ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 42, nº 3 (1985), p. 283-309
<http://www.numdam.org/item?id=AIHPA_1985__42_3_283_0>

Ann. Inst. Henri Poincaré,
Vol. 42, n° 3, 1985, p. 283-309. *Physique théorique*

From power pure point to continuous spectrum in disordered systems

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ABSTRACT. — We prove that two classes of random Schrödinger operators exhibit a transition from pure point spectrum with power decaying eigenfunctions to purely continuous spectrum when varying the coupling constant. These classes are *i*) a Schrödinger operator with a random Krönig-Penney potential and an electric field and *ii*) a Jacobi matrice with a random potential of strength decaying as $n^{-\alpha}$, $\alpha = 1/2$.

RÉSUMÉ. — On démontre que deux classes d'opérateurs de Schrödinger aléatoires présentent une transition d'un spectre purement ponctuel avec fonctions propres décroissant suivant une puissance à un spectre purement continu quand on fait varier la constante de couplage. Ces classes sont *i*) un opérateur de Schrödinger avec un potentiel de Krönig-Penney et un champ électrique, et *ii*) une matrice de Jacobi avec un potentiel aléatoire décroissant en $n^{-\alpha}$, $\alpha = 1/2$.

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I. INTRODUCTION AND STATEMENT OF THE RESULTS

A. Introduction.

Disordered systems have been and are being extensively studied because of their relevance to a large variety of physical situations; for a review of the physics of this problem as well as the present mathematical status, see Ref. [*1*]-[*3*]. It has been proven recently that, as predicted by Mott and Twose [*4*], one-dimensional Schrödinger Hamiltonians with disorder possess a complete set of exponentially localized states, i. e. have only a pure point spectrum with exponentially decaying eigenfunctions [*5*]-[*10*]. In higher dimensions d, at least for $d \geqslant 3$ a transition from a pure point spectrum with exponentially decaying eigenfunctions to a purely absolutely continuous spectrum is expected to occur at a given energy when varying the coupling constant: this is the Anderson-Mott transition. A transition of this kind has been proven [*11*] for the Anderson model on a Bethe lattice, which corresponds to the case of large enough d. The present paper intends to describe and prove a transition from purely continuous spectrum to pure point spectrum with power decaying wave functions.

This new type of transition will be proven in two kinds of models. The first one consists of Jacobi matrices (one-dimensional tight-binding model) with a random potential whose strength decays at infinity as $n^{-\alpha}$. When $\alpha = 0$, we have a usual homogeneous random potential, an Anderson model, and as proven in Ref. [*6*] [*10*], the spectrum is almost surely pure point with exponentially decaying eigenfunctions. It is useful to realize that the method of Ref. [*10*] is up to now the most powerful one among the one-dimensional methods in the sense that it is the only one which does not require homogeneity of the potential: it can accomodate an additional potential or non homogeneity of the randomness from site to site and it is a finding of Ref. [*12*] that the method of Ref. [*10*] also yields that the spectrum is pure point for $0 < \alpha < 1/2$. In the present paper we will show, among other results, that when $\alpha = 1/2$ we have a transition from purely continuous to pure point spectrum with power decaying wave functions when varying the coupling constant or the energy.

The second system for which we will prove this new type of transition concerns a one-dimensional Schrödinger equation with an electric field F and a random Krönig-Penney potential. There we prove that for small field the spectrum is pure point, with eigenfunctions decaying as $|x|^{-\alpha(\mathrm{F})}$ at $+\infty$ where $\alpha(\mathrm{F}) \sim \mathrm{C}^{\mathrm{te}}\,\mathrm{F}^{-1}$ for small F, whereas for large enough field the spectrum is purely continuous. These results are somewhat surprising since Ref. [13] had studied the model where the potential was an arbitrary (disordered) potential, sufficiently smooth e. g. with two bounded derivatives (some square integrable singularities can also be accomodated [14]), instead of a Krönig-Penney potential: there was no localized states; actually the spectrum was purely absolutely continuous, and initial wave packets would be uniformly accelerated. On the other hand, the results that we have proven for a Krönig-Penney potential, exhibiting a transition as described above, had been anticipated in Ref. [15] where such a behaviour was predicted in the case of an electron in a white noise potential and an electric field and in Ref. [16] some transmission coefficient had been numerically computed in an approximate model with a stair potential replacing the electric field.

Our results have been annonced in Ref. [17]. In the same time a precise numerical study of the electric field case has been performed [18] which in particular illustrates clearly some of the aspects discussed here, and moreover discusses the behaviour of resonances.

B. Statement of our results.

The first model (Model I) that we consider, consists of Jacobi matrices acting on $l^2(\mathbb{Z})$ defined by

$$(\mathrm{H}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda a_n \mathrm{V}_n \psi(n) \tag{I.1}$$

where the $\{ \mathrm{V}_n \}_{n\in\mathbb{Z}}$ are independant identically distributed random variables with a probability density $r(\mathrm{V})$, always supposed to have zero mean, and $| a_n |$ behaves as $| n |^{-\alpha}$ for $| n |$ large, namely for some C_1, C_2

$$\mathrm{C}_1 | n |^{-\alpha} < | a_n | < \mathrm{C}_2 | n |^{-\alpha}, \qquad n \neq 0 . \tag{I.2}$$

The operators H are self-adjoint and admit as a core the subset of $l^2(\mathbb{Z})$ consisting of those ψ with a finite support. In the following we will suppose that

$$\| r \|_\infty < \infty \tag{I.3}$$

and that

$$p \equiv \operatorname{Sup} \{ z > 0 : \mathbb{E}(| \mathrm{V} |^z) < \infty \} > \emptyset \tag{I.4}$$

Concerning this model, our main results are summarized in Theorem I.1 below:

THEOREM I.1. — *The following properties of* H *hold for almost all sequences* $\{ \mathrm{V}_n \}_{n\in\mathbb{Z}}$:

i) if $0 \leqslant \alpha < 1/2$, *the spectrum of* H *is pure point and the corresponding eigenfunctions* $\psi_{\mathrm{E}}(n)$ *satisfy*

$$| \psi_{\mathrm{E}}(n) | \leqslant \mathrm{C(E)} \exp \{ - \mathrm{C}' | n |^{1-2\alpha} \} \tag{I.5a}$$

and

$$(\psi_E(n)^2 + \psi_E(n+1)^2)^{1/2} \geqslant \tilde{C}(E) \exp\{ -\tilde{C}' \mid n \mid^{1-2\alpha} \} \qquad (I.5b)$$

ii) if $\alpha = \frac{1}{2}$ *and* $\lambda > \lambda_1(r)$, *the spectrum of* H *is pure point and the corresponding eigenfunctions* $\psi_E(n)$ *satisfy*

$$\mid \psi_E(n) \mid \leqslant C(E) \mid n \mid^{3/2+\varepsilon - C_1\lambda^2} \qquad (I.6)$$

and if $p > 2$, *the eigenfunctions for* $E \in]-2, 2[$ *satisfy*

$$C'(E) \mid n \mid^{-a\lambda^2(4-E^2)^{-1}} \leqslant \psi_E^2(n) + \psi_E^2(n+1) \qquad (I.7)$$

iii) if $\alpha = \frac{1}{2}$, $p > 2$, κ *any compact of* $]-2, 2[$, *and* $\lambda < \lambda_2(r, \kappa)$, *the spectrum of* H *is purely continuous in* κ.

iv) if $\alpha > \frac{1}{2}$, $p > 2$, *the spectrum of* H *in* $[-2, 2]$ *is purely continuous.*

v) if $\alpha p > 1$, *the essential spectrum of* H *is* $[-2, 2]$.

vi) if $\alpha < 0$ *and* $\mid \tilde{r}(k) \mid \leqslant C \mid k \mid^{-\gamma}$ *for some* $C, \gamma > 0$, *the spectrum of* H *is pure point and the eigenfunctions* $\psi_E(n)$ *satisfy*

$$\mid \psi_E(n) \mid \leqslant C_\varepsilon(E) \mid n \mid^{-\mid n \mid \mid \alpha \mid \gamma(1-\varepsilon)/\gamma+1}$$

for all $\varepsilon > 0$.

We have also some arguments for the following conjecture

CONJECTURE I.2. — *When* $\alpha = \frac{1}{2}$, *for any* λ, *the spectrum of* H *is almost surely pure point in a neighbourhood of* -2 *and of* $+2$, *neighbourhood depending on* λ *and the corresponding eigenfunctions decay with a power law, power diverging as* $\mid E \mid \to 2 - 0$. *Hence there is also a transition from power localized to continuous spectrum at weak disorder when varying* $\mid E \mid$.

Remarks. — While we were writing the present complete version, S. Kotani has proven [*33*] that *iv*) can be strengthened. Using a martingale inegality different from the one we use in Section II, and a result of Carmona [*24*], he proves that: if $\alpha > 1/2, p > 2$ and the distribution is of compact support, the essential spectrum of H, $[-2, 2]$, is almost-surely absolutely continuous.

— (I.5*a*) is proved in [*12*].

— All results concerning pure point spectrum and upper bounds on the eigenfunctions depend only on the lower bound on $\mid a_n \mid$ in (I.2); they also extend readily to the case where $\mathbb{E}(V) \neq 0$.

— All results concerning continuous spectrum and lower bounds on the eigenfunctions depend only on the upper bound on $\mid a_n \mid$ in (I.2), and do not depend on hypothesis (I.3), and in fact do not require the existence of the density $r(.)$.

— The lower bounds should also hold in some cases when $\mathbb{E}(V) \neq 0$. Then instead of a variation of parameters about the $V = 0$ solutions, one would need a variation of parameters about the long range potential $a_n\mathbb{E}(V)$.

— It follows from general results of Ref. [*6*] that the diffusion constant vanishes in the regions where we prove that the spectrum is pure point, and that the participation ratio vanishes when the spectrum is purely continuous.

— If $p < 2$, it can happen that there is localization and pure point spectrum for some α, $\alpha > \frac{1}{2}$. For example for a Cauchy distribution ($p = 1$), we will prove pure point spectrum for all $\alpha < 1$ and for $\alpha = 1$ pure point spectrum for large coupling constant λ.

— Our proofs accomodate easily the cases where the variables V_n are not identically distributed.

The reasons for the results described above, a sketch of the arguments and of proofs will be given latter in the next subsection, but we turn first to the results concerning our second model.

Our second model (Model II) is a Schrödinger operator with an electric field and a random Krönig-Penney potential, acting on $L^2(\mathbb{R})$ and defined by

$$(H\psi)(x) = -\psi''(x) + \sum_{n\in\mathbb{Z}} V_n\delta(x-n)\psi(x) - Fx\psi(x) \tag{I.8}$$

where $F \geqslant 0$ and the $\{V_n\}_{n\in\mathbb{Z}}$ are independant identically distributed random variables with a density $r(V)$ and zero mean. The operators H can be defined as self-adjoint operators using form methods and admit as a form-core the subset C_0^∞ of $L^2(\mathbb{R})$. In the following we will suppose that

$$\| r \|_\infty < \infty \tag{I.9}$$

and that

$$p \equiv \operatorname{Sup}\{ z > 0 : \mathbb{E}(|V|^z) < \infty \} > 0 . \tag{I.10}$$

THEOREM I.3. — *The following properties of* H *hold for almost all sequences* $\{V_n\}_{n\in\mathbb{Z}}$:

i) if $F = 0$, *the spectrum of* H *is pure point and the eigenfunctions are exponentially decaying.*

ii) If $F < F_1$, *the spectrum of* H *is pure point and the eigenfunctions satisfy.*

$$|\psi_E(n)| \leqslant C(E)n^{-C'(E)/F} \quad \text{for} \quad n > 0 \tag{I.11}$$

and if $p > 2$, *the eigenfunctions satisfy also*

$$C'F^{-1/2}n^{-a/F} \leqslant \int_n^{n+1} |\psi(x)|^2 dx \quad \text{for} \quad n > 0 \tag{I.12}$$

If $|\tilde{r}(k)| \leqslant C|k|^{-\gamma}$ *for some* $C, \gamma > 0$, *they also satisfy*

$$|\psi_E(n)| \leqslant C(E)\exp\{-C'(E)F^{1/2}|n|^{3/2}\} \quad \text{for} \quad n < 0 \quad (I.13)$$

iii) if $p > 2$ *and* $F > F_2$, *the spectrum of* H *is purely continuous.*
iv) if $p > 2$, $F \neq 0$, *the essential spectrum of* H *is* $\mathbb{R}$.

Remarks. — All results concerning continuous spectrum and lower bounds on eigenfunctions do not depend on hypothesis (I.9) and in fact do not require the existence of the density $r(.)$.

— It follows from general results of Ref. [*6*] that the diffusion constant vanishes in the regions where we prove that the spectrum is pure point, and that the participation ratio vanishes when the spectrum is purely continuous.

— If $p < 2$, we may have point spectrum even for large field, and for example in the case of a Cauchy distribution ($p = 1$), the spectrum of H is almost surely pure point for all field F.

— Our proofs accomodate easily the cases where the variables V_n are not identically distributed.

— For $F = 0$, the localization length (inverse rate of exponential decay of the wave functions) diverges at the energies $E = k^2\pi^2$, k an integer.

C. Explanation of the results. Beginning of the proofs.

Let us first give a heuristic explanation of our results [*17*]. Let us start with our Model II (eq. I.8) and consider a solution ψ of the equation $H\psi = E\psi$. Since we can integrate the equation between the δ functions, we can get a relation equivalent to $H\psi = E\psi$ but involving ψ only at points $x = n$. One then gets the following recursion relation on the $\psi(n)$:

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \begin{pmatrix} \alpha_n + V_n\gamma_n & \beta_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}$$

where the α_n, β_n, γ_n are expressed in terms of Airy functions. The important point is that γ_n will be mainly a product of two Airy functions, each of them decaying as $n^{-1/4}$ at $+\infty$. Hence γ_n will behave at $+\infty$ as $n^{-1/2}$ and more precisely one finds for n large and modulo oscillations that

$$\gamma_n \sim F^{-1/2}n^{-1/2}\,. \qquad (I.15)$$

By studying α_n and β_n, one can show that the qualitative behaviour is the same as if equation (I.14) was replaced by

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \begin{pmatrix} V_n\gamma_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix}. \qquad (I.16)$$

This is the equation associated to an Anderson model (our Model I), at zero energy, with potential $n^{-1/2}V_n$ at site n and coupling constant $F^{-1/2}$.

We can understand why model II and Model I for $\alpha = 1/2$ have power localized states for F small, or respectively λ large: since wave functions represent standing waves, we can imagine the wave being set up by transmissions from a central peak. In the approximation of ignoring multiple reflections and approximating the randomness of potentials at distinct sites by independant reflections, we see that

$$\psi(n) \sim \prod_{j=1}^{n} (1 - r_j) \tag{I.17}$$

where r_j is the reflection probability at site j. For weak coupling, the Born approximation says that r is proportional to the square of the potential strength, so when $\mathbb{E}(V^2) < \infty$

$$\psi(n) \sim \prod_{j=1}^{n} (1 - C(Fn)^{-1}) \sim n^{-C/F}. \tag{I.18}$$

We have thus power localized states for F small enough and we turn to extended states when F becomes large and the wave function ceases to be square integrable. The above heuristic argument shows also that for $\alpha < 1/2$ our Model I will have always localized states, whereas it will have only extended ones in $[-2, 2]$ for $\alpha > 1/2$. We also see why different behaviour is to be expected if $\mathbb{E}(V^2) = \infty$.

The results on the lower bounds on the eigenfunctions and on the continuous spectrum will be proven in Section II. Actually it will be proven there a lower bound at $x = +\infty$ on the decay of *all* solutions of the equation $H\psi = E\psi$ for some range of parameters and for all E in some interval. When we have non square integrable lower bounds, it ensures that in the corresponding region of energy the spectrum is purely continuous.

The results on pure point spectrum and on upper bounds on the eigenfunctions will be proven in Section III. They will follow from the study of a correlation function. We will basically use the method developped in Ref. [*10*] which was used in Ref. [*12*] to prove the upper bound half of part *i*) of our Theorem I.1. However the version we give here is a little simpler, and can be adapted to continuous equations and in particular to model II.

In Section III, the reader will also find our arguments for the conjecture I.2; the idea is that in a reasonable perturbation theory, the power behaviour of wave functions at infinity should diverge as $|E| \to 2 - 0$, ensuring always square integrability and thus localization and pure point spectrum near $|E| = 2$.

Finally let us explain point *v*) of Theorem I.1 and *iv*) of Theorem I.3.

If $\alpha p > 1$, the essential spectrum of Model I is $[-2, +2]$ because almost surely the sequence $\{ a_n V_n \}_{n\in\mathbb{Z}}$ goes to 0 as $|n| \to \infty$, as follows from the fact that $\sum_n \mathbb{E}([a_n V_n]^q) < \infty$ for some $q<p$, $\alpha q>1$ which implies by Fubini that $\Sigma \,|\, a_n V_n \,|^p$ is almost surely finite. The essential spectrum of Model II is $\mathbb{R}$ when $p > 2$, because then we prove in Section II that for given E (e. g. $E \in \mathbb{Q}$) almost surely no solution of $H\psi = E\psi$ can grow exponentially at $+\infty$, which implies that E belongs to the spectrum, since there is always a solution which is L^2 at $-\infty$.

II. THE LOWER BOUNDS AND THE CONTINUOUS SPECTRUM

In this Section we intend to study the solution of the equation

$$H\psi = E\psi \tag{II.1}$$

for operators H given by (I.1) and (I.8), and will prove the lower bounds which have been annonced in Theorem I.1 and I.3. Since a solution of equation (II.1) will oscillate, in order to acommodate the oscillations we will look to lower bounds on g_n where g_n is defined respectively for Model I and II by

$$g_n = \psi(n)^2 + \psi(n+1)^2 \tag{II.2}$$

$$g_n = \int_{n-1}^{n} |\,\psi(x)\,|^2 dx\,. \tag{II.3}$$

As we will see it is not too difficult to get lower bounds for a fixed E, almost surely with respect to the sequence V_n. However we want, in contrast, a lower bound almost surely, for all E in some interval or in $\mathbb{R}$. This difficulty will be solved by using an idea similar to the one used by Kolmogorov in his study of the regularity of stochastic process (see e. g. Ref. [20], p. 43-44). But let us first rewrite our problem in an equivalent but more convenient way.

In the case of the Jacobi matrices we choose two solutions $\varphi_\pm(n)$ of $\varphi(n+1) + \varphi(n-1) = E\varphi(n)$, for instance $\varphi_+(n) = \cos kn$, $\varphi_-(n) = \sin kn$, $E = 2\cos k$ and for any solution ψ of (II.1), we define (A_n, B_n), $n \in \mathbb{Z}$, by

$$\begin{pmatrix} \psi(n) \\ \psi(n-1) \end{pmatrix} = \begin{pmatrix} \varphi_+(n) & \varphi_-(n) \\ \varphi_+(n-1) & \varphi_-(n-1) \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \equiv \eta_n \begin{pmatrix} A_n \\ B_n \end{pmatrix}. \tag{II.4}$$

Equation (II.1) becomes on the variables (A_n, B_n)

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = M_n \begin{pmatrix} A_n \\ B_n \end{pmatrix} \tag{II.5}$$

where

$$M_n = \mathbb{1} + \frac{\lambda a_n V_n}{W(\varphi_+, \varphi_-)} \begin{pmatrix} \varphi_+(n)\varphi_-(n) & \varphi_-^2(n) \\ -\varphi_+^2(n) & -\varphi_+(n)\varphi_-(n) \end{pmatrix} \tag{II.6}$$

and $W(\varphi_+, \varphi_-)$ is the Wronskian of the two solutions $\varphi_\pm$

$$W(\varphi_+, \varphi_-) = \varphi_+(n)\varphi_-(n-1) - \varphi_+(n-1)\varphi_-(n) = \sin k\,.$$

Now $\operatorname{tr}(\eta_n^t \eta_n) \leqslant 4$ and $\det(\eta_n^t \eta_n) = \sin^2 k$. So the smallest eigenvalue of $\eta_n^t \eta_n$ is at least $\sin^2 k/4$. Thus equation (II.4) yields

$$g_n \geqslant \frac{\sin^2 k}{4}(A_n^2 + B_n^2) \tag{II.7}$$

so that we only need to get lower bounds on the norm of (A_n, B_n) when it satisfies (II.5).

For Model II we choose two solutions $\varphi_\pm(x)$ of $-\varphi''(x) - Fx\varphi(x) = E\varphi(x)$ and for any solution ψ of (II.1), we define (A_n, B_n), $n \in \mathbb{Z}$, again by (II.4), which implies that equation (II.1) in the variables (A_n, B_n) becomes again equation (II.5) where now

$$M_n = \mathbb{1} + \frac{V_n}{W(\varphi_+, \varphi_-)} \begin{pmatrix} \varphi_+(n)\varphi_-(n) & \varphi_-^2(n) \\ -\varphi_+^2(n) & -\varphi_+(n)\varphi_-(n) \end{pmatrix} \tag{II.8}$$

and the Wronskian $W(\varphi_+, \varphi_-)$ is now $W(\varphi_+, \varphi_-) = \varphi'_+(x)\varphi_-(x) - \varphi'_-(x)\varphi_+(x)$ independant of x. Furthermore we know that we can choose $\varphi_\pm$ such that asymptotically at $+\infty$

$$\begin{aligned} \varphi_+(x) &\sim |Fx|^{-1/4} \cos\{\sqrt{F}(x - E/F)^{3/2}\} \\ \varphi_-(x) &\sim |Fx|^{-1/4} \sin\{\sqrt{F}(x - E/F)^{3/2}\} \end{aligned} \tag{II.9}$$

and $W(\varphi_+, \varphi_-) = 1$. It is then easy to find a constant C such that

$$g_n \geqslant C(A_n^2 + B_n^2)(Fn)^{-1/2} \tag{II.10}$$

and again our problem is to find a lower bound on the norm of (A_n, B_n).

Let us now set for both cases $U_n = \begin{pmatrix} A_n \\ B_n \end{pmatrix}$; we have to study

$$U_{n+1} = M_n U_n = (\mathbb{1} + n^{-\alpha}\mathscr{V}_n T_n)U_n \tag{II.11}$$

where $\mathscr{V}_n$ is $\lambda(\sin k)^{-1}V_n$ for Model I and for Model II, $\mathscr{V}_n$ is $F^{-1/2}V_n$ and $\alpha = \frac{1}{2}$; the T_n are 2×2 matrices in both cases uniformly bounded in n, independantly of E, F. A crucial fact is that for both models

$$\operatorname{Trace} T_n = \det T_n = 0 \qquad \forall n \tag{II.12}$$

so that $\det M_n = 1$ and if we set $\tilde{M}_n = \prod_{i=1}^{n} M_i$, then also

$$\det \tilde{M}_n = 1 \tag{II.13}$$

In order to get a *lower* bound on the solutions of (II.11), we use the fact [*19*] that it is sufficient to get an *upper* bound on the growth of products of transfer matrices: indeed we have $U_{n+1} = \tilde{M}_n U_1$ so that $\| U_{n+1} \| \geqslant \| \tilde{M}_n^{-1} \|^{-1} \| U_1 \|$, but $\tilde{M}_n$ is a 2×2 matrix with determinant 1 and so $\| \tilde{M}_n^{-1} \| = \| \tilde{M}_n \|$ which implies

$$\| U_{n+1} \| \geqslant \| \tilde{M}_n \|^{-1} \| U_1 \| . \tag{II.14}$$

We are now left to obtain upper bounds on the norm of the products of transfer matrices $\tilde{M}_n$. This amounts also to obtain, with probability one, upper bounds on the norm of U_{n+1}, uniformly with respect to the energy E and to the initial condition U_1. The uniformity with respect to U_1 follows if we obtain upper bounds for two independant initial conditions, since the intersection of two sets of measure one is a set of measure one. So from now U_1 will be a fixed initial condition that is a given vector of $\mathbb{R}^2$ with norm unity, and we set

$$X_n = \| U_{n+1} \| = \| \tilde{M}_n U_1 \|$$

X_n is a random variable depending on the random variables $\{ \mathscr{V}_i \}_{1 \leqslant i \leqslant n}$. In the following, we shall denote by $\mathbb{E}(X_n \mid \mathscr{F}_{n-1})$ the conditional expectation of X_n given $\{ \mathscr{V}_i \}_{1 \leqslant i \leqslant n-1}$, that is the average of X_n with respect to $\mathscr{V}_n$ for $\mathscr{V}_{n-1}, \dots, \mathscr{V}_1$, fixed. ($\mathscr{F}_n$ is the σ-algebra generated by $\{ \mathscr{V}_i \}_{1 \leqslant i \leqslant n}$, X_n is $\mathscr{F}_n$ measurable and $\mathbb{E}(X_n \mid \mathscr{F}_{n-1})$ is the conditional expectation of X_n with respect to $\mathscr{F}_{n-1} \subset \mathscr{F}_n$).

In view of (I.4) and of the definition of $\mathscr{V}$ for Model I and Model II, we see that

$$p = \operatorname{Sup} \{ z > 0 : \mathbb{E}(| \mathscr{V} |^z) < \infty \} .$$

We note also that the matrices T_n appearing in Model I and II are bounded in norm independently of n, so we let T denote such a bound. We then have the

LEMMA II.1. — *If* $\mathbb{E}(| \mathscr{V} |^q) < \infty$ *for some* $q \geqslant 2$, *there exists a constant* D_q *depending only on* q *so that*

$$\mathbb{E}(X_n^q \mid \mathscr{F}_{n-1}) \leqslant X_{n-1}^q [1 + D_q(n^{-2\alpha} T^2 \mathbb{E}(\mathscr{V}^2) + n^{-q\alpha} T^q \mathbb{E}(| \mathscr{V} |^q))] . \tag{II.15}$$

Proof. — We have that

$$\mathbb{E}(X_n^q \mid \mathscr{F}_{n-1}) \leqslant X_{n-1}^q \operatorname*{Sup}_{\| U \| = 1} \mathbb{E}(\| M_n U \|^q) . \tag{II.16}$$

Note that since $M_n = 1 + n^{-\alpha}\mathscr{V}_n T_n$, we have that

$$\| M_n U \|^2 = 1 + a + b, \qquad a = 2n^{-\alpha}\mathscr{V}_n(U, T_n U), \quad b = n^{-2\alpha}\mathscr{V}_n^2 \| T_n U \|^2 .$$

For $q > 2$, the result follows directly from the fact that $\mathbb{E}(\mathscr{V}) = 0$ and that for a and b as above ($b \geqslant 0$, $1 + a + b > 0$) one has

$$(1 + a + b)^{q/2} \leqslant 1 + \frac{q}{2}a + C_q(b + b^{q/2} + a^2 + a^q). \qquad \text{(II.17)}$$

Inequality (II.17) itself is proven separately for $q \geqslant 4$ and $2 < q \leqslant 4$. For $q \geqslant 4$ it follows from the fact that if $1 + x > 0$

$$(1 + x)^{q/2} \leqslant 1 + \frac{q}{2}x + C_1(x^2 + | x |^{q/2})$$

since $(1 + x)^{q/2} - 1 - \frac{q}{2}x$ is $0(x^2)$ for x small and $0(x^{q/2})$ for x large, and taking into account that $b^2 \leqslant (b + b^{q/2})$ and $| a |^{q/2} \leqslant a^2 + | a |^q$ in this case. For $2 < q < 4$, inequality (II.17) follows from the fact that for $1 + x$ positive,

$$(1 + x)^{q/2} \leqslant 1 + \frac{q}{2}x + C_1 f(x), \qquad f(x) = \min (x^2, x^{q/2})$$

since $f(x) = 0(x^{q/2})$ for x large and $0(x^2)$ for x small, and from the fact that $f(a + b) \leqslant 4a^2 + 2^{q/2}b^{q/2}$ (for if $a \geqslant b$, $f(a + b) \leqslant f(2a) \leqslant (2a)^2$ and if $a \leqslant b$, $f(a + b) \leqslant f(2b) \leqslant (2b)^{q/2}$).

Finally we note that, since $\mathbb{E}(\mathscr{V}_n) = 0$, the inequality (II.15) is readily verified for $q = 2$ with $D_q = 1/2$, that is

$$\mathbb{E}(X_n^2 \mid \mathscr{F}_{n-1}) \leqslant X_{n-1}^2 [1 + n^{-2\alpha}T^2\mathbb{E}(\mathscr{V}^2)] . \qquad \text{(II.18)}$$

$\square$

Let us set $c_n = \prod_{i=1}^{n} [1 + D_q n^{-2\alpha}T^2\mathbb{E}(\mathscr{V}^2) + D_q n^{-q\alpha}T^q\mathbb{E}(| \mathscr{V} |^q)]$. Let us also define S_n as

$$S_n(r) = \left\{ \frac{k}{2^i} ; 0 \leqslant k < 2^i, k \in \mathbb{N} \right\}$$

where for a given $r > 0$, i is the integer such that $2^{ri} \leqslant n < 2^{r(i+1)}$. Using Lemma II.1, we are now going to prove the

LEMMA II.2. — *If $\mathbb{E}(| \mathscr{V} |^q) < \infty$ for some $q > 2$, then for any $\varepsilon > 0$, $r > 0$, there exists a random variable $C_E(\omega)$ depending on $\omega = \{ V_i \}_{i\in\mathbb{N}}$, almost-surely finite, such that*

Proof. — From Lemma II.1, and the definition of C_n, we get that

$$\forall n \in \mathbb{N}, \quad \forall E \in S_n(r), \qquad X_n^q \leqslant C_\varepsilon(\omega)n^{(1+\varepsilon)/r}c_n . \qquad \text{(II.19)}$$

$$\mathbb{E}\left(\frac{X_n^q}{c_n}\middle| \mathscr{F}_{n-1}\right) \leqslant \frac{X_{n-1}^q}{c_{n-1}}$$

which ensures that X_n^q/c_n is a (positive) supermartingale. A classical theorem on supermartingales [21] then tells us that

$$\mathbb{P}\left(\operatorname*{Sup}_{1\leqslant n\leqslant N}\frac{X_n^q}{c_n} > C\right) \leqslant \frac{1}{C}\operatorname*{Sup}_{1\leqslant n\leqslant N}\mathbb{E}\left(\frac{X_n^q}{c_n}\right)$$

from which

$$\mathbb{P}\left(\operatorname*{Sup}_{n\in\mathbb{N}}\frac{X_n^q}{c_n} > C\right) \leqslant \frac{1}{C}\mathbb{E}(\|u_1\|^q) = \frac{1}{C}.$$

By replacing C by $C2^{i(1+\varepsilon)}$ we can obtain that

$$\mathbb{P}\left(\operatorname*{Sup}_{2^{ri}\leqslant n<2^{r(i+1)}}\ \operatorname*{Sup}_{E\in S_n}\frac{X_n^q}{c_n} > Cn^{(1+\varepsilon)/r}\right) \leqslant (C2^{i\varepsilon})^{-1}$$

which is summable with respect to i, and so by the Borel-Cantelli Theorem we get Lemma II.2. □

Using Lemma II.1, we are now going to prove also the

LEMMA II.3. — *If $p > 2$, and if the 2×2 matrices T_n satisfy*

$$\|T_n(E) - T_n(E')\| \leqslant T'n^\beta |E - E'| \tag{II.20}$$

then

$$Z_n = \frac{1}{d_n}\left(\frac{\|U_{n+1}(E) - U_{n+1}(E')\|^2}{n^{2\beta}} + |E-E'|^2\|U_{n+1}(E)\|^2\right) \tag{II.21}$$

is a positive $\mathcal{F}_n$-supermartingale, with

$$\mathbb{E}(Z_0) = (E - E')^2$$

and

$$d_n = \prod_{i=1}^{n}(1 + 2i^{-2\alpha}(T^2 + T'^2)\mathbb{E}(\mathcal{V}^2)).$$

Proof. — Let us consider the random variables

$$Y_n^2 = \|U_{n+1}(E) - U_{n+1}(E')\|^2$$

and for the sake of notations, let us drop E and E′, keeping only the prime for distinguishing both energies. So

$$\begin{aligned}\mathbb{E}(Y_n^2\,|\,\mathcal{F}_{n-1}) &\equiv \mathbb{E}(\|M_nU_n - M_n'U_n'\|^2\,|\,\mathcal{F}_{n-1})\\ &= \mathbb{E}(\|M_n'(U_n - U_n') + (M_n - M_n')U_n\|^2\,|\,\mathcal{F}_{n-1})\\ &= \mathbb{E}(\|M_n'(U_n - U_n')\|^2\,|\,\mathcal{F}_{n-1}) + \mathbb{E}(\|(M_n - M_n')U_n\|^2\,|\,\mathcal{F}_{n-1})\\ &+ 2\mathbb{E}((U_n, (M_n^t - M_n^t)M_n'(U_n - U_n'))\,|\,\mathcal{F}_{n-1}).\end{aligned} \tag{II.22}$$

In view of (II.18), one can readily bound the first term of (II.22) by $(1 + n^{-2\alpha}T^2\mathbb{E}(\mathcal{V}^2))Y_{n-1}^2$. Using the hypothesis (II.20), and the definition of M_n, M_n' the second term is bounded by $X_{n-1}^2T'^2(E-E')^2n^{2(\beta-\alpha)}\mathbb{E}(\mathcal{V}'^2)$.

The third one is bounded by

$$2X_{n-1}Y_{n-1} \sup_{\|U\|=\|U'\|=1} \mathbb{E}\left(\frac{\mathscr{V}_n}{n^\alpha}\left(U, (T_n^t - T_n'^t)\left(1 + \frac{\mathscr{V}_n}{n^\alpha}T_n'\right)U'\right)\right)$$

and using $\mathbb{E}(\mathscr{V}_n) = 0$, this is equal to

$$\begin{aligned} 2X_{n-1}Y_{n-1}\frac{\mathbb{E}(\mathscr{V}^2)}{n^{2\alpha}} & \sup_{\|U\|=\|U'\|=1} (U, (T_n^t - T_n'^t)T_n'U') \\ & \leqslant 2X_{n-1}Y_{n-1}\frac{\mathbb{E}(\mathscr{V}^2)}{n^{2\alpha}}n^\beta \,|\, E - E' \,|\, T'T \\ & \leqslant Y_{n-1}^2 \frac{\mathbb{E}(\mathscr{V}^2)}{n^{2\alpha}}T^2 + X_{n-1}^2(E - E')^2 T'^2 n^{2\beta}\frac{\mathbb{E}(\mathscr{V}^2)}{n^{2\alpha}}, \end{aligned}$$

where we have used (II.20).

From these bounds on the three terms of (II.22) it follows that

$$\mathbb{E}(Y_n^2 \mid \mathscr{F}_{n-1}) \leqslant Y_{n-1}^2\left(1 + 2T^2\frac{\mathbb{E}(\mathscr{V}^2)}{n^{2\alpha}}\right) + 2X_{n-1}^2(E - E')^2 T'^2 n^{2(\beta-\alpha)},$$

from which we finally obtain by using (II.18) that

$$\begin{aligned} \mathbb{E}\left(\frac{Y_n^2}{(n+1)^{2\beta}} + (E - E')^2 X_n^2 \,\middle|\, \mathscr{F}_{n-1}\right) & \\ \leqslant \left(\frac{Y_{n-1}^2}{n^{2\beta}} + (E - E')^2 X_{n-1}^2\right)&\left(1 + 2\frac{T^2 + T'^2}{n^{2\alpha}}\mathbb{E}(\mathscr{V}^2)\right) \qquad \text{(II.23)} \end{aligned}$$

Now from (II.23) we see that if

$$Z_n = d_n^{-1}\left(\frac{Y_n^2}{(n+1)^{2\beta}} + (E - E')^2 X_n^2\right)$$

$$\mathbb{E}(Z_n \mid \mathscr{F}_{n-1}) \leqslant Z_{n-1}$$

and Z_n is a (positive) supermartingale. And of course $\mathbb{E}(Z_0) = (E - E')^2$. □

We prove now the

LEMMA II.4. — *Under the hypothesis of Lemma* II.3, *for any* ε, $0 < \varepsilon < 1$, $r > 0$, *there exists a random variable* $D_\varepsilon(\omega)$ *depending on* $\omega = \{ V_i \}_{i \in \mathbb{N}}$, *almost surely finite, such that*

$$\forall n \in \mathbb{N}, \quad \forall l \geqslant i, \quad \forall k, \qquad 0 \leqslant k < 2^l$$

$$\left| X_n\left(\frac{k}{2^l}\right) - X_n\left(\frac{k+1}{2^l}\right)\right|^2 \leqslant D_\varepsilon(\omega) d_n n^{2\beta} 2^{-l(1-\varepsilon)}$$

where $i = i(n)$ *is the integer such that* $2^{ri} \leqslant n < 2^{r(i+1)}$.

Proof. — Since Z_n is a positive supermartingale, we get that for all l and k

$$\mathbb{P}\left(\operatorname{Sup}_n Z_n\left(\frac{k}{2^l}, \frac{k+1}{2^l}\right) > C\right) \leqslant \frac{\mathbb{E}(Z_1)}{C} = \frac{2^{-2l}}{C}$$

where we have made explicit the two energies $k/2^l$ and $(k+1)/2^l$ that we consider in Z_n. It follows

$$\mathbb{P}\left(\operatorname{Sup}_{0 \leqslant k \leqslant 2^l} \operatorname{Sup}_n Z_n\left(\frac{k}{2^l}, \frac{k+1}{2^l}\right) > C\right) \leqslant (C2^l)^{-1}.$$

Choosing now $C = C'2^{-l(1-\varepsilon)}$, we get from it

$$\mathbb{P}\left(\operatorname{Sup}_{2^{ri} \leqslant n \leqslant 2^{r(i+1)}} \operatorname{Sup}_{l > i} \operatorname{Sup}_{0 \leqslant k < 2^l} Z_n\left(\frac{k}{2^l}, \frac{k+1}{2^l}\right) > C'2^{-l(1-\varepsilon)}\right)$$
$$\leqslant \sum_{l > i} (C'2^{l\varepsilon})^{-1} = (C'2^{i\varepsilon})^{-1}(2^\varepsilon - 1)^{-1}.$$

The right hand side is summable with respect to i so that we get by Borel-Cantelli that for all n, all $l > i(n)$, all k, $0 \leqslant k \leqslant 2^l$, $Z_n\left(\frac{k}{2^l}, \frac{k+1}{2^l}\right) < D_\varepsilon(\omega)2^{-l(1-\varepsilon)}$ where $D_\varepsilon(\omega)$ is a random variable almost surely finite. The Lemma follows then from the definition of Z_n and from the fact that

$$Y_n^2(E, E') \geqslant |X_n(E) - X_n(E')|^2. \qquad \square$$

We can now state and prove the main result of this Section:

THEOREM II.5. — *If* $\mathbb{E}(|\mathcal{V}|^q) < \infty$ *for some* $q > 2$, *and if* $\forall n \in \mathbb{N}$, $\forall E, E' \in [a, b]$

$$\| T_n(E) \| \leqslant T$$
$$\| T_n(E) - T_n(E') \| \leqslant T' n^\beta | E - E' |$$

then for any $\varepsilon > 0$, *there exists a random variable* $\Gamma_\varepsilon(\omega)$, *depending on* $\omega = \{ V_i \}_{i \in \mathbb{N}}$, *almost surely finite, such that*

$$\forall E \in [a, b], \quad \forall n \in \mathbb{N} \quad \| \tilde{M}_n(E) \| \leqslant \Gamma_\varepsilon(\omega).\operatorname{Max}\left(c_n^{1/q}, d_n^{1/2}\right).n^{\varepsilon + \beta\left(1 + \frac{q}{2}\right)^{-1}}$$

with c_n *and* d_n *defined above.*

Proof. — Let n be given and suppose first that $[a, b] \subset [0, 1]$. Any energy E in $[0, 1]$ can be written as

$$E = E_i + \sum_{l > i} \frac{\varepsilon_l}{2^l}$$

where $E_i \in S_n(r)$ and ε_l is 0 or 1 : $\varepsilon_l/2^l$ is the dyadic decomposition of $E - E_i$.

Clearly $E = \lim_{L\to\infty} E_L$, if we set $E_L = E_i + \sum_{l=i+1}^{L} \frac{\varepsilon_l}{2^l}$, and since $X_n(E)$ is a continuous function with respect to E

$$X_n(E) = \lim_{L\to\infty} X_n(E_i) + \sum_{l=i+1}^{L} (X_n(E_l) - X_n(E_{l-1})).$$

By Lemmas II.2 and II.4 we thus get

$$X_n(E) \leqslant C_\varepsilon(\omega)^{1/q} c_n^{1/q} n^{(1+\varepsilon)/rq} + D_\varepsilon(\omega)^{1/2} d_n^{1/2} n^{\beta - \frac{1-\varepsilon}{2r}} \sum_{m\geqslant 0} 2^{-m(1-\varepsilon)}$$

where in the last term we have made use of the fact that

$$2^{-l(1-\varepsilon)} = 2^{-(l-i-1)(1-\varepsilon)} 2^{-(i+1)(1-\varepsilon)} \leqslant 2^{-(l-i-1)(1-\varepsilon)} n^{-(1-\varepsilon)/r}$$

since $n < 2^{r(i+1)}$.

This result clearly does not depend on the special interval [0, 1] which has been choosen for simplicity and by a scale factor applies to the interval $[a, b]$ on which the hypothesis of the Theorem hold. Furthermore the result will be the same with another initial vector U_1^1 orthogonal to U_1, hence yielding the result on the norm of $\tilde{M}_n(E)$ stated in the Theorem if we choose $r = \beta^{-1}(q^{-1} + 1/2)$. □

We can now apply the result of Theorem II.5 to our models. For Model I, if $\alpha = 1/2$, c_n and d_n behave as $n^{c\mathbb{E}(\mathscr{V}^2)}$, that is like $n^{c\mathbb{E}(\mathscr{V}^2)\lambda^2(4-E^2)^{-1}}$ if the interval (a, b) of Theorem II.5 is choosen as $[-E, E]$, $E < 2$: in view of the discussion before eq. (II.14), we have thus obtained power decaying lower bounds on the solutions of the equation $H\psi = e\psi$, $|e| < E$ and so on the eigenfunctions when the spectrum is pure point in $[-E, E]$. Furthermore we can check that $\beta = 1$, so that, since $p > 2$, for ε and $\lambda^2(4 - E^2)^{-1}$ sufficiently small, $c\mathbb{E}(\mathscr{V}^2)\lambda^2(4-E^2)^{-1}$ will be smaller than 1/2: thus almost surely with respect to the potential, no solution of $H\psi = e\psi$ can be square integrable for any e, $|e| < E$ and the spectrum is purely continuous in $[-E, E]$. For Model I, $0 \leqslant \alpha < 1/2$, c_n and d_n behave like $\exp\{-cn^{1-2\alpha}\}$, which yields the lower bound claimed in (I.5*b*).

For Model II, $\mathbb{E}(\mathscr{V}^2) = \lambda^2\mathbb{E}(V^2)F^{-1}$, $\alpha = 1/2$ and from the asymptotic behavior of Airy functions, one can deduce that $\beta = 1/2$. Thus c_n and d_n behave like $n^{c\mathbb{E}(V^2)\lambda^2F^{-1}}$, and so we obtain power decaying lower bounds on the solutions of $H\psi = E\psi$. Actually we have

$$g_n \geqslant C'_\varepsilon(Fn)^{-1/2} n^{-\left[\varepsilon + 2\left(\frac{p}{2}+1\right)^{-1} + 2c\mathbb{E}(V^2)F^{-1}\right]}$$

so that g_n is not integrable for $p > 2$, and for sufficiently small λ^2F^{-1}. This result extends from compact energy interval to the whole spectrum $\mathbb{R}$

by countable intersection of sets of measure one. Incidently we have also proven that the spectrum of H is almost surely $\mathbb{R}$: this follows from the fact that almost surely for all E the solutions of the equation $H\psi = E\psi$ do not increase exponentially.

III. UPPER BOUNDS AND PURE POINT SPECTRUM

In this section we intend to prove that, as stated in Theorems I.1 and I.3, the spectrum of Models I and II is pure point in various cases and to provide the announced upper bounds on eigenfunctions. We intend also to give the heuristic arguments leading to Conjecture I.2.

We first discuss Model I, which is simpler. We follow the basic stategy of [6], studying the correlation function $\overline{\rho}(m, n; A)$ defined for an energy interval A as

$$\overline{\rho}(m, n; A) = \mathbb{E}\left(\int_A |E(m, n; d\lambda)|\right) \tag{III.1}$$

where $|E(m, n; d\lambda)|$ denotes the absolute value of the spectral measure of the Hamiltonian H, taken between the two vectors δ_m and δ_n of the canonical basis of $l^2(\mathbb{Z})$.

If for all m in $\mathbb{Z}$, $\overline{\rho}(m, n; A)$ is summable with respect to n, then for almost all potential the spectrum of H is pure point in A [6]. And if in addition it is bounded by a function $f(|m-n|)$ then for almost all V the eigenfunctions ψ_E for energies E in the interval A decay at least according to

$$|\psi(n)| \leqslant C(V) f(|n|) n^{1+\varepsilon} \tag{III.2}$$

for ε arbitrarily small; the constant C(V) depends only on V.

Actually it is sufficient [6] to obtain, uniformly in Λ, a bound on the correlation functions $\overline{\rho}^{\Lambda}(m, n; A)$ associated to a sequence of boxes $\Lambda = [-L, L]$ increasing to $\mathbb{Z}$. In this latter case, the correlation functions are given as

$$\overline{\rho}^{\Lambda}(m, n; A) = \mathbb{E}\left(\sum_{E \in A}\left(\frac{|\psi_E(m)\psi_E(n)|}{\sum_{i \in \Lambda} \psi_E^2(i)}\right)\right) \tag{III.3}$$

where the sum runs, for a given potential, over all eigenvalues in A of the operator H^{Λ}—restriction of H to the box Λ with Dirichlet boundary conditions at $\pm(N+1)$—and ψ_E denote the corresponding eigenfunctions (equation III.3 holds as an equality because for all V, the eigenvalues of H^{Λ} are non degenerate).

We are going to prove the

PROPOSITION III.1. — *If $p > 2$, the correlation function for Model I satisfies for all Λ, all m*

$$\begin{aligned}\bar{\rho}^{\Lambda}(m, n; \mathrm{A}) &\leqslant \mathrm{C}(m)\lambda^{-2} \exp\{-\mathrm{C}_1\lambda^2 |n|^{1-2\alpha}\}, \quad \alpha < 1/2 \\ &\leqslant \mathrm{C}(m)\lambda^{-2} |n|^{1/2-\mathrm{C}_1\lambda^2} \qquad , \quad \alpha = 1/2\end{aligned} \tag{III.4}$$

(The case $\alpha < 0$ will be treated later on in this Section).

Proof. — The basic method for proving these results is the one of Ref. [*10*] where the case $\alpha = 0$ was proven. It was used [*12*] to treat the case $\alpha < 1/2$, and the case $\alpha = 1/2$ can also be obtained from it. We give here a variation of this method which is a little simpler and extends more easily to other situations and in particular to continuous Schrödinger equations such as our Model II.

Let us first consider the expression (III.3) of $\bar{\rho}^{\Lambda}(m, n; \mathrm{A})$. The expectation holds there for $\displaystyle\int \prod_{i\in\Lambda} r(\mathrm{V}_i)d\mathrm{V}_i$, and we are going to perform first the integration with respect to V_n. The eigenvalue E is a function of V_n which is for almost every V_n a local diffeomorphism; this allows us to transform the integration with respect to V_n into an integration with respect to E if we note that, for $\{\mathrm{V}_m\}_{m\neq n}$ fixed

$$\lambda a_n \psi_{\mathrm{E}}^2(n) d\mathrm{V}_n = \Big(\sum_{i\in\Lambda} \psi_{\mathrm{E}}^2(i)\Big) d\mathrm{E} \tag{III.5}$$

and so

$$\bar{\rho}^{\Lambda}(m, n; \mathrm{A}) = \int_{\mathrm{A}} d\mathrm{E} \int \frac{1}{\lambda a_n} r(\mathrm{V}_n) \left| \frac{\psi_{\mathrm{E}}(m)}{\psi_{\mathrm{E}}(n)} \right| \prod_{\substack{i\in\Lambda \\ i\neq n}} r(\mathrm{V}_i) d\mathrm{V}_i \tag{III.6}$$

where V_n is now a function of E and $\{\mathrm{V}_i\}_{i\in\Lambda, i\neq n}$. This defines in a natural way a correlation function at energy E, $\bar{\rho}^{\Lambda}(m, n; \mathrm{E})$, that we are now going to study.

We suppose that for example $m \leqslant n$, and we set

$$\begin{aligned} \theta_i &= \frac{\psi_{\mathrm{E}}(i+1)}{\psi_{\mathrm{E}}(i)}, \qquad -\mathrm{L} \leqslant i < n\,; \qquad \frac{1}{\theta_{-\mathrm{L}-1}} = 0 \\ \theta_i &= \frac{\psi_{\mathrm{E}}(i-1)}{\psi_{\mathrm{E}}(i)}, \qquad n < i \leqslant \mathrm{L}\,; \qquad \frac{1}{\theta_{\mathrm{L}}} = 0\,. \end{aligned} \tag{III.7}$$

We have thus

$$\begin{aligned} \lambda a_i \mathrm{V}_i &= \mathrm{E} - \theta_{i+1}^{-1} - \theta_i, \qquad n+1 \leqslant i \leqslant \mathrm{L} \\ \lambda a_i \mathrm{V}_i &= \mathrm{E} - \theta_i - \theta_{i-1}^{-1}, \qquad -\mathrm{L} \leqslant i \leqslant n-1 \\ \lambda a_n \mathrm{V}_n &= \mathrm{E} - \theta_{n-1}^{-1} - \theta_{n+1}^{-1} \end{aligned} \tag{III.8}$$

and following [6] [10] we can pass from the integration variables $\{ V_i \}_{i \neq n, i \in \Lambda}$ to the variables $\{ \theta_i \}_{i \neq n, i \in \Lambda}$. This can be done easily by iteration and one can also check that this change of variables is valid. We obtain

$$\bar{\rho}^\Lambda(m, n; \mathrm{E}) = \int d\theta_{n-1} | \theta_{n-1} |^{-1} f_n(\theta_{n-1}^{-1})(\mathrm{T}_{n-1}\mathrm{T}_{n-2} \dots \mathrm{T}_{m+1} g_m)(\theta_{n-1}) \quad \text{(III.9)}$$

where

$$f_n(\theta_{n-1}^{-1}) = \int r_n(\theta_{n+1}^{-1} + \theta_{n-1}^{-1}) \prod_{i \geq n+1} r_i(\theta_i + \theta_{i+1}^{-1}) d\theta_i \quad \text{(III.10)}$$

$$g_m(\theta_m) = \int \prod_{i \leq m} r_i(\theta_i + \theta_{i-1}^{-1}) \prod_{i < m} d\theta_i \quad \text{(III.11)}$$

$$r_i(x) = \frac{1}{\lambda | a_i |} r\left(\frac{1}{\lambda a_i} (\mathrm{E} - x) \right) \quad \text{(III.12)}$$

$$(\mathrm{T}_i h)(x) = \int r_i(x + y^{-1}) | y |^{-1} h(y) dy \quad \text{(III.13)}$$

From these expressions, we get by iteration that

$$\begin{aligned} &\| f_n \|_1 = \| g_m \|_1 = \| r_i \|_1 = 1 \\ &\| f_n \|_\infty \leq \| r_n \|_\infty = (\lambda | a_n |)^{-1} \| r \|_\infty \\ &\| g_m \|_\infty \leq \| r_m \|_\infty = (\lambda | a_m |)^{-1} \| r \|_\infty . \end{aligned} \quad \text{(III.14)}$$

Finally we need the following estimate, which is crucial in this approach and which follows from the estimates of Ref. [10] [12] and from the upper bounds on the behaviour of $\{ a_i \}$ in (I.2):

$$\| \mathrm{T}_{i+1}\mathrm{T}_i \|_2 \leq 1 - \mathrm{C}_1' \lambda^2 | i |^{-2\alpha} . \quad \text{(III.15)}$$

From the expression (III.9) of $\bar{\rho}^\Lambda(m, n; \mathrm{E})$ and using the estimates (III.14), (III.15) we obtain now by

$$\bar{\rho}^\Lambda(m, n; \mathrm{E}) \leq \| f_n \|_2 \| \mathrm{T}_{n-1}\mathrm{T}_{n-2} \dots \mathrm{T}_{m+1} \|_2 \| g_m \|_2 \quad \text{(III.16)}$$

the estimates of Proposition III.1. □

We remark that for $p < 2$, one may get better estimates than (III.15) and thus localization and pure point spectrum also for some $\alpha > \frac{1}{2}$. For example in the case of a Cauchy distribution for the variables V's, $p = 1$ and (III.15) is replaced by $1 - \mathrm{C}_1' | \lambda | | i |^{-\alpha}$. This is because the estimate $| \hat{r}(k) | \leq 1 - \mathrm{C}k^2$ for k small which holds if $p > 2$ is replaced by the stronger $| \hat{r}(k) | \leq 1 - \mathrm{C} | k |$ for k small in the case of a Cauchy distribution. We thus have pure point spectrum for all $\alpha < 1$, with eigenfunctions decaying as a fractional exponential, and we have also pure point spectrum for $\alpha = 1$ at large enough coupling constant, with power decaying eigenfunctions.

The upper bound which has been proven in Proposition III.1 is independent of the energy. When $\alpha = 1/2$, it yields pure point spectrum only for λ large enough and we have seen in Section II that for small λ the spectrum is purely continuous within $[-E_0(\lambda), E_0(\lambda)]$, $E_0(\lambda) < 2$. The question arises thus to know what happens for small λ near $|E| = 2 - 0$. We have no rigorous result on this case, but we can argue heuristically in the following way, which yields our conjecture I.2: we can compute by perturbation the spectral radius of the operator T for λ small, $\alpha = 0$, at given E near 2; we obtain that it is of order $1 - C\lambda^2(4 - E^2)^{-1}$, which reinterpreted in our situation with $\alpha = \frac{1}{2}$ yields the result. The same singularity near the edge of the spectrum has been noticed for the Lyapunov exponent associate with the case $\alpha = 0$ both by perturbative argument and numerical computation [*22*].

We turn now to Model II and more precisely to the proof that under some circumstances the spectrum of H can be pure point and to the corresponding bounds on the eigenfunctions. Here we face a continuous Schrödinger equation, but the basic strategy will be the same as in the study of Model I, since Royer [*8*] has shown how to adapt the approach of Ref. [*6*] to the continuous case. In particular it is also possible to consider the correlation function $\overline{\rho}(x, y; A)$ defined in a way analogous to (III.1) and integrability with respect to y will ensure the pure point character of the spectrum [*8*], the rate of decay of the correlation function being again linked to the decay properties of the wave functions. It is thus sufficient for our purpose to prove the

PROPOSITION III.2. — *If $p > 2$, the correlation function for Model II satisfies for all* Λ

$$\overline{\rho}^{\Lambda}(m, n; A) \leqslant C(m)n^{-C'/F}, \qquad n > 0. \tag{III.17}$$

(The bound for $n < 0$ will be obtained later on in this Section).

Proof. — We are going to follow closely the version of the proof of Ref. [*10*] given above in our study of Model I. The correlation function for a finite box Λ is given by

$$\overline{\rho}^{\Lambda}(x, y; A) = \mathbb{E}\left(\sum_{E \in A} \frac{|\psi_E(x)\psi_E(y)}{\int_{-L-1}^{L+1} \psi_E^2(x)dx} \right) \tag{III.18}$$

where the sum runs, for a given sequence V_n, over all eigenvalues in the energy interval A, of the Hamiltonian H_Λ restriction of H to the box Λ

with Dirichlet boundary conditions at $-L-1$ and $L+1$; these eigenvalues are non degenerate. Actually it is easy to verify that we only need a bound on $\bar{\rho}^{\Lambda}(x, y; A)$ for x and y in $\mathbb{Z}$; hence we will restrict to study $\bar{\rho}^{\Lambda}(m, n; A)$. As previously the expectation holds for $\displaystyle\int \prod_{i=-L}^{L} r(V_i)dV_i$, and we again perform first the integration with respect to V_n. In this case we have

$$\psi_E^2(n)dV_n = \left(\int_{-L-1}^{L+1} \psi_E^2(x)dx\right)dE \tag{III.19}$$

so that the correlation function becomes

$$\bar{\rho}^{\Lambda}(m, n; A) = \int_A dE \int r(V_n)\left|\frac{\psi_E(m)}{\psi_E(n)}\right| \prod_{\substack{i\neq n\\ -L\leqslant i\leqslant L}} r(V_i)dV_i \tag{III.20}$$

which allows us to introduce a correlation function at energy E $\bar{\rho}^{\Lambda}(m, n; E)$ which we are going to study.

Let us now introduce two independent solutions $\varphi_{\pm}(x)$ of the equation $-\varphi''(x) - Fx\varphi(x) = E\varphi(x)$ and set

$$\psi(x) = A_i\varphi_+(x) + B_i\varphi_-(x), \qquad x \in [i, i+1]. \tag{III.21}$$

We thus have of course

$$\begin{aligned}
\psi_E(i+1) &= A_i\varphi_+(i+1) + B_i\varphi_-(i+1)\\
\psi_E(i) &= A_i\varphi_+(i) + B_i\varphi_-(i) = A_{i-1}\varphi_+(i) + B_{i-1}\varphi_-(i)\\
\psi_E(i-1) &= A_{i-1}\varphi_+(i-1) + B_{i-1}\varphi_-(i-1)\\
\psi'_E(i) &= A_i\varphi'_+(i) + B_i\varphi'_-(i) = A_{i-1}\varphi'_+(i) + B_i\varphi'_-(i) + V_i\psi_E(i)
\end{aligned} \tag{III.22}$$

and by elimination of A_i, A_{i-1}, B_i, B_{i-1} one gets

$$\psi_E(i+1) = -a_i\psi_E(i-1) + b_i\psi_E(i) + \gamma_i V_i\psi_E(i) \tag{III.23}$$

where

$$a_i = \frac{\varphi_+(i+1)\varphi_-(i) - \varphi_-(i+1)\varphi_+(i)}{\varphi_+(i)\varphi_-(i-1) - \varphi_-(i)\varphi_+(i-1)}, \quad b_i = \frac{\varphi_+(i+1)\varphi_-(i-1) - \varphi_+(i-1)\varphi_-(i+1)}{\varphi_+(i)\varphi_-(i-1) - \varphi_+(i-1)\varphi_-(i)}$$

$$\gamma_i = \frac{\varphi_+(i+1)\varphi_-(i) + \varphi_-(i+1)\varphi_+(i)}{\varphi_-(i)\varphi'_+(i-1) - \varphi_+(i)\varphi'_-(i-1)}. \tag{III.24}$$

We suppose for example $m \leqslant n$ and we set as in the previous case

$$\theta_i = \frac{\psi_E(i+1)}{\psi_E(i)}, \qquad -L \leqslant i < n; \qquad \frac{1}{\theta_{-L-1}} = 0$$

$$\theta_i = \frac{\psi_E(i-1)}{\psi_E(i)}, \qquad n < i \leqslant L; \qquad \frac{1}{\theta_L} = 0 \tag{III.25}$$

so that

$$\gamma_i V_i = -b_i + \theta_i + a_i\theta_{i-1}^{-1}, \qquad -L \leqslant i \leqslant n-1$$
$$\gamma_i V_i = -b_i + \theta_{i+1}^{-1} + a_i\theta_i, \qquad n+1 \leqslant i \leqslant L$$
$$\gamma_n V_n = -b_n + \theta_{n+1}^{-1} + a_n\theta_{n-1}^{-1} \tag{III.26}$$

and mimicking the discrete case studied previously we can pass from the integration variables $\{V_i\}_{i\neq n}$ to the variables $\{\theta_i\}_{i\neq n}$, and we obtain for the correlation function at energy E:

$$\bar{\rho}^{\Lambda}(m, n; E) = \int d\theta_{n-1} |\theta_{n-1}|^{-1} f_n(\theta_{n-1}^{-1}) \left|\frac{\gamma_n}{a_n}\right| (T_{n-1}T_{n-2}\dots T_{m+1}g_m)(\theta_{n-1}) \prod_{i=m+1}^{n-1} |a_i|^{-1/2} \tag{III.27}$$

where

$$f_n(\theta_{n-1}^{-1}) = \int r_n(\theta_{n+1}^{-1} + a_n\theta_{n-1}^{-1})\,|a_n| \prod_{i\geqslant n+1} r_i(\theta_{i+1}^{-1} + a_i\theta_i)\,|a_i|\,d\theta_i \tag{III.28}$$

$$g_m(\theta_m) = \int \prod_{i\leqslant m} r_i(\theta_i + a_i\theta_{i-1}^{-1}) \prod_{i<m} d\theta_i \tag{III.29}$$

$$r_i(x) = |\gamma_i^{-1}|\, r(\gamma_i^{-1}(-b_i + x)) \tag{III.30}$$

$$(T_i h)(x) = |a_i|^{1/2} \int r_i(x + a_i y^{-1})\,|y|^{-1} h(y)\,dy \tag{III.31}$$

and from these expressions we get by iteration that

$$\|f_n\|_1 = \|g_m\|_1 = \|r_i\|_1 = 1$$
$$\|f_n\|_\infty \leqslant |a_n|\,\|r_n\|_\infty = |a_n\gamma_n^{-1}|\,\|r\|_\infty$$
$$\|g_m\|_\infty \leqslant \|r_m\|_\infty\,|\gamma_m^{-1}|\,\|r\|_\infty\,. \tag{III.32}$$

From (III.27) we have, using (III.32) and (III.24)

$$\bar{\rho}^{\Lambda}(m,n;E) \leqslant \|f_n\|_2\,\|T_{n-1}T_{n-2}\dots T_{m+1}\|_2\,\|g_m\|_2 \left|\frac{\gamma_n}{a_n}\right| \prod_{i=m+1}^{n-1} |a_i|^{-1/2} \tag{III.33}$$

$$\leqslant \|r\|_\infty \left|\frac{\varphi_-(m)\varphi'_+(m-1) - \varphi_+(m)\varphi'_-(m-1)}{\varphi_-(n)\varphi'_+(n-1) - \varphi_+(n)\varphi'_+(n-1)}\right|^{1/2} \|T_{n-1}\dots T_{m+1}\|_2 \tag{III.34}$$

If we had started our proof by changing first the variable V_m into E instead of V_n, and proceeding similarly as above we would have got

$$\bar{\rho}^{\Lambda}(m, n; E) \leqslant \|r\|_\infty \left|\frac{\varphi_-(n)\varphi'_+(n-1) - \varphi_+(n)\varphi'_-(n-1)}{\varphi_-(m)\varphi'_+(m-1) - \varphi_+(m)\varphi'_-(m-1)}\right|^{1/2} \|T_{m+1}\dots T_{n-1}\|_2 \tag{III.35}$$

and thus

$$\bar{\rho}^{\Lambda}(m, n; \mathrm{E}) \leqslant \| r \|_{\infty} \operatorname{Max} (\| \mathrm{T}_{n-1} \ldots \mathrm{T}_{m+1} \|_2, \| \mathrm{T}_{m+1} \ldots \mathrm{T}_{n-1} \|_2) \quad (\mathrm{III}.36)$$

The operator norms in (III.36) are bounded now as for Model I, as in Ref. [10] by product of L^2-norms of the operators $T_{i+1}T_i$ or T_iT_{i+1}. Let us first note that T_iT_{i+1} (resp. $T_{i+1}T_i$) is conjugate through unitary dilatation and translation to the operator T^2, where T is defined by (III.30), (III.31) with b_i set equal to zero, a_i equal to 1 and γ_i replaced by $(\gamma_i\gamma_{i+1}/a_i)^{1/2}$ (resp. $(\gamma_i\gamma_{i+1}/a_{i+1})^{1/2}$. Now, as in (III.15) we get from Ref. [10] [12] that

$$\| \mathrm{T}_i\mathrm{T}_{i+1} \|_2 = \| \mathrm{T}^2 \|_2 \leqslant 1 - c' \, | \gamma_i\gamma_{i+1}/a_i | \quad (\mathrm{III}.37)$$

for $| \gamma_i\gamma_{i+1}/a_i |$ sufficiently small. We thus need only to know some informations on the behaviour of γ_i, γ_{i+1} and a_i with i as $i \to +\infty$. In fact we restrict to those i such that

$$\sqrt{\mathrm{F}}(i - \mathrm{E}/\mathrm{F})^{1/2} \notin [-\varepsilon, \varepsilon] \bmod \pi/2$$

for some fixed small enough ε. From the behaviour of the Airy functions (see II.9) we see that for those i, as $i \to +\infty$

$$| \gamma_i | \sim (\mathrm{F}i)^{1/2} \, | \sin \{ \sqrt{\mathrm{F}}(i - \mathrm{E}/\mathrm{F})^{1/2} \} |$$

and

$$a_i \sim 1$$

So that for a large density of pairs $(i, i+1)$ we get

$$c' \, | \gamma_i\gamma_{i+1}/a_i | \geqslant c''/\mathrm{F}^i$$

and thus

$$\| \mathrm{T}_i\mathrm{T}_{i+1} \|_2 \leqslant 1 - c(\mathrm{F}i)^{-1}, \quad (\mathrm{III}.38)$$

whereas for the other i we use that

$$\| \mathrm{T}_n \|_2 = 1$$

and we get the upper bound (III.17). □

We note that if $p < 2$, the bounds (III.38) may be strengthened and one can then have pure point spectrum for all field. For example in the case of a Cauchy distribution, $p = 1$ and (III.38) is replaced by $1 - \mathrm{C}(\mathrm{F}n)^{-1/2}$, and thus in this case H has almost-surely a pure point spectrum for all field F and with eigenfunctions decaying according to a fractional exponential.

As announced after Propositions III.1 and III.2, we are now going to study the case of Model I with $\alpha < 0$ and of Model II for $n \to -\infty$. Actually we are going to prove the

PROPOSITION III.3. — *If* $|\tilde{r}(k)| \leqslant C|k|^{-\gamma}$ *for some* C, $\gamma > 0$, *then*

i) the correlation of Model I with $\alpha < 0$ *satisfies*

$$\bar{\rho}^{\Lambda}(m, n; A) \leqslant C_{\varepsilon}(m)|n|^{-|n||\alpha|\gamma(1-\varepsilon)/(\gamma+1)} \tag{III.39}$$

for all $\varepsilon > 0$.

ii) the correlation function of Model II satisfies

$$\bar{\rho}^{\Lambda}(m, n; A) \leqslant C(m)\exp\{-C'F^{1/2}|n|^{3/2}\}, \qquad n < 0. \tag{III.40}$$

Proof. — In the beginning of this Section we have seen the usefulness of estimating the norm of the operator $T_{i+1}T_i$. Basically the results of Propositions III.1 and III.2 followed from the behaviour of this norm at fixed or small disorder. The present results will in contrast follow from the behaviour of this norm at large disorder, which we are going to work out now.

We are thus going to obtain bounds on the norm of the operator $T_{i+1}T_i$ at strong disorder, where T_i is defined from (III.12), (III.13). It is useful *[10]* to define two operators U and K_i acting on L^2-functions by

$$(Uf)(x) = \frac{1}{|x|} f\left(\frac{1}{x}\right) \tag{III.41}$$

$$(K_i f)(x) = \int r_i(x+y) f(y) dy. \tag{III.42}$$

Then U is a unitary operator on L^2 and we have

$$\| T_{i+1}T_i \|_2 = \| K_{i+1}UK_iU \|_2 = \| K_{i+1}UK_i \|_2 .$$

Since the Fourier transform is an isometry in L^2, we may as well estimate the norm of $S \equiv K_{i+1}UK_i$ in Fourier space

$$\tilde{S}\tilde{f}(k) = \tilde{K}_{i+1}\tilde{U}\tilde{K}_i\tilde{f}(k) = \int \tilde{r}_{i+1}(k) G(k, k') \bar{\tilde{r}}_i(k') \tilde{f}(k') dk'$$

where $|\tilde{r}_i(k)| \leqslant 1$, $\tilde{r}_i(k) = 1 \Leftrightarrow k = 0$ since it is the Fourier transform of the density probability r_i, and $G(k, k')$ is the kernel (in the distributional sense) of $\tilde{U}$:

$$G(k, k') = \int |y|^{-1} \exp\{i(ky^{-1} + k'y)\} dy$$

Let us now set

$$\varepsilon(k_0) = \sup_{|k| > k_0/2} (|\tilde{r}_i(k)|, |\tilde{r}_{i+1}(k)|) \tag{III.43}$$

and let χ denote the characteristic function of the interval $[-k_0/2, k_0/2]$. We then have

$$\begin{aligned} \| \tilde{S}\tilde{f} \|_2 &\leqslant \| \tilde{S}(1-\chi)\tilde{f} \|_2 + \| (1-\chi)\tilde{S}\chi\tilde{f} \|_2 + \| \chi\tilde{S}\chi\tilde{f} \|_2 \\ &\leqslant \varepsilon(k_0)(\| (1-\chi)\tilde{f} \|_2 + \| \chi\tilde{f} \|_2) + \| \chi\tilde{S}\chi\tilde{f} \|_2 . \end{aligned} \tag{III.44}$$

We are thus left to estimate

$$\| \chi \tilde{S} \chi \tilde{f} \|_2 \leqslant \left(\int_{|k|,|k'|,|k''| \leqslant k_0/2} | G(k, k') G(k, k'') \tilde{f}(k') \tilde{f}(k'') | \, dk dk' dk'' \right)^{1/2} .$$

But it was proved in Ref. [10] that for $| kk' | < 1$,

$$| G(k, k') | \leqslant - \text{Log} \, | kk' | + 10$$

so that

$$\| \chi \tilde{S} \chi \tilde{f} \|_2 \leqslant 4k_0 \left(5 + 2 \left| \text{Log} \frac{k_0}{2} \right| \right) \| \chi \tilde{f} \|_2$$

if $k_0/2 < 1$.

Equations (III.44) and (III.45) thus yield

$$\| \tilde{S} \|_2 \leqslant \varepsilon(k_0) + 4k_0 \left(5 + 2 \left| \text{Log} \frac{k_0}{2} \right| \right). \tag{III.46}$$

We are left with the choice of k_0 in order to get an optimal bound on this norm. In order to find it we have to recall that

$$r_i(x) = (\lambda \, | a_i |)^{-1} r((\lambda a_i)^{-1}(E - x))$$

so that

$$| \tilde{r}_i(k) | = | \tilde{r}(\lambda a_i k) | \tag{III.47}$$

and if $a = \min (| a_i | , | a_{i+1} |)$

$$\varepsilon(k_0) = \sup_{|k| > k_0/2} | \tilde{r}(\lambda a k) | . \tag{III.48}$$

Now, in view of the hypothesis on $\tilde{r}$ in the Proposition III.3 (which is satisfied by any natural density probability we are interested in), by choosing $k_0 = | \lambda a |^{-\gamma/(\gamma+1)}$ in (III.46) we obtain that if $| \lambda a | > 1$

$$\| \tilde{S} \|_2 \leqslant \left(C + 20 + \frac{4\gamma}{\gamma + 1} \log \frac{| \lambda a |}{2} \right) | \lambda a |^{-\gamma/(\gamma+1)} . \tag{III.49}$$

From this later bound and from the expression (III.9) of the correlation function $\bar{\rho}$ one can read directly the behaviour of $\bar{\rho}$ and so also of the eigenfunctions of Model I for $\alpha < 0$, if one remembers that $a = 0(n^{|\alpha|})$ and we obtain (III.39).

We turn now to the proof of (III.40), and we are going to bound (III.36). As $n \to \infty$, we see from the definitions of a_n, b_n and γ_n and from the properties of the Airy functions that

$$a_n \to 1$$
$$b_n \sim \exp | F_n |^{1/2}$$
$$\gamma_n \sim | F_n |^{-1/2} .$$

For n large enough, we are going to find a bound on the norm of the operator $T_nT_{n+1}T_{n+2}$. We have

$$(T_nT_{n+1}T_{n+2}f)(x) = |a_na_{n+1}a_{n+2}|^{+1/2}\int r_n(x + a_ny^{-1})|y|^{-1}$$
$$r_{n+1}(y + a_{n+1}z^{-1})|z|^{-1}r_{n+2}(z + a_{n+2}t^{-1})|t|^{-1}f(t)dydzdt$$
$$= |a_na_{n+1}a_{n+2}|^{+1/2}\int|\gamma_{n+1}|^{-1}r(\gamma_{n+1}^{-1}V)dV\int r_n(x + a_ny^{-1})|y|^{-1}$$
$$\delta(y + a_{n+1}z^{-1} - b_{n+1} - V)|z|^{-1}r_{n+2}(z + a_{n+2}t^{-1})|t|^{-1}f(t)dydzdt.$$

Thus by convexity of the norm

$$\|T_nT_{n+1}T_{n+2}f\|_2$$
$$\leqslant |a_na_{n+1}a_{n+2}|^{1/2}\int|\gamma_{n+1}|^{-1}r(\gamma_{n+1}^{-1}V)dV\Big\|\int r_n(x+a_ny^{-1})|y|^{-1}$$
$$\delta(y+a_{n+1}z^{-1}-b_{n+1}-V)|z|^{-1}r_{n+2}(z+a_{n+2}t^{-1})|t|^{-1}f(t)dydzdt\Big\|_2$$

and we are thus lead to estimate the norm of the operator T'_V defined by

$$(T'_Vf)(x)$$
$$= |a_na_{n+1}a_{n+2}|^{1/2}|\int r_n(x+a_n(V+b_{n+1}-a_{n+1}z^{-1})^{-1})|V+b_{n+1}$$
$$- a_{n+1}z^{-1}|^{-1}|z|^{-1}r_{n+2}(z + a_{n+1}t^{-1})|t|^{-1}f(t)dzdt$$
$$= |a_na_{n+1}a_{n+2}|^{1/2}\int r_n(x+a_n(V+b_{n+1})^{-1}(u+1)u^{-1})|u|^{-1}|V+b_{n+1}|^{-1}$$
$$r_{n+2}((u + 1)(V + b_{n+1})^{-1} + a_{n+2}t^{-1})|t|^{-1}f(t)dudt.$$

But now we can see that the operator T'_v is unitarily equivalent to an operator of the type T_nT_{n+2} except that the disorder in there has been multiplied by $|b_{n+1} + V|$. Then (III.40) will follow from (III.49) and the fact that $|b_{n+1}| \sim \exp|Fn|^{1/2}$. □

ACKNOWLEDGEMENTS

B. Souillard is glad to thank W. Kohn and the Institute for Theoretical Physics at Santa Barbara, where part of this work has been done. F. Delyon and B. Souillard are glad to thank E. Stone and W. Luxembourg for the hospitality of Caltech, where part of this work has been done. This work has been partially supported by a Nato Grant. B. Simon's research is partially supported under NSF grant MCS-81-20833.

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(Manuscrit reçu le 17 octobre 1984)