+(E_{11}) < L$. Moreover since $\check{Z}_1(0) = 0$, $\check{Z}_1^1(0) = 1$, $\check{Z}_1(x)$ is negative for $x^* < x < 2T_-(E_{11}) + T_+(E_{11})$.

If we can prove that $\check{Z}_1(L) < 0$, uniformly with respect to L, using again the theorem on separation of zeroes we get that there exists a zero of $\check{Z}_1(x)$, say x^{**}, which belongs to $]L, L + T_-(E_{11}[$, therefore $\check{Z}_1(x)$ is the second eigenvalue of the operator

$$-\frac{d^2}{dx^2} - W''(u_{11,-}) = L_D(0, x^{**})$$

with Dirichlet boundary condition on $(0, x^{**})$, zero is the corresponding second eigenvalue. It follows from the quadratic form inequality

$$L_D(0, x^{**}) \leqslant S''(u_{11,-})$$

that the second eigenvalue of $S^{11}(u_{11,-})$ is strictly positive.

We prove now $\check{Z}_1(L) < 0$ uniformly in L. It is not difficult to check that if

$$2T_-(E_{11}) + 2\tau_m(E_{11}) + \tau_+(E_{11}) < x < 2T_-(E_{11}) + 2T_+(E_{11})$$

then

$$x = 2T_-(E_{11}) + 2T_+(E_{11}) - \int_0^{u(x,v_0)} (E_{11} - W(u))^{-1/2} du \qquad \text{B.51}$$

therefore differentiating with respect to v_0 we get

$$\check{Z}_1(x) = v_0(E_{11} - W(0))^{1/2}\left[2\frac{dT_-}{dE}(E_{11}) + 2\frac{dT_+}{dE}(E_{11}) + \int_0^{u(x,v_0)} (E_{11} - W(u))^{-3/2} du\right] \qquad \text{B.52}$$

Since, when $x \to L$, $u(x, v_0) \equiv u_{11}(x)$ goes to zero, by lemma B.2, B.3 and Proposition B.4, we get $\check{Z}_1(L) \leqslant -\check{K}L$ for some constant $\check{K}$, independent on L.

The general case $u^* = u_{IJ,\sigma_{IJ}}$ for $I \leqslant K_2L$ is treated with the very same arguments and it is left to the reader.

Proposition B.8. — Calling λ_0 the lowest eigenvalue of $S''(u^*)$, the following inequalities hold.

For $u^* = u_{10}$ or u_{01}

$$\lambda_0 \geqslant A/(L \exp[\mu L/2]) \qquad \text{B.53}$$

and for $u^* = u_{11,\pm 1}$

$$\lambda_0 \leqslant -B/(L \exp[\mu L/2]) \qquad \text{B.54}$$

where A, B and μ are constant independent from L.

Proof. — We start considering the two equations

$$-\frac{d^2 y_1}{dx^2} - W''(u^*(x))y_1 = \lambda_0 y_1 \qquad y_1(0) = y_1(L) = 0 \qquad \text{B.55}$$

$$-\frac{d^2 y_1}{dx^2} - W''(u^*(x))y_2 = 0 \qquad y_2(0) = y_2(L) = 0 \qquad \text{B.56}$$

If we perform the so called Pfrüfer transformations

$$y_i = y_i; \qquad Z_i = \frac{dy_i}{dx} \qquad \text{B.57}$$

$$y_i = \rho_i \sin\theta_i; \qquad Z_i = \rho_i \cos\theta_i \qquad \text{B.58}$$

we get the following eq. for the θ_i's

$$\frac{d\theta_1}{dx} = \cos^2\theta_1 + (W'' + \lambda_0)\sin^2\theta_1 \qquad \theta_1(0) = 0 \qquad B.59$$

$$\frac{d\theta_2}{dx} = \cos^2\theta_2 + (W'')\sin^2\theta_2 \qquad \theta_2(0) = 0 \qquad B.60$$

The following are standard properties of the $\theta_i(x)$'s [16]

$$\begin{array}{lll} \text{for } \lambda_0 > 0 & \theta_1(x) - \theta_2(x) > 0 & \forall x > 0 \\ \text{for } \lambda_0 < 0 & \theta_1(x) - \theta_2(x) < 0 & \forall x < 0 \end{array} \qquad B.61$$

$\theta_1(L) = \pi$ if λ_0 is the lowest eigenvalue of eq. B.55 or B.56.

Calling $\Delta(x) = \theta_1(x) - \theta_2(x)$, substracting B.60 from B.59 and formally solving the non homogeneous linear differential equation of first order

$$\frac{d\Delta}{dx} = f(\theta_1, \theta_2, x)\Delta + \lambda_0 \sin^2\theta_1(x) \qquad B.62$$

where

$$f(\theta_1, \theta_2, x) = \frac{\cos^2\theta_1 - \cos^2\theta_2 + W''(u^*(x))(\sin^2\theta_1 - \sin^2\theta_2)}{\theta_1 - \theta_2} \qquad B.63$$

we get

$$\Delta(x) = \int_0^x dx^1 \lambda_0 \sin^2\theta_1(x^1) \exp\left[\int_{x^1}^{L} f(\theta_1, \theta_2, x^{11}) dx^{11}\right]. \qquad B.64$$

If we remark that $| f(\theta_1, \theta_2, x) | < \mu/2$ where μ is a constant independent from L

$$| \lambda_0 | \geqslant [\pi - \theta_2(L)]/(L \exp(\mu L/2)). \qquad B.65$$

To evaluate $\theta_2(L)$ we recall that

$$\operatorname{tg}\theta_2(L) = \frac{y_2(L)}{\left.\left(\dfrac{dy_2}{dx}\right)\right|_{x=L}} \qquad B.66$$

where

$$y_2(x) = \frac{du^*}{dv_0}.$$

From eq. B.48, for $u^* = u_{10}$ and the similar relations for the other cases we will have:

$$\operatorname{tg}\theta_2(L) = -\operatorname{sign}\left.\frac{du^*}{dx}\right|_{x=L} \cdot \frac{(E_{IJ} - W(0))^{1/2}}{W'(0)} \qquad B.67$$

Since $\exists\alpha > 0 : E_{IJ} - W(0) > \alpha$, uniformely in L, from B.65 and B.67 the bounds B.53 and B.54 will follow.

Proposition B.9. — if $\hat{m} > 0$ there exists a unique critical point u which is the absolute minimum of $S(u)$.

If $0 \leqslant m_-$ there exists, for L sufficiently large, three critical points u_a, u_b, u_c:

u_a is a local minimum, u_b is a saddle point with a unique direction of instability, u_c is the absolute minimum and

$$\lim_{L\to\infty} S(u_b) - S(u_a) > 0$$

$$\lim_{L\to\infty} [S(u_c) - S(u_a)]/L < 0$$

Proof. — The case $\hat{m} > 0$ is obvious (see fig. 1.B). For the case $m_- \geqslant 0$ the proof follows the very same line of the proof of Proposition B.4 and it is left to the reader.

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