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Scattering problem for nonlinear Schrödinger equations (*)

by

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ABSTRACT. — In this paper we study the asymptotic behavior as $t \to \pm\infty$ of solutions and the scattering theory for the following nonlinear Schrödinger equation with power interaction:

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + |u|^{p-1}u, \qquad \text{in} \quad \mathbb{R}^n, \qquad n \geqq 1\,.$$

We show the existence of the wave operators defined in $\Sigma = \{ v\,;\, v \in H^1(\mathbb{R}^n), xv \in L^2(\mathbb{R}^n) \}$ and their asymptotic completeness for $\gamma(n) < p < \dfrac{n+2}{n-2}$, where

$$\gamma(n) = \frac{n + 2 + \sqrt{n^2 + 12n + 4}}{2n}.$$

Our results are the extensions of the results due to Ginibre and Velo [*6*] in that our results cover the case of $\gamma(n) < p < 1 + \dfrac{4}{n}$.

RÉSUMÉ. — On étudie dans cet article le comportement des solutions

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quand $t \to \pm\infty$ et la théorie de la diffusion pour l'équation de Schrödinger non linéaire suivante, avec une interaction en puissance :

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + |u|^{p-1}u, \qquad \text{dans} \quad \mathbb{R}^n, \qquad n \geqq 1.$$

On montre l'existence des opérateurs d'onde définis dans $\Sigma = \{ v ; v \in H^1(\mathbb{R}^n), xv \in L^2(\mathbb{R}^n) \}$ et leur complétude asymptotique pour $\gamma(n) < p < \frac{n+2}{n-2}$, où

$$\gamma(n) = \frac{n + 2 + \sqrt{n^2 + 12n + 4}}{2n}.$$

Nos résultats sont des extensions de ceux de Ginibre et Velo [*6*], en ce sens qu'ils couvrent le cas où $\gamma(n) < p < 1 + \frac{4}{n}$.

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§ 1. INTRODUCTION AND THEOREMS

We consider the asymptotic behavior as $t \to \pm\infty$ of solutions for the following nonlinear Schrödinger equation with power interaction:

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + |u|^{p-1}u, \qquad t \in \mathbb{R}, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}^n. \tag{1.2}$$

Let $U(t)$ be an evolution operator associated with the free Schrödinger equation and let Σ denote the Hilbert space

$$\Sigma = \{ v \in L^2(\mathbb{R}^n) ; \| v \|_{L^2(\mathbb{R}^n)} + \| \nabla v \|_{L^2(\mathbb{R}^n)} + \| xv \|_{L^2(\mathbb{R}^n)} < \infty \}$$

with the norm

$$\| v \|_\Sigma^2 = \| v \|_{L^2(\mathbb{R}^n)}^2 + \| \nabla v \|_{L^2(\mathbb{R}^n)}^2 + \| xv \|_{L^2(\mathbb{R}^n)}^2.$$

We put

$$\alpha(n) = \begin{cases} \infty, & n = 1, 2, \\ \dfrac{n+2}{n-2}, & n \geqq 3 \end{cases}$$

$$\gamma(n) = \frac{n + 2 + \sqrt{n^2 + 12n + 4}}{2n}.$$

We note that $1 + \frac{2}{n} < \gamma(n) < 1 + \frac{4}{n} < \alpha(n)$.

In [*25*] Yajima and the author show that if $1 + \frac{2}{n} < p < \alpha(n)$, then all solutions $u(t)$ of (1.1)-(1.2) with $u_0 \in \Sigma$ have scattering states $u_\pm \in L^2(\mathbb{R}^n)$ satisfying

$$\| u_\pm - U(-t)u(t) \|_{L^2(\mathbb{R}^n)} \to 0 \qquad (t \to \pm \infty). \tag{1.3}$$

Naturally the following question arises: Can we develop the scattering theory for the equation (1.1)? In the present paper we discuss the construction of the wave operators $W_\pm : u_\pm \to u_0$, i. e., the mappings from the free states $u_\pm$ to the interacting states u_0 satisfying (1.3), and their asymptotic completeness, i. e., Range (W_+) = Range (W_-). These lead to the construction of the scattering operator $S = W_+^{-1} W_- : u_- \to u_+$.

Our main theorems in this paper are the following:

THEOREM 1.1. — Assume that $\gamma(n) < p < \alpha(n)$.

(*i*) For any $u_+ \in \Sigma$ there exists a unique $u_0 \in \Sigma$ such that

$$\| u_+ - U(-t)u(t) \|_\Sigma \to 0 \qquad (t \to +\infty), \tag{1.4}$$

where $u(t)$ is a solution in $C(\mathbb{R}; \Sigma)$ of (1.1) with $u(0) = u_0$.

(*ii*) For any $u_- \in \Sigma$ there exists a unique $u_0 \in \Sigma$ such that

$$\| u_- - U(-t)u(t) \|_\Sigma \to 0 \qquad (t \to -\infty), \tag{1.5}$$

where $u(t)$ is a solution in $C(\mathbb{R}; \Sigma)$ of (1.1) with $u(0) = u_0$.

THEOREM 1.2. — Assume that $\gamma(n) < p < \alpha(n)$. For any $u_0 \in \Sigma$ there exist unique scattering states $u_\pm \in \Sigma$ such that the solution $u(t) \in C(\mathbb{R}; \Sigma)$ of (1.1) with $u(0) = u_0$ satisfies

$$\| u_\pm - U(-t)u(t) \|_\Sigma \to 0 \qquad (t \to \pm \infty). \tag{1.6}$$

REMARK 1.1. — (*i*) For any $u_0 \in \Sigma$ there exists a unique weak solution in $C(\mathbb{R}; \Sigma)$ of (1.1)-(1.2) (see Ginibre and Velo [*4*, Theorem 3.1], [*5*, Proposition 3.5] and Proposition 2.7 in § 2).

(*ii*) Theorem 1.1 implies that if $\gamma(n) < p < \alpha(n)$, the wave operators $W_\pm$ are well defined as a mapping from Σ to Σ. Theorem 1.2 implies that Range (W_+) = Range (W_-) = Σ and that $W_\pm$ are one to one.

The following result is an immediate consequence of Theorems 1.1 and 1.2.

COROLLARY 1.3. — Assume that $\gamma(n) < p < \alpha(n)$. Then the wave operators $W_\pm$ are well defined in Σ and are bijections from Σ onto Σ. Accordingly, the scattering operator $S = W_+^{-1} W_-$ is well defined in Σ and is a bijection from Σ onto Σ.

REMARK 1.2. — It seems to be natural that the critical power $\gamma(n)$ should appear in Theorems 1.1, 1.2 and Corollary 1.3. $\gamma(n)$ appears in various papers (see, e. g., Dong and Li [*4*] and Strauss [*16*] [*17*]).

In [6] Ginibre and Velo show Theorems 1.1 and 1.2 for $1 + \frac{4}{n} \leqq p < \alpha(n)$. Recently, in [8] [9] they have also shown Theorems 1.1 and 1.2 with Σ replaced by $H^1(\mathbb{R}^n)$ for $1 + \frac{4}{n} < p < \alpha(n)$ and $n \geqq 3$ (see also Brenner [3]). When $\gamma(n) < p < \alpha(n)$, the construction of the scattering operator for small data in certain norms is discussed by many mathematicians (see, e. g., Dong and Li [4], Reed [14] and Strauss [16] [17]). In [10] N. Hayashi and M. Tsutsumi discuss the decay of the L^∞-norm of classical solutions for nonlinear Schrödinger equations (see also Pecher [12] [13]). In [1] Barab shows the decay of the L^{p+1} – norm of solutions for (1.1)-(1.2). The time decay estimates of solutions play an important role in nonlinear scattering theory. However, our Theorems 1.1 and 1.2 are not simple by-products of the time decay estimates of solutions, because Theorems 1.1 and 1.2 do not only construct the wave operators and the scattering operator but also show that they are bijections from Σ onto Σ.

Our Theorems 1.1 and 1.2 are the extensions of the results due to Ginibre and Velo [6] in that our Theorems 1.1 and 1.2 hold even for $\gamma(n) < p < 1 + \frac{4}{n}$. The proof in [6] is based on the pseudoconformal conservation law of (1.1)-(1.2):

$$\| xU(-t)u(t) \|^2_{L^2(\mathbb{R}^n)} + \frac{2}{p+1} t^2 \| u(t) \|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + \frac{np-n-4}{p+1} \int_0^t \tau \| u(\tau) \|^{p+1}_{L^{p+1}(\mathbb{R}^n)} d\tau = \| xu_0 \|^2_{L^2(\mathbb{R}^n)}, \ -\infty < t < +\infty, \quad (1.7)$$

(see Ginibre and Velo [6, § 3]). When $1 + \frac{4}{n} \leqq p < \alpha(n)$, the boundedness of $\| xU(-t)u(t) \|^2_{L^2(\mathbb{R}^n)}$ for $t \in \mathbb{R}$ follows directly from (1.7). This fact leads us to the proofs of Theorems 1.1 and 1.2 for $1 + \frac{4}{n} \leqq p < \alpha(n)$ (for details, see Ginibre and Velo [6]). But, when $p < 1 + \frac{4}{n}$, the circumstances are different. Even when $1 < p < 1 + \frac{4}{n}$, we can easily prove the time dacay estimate of the L^{p+1}-norm of solutions for (1.1)-(1.2) (see, e. g., Barab [1, Lemma 3]), which is not sufficient to construct the wave operators and the scattering operator as a bijection from Σ onto Σ. In the present paper we shall show that if $\gamma(n) < p < \alpha(n)$, $\| xU(-t)u(t) \|_{L^2(\mathbb{R}^n)}$ is bounded for $t \in \mathbb{R}$. This fact leads us to the proofs of Theorems 1.1 and 1.2. Our proofs of Theorems 1.1 and 1.2 are based on the pseudoconformal conservation law, the Strichartz estimate (see Strichartz [19, Corollary 1 in § 3) and the following transform:

$$u(t, x) = (it)^{-n/2} e^{\frac{ix^2}{2t}} \overline{v\left(\frac{1}{t}, \frac{x}{t}\right)} \tag{1.8}$$

The transform (1.8) was discovered independently by Ginibre and Velo [7] and Yajima [23]. The transform (1.8) is already applied to the scattering problem for linear Schrödinger equations in [23] and to the scattering problem for nonlinear Schrödinger equations in [22]. For the details of (1.8), see § 2.

Our plan in the present paper is as follows. In Section 2 we summarize some fundamental lemmas needed for the proofs of Theorems 1.1 and 1.2 and some properties of the transform (1.8). Furthermore, we briefly describe the results of Ginibre and Velo [5] [6] concerning the unique global existence of a weak solution of (1.1)-(1.2). In Section 3 we give the proofs of Theorems 1.1 and 1.2.

Finally we list some notations which will be used later. For $1 \leqq p \leqq \infty$, $L^p(\mathbb{R}^n)$ denotes the usual L^p function space defined on $\mathbb{R}^n$. By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of C^∞ functions of rapid decrease. By $\mathcal{S}'(\mathbb{R}^n)$ we denote the dual of $\mathcal{S}(\mathbb{R}^n)$, the space of tempered distributions. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we denote the Fourier transform of f and the inverse Fourier transform of f by $\hat{f}$ and $\check{f}$ or by $\mathcal{F} f$ and $\mathcal{F}^{-1} f$, respectively (for the definitions of the Fourier transform and the inverse Fourier transform, see Reed and Simon [15, Section 1 in Chapter IX]). For $1 < p < \infty$ and $s \in \mathbb{R}$, we put

$$H^{s,p}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) ; \| \mathcal{F}^{-1}(1 + |\xi|^2)^{s/2} \hat{f} \|_{L^p(\mathbb{R}^n)} \} . \tag{1.9}$$

The norm in $H^{s,p}(\mathbb{R}^n)$ is defined by

$$\| f \|_{H_\xi^{s,s}(\mathbb{R}^n)} = \| \mathcal{F}^{-1}(1 + |\xi|^2)^{s/2} \hat{f} \|_{L^p(\mathbb{R}^n)} . \tag{1.10}$$

We note that if s is a nonnegative integer, $H^{s,p}(\mathbb{R}^n)$ $(1 < p < \infty)$ is the standard Sobolev space on $\mathbb{R}^n$ and that if $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$, for $s \in \mathbb{R}$ $H^{q,-s}(\mathbb{R}^n)$ is equivalent to the dual of $H^{p,s}(\mathbb{R}^n)$ (see Bergh and Löfström [2, Chapter 6]). For simplicity, we abbreviate $H^{s,2}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$. For $1 \leqq p < \infty$ and a positive integer m, let $\Delta_y^m f(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k f(x+ky)$ and $\omega_p^m(t, f) = \sup_{|y| < t} \| \Delta_y^m f \|_{L^p(\mathbb{R}^n)}$. For $1 \leqq p, q < \infty$ and $s > 0$ we define the Besov space $B^s_{pq}(\mathbb{R}^n)$ by the completion of $\mathscr{S}(\mathbb{R}^n)$ in the following norm:

$$\| f \|_{B^s_{pq}(\mathbb{R}^n)} = \| f \|_{L^p(\mathbb{R}^n)} + \sum_{j=1}^{n} \left\{ \int_0^\infty (t^{N-s}\omega_p^m(t, \partial^N f/\partial x_j^N))^q \frac{dt}{t} \right\}^{\frac{1}{q}}, \quad (1.11)$$

where m and N are integers such that $m + N > s$ and $0 \leqq N < s$. For the details of the Besov space, see Bergh and Löfström [2, Chapter 6]. For simplicity, we abbreviate $L^p(\mathbb{R}^n)$, $H^s(\mathbb{R}^n)$, $H^{s,p}(\mathbb{R}^n)$, $B^s_{pq}(\mathbb{R}^n)$, $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ to L^p, H^s, $H^{s,p}$, B^s_{pq}, $\mathscr{S}$ and $\mathscr{S}'$, respectively. By $(.,.)$ we denote the scalar product in L^2. Let X be a Banach space with the norm $\| . \|_X$ and I be a closed interval in $\mathbb{R}$. For $f(t) \in C(I; X)$ we put

$$| f |_{I,X} = \sup_{t \in I} \| f(t) \|_X. \quad (1.12)$$

For $z \in \mathbb{C}$ we denote the complex conjugate of z by $\bar{z}$. For $a \in \mathbb{R}$ we denote by $[a]$ the greatest integer that is not larger than a. Let $h(x)$ be an even and positive function in $C_0^\infty(\mathbb{R}^n)$ with $\| h \|_{L^1} = 1$. We put $h_j(x) = j^n h(jx)$, $j = 1, 2, \ldots$. $*$ denotes the convolution (for the definition of the convolution, see Reed and Simon [15, Section 1 in Chapter IX]). We put

$$f(v) = | v |^{p-1} v, \quad (1.13)$$

$$f_j(v) = h_j * \{ f(h_j * v) \} \quad (1.14)$$

for any « nice » function v from $\mathbb{R}^n$ to $\mathbb{C}$. In the course of calculations below various constants will be simply denoted by C. $C(*, \ldots, *)$ denotes a constant depending only on the quantities appearing in parenthesis.

§ 2. PRELIMINARIES

We start with fundamental lemmas on the free evolution operator $U(t)$.

LEMMA 2.1. — (1) Let q and r be positive numbers such that $\frac{1}{q} + \frac{1}{r} = 1$ and $2 \leqq q \leqq \infty$. For any $t \neq 0$, $U(t)$ is a bounded operator from L^r to L^q satisfying

$$\| U(t)v \|_{L^q} \leqq (2\pi | t |)^{\frac{n}{q} - \frac{n}{2}} \| v \|_{L^r}, \qquad v \in L^r, \qquad t \neq 0, \quad (2.1)$$

and for any $t \neq 0$ the mapping $U(t)$ is strongly continuous. For $q = 2$, $U(t)$ is unitary and strongly continuous for all $t \in \mathbb{R}$.

(2) Let $t \in \mathbb{R}$. Then $U(t)$ maps Σ into Σ and for all $v \in \Sigma$,

$$xU(-t)v = U(-t)(x + it\nabla)v.$$

LEMMA 2.2. — Let $v \in L^2$. Then there exists a $C > 0$ depending only on n such that

$$\left(\int_{-\infty}^{+\infty} \| U(t)v \|_{L^q}^q dt \right)^{\frac{1}{q}} \leqq C \| v \|_{L^2},$$

where
$$q = \frac{2(n+2)}{n}.$$

Lemma 2.1 is well known (see, e. g., [*5*, Lemma 1.2]). For Lemma 2.2, see Strichartz [*19*, Corollary 1 in § 3].

LEMMA 2.3. — Let $0 \leqq s \leqq 1$ and let $v \in H^{2s}$. Then, for all $t \in \mathbb{R}$

$$\| \{ U(t) - 1 \} v \|_{L^2} \leqq C | t |^s \| v \|_{H^{2s}}, \tag{2.3}$$

where $C = C(s)$.

Proof. — We have

$$\| \{ U(t) - 1 \} v \|_{L^2} = \| (e^{\frac{1}{2} it|\xi|^2} - 1) \hat{v} \|_{L^2}$$
$$= \left\| \int_0^1 e^{\frac{1}{2} it|\xi|^2 \theta} d\theta \frac{1}{2} it | \xi |^2 \hat{v} \right\|_{L^2} \leqq \frac{1}{2} | t | \| v \|_{H^2}, \tag{2.4}$$
$$\| \{ U(t) - 1 \} v \|_{L^2} \leqq 2 \| v \|_{L^2}. \tag{2.5}$$

By interpolation we obtain (2.3). Q. E. D.

The following three lemmas will be useful later.

LEMMA 2.4. — Let $h(x)$ be an even and positive function in $\mathcal{S}$ with $\| h \|_{L^1} = 1$. Then the convolution with $h(x)$ is a contraction in L^q for all q, $1 \leqq q \leqq \infty$, and in H^s for all $s > 0$. Furthermore, it commutes with $U(t)$ and $(h * u, v) = (u, h * v)$ for all $u, v \in L^2$.

Lemma 2.4 is clear.

LEMMA 2.5. — Let α and β be such that $0 \geqq \alpha, \beta > -1$ and $\alpha + \beta > -1$. We put

$$K(t, s) = (t - s)^\alpha s^\beta$$

for $0 \leqq s < t$. Let $T > 0$ and $g(t)$ be a nonnegative function in $C([0, T])$. We assume that for some $a, b > 0$

$$g(t) \leqq a + b \int_0^t K(t, s) g(s) ds, \qquad t \in [0, T]. \tag{2.6}$$

Then $g(t)$ satisfies

$$g(t) \leqq Ca, \qquad t \in [0, T], \tag{2.7}$$

where $C = C(\alpha, \beta, b, T)$.

Proof. — Let $I = [0, T]$. Since $0 > \alpha, \beta > -1$ and $\alpha + \beta > -1$, we can choose $\overline{T} > 0$ such that

$$\sup_{t \in I} b \int_t^{t+\overline{T}} K(t + \overline{T}, \tau) d\tau \leqq \frac{1}{2}.$$

We note that by the fact that $\alpha, \beta < 0$

$$b \int_s^{s+\overline{T}} K(t + \overline{T}, \tau) d\tau \leqq b \int_s^{s+\overline{T}} K(s + \overline{T}, \tau) d\tau \leqq \frac{1}{2}, \quad 0 < s \leqq t \leqq T. \tag{2.8}$$

By (2.6) and the definition of $\overline{T}$ we have

$$g(t) \leqq a + \frac{1}{2} \sup_{s \in [0,t]} g(s), \qquad t \in [0, \overline{T}] \cap I. \tag{2.9}$$

(2.9) gives us

$$\sup_{s \in [0,\overline{T}] \cap I} g(s) \leqq 2a. \tag{2.10}$$

If $\overline{T} < T$, then we have by (2.6), (2.8) and (2.10)

$$\begin{aligned} g(t) &\leqq a + b \int_0^{\overline{T}} K(t, \tau) g(\tau) d\tau + b \int_{\overline{T}}^t K(t, \tau) g(\tau) d\tau \\ &\leqq 2a + \frac{1}{2} \sup_{s \in [\overline{T}, t]} g(s), \qquad t \in [\overline{T}, 2\overline{T}] \cap I. \end{aligned} \tag{2.11}$$

(2.11) gives us

$$\sup_{s \in [\overline{T}, 2\overline{T}] \cap I} g(s) \leqq 4a, \tag{2.12}$$

if $\overline{T} < T$. If $2\overline{T} < T$, by repeating this procedure we obtain

$$\sup_{s \in [N\overline{T}, (N+1)\overline{T}] \cap I} g(s) \leqq (N^2 + N + 2)a \tag{2.13}$$

for $0 \leqq N \leqq [T/\overline{T}]$. This completes the proof of Lemma 2.5. Q. E. D.

LEMMA 2.6. — (*i*) For any $s > 0$, $B^s_{22} = H^s$.

(*ii*) Let $s > 0$, $s_1 > 0$, $1 \leqq p \leqq p_1 < \infty$ and $1 \leqq q \leqq q_1 < \infty$. Assume that $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. Then the following inclusion holds:

$$B^s_{pq} \subset B^{s_1}_{p_1 q_1}.$$

(*iii*) (the Gagliardo-Nirenberg inequality). Let $1 \leqq p, q, r \leqq \infty$ and let m and j be nonnegative integers. For any nonnegative integers N, let $D^N u = \left(\left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n} u ; \alpha_1 + \dots + \alpha_n = N \right)$. Then

$$\| D^j u \|_{L^p} \leqq C \| D^m u \|^{\theta}_{L^r} \| u \|^{1-\theta}_{L^q}, \tag{2.14}$$

where $\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{n}{m}\right) + (1-\theta)\frac{1}{q}$ for all θ in the interval $\frac{j}{m} \leqq \theta \leqq 1$ with the following exceptional cases: (*a*) If $j = 0$, $rm < n$ and $q = \infty$, then we make the additional assumption that either u tends to zero at infinity or $u \in L^{\tilde{q}}$ for some finite $\tilde{q} > 0$. (*b*) If $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, then (2.14) holds for θ satisfying $j/m \leqq \theta < 1$. Here $C = C(n, m, j, p, q, r)$.

For Lemma 2.6, see Bergh and Löfström [*2*, Theorems 6.2.5 and 6.5.1].

Next we summarize the results of Ginibre and Velo [*5*] [*6*] concerning the unique global existence of weak solutions of (1.1)-(1.2). We first formulate our problem precisely. For an arbitrary initial time $t_0 \in \mathbb{R}$, we consider the integral equation

$$u(t) = U(t - t_0)u_0 - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))d\tau\,, \tag{2.15}$$

as the integral version of the initial value problem (1.1)-(1.2). For (2.15) we have the following result (see [*5*, Theorem 3.1] and [*6*, Proposition 3.5]).

PROPOSITION 2.7. — Assume that $1 < p < \alpha(n)$. For any $u_0 \in H^1$, (2.15) has a unique global solution $u(t)$ in $C(\mathbb{R}\,; H^1)$. Furthermore, if u_0 is in Σ, then the above solution $u(t)$ is in $C(\mathbb{R}\,; \Sigma)$.

REMARK 2.1. — Suppose that $1 < p < \alpha(n)$. Let $u(t)$ be a solution in $C(\mathbb{R}\,; H^1)$ of (2.15).

(*i*) The integral in the right hand side of (2.15) does not necessarily converge absolutely in H^1, but it converges absolutely in L^{p+1} (see, e. g., [*5*, Lemma 2.1 and Proposition 2.1]). Accordingly, $u(t)$ satisfies (2.15) in L^{p+1} for all $t \in \mathbb{R}$.

(*ii*) We note that $\partial u(t)/\partial t \in C(\mathbb{R}; H^{-1})$ and that $u(t)$ satisfies (1.1) in the distribution sense, which can be easily verified by a simple calculation. Therefore, $u(t)$ is a weak solution of (1.1).

Finally we describe some properties of the transform (1.8). We define the mapping J by

$$(Jv)(t, x) = (it)^{-n/2} e^{\frac{ix^2}{2t}} \overline{v\left(\frac{1}{t}, \frac{x}{t}\right)}. \tag{2.16}$$

Let $u(t, x)$ be a solution of (1.1). We put $v(t, x) = (J^{-1}u)(t, x)$. Then J^{-1} transforms the equation (1.1) into the new equations:

$$i\frac{\partial v}{\partial t} = -\frac{1}{2}\Delta v + |\,t\,|^{n(p-1)/2-2}\,|\,v\,|^{p-1}v, \quad t \in \mathbb{R}\backslash\{\,0\,\}, \quad x \in \mathbb{R}^n. \tag{2.17}$$

We note that the asymptotic behavior as $t \to \pm 0$ of solutions of (2.17) corresponds to the asymptotic behavior as $t \to \pm\infty$ of solutions of the original equation (1.1) and that the asymptotic behavior as $t \to \pm\infty$ of solutions of (2.17) corresponds to the asymptotic behavior as $t \to \pm 0$ of solutions of the original equation (1.1). (2.17) has almost the same form as (1.1). However, for $p < 1 + \frac{4}{n}$ the nonlinear term $|t|^{n(p-1)/2-2}|v|^{p-1}v$ has the singularity at $t=0$, since $\frac{n(p-1)}{2} - 2 < 0$ for $p < 1 + \frac{4}{n}$. This fact makes it difficult to consider the scattering theory for $p < 1 + \frac{4}{n}$.

The mapping J has the following properties.

LEMMA 2.8. — (*i*) Let $u(x)$, $v(x) \in L^2$. Then,

$$U(-t)\left\{\left(\frac{1}{it}\right)^{\frac{n}{2}} e^{\frac{ix^2}{2t}} \hat{v}\left(\frac{x}{t}\right)\right\} = U(-t)(J\bar{\hat{v}}) = e^{-\frac{ix^2}{2t}} v(x), \quad t \neq 0, \quad x \in \mathbb{R}^n, \tag{2.18}$$

$$\| U(t)v - J\bar{\hat{v}} \|_{L^2} \to 0 \qquad (t \to \pm\infty), \tag{2.19}$$

$$\| (Jv)(t) \|_{L^2} = \left\| (it)^{-n/2} e^{\frac{ix^2}{2t}} \overline{v\left(\frac{x}{t}\right)} \right\|_{L^2} = \| v \|_{L^2}, \qquad t \neq 0, \tag{2.20}$$

$$(Ju)(t, x) + (Jv)(t, x) = (J(u+v))(t, x), \qquad t \neq 0, \qquad x \in \mathbb{R}^n. \tag{2.21}$$

(*ii*) Assume that $1 < p < \alpha(n)$. Let T_1 and T_2 be any two constants with $T_1T_2 > 0$ and $T_2 > T_1$. Let $u(t)$ be a weak solution of (1.1) such that $u(t) \in C([T_1, T_2]; \Sigma)$. Then J^{-1} translates $u(t)$ into a weak solution $v(t) = (J^{-1}u)(t)$ of (2.17) such that $v(t) \in C\left(\left[\frac{1}{T_2}, \frac{1}{T_1}\right]; \Sigma\right)$. For J we also have a reverse result.

Proof. — We first prove (*i*). A direct calculation gives us (2.18). By the dominated convergence theorem we easily see that $\exp(-ix^2/2t)v(x) \to v(x)$ in L^2 as $t \to \pm\infty$. This fact and (2.18) give us (2.19). (2.20) and (2.21) are clear.

Next we prove (*ii*). Since $u(t)$ is a weak solution of (1.1) in $C([T_1, T_2]; \Sigma)$, $u(t)$ can be represented as follows:

$$u(t) = U(t - T_1)u(T_1) - i\int_{T_1}^{t} U(t-\tau)f(u(\tau))d\tau \quad \text{in} \quad L^{p+1} \tag{2.22}$$

for all $t \in [T_1, T_2]$. By the definition of J^{-1} it is clear that if $u(t) \in C([T_1, T_2]; \Sigma)$,

$$v(t) = (J^{-1}u)(t) \in C\left(\left[\frac{1}{T_2}, \frac{1}{T_1}\right]; \Sigma\right). \tag{2.23}$$

Substituting $u(t) = (\mathrm{J}v)(t)$ into (2.22), we have

$$v(t)=\mathrm{U}\left(t-\frac{1}{\mathrm{T}_1}\right)v\left(\frac{1}{\mathrm{T}_1}\right) - i\int_{1/\mathrm{T}_1}^{t} |\tau|^{n(p-1)/2-2}\mathrm{U}(t-\tau)f(v(\tau))d\tau \quad \text{in} \quad \mathrm{L}^{p+1} \tag{2.24}$$

for all $t \in [1/\mathrm{T}_2, 1/\mathrm{T}_1]$. Since $0 \notin [1/\mathrm{T}_2, 1/\mathrm{T}_1]$, (2.23) and (2.24) show (*ii*) for J^{-1}. In the same way we can also prov (*ii*) for J. Q. E. D.

§ 3. PROOFS OF THEOREMS 1.1 AND 1.2

In this section we give the proofs of Theorems 1.1 and 1.2. By using (2.16) we can reduce the scattering problem for (1.1) to the problems of existence and regularity of solutions for (2.17).

We first consider Theorem 1.1 (*i*). For any $u_+ \in \Sigma$ we consider

$$v(t) = \mathrm{U}(t)\bar{\hat{u}}_+ - i\int_0^t \tau^{n(p-1)/2-2}\mathrm{U}(t-\tau)f(v(\tau))d\tau, \qquad t \geqq 0. \tag{3.1}$$

(3.1) is the integral version of the initial value problem for (2.17). Suppose that for some $\mathrm{T} > 0$ (3.1) has a unique solution $v(t) \in \mathrm{C}([0, \mathrm{T}]; \Sigma)$. Then it follows from Lemma 2.8 (*ii*) that $u(t) = (\mathrm{J}v)(t)$ is a weak solution of (1.1) such that $u(t) \in \mathrm{C}([1/\mathrm{T}, \infty); \Sigma)$. Furthermore, since $v(t) = (\mathrm{J}^{-1}u)(t) \to \bar{\hat{u}}_+$ in Σ $(t \to +0)$, we have by (2.18)

$$\begin{aligned}
&\| xu_+ - x\mathrm{U}(-t)u(t)\|_{\mathrm{L}^2} \\
&\leqq \| xu_+ - xe^{-ix^2/2t}u_+ \|_{\mathrm{L}^2} + \| xe^{-ix^2/2t}u_+ - x\mathrm{U}(-t)(\mathrm{J}v)(t)\|_{\mathrm{L}^2} \\
&= \| xu_+ - xe^{-ix^2/2t}u_+ \|_{\mathrm{L}^2} + \left\| xe^{-ix^2/2t}u_+ - xe^{-ix^2/2t}\check{\bar{v}}\left(\frac{1}{t}\right)\right\|_{\mathrm{L}^2} \\
&= \| xu_+ - xe^{-ix^2/2t}u_+ \|_{\mathrm{L}^2} + \left\| xu_+ - x\check{\bar{v}}\left(\frac{1}{t}\right)\right\|_{\mathrm{L}^2} \\
&= \| (1 - e^{-ix^2/2t})xu_+ \|_{\mathrm{L}^2} + \left\| \nabla\bar{\hat{u}}_+ - \nabla v\left(\frac{1}{t}\right)\right\|_{\mathrm{L}^2} \to 0 \ (t \to +\infty),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\| \nabla u_+ - \nabla\mathrm{U}(-t)u(t)\|_{\mathrm{L}^2} \\
&= \| \nabla u_+ - \nabla\mathrm{U}(-t)(\mathrm{J}v)(t)\|_{\mathrm{L}^2} = \left\| \nabla u_+ - \nabla\left\{ e^{-ix^2/2t}\check{\bar{v}}\left(\frac{1}{t}\right)\right\}\right\|_{\mathrm{L}^2} \\
&\leqq \frac{1}{t}\left\| xe^{-ix^2/2t}\check{\bar{v}}\left(\frac{1}{t}\right)\right\|_{\mathrm{L}^2} + \| \nabla u_+ - e^{-ix^2/2t}\nabla u_+ \|_{\mathrm{L}^2} \\
&\qquad + \left\| e^{-ix^2/2t}\nabla u_+ - e^{-ix^2/2t}\nabla\check{\bar{v}}\left(\frac{1}{t}\right)\right\|_{\mathrm{L}^2}
\end{aligned}$$

$$\leqq \frac{1}{t}\left\|\nabla v\left(\frac{1}{t}\right)\right\|_{L^2} + \|(1 - e^{-ix^2/2t})\nabla u_+\|_{L^2}$$
$$+ \left\|x\bar{\hat{u}}_+ - xv\left(\frac{1}{t}\right)\right\|_{L^2} \to 0 \ (t \to +\infty), \tag{3.3}$$

$$\|u_+ - U(-t)u(t)\|_{L^2} = \|u_+ - U(-t)(Jv)(t)\|_{L^2}$$
$$= \left\|u_+ - e^{-ix^2/2t}\check{\bar{v}}\left(\frac{1}{t}\right)\right\|_{L^2}$$
$$\leqq \|u_+ - e^{-ix^2/2t}u_+\|_{L^2} + \left\|e^{-ix^2/2t}u_+ - e^{-ix^2/2t}\check{\bar{v}}\left(\frac{1}{t}\right)\right\|_{L^2}$$
$$\leqq \|(1 - e^{-ix^2/2t})u_+\|_{L^2} + \left\|\bar{\hat{u}}_+ - v\left(\frac{1}{t}\right)\right\|_{L^2} \to 0 \ (t \to +\infty). \tag{3.4}$$

The solution $u(t) = (Jv)(t)$ of (1.1) can be uniquely extended from $[1/T, \infty)$ to $(-\infty, +\infty)$ by Proposition 2.7. Therefore, if we choose $u_0 = u(0) \in \Sigma$, then we obtain the desired interacting state u_0 satisfying (1.4). Accordingly, in order to prove Theorem 1.1 (*i*), it is sufficient to show that (3.1) has a unique solution $v(t) \in C([0, T]; \Sigma)$ for some $T > 0$. The proof of Theorem 1.1 (*ii*) is the same as that of Theorem 1.1 (*i*).

On the other hand, for any $u_0 \in \Sigma$ we have a unique global weak solution $u(t) \in C(\mathbb{R}; \Sigma)$ of (1.1)-(1.2) by Proposition 2.7. Then it follows from Lemma 2.8 (*ii*) that $v(t) = (J^{-1}u)(t)$ is a weak solution of (2.17) in $C(\mathbb{R}\setminus\{0\}; \Sigma)$. If the limits

$$v_\pm = \lim_{t \to \pm 0} v(t) \quad \text{in} \quad \Sigma \tag{3.5}$$

exist, then we see by (3.2-4) that

$$\|\bar{\check{v}}_\pm - U(-t)u(t)\|_\Sigma \to 0 \qquad (t \to \pm\infty). \tag{3.6}$$

Therefore, if we choose $u_\pm = \bar{\check{v}}_\pm$, then we obtain the desired scattering states $u_\pm$ satisfying (1.6). Now we consider

$$v_\pm(t) = U(t \mp 1)v_{\pm 1} - i\int_{\pm 1}^{t} |\tau|^{n(p-1)/2-2}U(t - \tau)f(v_\pm(\tau))d\tau. \tag{3.7$\pm$}$$

In order to prove Theorem 1.2, it is sufficient to show that for any $v_{\pm 1} \in \Sigma$ (3.7$\pm$) have unique solutions $v_+(t) \in C([0, 1]; \Sigma)$ and $v_-(t) \in C([-1, 0]; \Sigma)$, respectively.

Thus, we first consider.

$$v(t) = U(t - t_0)v_0 - i\int_{t_0}^{t} |\tau|^{n(p-1)/2-2}U(t-\tau)f(v(\tau))d\tau, \tag{3.8}$$

$$v_j(t) = U(t - t_0)h_j * v_0 - i\int_{t_0}^{t} |\tau|^{n(p-1)/2-2}U(t-\tau)f_j(v_j(\tau))d\tau, \ j = 1, 2, \ldots, \tag{3.9}$$

where $t_0 \in [-1, 1]$. (3.8) is the integral version of the initial value problem for (2.17) and (3.9) is the regularized problem associated with (3.8).

For (3.8) and (3.9) we have the following results.

PROPOSITION 3.1. — Assume that $\gamma(n) < p < \alpha(n)$. For any $\rho > 0$ there exists a $\mathrm{T}(\rho) > 0$ depending only on ρ, n and p (but independent of j and t_0) such that for any $t_0 \in [-1, 1]$ and any $v_0 \in \mathcal{S}'$ with

$$| \mathrm{U}(. - t_0)v_0 |_{\mathrm{I},\mathrm{L}^{p+1}} \leqq \rho,$$

(3.8) and (3.9) have unique solutions $u(t)$ and $u_j(t)$ in $\mathrm{C}(\mathrm{I}; \mathrm{L}^{p+1})$ with $| u |_{\mathrm{I},\mathrm{L}^{p+1}} \leqq 2\rho$ and $| u_j |_{\mathrm{I},\mathrm{L}^{p+1}} \leqq 2\rho$, where $\mathrm{I} = [t_0 - \mathrm{T}(\rho), t_0 + \mathrm{T}(\rho)]$.

LEMMA 3.2. — Assume that $\gamma(n) < p < \alpha(n)$. For any $\rho > 0$, let $\mathrm{T}(\rho)$ be defined as in Proposition 3.1, let $\mathrm{I} = [t_0 - \mathrm{T}(\rho), t_0 + \mathrm{T}(\rho)]$ and let $v_0 \in \mathcal{S}'$ be such that $| \mathrm{U}(. - t_0)v_0 |_{\mathrm{I},\mathrm{L}^{p+1}} \leqq \rho$. Let $v(t)$ and $v_j(t)$, $j = 1, 2, \ldots$, be the solutions in $\mathrm{C}(\mathrm{I}; \mathrm{L}^{p+1})$ of (3.8) and (3.9), respectively. Then $\{ v_j(t) \}_{j=1}^{\infty}$ tends to $v(t)$ in $\mathrm{C}(\mathrm{I}; \mathrm{L}^{p+1})$ as $j \to \infty$.

LEMMA 3.3. — Assume that $\gamma(n) < p < \alpha(n)$. For $\mathrm{T} > 0$, we put $\mathrm{I} = [t_0 - \mathrm{T}, t_0 + \mathrm{T}]$. Let $v_0 \in \mathrm{H}^1$ and let $v_j(t)$, $j = 1, 2, \ldots$, be the solutions of (3.9) in $\mathrm{C}(\mathrm{I}; \mathrm{L}^{p+1})$. Then, for any nonnegative integer N $v_j(t) \in \mathrm{C}(\mathrm{I}; \mathrm{H}^{\mathrm{N}})$, $j = 1, 2, \ldots$, and for any multi-index α, $v_j(t)$, $j = 1, 2, \ldots$, satisfy the following equation in L^2:

$$i \frac{\partial}{\partial t} \mathrm{D}^{\alpha} v_j(t) = -\frac{1}{2} \Delta \mathrm{D}^{\alpha} v_j(t) + | t |^{n(p-1)/2 - 2} \mathrm{D}^{\alpha} f_j(v_j(t)), \tag{3.10}$$

for $t \in \mathrm{I}$ if $1 + \frac{4}{n} \leqq p < \alpha(n)$ and for $t \in \mathrm{I}$, $t \neq 0$, if $\gamma(n) < p < 1 + \frac{4}{n}$, where

$$\mathrm{D}^{\alpha} = \prod_{i=1}^{n} \partial^{\alpha_i} / \partial x_i^{\alpha_i} \text{ and } | \alpha | = \alpha_1 + \ldots + \alpha_n.$$

REMARK 3.1. — Let $a(p)$ be the Sobolev constant with

$$\| v \|_{\mathrm{L}^{p+1}} \leqq a(p) \| v \|_{\mathrm{H}^1}.$$

If we choose $v_0 \in \mathrm{H}^1$ with $\| v_0 \|_{\mathrm{H}^1} \leqq \dfrac{\rho}{a(p)}$ in Proposition 3.1 and Lemma 3.2, then v_0 satisfies $\| \mathrm{U}(t - t_0)v_0 \|_{\mathrm{L}^{p+1}} \leqq \rho$ $(t \in \mathbb{R})$, that is, $| \mathrm{U}(. - t_0)v_0 |_{\mathrm{I},\mathrm{L}^{p+1}} \leqq \rho$.

We can prove Proposition 3.1 and Lemma 3.2 by using the contraction mapping principle. The proofs of Proposition 3.1 and Lemma 3.2 are the same as those of Proposition 2.1 and Proposition 3.1 in [5]. In the proofs of Proposition 3.1 and Lemma 3.2 the integral of the following type appears:

$$\int_{t_0}^{t} | t - \tau |^{\alpha} | \tau |^{\beta} d\tau , \tag{3.11}$$

where $\alpha = \frac{n}{p+1} - \frac{n}{2}$ and $\beta = \frac{n(p-1)}{2} - 2$. When $\gamma(n) < p < \alpha(n)$, the integrand of (3.11) is integrable near $\tau = t$ and (3.11) tends to 0 uniformly in $t_0 \in [-1, 1]$ as $t \to t_0$. Therefore, the assumption that $\gamma(n) < p < \alpha(n)$ is needed for the proofs of Proposition 3.1 and Lemma 3.2. The proof of Lemma 3.3 is the same as that of Proposition 3.2 in [5]. We note that if $\gamma(n) < p < 1 + \frac{4}{n}$, (3.10) makes no sense at $t = 0$ because of the singularity of $| t |^{n(p-1)/2-2}$.

In order to show Theorem 1.1, we have only to prove the following lemma.

LEMMA 3.4. — Assume that $\gamma(n) < p < \alpha(n)$ and that $t_0 = 0$ in (3.8) and (3.9). For any $\rho > 0$, let $T(\rho)$ be defined as in Proposition 3.1, let $I = [-T(\rho), T(\rho)]$ and let $v_0 \in \Sigma$ be such that $| U(.)v_0 |_{I,L^{p+1}} \leqq \rho$. Let $v(t)$ be the solution in $C(I; L^{p+1})$ of (3.8). Then $v(t) \in C(I; \Sigma)$.

Proof. — By Proposition 3.1 we have the solutions $v_j(t)$, $j=1, 2, \ldots$, in $C(I; L^{p+1})$ of (3.9) such that

$$| v_j |_{I,L^{p+1}} \leqq 2\rho, \qquad j = 1, 2, \ldots . \tag{3.12}$$

We note that by (3.9) and (3.12) we have

$$\begin{aligned} &\| U(t-s)v_j(s) \|_{L^{p+1}} \\ &\leqq \rho + C \left| \int_0^s | \tau |^{n(p-1)/2-2} | t - \tau |^{-n/2+n/(p+1)} (2\rho)^p d\tau \right| \\ &\leqq C, \qquad t, s \in I, \qquad j = 1, 2, \ldots , \end{aligned} \tag{3.13}$$

where $C = C(\rho, T(\rho), n, p)$. We put $I_1 = [0, T(\rho)]$. We prove only $v(t) \in C(I_1; \Sigma)$, since we can prove $v(t) \in C([-T(\rho), 0]; \Sigma)$ in the same way. We divide the proof into two parts.

(*Part I*). We shall first prove $v(t) \in C(I_1; H^1)$. Multiplying (3.10) with $\alpha = 0$ by $\bar{v}_j(t)$ and taking the imaginary part, we have

$$\| v_j(s) \|_{L^2}^2 = \| v_j(t) \|_{L^2}^2, \quad 0 < s \leqq t \leqq T(\rho), \quad j = 1, 2, \ldots . \tag{3.14}$$

Letting $s \to +0$ in (3.14), we obtain by Lemma 3.3

$$\| v_j(t) \|_{L^2}^2 = \| h_j * v_0 \|_{L^2}^2 \leqq \| v_0 \|_{L^2}^2, \quad 0 < t \leqq T(\rho), \quad j = 1, 2, \ldots . \tag{3.15}$$

Multiplying (3.10) with $\alpha=0$ by $\partial\bar{v}_j(t)/\partial t$ and taking the real part, we have

$$\begin{aligned}
&\| \nabla v_j(s) \|_{L^2}^2 + \frac{4}{p+1} s^{n(p-1)/2-2} \| h_j * v_j(s) \|_{L^{p+1}}^{p+1} \\
&\qquad - \frac{8-2n(p-1)}{p+1} \int_s^t \tau^{n(p-1)/2-3} \| h_j * v_j(\tau) \|_{L^{p+1}}^{p+1} d\tau \\
&\qquad = \| \nabla v_j(t) \|_{L^2}^2 + \frac{4}{p+1} t^{n(p-1)/2-2} \| h_j * v_j(t) \|_{L^{p+1}}^{p+1}, \\
&\qquad\qquad 0 < s \leqq t \leqq T(\rho), \quad j = 1, 2, \ldots . \qquad (3.16)
\end{aligned}$$

We shall show that

$$\| \nabla v_j(t) \|_{L^2} \leqq C(1 + \| v_0 \|_{H^1}), \quad t \in I_1, \quad j = 1, 2, \ldots, \qquad (3.17)$$

where $C = C(\rho, T(\rho), p, n)$. In the case of $1 + \frac{4}{n} \leqq p < \alpha(n)$ we can easily prove (3.17). In fact, since $\frac{n(p-1)}{2} - 2 \geqq 0$ for $p \geqq 1 + \frac{4}{n}$ and $\frac{8-2n(p-1)}{p+1} = 0$ for $p = 1 + \frac{4}{n}$, by letting $s \to +0$ in (3.16) and using Lemma 3.3 and (3.12) we obtain (3.17) for $1 + \frac{4}{n} \leqq p < \alpha(n)$. We next assume that $\gamma(n) < p < 1 + \frac{4}{n}$. In this case $s^{n(p-1)/2-2}$ tends to ∞ as $s \to +0$ and $|\tau|^{n(p-1)/2-3}$ is not integrable near $\tau = 0$. These facts make it difficult to consider the case of $\gamma(n) < p < 1 + \frac{4}{n}$. We can rewrite (3.16) as follows:

$$\begin{aligned}
&\| \nabla v_j(s) \|_{L^2}^2 + \frac{8-2n(p-1)}{p+1} \int_s^t \tau^{n(p-1)/2-3} \\
&\quad \times \Big[\{ \| h_j * v_j(s) \|_{L^{p+1}}^{p+1} - \| h_j * U(\tau - s) v_j(s) \|_{L^{p+1}}^{p+1} \} \\
&\quad\quad + \{ \| h_j * U(\tau - s) v_j(s) \|_{L^{p+1}}^{p+1} - \| h_j * v_j(\tau) \|_{L^{p+1}}^{p+1} \} \Big] d\tau \\
&\quad + \frac{4}{p+1} t^{n(p-1)/2-2} \Big[\{ \| h_j * v_j(s) \|_{L^{p+1}}^{p+1} - \| h_j * U(t-s) v_j(s) \|_{L^{p+1}}^{p+1} \} \\
&\quad\quad + \{ \| h_j * U(t-s) v_j(s) \|_{L^{p+1}}^{p+1} - \| h_j * v_j(t) \|_{L^{p+1}}^{p+1} \} \Big] \\
&\quad = \| \nabla v_j(t) \|_{L^2}^2, \qquad 0 < s \leqq t \leqq T(\rho), \qquad j = 1, 2, \ldots . \qquad (3.18)
\end{aligned}$$

We evaluate (3.18) in order to show (3.17) for $\gamma(n) < p < 1 + \frac{4}{n}$. We divide the proof of (3.17) for $\gamma(n) < p < 1 + \frac{4}{n}$ into four steps.

STEP 1. — By integration by parts, Hölder's inequality, Lemma 2.4 and (3.13) we have

$$
\begin{aligned}
& \left| \| h_j * v_j(s) \|_{L^{p+1}}^{p+1} - \| h_j * U(\tau-s)v_j(s) \|_{L^{p+1}}^{p+1} \right| \\
& \leqq \left| \int_\tau^s \frac{d}{ds_1} \| h_j * U(s_1 - s)v_j(s) \|_{L^{p+1}}^{p+1} ds_1 \right| \\
& \leqq (p+1) \left| \int_s^\tau \operatorname{Re} \int_{\mathbb{R}^n} | h_j * U(s_1-s)v_j(s) |^{p-1} h_j * U(s_1-s)v_j(s) \right. \\
& \qquad \left. \times \overline{\frac{i}{2} \Delta h_j * U(s_1 - s)v_j(s)} dx ds_1 \right| \\
& \leqq C \int_s^\tau \int_{\mathbb{R}^n} | h_j * U(s_1-s)v_j(s) |^{p-1} | h_j * \nabla U(s_1-s)v_j(s) |^2 dx ds_1 \\
& \leqq C \int_s^\tau \| U(s_1-s)v_j(s) \|_{L^{p+1}}^{p-1} \| \nabla U(s_1 - s)v_j(s) \|_{L^{p+1}}^2 ds_1 \\
& \leqq C \int_s^\tau \| \nabla U(s_1 - s)v_j(s) \|_{L^q}^{2\theta} \| \nabla U(s_1 - s)v_j(s) \|_{L^2}^{2(1-\theta)} ds_1 \\
& \leqq C \| \nabla v_j(s) \|_{L^2}^{2(1-\theta)} \int_s^\tau \| \nabla U(s_1 - s)v_j(s) \|_{L^q}^{2\theta} ds_1 \\
& \leqq C \| \nabla v_j(s) \|_{L^2}^{2(1-\theta)} \left(\int_s^\tau ds_1 \right)^{\bar\theta} \\
& \qquad \times \left(\int_s^\tau \| \nabla U(s_1 - s)v_j(s) \|_{L^q}^q ds_1 \right)^{\frac{n\theta}{n+2}} \\
& \leqq C(\tau-s)^{\bar\theta} \| \nabla v_j(s) \|_{L^2}^2, \quad 0 < s < \tau \leqq T(\rho), \ j = 1, 2, \ldots,
\end{aligned} \tag{3.19}
$$

where $\theta = \dfrac{(p-1)(n+2)}{2(p+1)}$, $\bar\theta = \dfrac{n+2-(n-2)p}{2(p+1)}$ and $q = \dfrac{2(n+2)}{n}$. We have used the interpolation inequality at the fifth inequality and have used Lemma 2.2 at the last inequality. We note that $\theta < 1$ for $p < 1 + \dfrac{4}{n}$.

STEP 2. — If $n \geqq 3$, we have by (3.10) with $\alpha = 0$, Lemma 2.4, (2.1), (3.12) and (3.13)

$$
\begin{aligned}
& \left| \| h_j * U(\tau-s)v_j(s) \|_{L^{p+1}}^{p+1} - \| h_j * v_j(\tau) \|_{L^{p+1}}^{p+1} \right| \\
& \leqq \left| \int_\tau^s \frac{d}{s_1} \| h_j * U(\tau-s_1)v_j(s_1) \|_{L^{p+1}}^{p+1} ds_1 \right|
\end{aligned}
$$

$$\leqq (p+1)\left|\int_{s_1}^{\tau} s_1^{n(p-1)/2-2}\mathrm{Re}\int_{\mathbb{R}^n} |h_j * \mathrm{U}(\tau-s_1)v_j(s_1)|^{p-1}\right.$$
$$\left.\times\, h_j * \mathrm{U}(\tau-s_1)v_j(s_1).\overline{h_j * \mathrm{U}(\tau-s_1)if_j(v_j(s_1))}dxds_1\right|$$
$$\leqq \mathrm{C}\int_s^{\tau} s_1^{n(p-1)/2-2}\,\|\mathrm{U}(\tau-s_1)v_j(s_1)\|_{\mathrm{L}^r}^p\,\|\mathrm{U}(\tau-s_1)f_j(v_j(s_1))\|_{\mathrm{L}^{\bar r}}ds_1$$
$$\leqq \mathrm{C}\int_s^{\tau} s_1^{n(p-1)/2-2}\,\|\mathrm{U}(\tau-s_1)v_j(s_1)\|_{\mathrm{L}^r}^p$$
$$\times\,(\tau-s_1)^{-\frac{(n-1)p-n-1}{p+1}}\,\|f_j(v_j(s_1))\|_{\mathrm{L}^{r/p}}ds_1$$
$$\leqq \mathrm{C}\int_s^{\tau} s_1^{n(p-1)/2-2}(\tau-s_1)^{-\frac{(n-1)p-n-1}{p+1}}$$
$$\times\,\|\mathrm{U}(\tau-s_1)v_j(s_1)\|_{\mathrm{L}^r}^p\,\|v_j(s_1)\|_{\mathrm{L}^r}^p ds_1$$
$$\leqq \mathrm{C}\int_s^{\tau} s_1^{n(p-1)/2-2}(\tau-s_1)^{-\frac{(n-1)p-n-1}{p+1}}$$
$$\times\,\|\mathrm{U}(\tau-s_1)v_j(s_1)\|_{\mathrm{L}^{p+1}}^{p-1}\,\|\nabla\mathrm{U}(\tau-s_1)v_j(s_1)\|_{\mathrm{L}^2}$$
$$\times\,\|v_j(s_1)\|_{\mathrm{L}^{p+1}}^{p-1}\,\|\nabla v_j(s_1)\|_{\mathrm{L}^2}ds_1$$
$$\leqq \mathrm{C}\int_s^{\tau} s_1^{n(p-1)/2-2}(\tau-s_1)^{-\frac{(n-1)p-n-1}{p+1}}\,\|\nabla v_j(s_1)\|_{\mathrm{L}^2}^2 ds_1\,,$$
$$0 < s < \tau \leqq \mathrm{T}(\rho), \qquad j = 1, 2, \ldots, \qquad (3.20)$$

where $r = \dfrac{2n(p+1)p}{(3n-2)p-n-2}$, $\bar r = \dfrac{2n(p+1)}{3n+2-(n-2)p}$ and $\mathrm{C}=\mathrm{C}(\rho,\ \mathrm{T}(\rho),\ p,\ n)$.

We have used Hölder's inequality with $p/r+1/\bar r=1$ at the third inequality and have used Lemma 2.6 (*iii*) at the last inequality but one. Since

$$\frac{n+2}{n-2} > p > \gamma(n) \geqq \frac{n+1}{n-1} > \frac{n+2}{3n-2} \qquad \text{for} \qquad n \geqq 3\,,$$

we note that $r > p + 1$ and $\bar r > 2$.

STEP 3. — If $1 \leqq n \leqq 2$, we have by (3.10) with $\alpha = 0$, Lemma 2.4, (3.12) and (3.13)

$$\left|\,\|h_j * \mathrm{U}(\tau-s)v_j(s)\|_{\mathrm{L}^{p+1}}^{p+1} - \|h_j * v_j(\tau)\|_{\mathrm{L}^{p+1}}^{p+1}\right|$$
$$\leqq \left|\int_{\tau}^{s} \frac{d}{ds_1}\|h_j * \mathrm{U}(\tau-s_1)v_j(s_1)\|_{\mathrm{L}^{p+1}}^{p+1}ds_1\right|$$
$$\leqq (p+1)\left|\int_s^{\tau} s_1^{n(p-1)/2-2}\mathrm{Re}\int_{\mathbb{R}^n} |h_j * \mathrm{U}(\tau-s_1)v_j(s_1)|^{p-1}\right.$$
$$\left.\times\, h_j * \mathrm{U}(\tau-s_1)v_j(s_1).\overline{h_j * \mathrm{U}(\tau-s_1)if_j(v_j(s_1))}dxds_1\right|$$

$$
\begin{aligned}
&\leqq C\left|\int_s^\tau s_1{}^{n(p-1)/2-2}\operatorname{Im}\int_{\mathbb{R}^n} f_j(U(\tau-s_1)v_j(s_1))\right.\\
&\qquad\left.\times\,\overline{U(\tau-s_1)f_j(v_j(s_1))}dxds_1\right|\\
&\leqq C\left|\int_s^\tau s_1{}^{n(p-1)/2-2}\right.\\
&\qquad\times\left[\operatorname{Im}\int_{\mathbb{R}^n} f_j(U(\tau-s_1)v_j(s_1)).\overline{\{U(\tau-s_1)-1\}f_j(v_j(s_1))}dx\right.\\
&\qquad\quad\left.\left.+\operatorname{Im}\int_{\mathbb{R}^n}\{f_j(U(\tau-s_1)v_j(s_1))-f_j(v_j(s_1))\}.\overline{f_j(v_j(s_1))}dx\right]ds_1\right|\\
&\leqq C\int_s^\tau s_1{}^{n(p-1)/2-2}\\
&\qquad\times\left[\|U(\tau-s_1)v_j(s_1)\|_{L^{2p}}^p\|\{U(\tau-s_1)-1\}f_j(v_j(s_1))\|_{L^2}\right.\\
&\qquad\quad+\|\{U(\tau-s_1)-1\}v_j(s_1)\|_{L^2}^\delta\||h_j*\{U(\tau-s_1)-1\}v_j(s_1)|^{1-\delta}\\
&\qquad\left.\times(|h_j*U(\tau-s_1)v_j(s_1)|+|h_j*v_j(s_1)|)^{p-1}h_j*f_j(v_j(s_1))\|_{L^q}\right]ds_1\\
&\leqq C\int_s^\tau s_1{}^{n(p-1)/2-2}(\tau-s_1)^{\delta/2}\\
&\qquad\times\left[\|U(\tau-s_1)v_j(s_1)\|_{L^{2p}}^p\|f_j(v_j(s_1))\|_{H^\delta}\right.\\
&\qquad\quad\left.+\|v_j(s_1)\|_{H^1}^\delta(\|U(\tau-s_1)v_j(s_1)\|_{L^{\bar q}}^r+\|v_j(s_1)\|_{L^{\bar q}}^r)\right]ds_1\\
&\leqq C\int_s^\tau s_1{}^{n(p-1)/2-2}(\tau-s_1)^{\delta/2}\\
&\qquad\times\left[\|\nabla U(\tau-s_1)v_j(s_1)\|_{L^2}^{p\theta}\|U(\tau-s_1)v_j(s_1)\|_{L^{p+1}}^{p(1-\theta)}\|f_j(v_j(s_1))\|_{H^\delta}\right.\\
&\qquad\quad+\|v_j(s_1)\|_{H^1}^\delta(\|\nabla U(\tau-s_1)v_j(s_1)\|_{L^2}^{r\tilde\theta}\|U(\tau-s_1)v_j(s_1)\|_{L^{p+1}}^{r(1-\tilde\theta)}\\
&\qquad\quad\left.+\|\nabla v_j(s_1)\|_{L^2}^{r\tilde\theta}\|v_j(s_1)\|_{L^{p+1}}^{r(1-\tilde\theta)})\right]ds_1\\
&\leqq C\int_s^\tau s_1{}^{n(p-1)/2-2}(\tau-s_1)^{\delta/2}\\
&\qquad\times\left[\|\nabla v_j(s_1)\|_{L^2}^{p\theta}\|f_j(v_j(s_1))\|_{H^\delta}+\|v_j(s_1)\|_{H^1}^2\right]ds_1,\\
&\qquad\qquad 0<s<\tau\leqq T(\rho),\qquad j=1,2,\ldots,
\end{aligned}\tag{3.21}
$$

where $\delta=\dfrac{2(n+1)-2(n-1)p}{p+1}$, $q=\dfrac{p+1}{n(p-1)}$, $\bar q=\dfrac{2(p+n+1)}{n}$, $r=\dfrac{2(p-1)(p+n+1)}{p+1}$, $\theta=\dfrac{n(p-1)}{p\{(2-n)p+n+2\}}$ and $\tilde\theta=\dfrac{n}{p+n+1}$. At the fifth inequality we have used Schwarz's inequality and Hölder's inequality with $\dfrac{1}{2/\delta}+\dfrac{1}{q}=1$. We have used Lemma 2.3 at the sixth inequality and have used Lemma 2.6 (*iii*) at the last inequality but one. At the last inequality we have used the fact that $\delta + r\tilde{\theta} = 2$. Since

$$\frac{2n+1}{2n-1} \leqq 1+\frac{2}{n} < \gamma(n) < p < 1+\frac{4}{n} \leqq \frac{5n+2}{5n-2} \leqq \frac{n+1}{n-1} \qquad \text{for} \quad 1 \leqq n \leqq 2,$$

we note that $0 < p\theta < 1$ and $0 < \delta < 1$. On the other hand, we have by Lemma 2.6 (*i*) and (3.12)

$$\begin{aligned}
\| f_j(v_j(s_1)) \|_{H^\delta} &\leqq \| \, | h_j * v_j(s_1) |^{p-1} h_j * v_j(s_1) \|_{H^\delta} \\
&\leqq C \| f(h_j * v_j(s_1)) \|_{L^2} \\
&\quad + C\left(\int_0^\infty \{ t^{-\delta} \sup_{|y|<t} \| f((h_j * v_j)(s_1, x)) - f((h_j * v_j)(s_1, x+y)) \|_{L^2} \} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\leqq C \| v_j(s_1) \|_{L^{2p}}^{p} \\
&\quad + C\left(\int_0^\infty [t^{-\delta} \sup_{|y|<t} \{ (\| (h_j * v_j)(s_1, x) \|_{L^q}^{p-1} + \| (h_j * v_j)(s_1, x+y) \|_{L^q}^{p-1}) \right. \\
&\qquad \left. \times \| (h_j * v_j)(s_1, x) - (h_j * v_j)(s_1, x+y) \|_{L^{\bar{q}}} \}]^2 \frac{dt}{t} \right)^{1/2} \\
&\leqq C \| v_j(s_1) \|_{L^{p+1}}^{p(1-\theta)} \| \nabla v_j(s_1) \|_{L^2}^{p\theta} \\
&\quad + C\left(\int_0^\infty [t^{-\delta} \sup_{|y|<t} \{ \| \nabla v_j(s_1) \|_{L^2}^{(p-1)\bar{\theta}} \| v_j(s_1) \|_{L^{p+1}}^{(p-1)(1-\bar{\theta})} \right. \\
&\qquad \left. \times \| (h_j * v_j)(s_1, x) - (h_j * v_j)(s_1, x+y) \|_{L^{\bar{q}}} \}]^2 \frac{dt}{t} \right)^{1/2} \\
&\leqq C \| \nabla v_j(s_1) \|_{L^2}^{p\theta} + C \| \nabla v_j(s_1) \|_{L^2}^{(p-1)\bar{\theta}} \\
&\quad \times \left(\int_0^\infty [t^{-\delta} \sup_{|y|<t} \| v_j(s_1, x) - v_j(s_1, x+y) \|_{L^{\bar{q}}}]^2 \frac{dt}{d} \right)^{1/2} \\
&\leqq C \| \nabla v_j(s_1) \|_{L^2}^{p\theta} + C \| \nabla v_j(s_1) \|_{L^2}^{(p-1)\bar{\theta}} \| v_j(s_1) \|_{B^{\delta}_{\bar{q}2}} \\
&\leqq C \| \nabla v_j(s_1) \|_{L^2}^{p\theta} + C \| \nabla v_j(s_1) \|_{L^2}^{(p-1)\bar{\theta}} \| v_j(s_1) \|_{H^1}, \qquad (3.22)
\end{aligned}$$

where $\theta = \dfrac{n(p-1)}{p\{(2-n)p+n+2\}}$, $\bar{\theta} = \dfrac{2(n+1)-2(n-1)p}{\{(2-n)p+n+2\}(p-1)}$, $q = \dfrac{n(p^2-1)}{(2n-1)p-(2n+1)}$ and $\bar{q} = \dfrac{2n(p+1)}{5n+2-(3n-2)p}$. At the third inequality we have used the fact that

$$| f(z_1) - f(z_2) | \leqq C(| z_1 |^{p-1} + | z_2 |^{p-1}) | z_2 - z_1 |$$

$(z_1, z_2 \in \mathbb{C})$ and Hölder's inequality with $(p-1)/q + 1/\bar{q} = 1/2$. We have used Lemma 2.6 (*iii*) at the fourth inequality and Lemma 2.6 (*ii*) at the last inequality. Since $\dfrac{2n+1}{2n-1} < \gamma(n) < p < 1 + \dfrac{4}{n} \leqq \dfrac{5n+2}{3n-2} \leqq \dfrac{n+1}{n-1}$ for $1 \leqq n \leqq 2$, we note that $p\theta < 1$, $\bar{q} > 2$ and

$$p\theta + (p-1)\bar{\theta} = 1 \qquad (3.23)$$

Therefore, combining (3.21), (3.22) and (3.23), we obtain for $1 \leqq n \leqq 2$

$$\begin{aligned} &\left| \| h_j * U(\tau - s)v_j(s) \|_{L^{p+1}}^{p+1} - \| h_j * v_j(\tau) \|_{L^{p+1}}^{p+1} \right| \\ &\qquad \leqq C \int_s^\tau {s_1}^{n(p-1)/2-2}(\tau - s_1)^{\frac{(n+1)-(n-1)p}{p+1}} \\ &\qquad \times (1 + \| v_j(s_1) \|_{L^2}^2 + \| \nabla v_j(s_1) \|_{L^2}^2) ds_1, \\ &\qquad 0 < s \leqq \tau \leqq T(\rho), \qquad j = 1, 2, \ldots, \qquad (3.24) \end{aligned}$$

where $C = C(\rho, T(\rho), p, n)$.

STEP 4. — Since we can evaluate the third term in the left hand side of (3.18) in the same way as (3.20) and (3.24), we obtain by (3.18), (3.19), (3.20), (3.24) and the fact that $\frac{n}{2}(p - 1) - 2 < 0$ for $p < 1 + \frac{4}{n}$

$$\begin{aligned} \| \nabla v_j(t) \|_{L^2}^2 &\leqq \| \nabla v_j(s) \|_{L^2}^2 \\ &+ C \int_s^t \tau^{n(p-1)/2-3} d\tau \int_s^\tau {s_1}^{n(p-1)/2-2}(\tau - s_1)^{-\frac{(n-1)p-(n+1)}{p+1}} \\ &\times (1 + \| v_j(s_1) \|_{L^2}^2 + \| \nabla v_j(s_1) \|_{L^2}^2) ds_1 \\ &+ C t^{n(p-1)/2-2} \int_s^t \tau^{n(p-1)/2-2}(t - \tau)^{-\frac{(n-1)p-(n+1)}{p+1}} \\ &\times (1 + \| v_j(\tau) \|_{L^2}^2 + \| \nabla v_j(\tau) \|_{L^2}^2) d\tau \\ &\leqq \| \nabla v_j(s) \|_{L^2}^2 \\ &+ C \int_s^t {s_1}^{n(p-1)/2-2}(1 + \| v_j(s_1) \|_{L^2}^2 + \| \nabla v_j(s_1) \|_{L^2}^2) \\ &\times \int_{s_1}^t \tau^{n(p-1)/2-3}(\tau - s_1)^{-\frac{(n-1)p-(n+1)}{p+1}} d\tau ds_1 \\ &+ \int_s^t \{ \min(\tau, (t-\tau)) \}^{n(p-1)/2-2} \tau^{n(p-1)/2-2}(t-\tau)^{-\frac{(n-1)p-(n+1)}{p+1}} \\ &\times (1 + \| v_j(\tau) \|_{L^2}^2 + \| \nabla v_j(\tau) \|_{L^2}^2) d\tau, \\ &\qquad 0 < s \leqq t \leqq T(\rho), \quad j = 1, 2, \ldots . \qquad (3.25) \end{aligned}$$

On the other hand, if $1 \leqq n \leqq 2$, we have

$$\begin{aligned} &\int_{s_1}^t \tau^{n(p-1)/2-3}(\tau - s_1)^{-\frac{(n-1)p-(n+1)}{p+1}} d\tau \\ &\qquad \leqq \int_{s_1}^t \tau^{\frac{n}{2}(p-1)-3-\frac{(n-1)p-(n+1)}{p+1}} d\tau \\ &\qquad \leqq C {s_1}^{\frac{n}{2}(p-1)-2-\frac{(n-1)p-(n+1)}{p+1}}, \qquad 0 < s_1 < t. \qquad (3.26) \end{aligned}$$

At the first inequality we have used the fact that if $1 \leqq n \leqq 2$, then $-\dfrac{(n-1)p-(n+1)}{p+1} > 0$ for $\gamma(n) < p < 1 + \dfrac{4}{n}$. If $n \geqq 3$, we have

$$\begin{aligned}
&\int_{s_1}^{t} \tau^{n(p-1)/2-3}(\tau - s_1)^{-\frac{(n-1)p-(n+1)}{p+1}} d\tau \\
&\leqq \left(\frac{1}{2}(t+s_1)\right)^{\frac{n}{2}(p-1)-3} \int_{(t+s_1)/2}^{t} (\tau-s_1)^{-\frac{(n-1)p-(n+1)}{p+1}} d\tau \\
&\quad + \left(\frac{1}{2}(t-s_1)\right)^{-\frac{(n-1)p-(n+1)}{p+1}} \int_{s_1}^{(t+s_1)/2} \tau^{\frac{n}{2}(p-1)-3} d\tau \\
&\leqq C\left(\frac{1}{2}(t+s_1)^{\frac{n}{2}(p-1)-2}\left(\frac{1}{2}(t+s_1)\right)^{-1}\left(\frac{1}{2}\,(t-s_1)\right)^{-\frac{(n-1)p-(n+1)}{p+1}+1}\right. \\
&\quad + C\left(\frac{1}{2}(t - s_1)\right)^{-\frac{(n-1)p-(n+1)}{p+1}} s_1^{\frac{n}{2}(p-1)-2} \\
&\leqq Cs_1^{\frac{n}{2}(p-1)-2}(t-s_1)^{-\frac{(n-1)p-(n+1)}{p+1}}, \qquad 0 < s_1 < t \leqq T(\rho). \qquad (3.27)
\end{aligned}$$

Here we have used the facts that if $n \geqq 3$, then for $\gamma(n) < p < 1 + \dfrac{4}{n}$

$$\frac{n}{2}(p-1) - 3 < 0, \quad \frac{n}{2}(p-1) - 2 < 0 \quad \text{and} \quad -\frac{(n-1)p-(n+1)}{p+1} < 0.$$

By (3.25), (3.26) and (3.27) we obtain for $\gamma(n) < p < 1 + \dfrac{4}{n}$

$$\begin{aligned}
&\| \nabla v_j(t) \|_{L^2}^2 \leqq \| \nabla v_j(s) \|_{L^2}^2 \\
&\quad + C\int_s^t K(t, \tau)(1 + \| v_j(\tau) \|_{L^2}^2 + \| \nabla v_j(\tau) \|_{L^2}^2)d\tau, \quad 0<s\leqq t\leqq T(\rho), \\
&\hspace{20em} j = 1, 2, \ldots . \qquad (3.28)
\end{aligned}$$

$$K(t, \tau) = \begin{cases} \tau^{\beta_1}\{\tau^{\beta_2} + (t-\tau)^{\beta_2}\}, & \text{if } 1 \leqq n \leqq 2, \\ \tau^{2\beta_1}(t-\tau)^{\beta_3}, & \text{if } n \geqq 3, \end{cases} \qquad (3.29)$$

$\beta_1 = \dfrac{n}{2}(p-1)-2$, $\beta_2 = \dfrac{n}{2}(p-1)-2-\dfrac{(n-1)p-(n+1)}{p+1}$ $\beta_3 = \beta_2 - \beta_1$ and $C = C(\rho, T(\rho), p, n)$. We note that if $1 \leqq n \leqq 2$, $\beta_1 + \beta_2 > -1$ and $\beta_2 > -1$ and that if $n \geqq 3$, $2\beta_1 > -1$ and $\beta_3 > -1$. Since $\beta_1 + \beta_2 = 2\beta_1 + \beta_3 > -1$

for $p > \gamma(n)$, letting $s \to +0$ in (3.28), we obtain by Lemma 3.3 and (3.15)

$$\|\nabla v_j(t)\|_{L^2}^2 \leqq C(1 + \|v_0\|_{H^1}^2) + C\int_0^t K(t,\tau)\|\nabla v_j(\tau)\|_{L^2}^2 d\tau, \qquad t \in I_1 = [0, T(\rho)], \qquad j = 1, 2, \ldots, \tag{3.30}$$

where $C = C(\rho, T(\rho), p, n)$. (3.30) and Lemma 2.5 give us (3.17) for $\gamma(n) < p < 1 + \frac{4}{n}$. Therefore, the proof of (3.17) is complete.

We obtain by (3.15), (3.17) and Lemma 3.2.

$$\|v(t)\|_{H^1} \leqq C(1 + \|v_0\|_{H^1}), \qquad t \in I_1, \tag{3.31}$$

where $C = C(\rho, T(\rho), p, n)$. Let t_0 be any time in I_1. For an arbitrary sequence $\{t_n\} \subset I_1$ with $t_n \to t_0$ $(n \to \infty)$, $v(t_n) \to v(t_0)$ in $\mathcal{S}'$ $(n \to \infty)$, since $v(t) \in C(I_1; L^{p+1})$. Combining this fact and (3.31), we have $v(t) \to v(t_0)$ weakly in H^1 $(t \to t_0)$. Since $t_0 \in I_1$ is arbitrary, $v(t)$ is weakly continuous in H^1. On the other hand, for any $t_0 \in I_1$, Proposition 3.1 and (3.31) enable us to solve (3.8) with the initial time and the initial datum replaced by t_0 and $v(t_0)$, respectively. Therefore, by using the regularizing technique of Ginibre and Velo [5, Proposition 3.4] we obtain

$$\|v(s)\|_{L^2} = \|v(t)\|_{L^2}, \tag{3.32}$$

for $0 \leqq s \leqq t \leqq T(\rho)$ and

$$\|\nabla v(s)\|_{L^2}^2 + \frac{4}{p+1} s^{\frac{n}{2}(p-1)-2}\|v(s)\|_{L^{p+1}}^{p+1} - \frac{8 - 2n(p-1)}{p+1}\int_s^t \tau^{\frac{n}{2}(p-1)-3}\|v(\tau)\|_{L^{p+1}}^{p+1} d\tau = \|\nabla v(t)\|_{L^2}^2 + \frac{4}{p+1} t^{\frac{n}{2}(p-1)-2}\|v(t)\|_{L^{p+1}}^{p+1}, \tag{3.33}$$

for $0 < s \leqq t \leqq T(\rho)$ if $\gamma(n) < p < 1 + \frac{4}{n}$ and for $0 \leqq s \leqq t \leqq T(\rho)$ if $1 + \frac{4}{n} \leqq p < \alpha(n)$. From (3.32), (3.33), the weak continuity in H^1 of $v(t)$ and the strong continuity in L^{p+1} of $v(t)$ it follows that if $\gamma(n) < p < 1 + \frac{4}{n}$, $v(t) \in C((0, T(\rho)]; H^1)$ and that if $1 + \frac{4}{n} \leqq p < \alpha(n)$, $v(t) \in C([0, T(\rho)]; H^1)$. It remains only to prove the strong continuity in H^1 of $v(t)$ at $t = 0$ for $\gamma(n) < p < 1 + \frac{4}{n}$. Letting $s \to +0$ and next $j \to \infty$ in (3.28), we have by Lemmas 3.2, 3.3 and (3.31)

$$\|\nabla v(t)\|_{L^2}^2 \leqq \varliminf_{j\to\infty} \|\nabla v_j(t)\|_{L^2}^2$$
$$\leqq \|\nabla v_0\|_{L^2}^2 + CL(t), \qquad 0 < t \leqq T(\rho), \tag{3.34}$$

where $L(t) = \int_0^t K(t, \tau)d\tau$ and $L(t) \to 0\,(t \to +0)$. (3.34) gives us

$$\varlimsup_{t\to+0} \|\nabla v(t)\|_{L^2}^2 \leqq \|\nabla v_0\|_{L^2}^2. \tag{3.35}$$

On the other hand, the weak continuity in H^1 of $v(t)$ at $t = 0$ gives us

$$\|\nabla v_0\|_{L^2}^2 \leqq \varliminf_{t\to+0} \|\nabla v(t)\|_{L^2}^2. \tag{3.36}$$

From (3.32), (3.35), (3.36) and the weak continuity in H^1 of $v(t)$ at $t = 0$ it follows that $v(t) \to v_0$ in H^1 as $t \to +0$. Therefore, $v(t) \in C(I_1; H^1)$.

(*Part II*). We shall next prove $v(t) \in C(I_1; \Sigma)$. By (2.17) and Lemma 2.1 (2) a formal calculation gives

$$\begin{aligned}
&\frac{d}{dt}\|xU(-t)v(t)\|_{L^2}^2 \\
&= -2\,\mathrm{Im}\,((x+it\nabla)v(t), (x+it\nabla)t^{n(p-1)/2-2}f(v(t))) \\
&= -2t^{n(p-1)/2-2}[-\mathrm{Im}\,(xv(t), xf(v(t))) - t^2\,\mathrm{Im}\,(-\Delta v(t), f(v(t))) \\
&\quad + t\,\mathrm{Re}\,\{(xv(t), \nabla f(v(t))) - (\nabla v(t), xf(v(t)))\}] \\
&= 2t^{n(p-1)/2-2}[-\mathrm{Im}\,(xv(t), xf(v(t))) - t^2\,\mathrm{Im}\,(-\Delta v(t), f(v(t))) \\
&\quad - 2t\,\mathrm{Re}\,(\nabla v(t), xf(v(t))) - nt(v(t), f(v(t)))].
\end{aligned} \tag{3.37}$$

Since

$$\mathrm{Im}\,(xv(t), xf(v(t))) = \mathrm{Im}\int_{\mathbb{R}^n} x^2\,|v(t,x)|^{p+1}dx = 0$$

and

$$\mathrm{Re}(\nabla v(t), xf(v(t))) = \frac{1}{p+1}\int_{\mathbb{R}^n} x.\nabla\,|v(t,x)|^{p+1}dx = -\frac{n}{p+1}\|v(t)\|_{L^{p+1}}^{p+1},$$

we have by (2.17) and (3.37)

$$\begin{aligned}
&\frac{d}{dt}\|xU(-t)v(t)\|_{L^2}^2 \\
&= 2t^{n(p-1)/2-2}[-t^2\,\mathrm{Im}\,(2i\frac{\partial v}{\partial t}(t) - 2f(v(t)), f(v(t))) - \frac{n}{p+1}t\,\|v(t)\|_{L^{p+1}}^{p+1}] \\
&= -\frac{4}{p+1}t^{\frac{n}{2}(p-1)}\frac{d}{dt}\|v(t)\|_{L^{p+1}}^{p+1} - \frac{2n(p-1)}{p+1}t^{\frac{n}{2}(p-1)-1}\|v(t)\|_{L^{p+1}}^{p+1}.
\end{aligned} \tag{3.38}$$

Integrating (3.38) in t, we obtain by integration by parts

$$\|xU(-t)v(t)\|_{L^2}^2 + \frac{4}{p+1}t^{n(p-1)/2}\|v(t)\|_{L^{p+1}}^{p+1} = \|xv_0\|_{L^2}^2, \quad t \in I_1. \tag{3.39}$$

The above calculation is rather formal, but it can be justified by the regularizing technique of Ginibre and Velo [6, § 3]. By Lemma 2.1 (2) we can rewrite (3.39) as follows:

$$\| xv(t) \|_{L^2}^2 + t \operatorname{Im}(xv(t), \nabla v(t)) + t^2 \| \nabla v(t) \|_{L^2}^2 + \frac{4}{p+1} t^{n(p-1)/2} \| v(t) \|_{L^{p+1}}^{p+1} = \| xv_0 \|_{L^2}^2, \qquad t \in I_1. \tag{3.40}$$

By (3.31) and (3.40) we have

$$\| xv(t) \|_{L^2} \leqq C(1 + \| v_0 \|_\Sigma), \qquad t \in I_1, \tag{3.41}$$

where $C = C(\rho, T(\rho), p, n)$. From (3.41) and the fact that $v(t) \in C(I_1; H^1)$ it follows that $v(t)$ is weakly continuous from I_1 to Σ. (3.40), the strong continuity in H^1 of $v(t)$ and the weak continuity in Σ of $v(t)$ imply that $v(t) \in C(I_1; \Sigma)$. Q. E. D.

Now Theorem 1.1 is an immediate consequence of Proposition 2.7 and Lemma 3.4.

Proof of Theorem 1.1. — In (3.8) we choose $t_0 = 0$ and $v_0 = \bar{\hat{u}}_+$ or $v_0 = \bar{\hat{u}}_-$. By Proposition 2.7, Lemma 3.4 and (3.2-4) we obtain Theorem 1.1. Q. E. D.

In order to show Theorem 1.2, we have only to prove the following lemma.

LEMMA 3.5. — Assume that $\gamma(n) < p < \alpha(n)$. By (3.8+) and (3.8−) we denote (3.8) with $t_0 = 1$ and (3.8) with $t_0 = -1$, respectively. For any $v_0 \in \Sigma$, (3.8±) have unique solutions $v_+(t) \in C([0, 1]; \Sigma)$ and $v_-(t) \in C([-1,0]; \Sigma)$, respectively.

Proof. — The proof of Lemma 3.5 is almost the same as that of Lemma 3.4. We consider only (3.8+), since we can treat (3.8−) in the same way.

By Proposition 3.1 and the regularizing technique of Ginibre and Velo [5] [6] we have a unique solution $v_+(t)$ of (3.8+) such that

$$v_+(t) \in C((0, 1]; \Sigma), \tag{3.42}$$

$$\| v_+(t) \|_{L^2} = \| v_0 \|_{L^2}, \qquad t \in (0, 1], \tag{3.43}$$

$$\| \nabla v_+(t) \|_{L^2}^2 + \frac{4}{p+1} t^{\frac{n}{2}(p-1)-2} \| v_+(t) \|_{L^{p+1}}^{p+1} - \frac{8 - 2n(p-1)}{p+1} \int_t^1 \tau^{\frac{n}{2}(p-1)-3} \| v_+(\tau) \|_{L^{p+1}}^{p+1} d\tau = \| \nabla v_0 \|_{L^2}^2 + \frac{4}{p+1} \| v_0 \|_{L^{p+1}}^{p+1}, \qquad t \in (0, 1], \tag{3.44}$$

$$\| xU(-t)v_+(t) \|_{L^2}^2 + \frac{4}{p+1} t^{n(p-1)/2} \| v_+(t) \|_{L^{p+1}}^{p+1} = \| xU(-1)v_0 \|_{L^2}^2 + \frac{4}{p+1} \| v_0 \|_{L^{p+1}}^{p+1}, \qquad t \in (0, 1]. \tag{3.45}$$

In addition, if $\gamma(n) < p < 1 + \frac{4}{n}$, by multiplying (2.17) by $t^{2-n(p-1)/2}\partial\bar{v}/\partial t$ and taking the real part we have

$$t^{2-\frac{n}{2}(p-1)} \| \nabla v_+(t) \|_{L^2}^2 + \frac{4}{p+1} \| v_+(t) \|_{L^{p+1}}^{p+1}$$
$$\leqq \| \nabla v_0 \|_{L^2}^2 + \frac{4}{p+1} \| v_0 \|_{L^{p+1}}^{p+1}, \qquad t \in (0, 1]. \quad (3.46)$$

(3.44)-(3.46) imply the following *a priori* bounds:

$$\| v_+(t) \|_{L^{p+1}} \leqq C, \quad (3.47)$$

$$\| \nabla v_+(t) \|_{L^2} \leqq C, \quad \text{if} \quad 1 + \frac{4}{n} \leqq p < \alpha(n), \quad (3.48)$$

$$t^{1-\frac{n}{4}(p-1)} \| \nabla v_+(t) \|_{L^2} \leqq C, \quad \text{if} \quad \gamma(n) < p < 1 + \frac{4}{n}, \quad (3.49)$$

$$\| x v_+(t) \|_{L^2} \leqq C, \quad (3.50)$$

for $t \in (0, 1]$, where $C = C(\| v_0 \|_\Sigma, p, n)$. If $1 + \frac{4}{n} \leqq p < \alpha(n)$, (3.43-3.45), (3.48), (3.50) and Proposition 3.1 enable us to extend the solution $v_+(t)$ of (3.8+) uniquely from $(0, 1]$ to $[0, 1]$. Therefore, the proof for

$$1 + \frac{4}{n} \leqq p < \alpha(n)$$

is complete. For $\gamma(n) < p < 1 + \frac{4}{n}$ we have by (3.47) and the fact that for $\gamma(n) < p < 1 + \frac{4}{n}$, $\frac{n}{2}(p-1) - 2 - \frac{n}{2} + \frac{n}{p+1} > -1$

$$\| U(t-s) v_+(s) \|_{L^{p+1}}$$
$$\leqq C + C \int_s^1 \tau^{\frac{n}{2}(p-1)-2} | t - \tau |^{-\frac{n}{2}+\frac{1}{p+1}} d\tau \leqq K, \qquad 0 < t, \quad s \leqq 1, \quad (3.51)$$

where $K = K(\| v_0 \|_\Sigma, p, n)$. By Proposition 3.1 and (3.51) we can choose $T(K) > 0$ depending only on K, p and n such that by Proposition 3.1 we can construct the unique solution $v_+(t)$ in $C([0, T(K)]; L^{p+1})$ of (3.8) with the initial time and the initial datum replaced by T(K) and $v_+(T(K))$. Therefore, considering the regularized equation on $[0, T(K)]$ and using the same argument as in the proof of Lemma 3.4; we can prove Lemma 3.5 for $\gamma(n) < p < 1 + \frac{4}{n}$. Q. E. D.

Now Theorem 1.2 is an immediate consequence of Lemma 3.5.

Proof of Theorem 1.2. — For any $u_0 \in \Sigma$ we have a unique global weak solution $u(t)$ in $C(\mathbb{R}; \Sigma)$ of (1.1)-(1.2) by Proposition 2.7. We put $v(t) = (J^{-1}u)(t)$. If we choose $v_0 = v(+1)$ for (3.8+) and $v_0 = v(-1)$ for (3.8−) in Lemma 3.5, then Lemma 3.5 and (3.2-3.4) imply Theorem 1.2.

Q. E. D.

Concluding Remarks. — (1) (3.33) and (3.44) correspond to the pseudo-conformal conservation law of the original equation (1.1), and (3.39) and (3.45) correspond to the energy conservation law of the original equation (1.1).

(2) In [*25*] it is shown that if $1 + \frac{2}{n} < p < \alpha(n)$, then all solutions $u(t)$ of (1.1)-(1.2) with $u_0 \in \Sigma$ have scattering states satisfying (1.3). On the other hand, it is already known that if $1 < p \leqq 1 + \frac{2}{n}$, the non-trivial solutions $u(t)$ of (1.1)-(1.2) with $u_0 \in \Sigma$ do not have any scattering states satisfying (1.3) (see, e. g., [*1*] and [*21*]). Accordingly, the following natural question arises: Can we construct the scattering theory for $1 + \frac{2}{n} < p \leqq \gamma(n)$? It is an open problem. But, for example, if $n = 1$ and $3 < p \leqq \gamma(1)$, we easily see that the scattering operator can be constructed as a mapping from Σ into L^2, that is, for any $u_- \in \Sigma$ there exist a weak solution $u(t) \in C(\mathbb{R}; \Sigma)$ of (1.1) and a $u_+ \in L^2$ such that

$$\| u_- - U(-t)u(t) \|_\Sigma \to 0 \quad (t \to -\infty), \tag{3.52}$$

$$\| u_+ - U(-t)u(t) \|_{L^2} \to 0 \quad (t \to +\infty). \tag{3.53}$$

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