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E. B. DAVIES

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Criteria for ultracontractivity

by

E. B. DAVIES

Department of Mathematics, King's College, Strand,
London WC2R 2LS, G. B.

ABSTRACT. — In an earlier joint paper with B. Simon, we defined a notion of intrinsic ultracontractivity for Schrödinger operators, and obtained a lower bound on the ground state eigenfunction which implies it. In this paper we present a method of verifying this condition which applies to a variety of particular cases. We also prove another generalization of the uncertainty principle lemma which has been so important in the study of ultracontractivity and other spectral properties of Schrödinger operators.

RÉSUMÉ. — Dans un article précédent en commun avec B. Simon, on a défini la notion d'ultracontractivité intrinsèque pour les opérateurs de Schrödinger, et on a obtenu une borne inférieure sur la fonction propre de l'état fondamental qui entraîne cette propriété. Dans cet article, on présente une méthode pour vérifier cette condition qui s'applique à un ensemble varié de cas particuliers. On prouve aussi une autre généralisation du lemme exprimant le principe d'incertitude qui a été important dans l'étude de l'ultracontractivité et des autres propriétés spectrales des opérateurs de Schrödinger.

§ 1. INTRODUCTION

In an earlier paper [*6*], B. Simon and the author introduced a notion of ultracontractivity which makes sense for a variety of second order elliptic operators. In particular we considered the case of a Schrödinger operator

$$H = -\Delta + V$$

on $L^2(\mathbb{R}^N)$, where V is a non-negative potential in L^1_{loc} and H is assumed to possess a ground state eigenvalue E with corresponding eigenfunction ϕ normalised by $\phi > 0$ and $\| \phi \|_2 = 1$. From this operator one may construct the intrinsic semigroup $e^{-\tilde{H}t}$ on $L^2(\mathbb{R}^N, \phi(x)^2 dx)$, where $\tilde{H} = \tilde{H}^* \geq 0$ is defined by

$$\tilde{H} = U^*(H - E)U$$

and the unitary operator U from $L^2(\mathbb{R}^N, \phi^2 dx)$ to $L^2(\mathbb{R}^N, dx)$ is defined by

$$Uf = \phi . f .$$

It is well-known that $e^{-\tilde{H}t}$ is contractive on $L^p(\mathbb{R}^N, \phi^2 dx)$ for all $1 \leq p \leq \infty$, and we say that H is intrinsically ultracontractive if $e^{-\tilde{H}t}$ maps L^2 into L^∞ for all $t > 0$. It was shown in [*6*, Th. 5.2] that this property holds if

$$- \log \phi \leq \delta H + g(\delta) \tag{1.1}$$

in the sense of quadratic forms on $L^2(\mathbb{R}^N, dx)$, for all $0 < \delta < 1$, where $g(\delta)$ does not increase too rapidly as $\delta \downarrow 0$. In this paper we shall specify for definiteness that g should satisfy the bound

$$g(\delta) \leq a + b\delta^{-M} \tag{1.2}$$

for certain positive a, b, M.

Although the condition (1.1) is of a very general nature, its validity was only confirmed in [*6*] for comparatively few cases. In particular if $V(x) = c\,|\,x\,|^\alpha$ where $c > 0$ and $\alpha > 0$ then by [*6*, Th. 6.1], (1.1) holds if and only if $\alpha > 2$. In this paper we present a new treatment of some of the examples of [*6*]. We also prove uniform intrinsic ultracontractivity for an example which necessitates the development of a generalized uncertainty principle. Because of its independent interest this generalization is given separately in Section 4.

We remark that (1.1) is equivalent to the construction of a function W such that

$$\phi \geq e^{-W}, \tag{1.3}$$

$$W \leq \delta H + g(\delta). \tag{1.4}$$

In view of the recent discoveries by Agmon and coworkers [*1*] [*2*] [*3*] of explicit upper and lower bounds to ϕ which are asymptotically nearly equal as $|\,x\,| \to \infty$, it seems natural to consider (1.3) first and then to check whether or not (1.4) holds. Our first observation is that it is frequently easier to choose W so that (1.4) is only just true, and then to check (1.3) by using standard subharmonic comparison inequalities. The precise version we shall need is as follows.

LEMMA 1. — Suppose that $W : \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies

(*a*) $W(x) \to +\infty$ as $|\,x\,| \to \infty$

(*b*) $|\,\nabla W\,|^2 - \Delta W \geq (V - E)_+$

in the distribution sense, outside some compact set K,

(c) $e^{-W} \le \phi$

in the set K. Then

$$e^{-W(x)} \le \phi(x)$$

for all $x \in \mathbb{R}^N$.

Proof. — Putting $\Omega = \mathbb{R}^N \backslash K$ we see that $\psi = e^{-W}$ satisfies $\psi \le \phi$ on $\partial\Omega \cup \{ \infty \}$ and $\Delta\psi = X\psi$ on Ω where

$$|\nabla W|^2 - \Delta W = X \ge (V - E)_+ .$$

Subharmonic comparison now implies that $\psi \le \phi$ on Ω.

As a first application and for the sake of completeness we restate Theorem 6.3 of [*6*].

LEMMA 2. — If there exist $c_i > 0$, $b_i \in \mathbb{R}$ and $a_i > 2$ such that

$$a_2 + 2 < 2a_1 \le 2a_2$$

and

$$c_1 |x|^{a_1} + b_1 \le V(x) \le c_2 |x|^{a_2} + b_2 \tag{1.5}$$

for all $x \in \mathbb{R}^N$, then (1.1) and (1.2) are valid.

Proof. — If we choose

$$W(x) = k |x|^{a_1 - \varepsilon}$$

where $\varepsilon > 0$, then (1.4) holds, and Lemma 1 is applicable with

$$K = \{ x : |x| \le R \}$$

provided ε is small enough, and R, k are large enough.

Lemma 2 suggests that the rate of divergence of V(x) as $|x| \to \infty$ along rays through the origin cannot vary too much between different rays. This is not correct.

LEMMA 3. — Let $H = -\Delta + V$ on $L^2(\mathbb{R}^2)$ where

$$V(r, \theta) = A(\theta) r^{B(\theta)}$$

and A, B are strictly positive periodic C^∞ functions on $[0, 2\pi]$. Then H is intrinsically ultracontractive if there exists $\beta > 2$ such that $B(\theta) \ge \beta$ for all $\theta \in [0, 2\pi]$.

Proof. — We put

$$W = c(V^\lambda + 1)$$

where $c > 0$ and $0 < \lambda < 1$. Then if $\varepsilon > 0$ and $\delta = c\varepsilon$ we have

$$\begin{aligned} W &\le c(\varepsilon V + c_\lambda \varepsilon^{-\lambda/(1-\lambda)} + 1) \\ &\le \delta V + c'_\lambda \delta^{-\lambda/(1-\lambda)} + c \\ &\le \delta H + c'_\lambda \delta^{-\lambda/(1-\lambda)} + c \end{aligned}$$

so (1.4) holds. To prove (1.3) we verify the conditions of Lemma 1 when K is $\{ x : |x| \leq 1 \}$.

Condition (*a*) is elementary. Also

$$\nabla W = c\lambda V^{\lambda-1}\nabla V$$

and

$$\Delta W = c\lambda V^{\lambda-1}\Delta V - c\lambda(1-\lambda)V^{\lambda-2}|\nabla V|^2$$

so

$$\begin{aligned} |\nabla W|^2 - \Delta W &\geq c^2\lambda^2 V^{2\lambda-2}|\nabla V|^2 - c\lambda V^{\lambda-1}\Delta V \\ &\geq c^2\lambda^2 V^{2\lambda-2}\left(\frac{\partial V}{\partial r}\right)^2 - c\lambda V^{\lambda-1}\Delta V. \end{aligned}$$

Now

$$\frac{\partial V}{\partial r} = ABr^{B-1}$$

and

$$\begin{aligned} \Delta V &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} \\ &= AB^2 r^{B-2} + r^{-2}(A'' r^B + 2A'B' \log r . r^B + AB'' \log r . r^B + A(B')^2(\log r)^2 . r^B) \end{aligned}$$

so

$$|\Delta V| \leq c_1 r^{B-1}$$

if $r \geq 1$. For such r we conclude that

$$\begin{aligned} |\nabla W|^2 - \Delta W &\geq c^2\lambda^2 A^{2\lambda-2} r^{(2\lambda-2)B} A^2 B^2 r^{2B-2} - c\lambda A^{\lambda-1} r^{(\lambda-1)B} c_1 r^{B-1} \\ &\geq c^2 c_2 r^{2\lambda B-2} - c c_3 r^{\lambda B-1} \end{aligned}$$

where $c_2 > 0$. If we put

$$\lambda = \frac{1}{\beta} + \frac{1}{2}$$

then $0 < \lambda < 1$ and

$$\lambda \geq \frac{1}{B} + \frac{1}{2}$$

so

$$2\lambda B \geq B + 2$$

and

$$|\nabla W|^2 - \Delta W \geq V$$

for all $r \geq 1$, provided c is large enough. Since ϕ has a strictly positive lower bound if $r \leq 1$ and $W \geq c$, condition (*c*) also holds if c is large enough.

In order to prove contractivity theorems for multiple well Schrödinger operators, it is useful to have a localized version of Lemma 1. In the following lemma one chooses K_i to be neighbourhoods of the various minima of V, and chooses W_i to have only one minimum and that one inside K_i.

LEMMA 4. — Suppose $W_i : \mathbb{R}^N \to \mathbb{R}$ are continuous for $1 \leq i \leq M$ and that they satisfy

(a) $W_i(x) \to +\infty$ as $|x| \to \infty$,
(b) $|\nabla W_i|^2 - \Delta W_i \geq (V - E)_+$ outside some compact subset K_i,
(c) $e^{-W_i} \leq \phi$ inside K_i,
(d) $\min_{1 \leq i \leq M} (W_i) \leq \delta H + g(\delta)$

for all $0 < \delta < 1$, where g satisfies (1.2). Then $-\log \phi \leq \delta H + g(\delta)$ so H is intrinsically ultracontractive.

Proof. — Lemma 1 implies that $e^{-W_i} \leq \phi$, so if

$$W = \min_{1 \leq i \leq M} W_i$$

then $e^{-W} \leq \phi$ and the lemma follows from (d).

§ 2. THE SEMICLASSICAL LIMIT

In order to prove uniform contractivity results in the semiclassical limit a further extension is necessary. Let us suppose that

$$H_\lambda = -\Delta + V_\lambda$$

where $\lambda \uparrow +\infty$ and

$$V_\lambda(x) = \lambda^2 V(x/\lambda).$$

We assume that V is non-negative and continuous and vanishes at x_i, $1 \leq i \leq M$. We suppose that H_λ has ground state ϕ_λ with corresponding eigenvalue $E_\lambda > 0$. Then uniform intrinsic *hypercontractivity* of H_λ as $\lambda \uparrow +\infty$ follows from the validity of the estimate

$$-\log \phi_\lambda \leq \delta H_\lambda + g(\delta)$$

for *some* $\delta > 0$ and all large enough λ. See Theorem 6.5 of [*6*] for an earlier treatment of this problem.

THEOREM 5. — Let $W_i : \mathbb{R}^N \to \mathbb{R}$ be continuous for $1 \leq i \leq M$, and suppose that the following conditions hold for large enough $\lambda > 0$.

(a) $W_i(x) \to +\infty$ as $|x| \to \infty$.
(b) There exists $a_i > 0$ such that

$$|\nabla W_i(y)|^2 - \lambda^{-2} \Delta W_i(y) \geq V(y)$$

provided $|\lambda y - \lambda x_i| \geq a_i$.
(c) We have

$$e^{-\lambda^2 W_i(x/\lambda)} \leq e^{c_i} \phi_\lambda(x)$$

provided $|x - \lambda x_i| \leq a_i$.

(*d*) There exists $\delta > 0$ such that

$$\min_{1 \leq i \leq M} W_i(y) \leq \delta V(y)$$

for all $y \in \mathbb{R}^N$.

Then we have the form bound

$$-\log \phi_\lambda \leq \delta H_\lambda + \max_{1 \leq i \leq M} c_i .$$

Proof. — This is an immediate consequence of Lemma 4 if we put

$$W_{i,\lambda}(x) = \lambda^2 W_i(x/\lambda) + c_i ,$$
$$K_{i,\lambda} = \{ x : | x - \lambda x_i | \leq a_i \} .$$

The above theorem can be used to give a new proof of Theorem 6.5 of [*6*], which is more elementary in that it makes no use of functional integration. To economise on notation we only consider one typical case.

THEOREM 6. — If

$$H_\lambda = \frac{d^2}{dx^2} + x^2(1 - x/\lambda)^2$$

on $L^2(\mathbb{R})$ then the conditions of Theorem 5 are satisfied with $x_1 = 0$, $x_2 = 1$, $a_1 = a_2 = 1$ if we put

$$W_1(x) = b(x^4 + x^2)$$

for suitable $b > 0$, and

$$W_2(x) = W_1(1 - x) .$$

Proof. — It is trivial that conditions (*a*) and (*d*) hold for any choice of $b > 0$, provided δ is large enough. By symmetry we need only prove (*b*) and (*c*) for $i = 1$. Condition (*b*) states that $| x | \geq \lambda^{-1}$ implies

$$b^2(4x^3 + 2x)^2 - \lambda^{-2} b(12x^2 + 2) \geq x^2(1 - x)^2 .$$

This follows from the bounds

$$b^2(4x^3 + 2x)^2 \geq x^2(1 - x)^2$$

which holds for all $x \in \mathbb{R}$ if b is large enough, and

$$b^2(4x^3 + 2x)^2 \geq \lambda^{-2} b(12x^2 + 2)$$

which holds for all $| x | \geq \lambda^{-1}$ provided b is large enough.

Condition (*c*) of Theorem 5 states that

$$e^{-b(x^2 + x^4/\lambda^2)} \leq e^{c_1} \phi_\lambda(x)$$

if $| x | \leq 1$ and λ is large enough. This is a consequence of the fact [*7*] that $\phi_\lambda(x)$ converges to $ke^{-x^2/2}$ uniformly on compact subsets of $\mathbb{R}$ as $\lambda \to \infty$, where $0 < k < \infty$.

We next turn to a completely new example which exhibits *uniform intrinsic ultracontractivity* in a semiclassical limit. We consider the Schrödinger operator

$$H_\lambda = -\Delta + \lambda X\chi_{X \geq \beta} \tag{2.1}$$

on $L^2(\mathbb{R}^N)$, as $\lambda \to +\infty$, where $\beta > 0$ and the function X satisfies

(H1) X is a non-negative C^∞ function on $\mathbb{R}^N$,
(H2) $X(x) = c\,|\,x\,|^\rho$ for some $c > 0$, $\rho > 2$ and $|\,x\,| \geq R$, where $0 < R < \infty$,
(H3) There exist β_1 with $0 < \beta_1 < \beta$ and $\alpha > 0$ such that $X(x) \geq \beta_1$ implies $|\,\nabla X(x)\,| \geq \alpha$.

An examination of the calculations below shows that all of the above conditions can be very much weakened without affecting the conclusions. In particular (H2) can be replaced by a condition such as (1.5).

It is evident that H converges in the strong resolvent sense to H_∞ as $\lambda \uparrow +\infty$, where H_∞ is minus the Dirichlet Laplacian of the bounded open set

$$\Omega = \{\, x \in \mathbb{R}^N : X(x) < \beta \,\}.$$

Since both H_λ and H_∞ are intrinsically ultracontractive, one might hope to be able to prove uniform intrinsic ultracontractivity of H_λ as $\lambda \uparrow +\infty$. For this example one cannot hope to deduce the uniform bound

$$-\log \phi_\lambda \leq \delta H_\lambda + g(\delta) \tag{2.2}$$

for all $0 < \delta < 1$ and all large enough λ from a bound

$$-\log \phi_\lambda \leq \delta\lambda X\chi_{X > \beta} + g(\delta)$$

because such a bound would imply

$$\phi_\lambda(x) \geq e^{-g(\delta)}$$

for all $x \in \Omega$, which contradicts the expectation that

$$\lim_{\lambda \to +\infty} \phi_\lambda(x) = 0$$

for all $x \in \partial\Omega$. For this example, therefore, we must not neglect the kinetic energy term in (2.2).

Although the bounds of the following theorem are much weaker than they need to be, they suffice for our purposes, and are simpler to compute with than more accurate bounds.

THEOREM 7. — If X satisfies (H1-3) then there exists $0 < \tau_6 < 1$ such that if λ is large enough then the operator H_λ of (2.1) satisfies

$$\langle H_\lambda f, f \rangle \geq \tau_6 \langle V_\lambda f, f \rangle$$

for all $f \in C_c^\infty(\mathbb{R}^N)$, where

$$V_\lambda(x) = \begin{cases} \lambda X(x) & \text{if } X(x) \geq \beta, \\ \dfrac{\lambda X(x)}{1 + \lambda(\beta - X(x))} & \text{if } \beta_1 \leq X(x) \leq \beta, \\ 0 & \text{otherwise} \end{cases}$$

Proof. — We first observe that (H1-3) imply that Ω is a bounded region with C^∞ boundary, so the conditions of Section 3 are all satisfied. Without serious loss of generality we assume that β_1 is close enough to β so that

$$\{ x : \beta_1 < X(x) < \beta \} \subseteq \Omega' . \tag{2.3}$$

The lower bound on $|\nabla X|$ implies that if $\beta_1 < X(x) < \beta$ then

$$\alpha\sigma(x) \leq \beta - X(x) \leq \beta .$$

If $f \in C_c^\infty(\mathbb{R}^N)$ then Theorem 12 implies

$$\begin{aligned} \langle Hf, f \rangle &\geq \frac{1}{2}\langle Hf, f \rangle + \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega} \lambda X |f|^2 dx \\ &\geq \int_{\Omega'} \frac{1}{2}\frac{\tau_3\lambda\beta |f|^2}{1 + \lambda\beta\sigma^2} dx + \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega} \lambda X |f|^2 dx \\ &\geq \int_{\Omega'} \frac{1}{2}\frac{\tau_3\lambda X |f|^2}{1 + \lambda\beta(\beta - X)^2/\alpha^2} dx + \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega} \lambda X |f|^2 dx \\ &\geq \int_{\Omega'} \frac{\tau_5\lambda X |f|^2}{1 + \lambda(\beta - X)} dx + \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega} \lambda X |f|^2 dx \end{aligned}$$

for some $\tau_5 > 0$. The theorem now follows by using (2.3).

The above quadratic form inequality is the crucial result for our main result concerning this example.

THEOREM 8. — If X satisfies (H1-3) then the operator H_λ defined in (2.1) is uniformly intrinsically ultracontractive as $\lambda \uparrow +\infty$.

Proof. — If we put

$$W_\lambda = c(V_\lambda^\theta + 1)$$

where $0 < \theta < 1$ and $c > 0$ are to be chosen later and V_λ is as in Theorem 7, then

$$\begin{aligned} W_\lambda &\leq c(\varepsilon V_\lambda + c_\theta \varepsilon^{-\theta/(1-\theta)} + 1) \\ &= \delta\tau_6 V_\lambda + c'_\theta \delta^{-\theta/(1-\theta)} + c \\ &\leq \delta H_\lambda + c'_\theta \delta^{-\theta/(1-\theta)} + c \end{aligned}$$

where $c\varepsilon = \tau_6\delta$. Thus (1.2) and (1.4) are satisfied uniformly as $\lambda \uparrow +\infty$.

We prove that (1.3) also holds uniformly by verifying the conditions of Lemma 1 for the compact set

$$K = \{ x : X(x) \leq \beta_1 \}.$$

We first note that condition (*a*) of Lemma 1 is trivial. To check condition (*b*) for points x such that

$$\beta_1 < X(x) < \beta$$

we note that at such points

$$W = c(f(X)^\theta + 1)$$

where

$$f(t) = \frac{\lambda t}{1 + \lambda(\beta - t)}$$

We see that f is an increasing convex function on $[\beta_1, \beta]$ whose derivatives are

$$f'(t) = \frac{\lambda(1 + \lambda\beta)}{(1 + \lambda(\beta - t))^2}$$

and

$$f''(t) = \frac{2\lambda^2(1 + \lambda\beta)}{(1 + \lambda(\beta - t))^3}.$$

Now using (H3) we see that

$$\begin{aligned} |\nabla W|^2 - \Delta W &= | c\theta f(X)^{\theta-1} f'(X) \nabla X |^2 - c\theta f(X)^{\theta-1} f'(X) \Delta X \\ &\quad - (c\theta(\theta - 1) f(X)^{\theta-2}(f'(X))^2 + c\theta f(X)^{\theta-1} f''(X)) | \nabla X |^2 \\ &\geq c^2\theta^2\alpha^2 f^{2\theta-2}(f')^2 - c\theta\alpha_1 f^{\theta-1} f' - c\theta\alpha_2 f^{\theta-1} f'' \end{aligned}$$

where $\alpha > 0$ and $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. Now

$$0 \leq \frac{f^{\theta-1} f'}{f^{2\theta-2} f'^2} = f^{-\theta} \frac{f}{f'} = f^{-\theta} \frac{t(1 + \lambda(\beta - t))}{1 + \lambda\beta} \leq f(\beta_1)^{-\theta}\beta = \alpha_3 < \infty$$

if $t \in [\beta_1, \beta]$, and

$$0 \leq \frac{f^{\theta-1} f''}{f^{2\theta-2}(f')^2} = f^{-\theta} \frac{f f''}{(f')^2} = f^{-\theta} \frac{2\lambda^3 t(1 + \lambda\beta)}{\lambda^2(1 + \lambda\beta)} \leq f(\beta_1)^{-\theta}\beta = \alpha_4 < \infty$$

if $t \in [\beta_1, \beta]$. Therefore

$$| \nabla W |^2 - \Delta W \geq f^{2\theta-2}(f')^2 [c^2\theta^2\alpha^2 - c\theta\alpha_1\alpha_3 - c\theta\alpha_2\alpha_4] \geq 0$$

provided $c > 0$ is large enough.

To check condition (*b*) of Lemma 1 for points x such that $\beta < X(x)$ we note that at such points

$$W = c(\lambda^\theta X^\theta + 1)$$

so

$$|\nabla W|^2 - \Delta W - (\lambda X \chi_{X \geq \beta} - E_\lambda)_+$$
$$\geq |c\lambda^\theta X^{\theta-1} \nabla X|^2 - c\lambda^\theta X^{\theta-1} \Delta X + c\theta(1-\theta)\lambda^\theta X^{\theta-2} |\nabla X|^2 - \lambda X. \quad (2.4)$$

We now choose $\theta = \frac{1}{2} + \frac{1}{\rho}$ where $\rho > 2$ is the constant in (H2). Then (2.4) holds uniformly as $\lambda \uparrow +\infty$ provided

$$cX^{\theta-1}\Delta X \leq c^2 X^{2\theta-2} |\nabla X|^2$$

and

$$X \leq c^2 X^{2\theta-2} |\nabla X|^2$$

for some $c > 0$ and all $X(x) > \beta$. This is proved for $X(x) > \beta$ and $|x| \leq R$ by using (H3) and compactness, while it is proved for $X(x) > \beta$ and $|x| > R$ by using (H2).

We next observe that there is no problem concerning a singularity in condition (*b*) of Lemma 1 as one passes through the surface $X(x) = \beta$, because W is actually defined as the minimum of two functions which are smooth around this surface, so ∇W can only have a jump discontinuity across the surface and ΔW can only have an infinite *negative* singularity, which does not affect condition (*b*).

We finally consider condition (*c*) of Lemma 1. If $x \in K$ then $V_\lambda \geq 0$ so $W_\lambda \geq c$. Thus we need only choose $c > 0$ so that

$$e^{-c} \leq \phi_\lambda(x)$$

for all $x \in K$ and all large enough λ. This can be done since ϕ_λ converges uniformly on K to the ground state ϕ_∞ of the Dirichlet Laplacian of Ω, which is strictly positive and continuous on K.

§ 3. A GENERALIZED UNCERTAINTY PRINCIPLE

In this section we present a new quadratic form inequality for operators on $L^2(\mathbb{R}^N)$, which is closely related to the uncertainty principle and to inequalities in [*4*] [*5*]. For the sake of precision we comment that all the form inequalities below are proved for test functions in $C_c^\infty(\mathbb{R}^N)$, which is a form core for all the operators we consider.

We start by studying an operator of the form

$$H = -\Delta + \beta\chi_{\{x : x \notin \Omega\}}$$

where $\beta > 0$ and Ω is an open subset of $\mathbb{R}^N$ which is uniformly locally Lipschitz in the following sense. If

$$\sigma(x) = \min\{ |x - y| : y \notin \Omega \}$$

then there exists $\tau_1 > 0$ and a finite covering of

$$\Omega' = \{ x \in \Omega : 0 < \sigma(x) < \tau_1 \}$$

by open sets Ω_i, $1 \leq i \leq M$. We assume that after an isometric motion each Ω_i has the form

$$\Omega_i = \{ (y, s) : y \in V_i \quad \text{and} \quad 0 < s < \omega_i(y) \}$$

where V_i is an open subset of $\mathbb{R}^{N-1}$, and ω_i is a continuous function on V_i with values in $(0, \infty)$, and

$$\partial\Omega_i \cap \partial\Omega = \{ (y, s) : y \in V_i \quad \text{and} \quad s = \omega_i(y) \} .$$

If we define σ_i on $V_i \times \mathbb{R}$ by

$$\sigma_i(y, s) = \omega_i(y) - s$$

then our Lipschitz hypothesis is made in the form

$$\sigma(x) \leq \sigma_i(x) \leq \tau_2 \sigma(x)$$

for all $x \in \Omega_i$, where $\tau_2 < \infty$ and τ_2 is independent of i. We finally assume that if we define

$$\Omega_i^\gamma = \{ (y, s) : y \in V_i \quad \text{and} \quad 0 < s < \omega_i(y) + \gamma \}$$

then

$$\Omega_i^\gamma \cap \Omega = \Omega_i \cap \Omega$$

for some $\gamma > 0$ and all $1 \leq i \leq M$.

In spite of the complicated form of the above conditions, it is easy to check their validity in many cases, for example if Ω has a compact C^∞ boundary.

LEMMA 9. — If we define the form Q_i by

$$Q_i(f) = \int_{V_i \times \mathbb{R}} \left(\left| \frac{\partial f}{\partial s} \right|^2 + \beta \chi_{\Omega_i^\gamma \setminus \Omega_i} | f |^2 \right) ds d^{N-1} y$$

then

$$\langle Hf, f \rangle \geq M^{-1} \sum_{i=1}^{M} Q_i(f)$$

for all $f \in C_c^\infty(\mathbb{R}^N)$.

We omit the elementary proof. The point of our list of conditions on Ω is that it enables us to find a good lower bound on Q_i. We start with a computation in one dimension.

LEMMA 10. — If $\beta > 0$ and $\gamma > 0$ then

$$-\frac{d^2}{ds^2} + \beta \chi_{(0,\gamma)} \geq \beta^{1/2} \tanh(\gamma \beta^{1/2}) \delta_0$$

as forms on $C_c^\infty(\mathbb{R})$, where δ_0 is the Dirac delta function. In particular if $\gamma > \beta^{-1/2}$ then

$$-\frac{d^2}{ds^2} + \beta\chi_{(0,\gamma)} \geq \frac{3}{4}\beta^{1/2}\delta_0 .$$

Proof. — If

$$H = -\frac{d^2}{ds^2} + \beta\chi_{(0,\gamma)} - \alpha\delta_0$$

is regarded as a function of α, then the threshold for the existence of a negative eigenvalue occurs at

$$\alpha = \beta^{1/2}\tanh(\gamma\beta^{1/2})$$

the zero energy resonance eigenfunction for this value of α being

$$\phi(x) = \begin{cases} 1 & \text{if } -\infty < s \leq 0, \\ \dfrac{\cosh \beta^{1/2}(\gamma - x)}{\cosh \beta^{1/2}\gamma} & \text{if } 0 \leq s \leq \gamma, \\ 1/\cosh \beta^{1/2}\gamma & \text{if } \gamma \leq s < \infty . \end{cases}$$

Thus

$$-\frac{d^2}{ds^2} + \beta\chi_{(0,\gamma)} - \beta^{1/2}\tanh(\gamma\beta^{1/2})\delta_0 \geq 0$$

as stated.

LEMMA 11. — If $\lambda > 0$ and $f \in C_c^\infty(\mathbb{R})$ then

$$\int_0^\infty \left|\frac{df}{ds}\right|^2 ds + \lambda\,|f(0)|^2 \geq \int_0^\infty \frac{|f(s)|^2}{(\lambda^{-1} + 2s)^2}\,ds .$$

Proof. — We first reduce to the case where f is real, and put $\rho(s) = s + 1/2\lambda$. The identity

$$(\rho^{-1/2}f)' = -\frac{1}{2}\rho^{-3/2}f + \rho^{-1/2}f'$$

implies that

$$(f')^2 = \left(\rho^{1/2}(\rho^{-1/2}f)' + \frac{1}{2}\rho^{-1}f\right)^2$$

$$\geq \rho^{-1/2}f(\rho^{-1/2}f)' + \frac{f}{4\rho^2} .$$

Therefore

$$\int_0^\infty (f')^2 ds \geq \frac{1}{2}[\rho^{-1}f^2]_0^\infty + \int_0^\infty \frac{f^2}{4\rho^2}\,ds$$

$$= -\lambda(f(0))^2 + \int_0^\infty \frac{f^2}{4\rho^2}\,ds$$

as stated.

THEOREM 12. — If Ω satisfies the conditions of this section then there exists $\tau_3 > 0$ such that

$$\langle \mathrm{H}f, f \rangle \geq \int_{\Omega'} \frac{\tau_3 \beta \mid f \mid^2}{1 + \beta\sigma^2} dx$$

for all $\gamma > \beta^{-1/2}$ and all $f \in C_c^\infty(\mathbb{R}^N)$.

Proof. — For each $y \in V_i$ we apply the last two lemmas to the variable $(s - \omega_i(y))$ to obtain

$$\begin{aligned} Q_i(f) &\geq \int_{\Omega_i} \frac{\mid f \mid^2}{(4/3\beta^{1/2} + \sigma_i(y, s))^2} ds d^{N-1}y \\ &= \int_{\Omega_i} \frac{\beta \mid f \mid^2}{(4/3 + \beta^{1/2}\sigma_i)^2} dx \\ &\geq \int_{\Omega_i} \frac{\beta \mid f \mid^2}{(4/3 + \beta^{1/2}\tau_2\sigma)^2} dx \\ &\geq \int_{\Omega_i} \frac{M\tau_3\beta \mid f \mid^2}{1 + \beta\sigma^2} dx . \end{aligned}$$

Therefore

$$\begin{aligned} \langle \mathrm{H}f, f \rangle &\geq \sum_{i=1}^{M} \int_{\Omega_i} \frac{\tau_3\beta \mid f \mid^2}{1 + \beta\sigma^2} dx \\ &\geq \int_{\Omega'} \frac{\tau_3\beta \mid f \mid^2}{1 + \beta\sigma^2} dx . \end{aligned}$$

Note 13. — An attractive feature of this bound is that if we let $\beta \uparrow +\infty$ for fixed $\gamma > 0$, we get the bound

$$\langle -\Delta f, f \rangle \geq \int_{\Omega'} \frac{\tau_3 \mid f \mid^2}{\sigma^2} dx$$

for all $f \in C_c^\infty(\Omega)$, which was discovered in [*4*]. See also [*5*] for a generalization in another direction.

An alternative form of the above theorem is as follows.

THEOREM 14. — If Ω satisfies the conditions of this section then $\mathrm{H} \geq \tau_4 \mathrm{Y}$ on $L^2(\mathbb{R}^N)$ where $0 < \tau_4 < 1$ and

$$\mathrm{Y}(x) = \begin{cases} \beta & \text{if } x \notin \Omega , \\ \dfrac{\beta}{1 + \beta\sigma(x)^2} & \text{if } x \in \Omega' \\ 0 & \text{otherwise} . \end{cases}$$

Proof. — If $f \in C_c^\infty(\mathbb{R}^N)$ then

$$\langle Hf, f \rangle \geq \frac{1}{2}\langle Hf, f \rangle + \frac{\beta}{2}\int_{x \notin \Omega} |f|^2 dx$$
$$\geq \frac{1}{2}\int_{\Omega'} \frac{\tau_3 \beta |f|^2}{1 + \beta\sigma^2} dx + \frac{\beta}{2}\int_{x \notin \Omega} |f|^2 dx$$

so the result follows with $\tau_4 = \min(\tau_3/2, 1/2)$.

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