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On the relation between the Einstein and the Komar expressions for the energy of the gravitational field

by

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ABSTRACT. — It is shown, that the interpretation of the Einstein energy-momentum « pseudo-tensor », « covariantized » with the help of a background metric, as the energy-momentum tensor of the gravitational field with respect to a background field, is consistent with a geometric hamiltonian analysis. It is also shown, that the von Freud superpotential and the Komar superpotential describe the dynamics of the gravitational field in different function spaces, subject to different boundary conditions. One can pass from one superpotential to the other by performing a Legendre transformation on the boundary. It is explained why the ADM and the von Freud energy expressions are the same, for asymptotically flat space-times.

RÉSUMÉ. — On démontre que l'interprétation du « pseudo-tenseur » d'énergie-impulsion d'Einstein, « covariantisé » à l'aide d'une métrique de fond, comme tenseur d'énergie-impulsion du champ gravitationnel par rapport à une métrique de fond, est consistante avec une analyse hamiltonienne géométrique (développée par Kijowski et Tulczyjew [10]). On démontre que les superpotentiels d'Einstein et de Komar sont des hamiltoniens pour la dynamique du champ gravitationnel dans des espaces fonctionnels où l'on impose des conditions de bord différentes. Le passage

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d'un superpotentiel à l'autre se fait à l'aide d'une sorte de transformation de Legendre. On explique pourquoi l'hamiltonien ADM coïncide avec celui d'Einstein dans le cas de métriques asymptotiquement Minkowskienne.

INTRODUCTION

It is well known, that the hamiltonian for general relativity (or, in fact, for every theory invariant under the action of the group of diffeomorphisms) depends only upon the boundary values of the fields, and the values of their derivatives on the boundary of the region in which dynamics takes place. Since there exist different expressions for this hamiltonian, before starting a hamiltonian analysis of general relativity one has to decide which of the existing hamiltonians is the correct one.

The theory of geometric hamiltonian analysis developed by Kijowski and Tulczyjew [10] allows to derive unambiguously the hamiltonian for any Lagrangian theory. This approach, applied to general relativity, gives the following results:

a) if one adopts the Hilbert-Palatini variational principle, one obtains the « Komar's energy expression » [3] [7] [11] [15] as the hamiltonian of the theory.

b) If one adopts the second order Hilbert lagrangian, one obtains the Komar hamiltonian ⁽¹⁾.

c) In the purely affine first order formulation of general relativity [9] once again one obtains the Komar expression.

These three examples suggest that the correct hamiltonian for general relativity is given by Komar's expression. However, as shown by Regge and Teitelboim [13], the so-called ADM expression for the energy [2] may also be used as the hamiltonian for the gravitational field, and this expression is not the same as the Komar expression, even for asymptotically flat space-times. Since the ADM hamiltonian is a special case of the Einstein energy expression [3] [5] [7] [15], or, rather, of the so-called von Freud superpotential [3] [6] [7] [15], which can be derived from the Einstein « pseudo-tensor », the work of Regge and Teitelboim shows that the von Freud superpotential can be in fact used as the hamiltonian for general

⁽¹⁾ A. Smólski, Ph. D. thesis, Warsaw University 1984 (unpublished). The derivation of this formula requires the generalization of the Kijowski-Tulczyjew formalism to higher order field theories. This has been done by Tulczyjew (unpublished), cf. also the Ph. D. thesis of A. Smólski.

relativity. The argument used in [9] to reject this expression, namely that the von Freud superpotential is obtained by arbitrarily splitting the « true » hamiltonian vector density of the gravitational field into two non-covariant parts and discarding a total divergence, is not entirely convincing, since:

a) as pointed out above, the von Freud's expression can be used as the hamiltonian for general relativity, and

b) it has the property of being positive definite for reasonable physical matter fields [4] [14] [16].

The Einstein's energy « pseudo-tensor » was originally derived [5] using the standard flat space-time hamiltonian formulation, starting from the non-invariant first order gravitational lagrangian obtained from the Hilbert lagrangian by dropping a divergence. Such an approach violates the spirit of Einstein's theory—the geometrical interpretation of the resulting quantities being rather obscure—however it suggests that the final expression for the energy « has something to do with hamiltonian dynamics ». It has been noted by several authors, that the so called « pseudo-expressions » for the energy have tensorial transformation properties, if one interprets them as « the energy of the gravitational field measured with respect to a tetrad field » [7], or as « the energy of the gravitational field measured with respect to a background metric » [1] [4]. If one adopts the « background metric » approach, one can write a first order invariant lagrangian for the gravitational field, depending upon the physical metric, its first derivatives and the background metric, and one can use the geometrical techniques developed by Kijowski and Tulczyjew [10] to derive the hamiltonian for such a theory (a non-invariant lagrangian can be used in their theory, but the transformation properties of the infinitesimal dynamics are then inconsistent with the geometrical properties of the infinitesimal phase space, and the whole construction loses its nice geometrical character). The background metric approach will be adopted in this paper.

It is shown, that the interpretation of (a slight modification of) the Einstein energy-momentum as the energy of the gravitational field measured with respect to a background metric fits nicely into a manifestly covariant hamiltonian analysis. The final expression for the hamiltonian—a modification of the von Freud superpotential—can therefore be consistently used as the hamiltonian for the gravitational field. If the background is asymptotically Minkowskian, one recovers the standard von Freud expression for the energy.

The most important result of this paper is the derivation of a very simple relation between the Komar hamiltonian and the von Freud hamiltonian. It is shown, that they describe the dynamics of the gravitational field in spaces of solutions of Einstein's equations, where different boundary conditions are imposed. The use of one or of the other expression for the hamil-

tonian is therefore subject to the choice of boundary conditions, which a specific physical situation forces us to adopt.

To change the boundary conditions one has to perform a boundary Legendre transformation. In the « von Freud mode » one fixes some components of the metric—the gravitational potential—as having prescribed boundary values in the relevant function space, with some combinations of the connection as response parameters, and this is at the origin of the non-tensorial character of the von Freud superpotential (without the introduction of a background metric). The relation between both hamiltonians for general relativity is very similar to the relation that exists between the symmetrical (gauge-invariant) and the canonical (non gauge-invariant) energy momentum tensors in electrodynamics. As has been shown by Kijowski ⁽²⁾, both can be used as hamiltonians for the electromagnetic field, depending upon what boundary conditions one is forced to choose in a specific physical situation. It must be emphasized that this observation of prof. Kijowski is at the origin of this work.

For conceptual simplicity, the vacuum Einstein theory only is considered in this paper. All the results presented here may easily be generalized to the matter case. The generalization is straightforward in the case of matter fields having vanishing the second energy-momentum tensor [9] [10], such as Yang-Mills fields, for example, the resulting formula for the total energy being the same as in the vacuum case. If the second energy-momentum tensor does not vanish, the generalization requires a modification of the superpotentials, and the use of Belinfante-Rosenfeld theorem, along very similar lines as in [9].

1. THE FIRST ORDER METRIC LAGRANGIAN

Let $g_{\mu\nu}$ be the physical metric, $f_{\mu\nu}$ the background metric, $g^{\mu\nu}$ and $f^{\mu\nu}$ their respective inverses. One easily derives the following relation between $\Gamma_{\mu\nu}^{\lambda}$ (the Christoffel symbol of the metric $g_{\mu\nu}$) and $\Phi_{\mu\nu}^{\lambda}$ (the Christoffel symbol of $f_{\mu\nu}$):

$$\begin{aligned}\Gamma_{\mu\nu}^{\lambda} &= \Phi_{\mu\nu}^{\lambda} + C_{\mu\nu}^{\lambda} \\ C_{\mu\nu}^{\lambda} &= (e_{\mu\nu}^{;\lambda} - e^{\lambda}_{\mu;\nu} - e^{\lambda}_{\nu;\mu})/2,\end{aligned}\quad (1.1)$$

where

$$e^{\mu\nu} = g^{\mu\nu} - f^{\mu\nu}, \quad (1.2)$$

a semi-colon or a ∇ denotes covariant differentiation with respect to the

⁽²⁾ J. Kijowski, seminar of the Institute for Theoretical Physics of the Polish Academy of Sciences, unpublished.

background metric, all the indices (but in $f^{\mu\nu}$) are raised and lowered with $g^{\mu\nu}$, for example

$$e_{\alpha\beta}{}^{;\gamma} = g_{\alpha\mu}g_{\beta\nu}g^{\gamma\sigma}e^{\mu\nu}{}_{;\sigma} = g_{\alpha\mu}g_{\beta\nu}g^{\gamma\sigma}g^{\mu\nu}{}_{;\sigma}$$

(notice that raising and lowering of indices does not commute with covariant differentiation with respect to $f_{\mu\nu}$).

From (1.1) one can obtain the relation between the curvature tensors

$$R^{\alpha}{}_{\beta\gamma\delta} = \Omega^{\alpha}{}_{\beta\gamma\delta} + 2C^{\alpha}_{\beta[\delta;\gamma]} + C^{\alpha}_{\sigma\gamma}C^{\sigma}_{\beta\delta} - C^{\alpha}_{\sigma\delta}C^{\sigma}_{\beta\gamma}, \tag{1.3}$$

where

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma}$$

is the curvature tensor of $\Gamma^{\alpha}_{\beta\gamma}$, and $\Omega^{\alpha}_{\beta\gamma\delta}$ is the curvature tensor of $\Phi^{\alpha}_{\beta\gamma}$.

The Hilbert lagrangian

$$\mathcal{L} = \sqrt{-\det g_{\mu\nu}}R/16\pi = \sqrt{-\det g}g^{\mu\nu}R_{\mu\nu}/16\pi, \\ R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu},$$

(units $G = c = 1$ are used, the signature of the metric is $-+++$) may be written in the following form

$$\mathcal{L} = \sqrt{-\det g} g^{\beta\delta}(\Omega_{\beta\delta} + 2C^{\alpha}_{\beta[\delta;\alpha]} + C^{\alpha}_{\sigma\alpha}C^{\sigma}_{\beta\delta} - C^{\alpha}_{\sigma\delta}C^{\sigma}_{\beta\alpha})/16\pi. \tag{1.4}$$

From the identity

$$2eg^{\beta\delta}C^{\alpha}_{\beta[\delta;\alpha]} = -Z^{\alpha}{}_{;\alpha} + 2eg^{\beta\delta}(C^{\lambda}_{\beta\alpha}C^{\alpha}_{\lambda\delta} - C^{\alpha}_{\lambda\alpha}C^{\lambda}_{\beta\delta}), \tag{1.5}$$

where

$$e = (\det(f^{\alpha\mu}g_{\mu\beta}))^{1/2} \\ Z^{\alpha} = e(g^{\alpha\mu}C^{\beta}_{\mu\beta} - g^{\mu\nu}C^{\alpha}_{\mu\nu}) \tag{1.6}$$

(to derive (1.5), the identities:

$$e_{;\alpha} = eC^{\beta}_{\beta\alpha}, \quad g^{\mu\nu}{}_{;\alpha} = -2C^{(\mu\nu)}{}_{\alpha}, \tag{1.7}$$

may be used), one obtains

$$\mathcal{L} = \mathcal{L}_1 - \sqrt{|\det f_{\mu\nu}|}Z^{\alpha}{}_{;\alpha}/16\pi, \tag{1.8}$$

$$\mathcal{L}_1 = \sqrt{|\det f_{\mu\nu}|}L_1/16\pi, \tag{1.9}$$

$$L_1 = eR + Z^{\alpha}{}_{;\alpha} = eA,$$

$$A = g^{\beta\delta}(\Omega_{\beta\delta} + C^{\alpha}_{\beta\gamma}C^{\gamma}_{\alpha\delta} - C^{\alpha}_{\alpha\gamma}C^{\gamma}_{\beta\delta}).$$

We can now consider the theory described by the coordinate invariant lagrangian \mathcal{L}_1 , which depends on the metric $g_{\mu\nu}$ and its derivatives up to first order. The field equations for such a theory

$$\partial_{\lambda}(\partial\mathcal{L}_1/\partial g^{\mu\nu}{}_{,\lambda}) = \partial\mathcal{L}_1/\partial g^{\mu\nu} \tag{1.10}$$

may be written in a manifestly invariant form

$$\nabla_\lambda \Pi_{\mu\nu}^\lambda = \partial^* \mathcal{L}_1 / \partial g^{\mu\nu}, \quad (1.11)$$

where

$$\Pi_{\mu\nu}^\lambda = \partial \mathcal{L}_1 / \partial g^{\mu\nu; \lambda} = \partial \mathcal{L}_1 / \partial g^{\mu\nu; \lambda} \quad (1.12)$$

is a tensor density symmetric in the first two indices, $\partial^* / \partial g^{\mu\nu}$ means differentiation with respect to $g^{\mu\nu}$, if $g^{\mu\nu; \lambda}$ and $g^{\mu\nu}$ are considered as independent variables ⁽³⁾. In Appendix A, it is shown that the canonical momenta $\Pi_{\mu\nu}^\lambda$ take the following form:

$$\begin{aligned} \Pi_{\mu\nu}^\lambda &= \sqrt{|\det f_{\alpha\beta}|} P_{\mu\nu}^\lambda \\ P_{\mu\nu}^\lambda &= -D_{\mu\nu}^\lambda - g_{\mu\nu} Z^\lambda / 2 \\ D_{\mu\nu}^\lambda &= e(C_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda C_{\nu)\alpha}^\alpha) \end{aligned} \quad (1.13)$$

Equations (1.11) are, of course, equivalent to Einstein equations (\mathcal{L}_1 differs from the Hilbert lagrangian by a divergence, irrespective of the background metric chosen). For the sake of completeness, it is shown explicitly in Appendix A.

2. THE HAMILTONIAN FORMULATION OF THE THEORY

We present here (in a vulgarized version) the results of Kijowski and Tulczyjew [10], in their application to the theory described by the lagrangian (1.9). In order to put a theory in a time-evolution form, we need to define a time in the space-time considered. Since in a general Lorentz manifold there is no preferred time coordinate, the whole construction will depend upon an arbitrary chosen vector field X , the flow of which will provide the required time coordinate. Consider now the following functional defined on the space of Lorentz metrics on some manifold:

$$I = \int_{\Sigma} \mathcal{L}_1 X \lrcorner \eta, \quad (2.1)$$

where Σ is any (also arbitrarily chosen) hypersurface transversal to the flow of X (the time-zero hypersurface), and

$$\eta = dx^0 \wedge \dots \wedge dx^3 \quad X \lrcorner \eta = X^\alpha \eta_\alpha = X^\alpha \partial_\alpha \lrcorner \eta.$$

⁽³⁾ In this paper the convention, that the derivative with respect to the set of independent variables $\{g_{\mu\nu}, \mu \leq \nu\}$ is one half of the usual one, is used. Therefore, $df(g^{\alpha\beta}) = \partial f / \partial g^{\alpha\beta} dg^{\alpha\beta}$ and not $1/2 \partial f / \partial g^{\alpha\beta} dg^{\alpha\beta}$. The same remark applies to differentiation with respect to any *a priori* symmetric quantities.

The variation of I in a point $g^{\mu\nu}$ satisfying the field equations yields:

$$\begin{aligned} \delta I &= \int_{\Sigma} (\Pi_{\mu\nu}{}^{\lambda} \delta g^{\mu\nu})_{,\lambda} X \lrcorner \eta = \int_{\Sigma} \{ (\Pi_{\alpha\beta}{}^{\mu} \delta g^{\alpha\beta} X^{\lambda})_{,\lambda} - \Pi_{\alpha\beta}{}^{\lambda} X^{\mu}{}_{,\lambda} \delta g^{\alpha\beta} \} \eta_{\mu} \\ &\quad + \int_{\partial\Sigma} X^{[\mu} \Pi_{\alpha\beta}{}^{\nu]} \delta g^{\alpha\beta} \eta_{\nu\mu} \\ &= \int_{\Sigma} \{ \Pi_{\alpha\beta}{}^{\mu} \delta L_X g^{\alpha\beta} + L_X \Pi_{\alpha\beta}{}^{\mu} \delta g^{\alpha\beta} \} \eta_{\mu} + \int_{\partial\Sigma} X^{[\mu} \Pi_{\alpha\beta}{}^{\nu]} \delta g^{\alpha\beta} \eta_{\nu\mu} \end{aligned} \tag{2.2}$$

where $\eta_{\mu\nu} = \partial_{\mu} \lrcorner \eta_{\nu}$, L_X denotes the Lie derivative with respect to X , and to derive (2.2) the identity

$$d(X^{\mu\nu} \eta_{\mu\nu}) = 2X^{\mu\nu}{}_{,\mu} \eta_{\nu}$$

has been used, which holds for any antisymmetric tensor density $X^{\mu\nu}$.

To obtain the hamiltonian of the theory, we perform a Legendre transformation which exchanges $\Pi_{\alpha\beta}{}^{\mu} \eta_{\mu}$ with $L_X g^{\mu\nu}$:

$$H = \int_{\Sigma} (\Pi_{\alpha\beta}{}^{\mu} L_X g^{\alpha\beta} - X^{\mu} \mathcal{L}_1) \eta_{\mu} = \int_{\Sigma} E^{\mu}(X) \eta_{\mu} \tag{2.3}$$

From (2.2) it follows, that

$$\delta H = \int_{\Sigma} (L_X g^{\alpha\beta} \delta \Pi_{\alpha\beta}{}^{\mu} - L_X \Pi_{\alpha\beta}{}^{\mu} \delta g^{\alpha\beta}) \eta_{\mu} + \int_{\partial\Sigma} X^{[\mu} \Pi_{\alpha\beta}{}^{\nu]} \delta g^{\alpha\beta} \eta_{\mu\nu}. \tag{2.4}$$

In order that the functional H defines a hamiltonian vector field on the space of solutions of Einstein equations, we must restrict ourselves to the space of functions satisfying

$$\int_{\partial\Sigma} X^{[\mu} \Pi_{\alpha\beta}{}^{\nu]} \delta g^{\alpha\beta} \eta_{\mu\nu} = 0. \tag{2.5}$$

If, for some physical reasons, we would be interested in the evolution of a system where certain components of the connection and not of the metric are fixed on the boundary, we have to perform a Legendre transformation on the boundary. From

$$Z^{\mu} = -g^{\alpha\beta} P_{\alpha\beta}{}^{\mu}, \tag{2.6}$$

one easily obtains

$$g^{\alpha\beta} \delta \Pi_{\alpha\beta}{}^{\mu} = -2\Pi_{\alpha\beta}{}^{\mu} \delta g^{\alpha\beta} + \sqrt{-\det g_{\alpha\beta}} (\delta \Gamma^{\mu}_{\alpha\beta} - \delta^{\mu}_{(\alpha} \delta \Gamma^{\gamma}_{\beta)\gamma}) / 16\pi \tag{2.7}$$

therefore

$$\sqrt{|\det f_{\alpha\beta}|} \delta Z^{\mu} = 16\pi \Pi_{\alpha\beta}{}^{\mu} \delta g^{\alpha\beta} - \sqrt{-\det g_{\alpha\beta}} g^{\alpha\beta} (\delta \Gamma^{\mu}_{\alpha\beta} - \delta^{\mu}_{(\alpha} \delta \Gamma^{\gamma}_{\beta)\gamma}) \tag{2.8}$$

Equation (2.8) shows, that

$$\delta \left(H - \int_{\partial\Sigma} X^{[\mu} Z^{\nu]} \sqrt{|\det f|} \eta_{\mu\nu} / 16\pi \right) = \delta H' = \int_{\Sigma} (L_X g^{\alpha\beta} \delta \Pi_{\alpha\beta}{}^{\mu} - L_X \Pi_{\alpha\beta}{}^{\mu} \delta g^{\alpha\beta}) \eta_{\mu} \\ - \int_{\partial\Sigma} \sqrt{-\det g_{\alpha\beta}} g^{\alpha\beta} X^{[\mu} (\delta \Gamma_{\alpha\beta}^{\nu]} - \delta_{(\alpha}^{\nu]} \delta \Gamma_{\beta\gamma)}^{\gamma}) \eta_{\nu\mu} / 16\pi. \quad (2.9)$$

Therefore, H' is the appropriate hamiltonian in the space of functions satisfying

$$\int_{\partial\Sigma} \sqrt{-\det g_{\alpha\beta}} g^{\alpha\beta} X^{[\mu} (\delta \Gamma_{\alpha\beta}^{\nu]} - \delta_{(\alpha}^{\nu]} \delta \Gamma_{\beta\gamma)}^{\gamma}) \eta_{\nu\mu} = 0. \quad (2.10)$$

It must be noted, that the transformation leading from one superpotential to the other is a Legendre transformation on the boundary only if one « forgets » about the interior degrees of freedom. This is due to the fact, that $g^{\alpha\beta}$ appears as a control parameter in the volume integral (2.9), and therefore should not appear as a response parameter in the boundary part of (2.9). The mixing of the interior degrees of freedom and of the boundary degrees of freedom is related to the gauge freedom, which makes the symplectic structure associated with the special symplectic form « given » by the right hand side of (2.9) degenerate. We conjecture, that the transformation relating both superpotentials is a Legendre transformation in the phase space of « true gravitational degrees of freedom », a phase space which has to include the boundary degrees of freedom.

3. THE HAMILTONIAN SUPERPOTENTIAL

The hamiltonian vector density

$$E^{\lambda}(\mathbf{X}) = \Pi_{\mu\nu}{}^{\lambda} L_X g^{\mu\nu} - X^{\lambda} \mathcal{L}_1 \quad (3.1)$$

takes the following form:

$$E^{\lambda}(\mathbf{X}) = \tilde{t}^{\lambda}{}_{\mu} X^{\mu} + \tilde{t}^{\lambda}{}_{\mu}{}^{\nu} X^{\mu}{}_{;\nu}, \quad (3.2)$$

where

$$\tilde{t}^{\lambda}{}_{\mu} = \sqrt{|\det f|} t^{\lambda}{}_{\mu} / 16\pi, \\ \tilde{t}^{\lambda}{}_{\mu}{}^{\nu} = \sqrt{|\det f|} t^{\lambda}{}_{\mu}{}^{\nu} / 16\pi, \\ t^{\lambda}{}_{\mu} = P_{\alpha\beta}{}^{\lambda} g^{\alpha\beta}{}_{;\mu} - \delta_{\mu}^{\lambda} L_1, \\ t^{\lambda}{}_{\mu}{}^{\nu} = -2P_{\mu\alpha}{}^{\lambda} g^{\alpha\nu}. \quad (3.3)$$

In the terminology of Kijowski and Tulczyjew [10], $\tilde{t}^{\lambda}{}_{\mu}$ and $\tilde{t}^{\lambda}{}_{\mu}{}^{\nu}$ are, respectively, the first and second energy-momentum tensors of the gravitational field, as measured with respect to the background metric.

From (1.13), one obtains

$$t^\lambda{}_\mu = -2e(C^{(\lambda\alpha)}{}_\mu C^\beta{}_{\alpha\beta} - C^{\alpha\beta}{}_\mu C^\lambda{}_{\alpha\beta}) + Z^\lambda C^\alpha{}_{\alpha\mu} - \delta^\lambda_\mu L_1 \tag{3.4}$$

If one takes the background to be the Minkowski metric, and chooses the standard rectangular coordinates, from the resulting identities

$$C^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} \quad e = \sqrt{-\det g_{\alpha\beta}}$$

expression (3.4) reduces to the classical expression for the Einstein energy « pseudo-tensor ».

PROPOSITION 1. — If X^α is chosen to be a Killing vector of the background metric, and the field equations for $g^{\mu\nu}$ are satisfied, then

$$E^\lambda{}_{;\lambda}(X) = E^\lambda{}_{,\lambda}(X) = 0. \tag{3.5}$$

Proof. — Choose coordinates in which $X = \partial/\partial t$ (it is $X^0 = 1, X^i = 0$). Since equation (3.5) has an invariant character, it is sufficient to show that it holds in this coordinate system. Since X is a Killing vector, the only dependence of \mathcal{L}_1 upon t will be through the field $g^{\mu\nu}$. One has

$$\begin{aligned} E^\lambda{}_{,\lambda} &= \Pi_{\mu\nu}{}^\lambda{}_{,\lambda} \partial g^{\mu\nu} / \partial t + \Pi_{\mu\nu}{}^\lambda \partial^2 g^{\mu\nu} / \partial x^\lambda \partial t - \partial \mathcal{L}_1 / \partial t \\ &= (\Pi_{\mu\nu}{}^\lambda{}_{,\lambda} - \partial \mathcal{L}_1 / \partial g^{\mu\nu}) \partial g^{\mu\nu} / \partial t + (\Pi_{\mu\nu}{}^\lambda - \partial \mathcal{L}_1 / \partial g^{\mu\nu}{}_{,\lambda}) \times \partial^2 g^{\mu\nu} / \partial x^\lambda \partial t = 0, \end{aligned}$$

in virtue of the field equations (1.10). We have also used the fact, that

$$L_X g^{\mu\nu} = g^{\mu\nu}{}_{,\alpha} X^\alpha - 2g^{\alpha(\mu} X^{\nu)}{}_{,\alpha} = \partial g^{\mu\nu} / \partial t$$

in the coordinate system chosen.

The vanishing of the divergence of E^λ is a necessary condition for the existence of a superpotential for E^λ (note that prop. 1 guarantees the vanishing of $E^\lambda{}_{,\lambda}$ for background Killing vector fields only, $E^\lambda{}_{,\lambda}$ may vanish for a larger class of vector fields, but certainly not for arbitrary X):

$$E^\lambda = E^{\nu\lambda}{}_{;\nu} = E^{\nu\lambda}{}_{,\nu}, \tag{3.6}$$

where $E^{\nu\lambda}$ is an antisymmetric tensor density. It is proved in the Appendix B, that the straightforward generalization of the von Freud superpotential [3] [6] [7] [15]

$$\sqrt{-\det g_{\alpha\beta}} U_\nu{}^{\mu\beta} = 2g_{\nu\alpha} (\det g g^{\alpha\beta} g^{\mu\lambda}){}_{,\lambda}, \tag{3.7}$$

namely (4) :

$$e U_\nu{}^{\mu\beta} = 2g_{\nu\alpha} (e^2 g^{\alpha\mu} g^{\beta\lambda}){}_{,\lambda}, \tag{3.8}$$

provides a superpotential for X covariantly constant with respect to the background

$$E_E^\lambda = U_\mu{}^{\nu\lambda} X^\mu \sqrt{|\det f_{\alpha\beta}|} / 16\pi. \tag{3.9}$$

(4) A similar superpotential has already been considered in [8], see also [1] for a « covariantization » of the Papapetrou-Gupta « pseudotensor » [12] with the use of a background metric.

Since proposition 1 is true for any Killing vectors (not only covariantly constant), one may expect that there should exist a superpotential for any Killing vector. The corresponding superpotential, as shown in Appendix B, is given by the following formula:

$$E^{v\lambda} = \sqrt{|\det f_{\alpha\beta}|} (U_{\kappa}{}^{v\lambda} X^{\kappa} + 2eg^{\alpha[v}\delta_{\beta}^{\lambda]} X^{\beta}{}_{;\alpha}) / 16\pi. \tag{3.10}$$

If one takes X to be covariantly constant with respect to a Minkowski background, and takes rectangular coordinates, expression (3.10) reduces obviously to the von Freud superpotential (3.7).

(In fact, formula (3.10) provides a hamiltonian superpotential for all vector fields X^α satisfying (as is easily seen from the formulae in Appendix B) the following equation:

$$2g^{\alpha[v} X^{\lambda]}{}_{;\alpha\nu} = (g^{\beta\gamma} \Omega^{\lambda}{}_{\beta\gamma\kappa} - g^{\lambda\beta} \Omega_{\beta\kappa}) X^{\kappa}.$$

This equation, in the flat background case, admits as solutions all the vector fields linear in the flat coordinates:

$$X^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + B^{\alpha},$$

with A^α_β—an arbitrary matrix (not necessarily antisymmetric) ⁽⁵⁾ with constant coefficients, and B^α—a constant vector.)

The formula (2.3) can be written now in the following form:

$$H = \frac{1}{2} \int_{\partial\Sigma} E^{v\lambda} \eta_{v\lambda} \tag{3.11}$$

H is the appropriate hamiltonian for the space of functions satisfying (2.5). The hamiltonian for functions satisfying (2.10) is given by H'

$$H' = \frac{1}{2} \int_{\partial\Sigma} \left(E^{v\lambda} - \frac{1}{8\pi} X^{[v} Z^{\lambda]} \sqrt{|\det f_{\alpha\beta}|} \right) \eta_{v\lambda}. \tag{3.12}$$

From (B.2), one obtains

$$\begin{aligned} E_K^{v\lambda} &= E^{v\lambda} - \frac{1}{8\pi} X^{[v} Z^{\lambda]} \sqrt{|\det f|} \\ &= -\sqrt{|\det f_{\alpha\beta}|} (2eC^{[v\lambda]}{}_{\kappa} X^{\kappa} + 2eg^{\alpha[\lambda}\delta_{\beta}^{v]} X^{\beta}{}_{;\alpha}) / 16\pi. \end{aligned} \tag{3.13}$$

This is exactly the Komar superpotential [11]:

$$E_K^{v\lambda} = \sqrt{-\det g_{\alpha\beta}} (-X^{v\lambda} + X^{\lambda[v} X^{\lambda]v}) / 16\pi, \tag{3.14}$$

where a bar denotes covariant differentiation with respect to the physical metric g_{μν}.

⁽⁵⁾ This has been pointed out to the author by A. Smólski, and is due to the scalar density transformation properties of the first order lagrangian with respect to the transformations generated by such vector fields.

4. BACKGROUND DEPENDENCE, ADM ENERGY VERSUS EINSTEIN ENERGY

It is easy to understand why all reference to a background metric has disappeared in (3.14). Let us show, that the « volume » contribution to the special symplectic structure « given » by the right hand side of (2.4) is in fact background independent. In order to do this, we perform a canonical transformation to the variables

$$\begin{aligned}\Pi^{\mu\nu} &= \sqrt{-\det g_{\alpha\beta}} g^{\mu\nu} / 16\pi, \\ D_{\mu\nu}^\lambda &= C_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda C_{\nu)\sigma}^\sigma.\end{aligned}\quad (4.1)$$

An easy calculation shows, that

$$\theta_\Sigma^\mu = L_X g^{\alpha\beta} \delta \Pi_{\alpha\beta}^\mu - L_X \Pi_{\alpha\beta}^\mu \delta g^{\alpha\beta} = L_X \Pi^{\alpha\beta} \delta D_{\alpha\beta}^\mu - L_X D_{\alpha\beta}^\mu \delta \Pi^{\alpha\beta}. \quad (4.2)$$

We have

$$\delta D_{\mu\nu}^\lambda = \delta A_{\mu\nu}^\lambda, \quad (4.3)$$

where

$$A_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \Gamma_{\nu)\sigma}^\sigma,$$

because the background is fixed, and

$$L_X D_{\mu\nu}^\lambda = L_X A_{\mu\nu}^\lambda, \quad (4.4)$$

because X is a Killing vector of the background. Therefore

$$\theta_\Sigma^\mu = L_X \Pi^{\alpha\beta} \delta A_{\alpha\beta}^\mu - L_X A_{\alpha\beta}^\mu \delta \Pi^{\alpha\beta} \quad (4.5)$$

The hamiltonian H' is the generating function of the dynamics with respect to the special symplectic structure « given » by the right hand side of (2.9). This special symplectic structure is background independent, so is the dynamics, therefore no background quantities are required in the definition of H' .

Eq. (4.5) explains also why H depends only upon the boundary values of the metric (and its first derivatives)—the special symplectic structure « given » by the right hand side of (2.4) depends upon the background on the boundary only.

In order to show, that the equality of the von Freud and the ADM hamiltonians is not accidental, let us suppose that Σ is a slice of an asymptotically flat space-time, diffeomorphic to R^3 . Let us choose any global set of coordinates in a neighbourhood of Σ , in which the physical metric $g^{\mu\nu}$ satisfies the usual asymptotic conditions at spatial infinity

$$g_{\mu\nu} - \eta_{\mu\nu} = O(r^{-1}), \quad \partial_\sigma g_{\mu\nu} = O(r^{-2}) \quad (4.6)$$

The background metric will be chosen as the rectangular flat metric

defined by this set of coordinates. The hamiltonian analysis with respect to this background, as presented above, reduces to the standard special relativistic one. Because the Einstein pseudo-tensor was derived in such a framework, it follows that the final hamiltonian, as derived here, must be equal to the von Freud superpotential (for the boundary conditions (4. 6)). On the other hand, it can be shown ⁽⁶⁾ that, by reduction of constraints, the formalism of Kijowski and Tulczyjew reduces to the standard ADM analysis [2] modulo boundary terms, and the hamiltonians survive unchanged the reduction process, in spaces with suitable boundary conditions. For the boundary conditions (4. 1) we have

$$\begin{aligned} C_{\beta\gamma}^{\alpha} &= \Gamma_{\beta\gamma}^{\alpha} = 0(r^{-2}) \Rightarrow \Pi_{\alpha}^{\beta\gamma} = 0(r^{-2}), \\ \delta g^{\alpha\beta} &= 0(r^{-1}), \quad \delta\Gamma_{\beta\gamma}^{\alpha} = 0(r^{-2}), \end{aligned} \quad (4. 7)$$

and therefore (2. 5) is satisfied (note that (2. 10) is not). This shows that for asymptotically flat space-times, in the sense of (4. 6), the correct hamiltonian in the Kijowski-Tulczyjew framework is given by the von Freud superpotential. Because the ADM expression provides a hamiltonian, in the ADM framework, for asymptotically flat space-times, and because the Kijowski-Tulczyjew framework reduces to the ADM one, both hamiltonians must be equal up to a constant (a hamiltonian is always uniquely defined up to a constant). This constant must be zero because both expressions give zero, when evaluated on the Minkowski space-time metric.

CONCLUSIONS

It has been shown that a generalization of the Einstein energy momentum « pseudotensor », « covariantized » with the help of a background metric, can be derived by geometrical hamiltonian methods. It has been also shown that the Komar hamiltonian and the von Freud hamiltonian differ by a Legendre transformation, and describe the dynamics of the gravitational field on spaces of data satisfying different boundary conditions. This observation provides the missing link between the methods of Kijowski [9], and the results of Regge and Teitelboim [13]. The methods used here allow also to explain why the ADM hamiltonian, as derived by Regge and Teitelboim [13], is equal to the von Freud superpotential.

It must be emphasized, that the superpotential (3. 10) provides also a hamiltonian for rotational background Killing vectors, therefore an expression for angular momentum. Due to the presence of the derivatives of X

⁽⁶⁾ A. Smólski, M. Sc. thesis, Warsaw University (unpublished). See also A. Smólski, *Bull. Acad. Pol. Sci., ser. sci. phys. mat.*, t. 3, 1979, p. 187-194, and J. Kijowski, lectures at Journées Relativistes 1983 and 1984, to be published in the Proceedings.

in (3.10), this expression differs from the known expressions for angular momentum, even for asymptotically flat space-times.

It may be of some interest to mention, that the Abbott-Deser energy [1] and the superpotential (3.10) may or may not give the same number for the energy of a given field configuration, depending upon the exact meaning of the notion of « a metric asymptotic to another metric ».

Once this work was completed, the author has learned that the idea of using a background metric to define the energy density in general relativity has already been considered a long time ago (⁷). The author wishes to point out that he does not attach any physical significance to the background metric in the interior of space-time, unless a physical situation singles out this background metric (as is the case for weak perturbations around a given metric, for example). The same remark applies to the notion of energy density with respect to a background metric.

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⁽⁷⁾ N. Rosen, *Phys. Rev.*, t. 57, 1940, p. 147-150, See also N. Rosen, *GRG*, t. 9, 1978, p. 339-351, and references given therein. The author is grateful to prof. A. Trautman for pointing him out these references.

APPENDIX A

Some algebra leads from (1.1) to the following expression

$$\partial C_{\beta\gamma}^{\alpha} / \partial g^{\mu\nu};_{\lambda} = g_{\beta(\mu} g_{\nu)\gamma} g^{\alpha\lambda} / 2 - \delta_{(\mu}^{\alpha} g_{\nu)(\beta} \delta_{\gamma)}^{\lambda} \quad (\text{A.1})$$

From (A.1) one obtains

$$\begin{aligned} C_{\beta\delta}^{\gamma} \partial C_{\alpha\gamma}^{\alpha} / \partial g^{\mu\nu};_{\lambda} &= -C_{\beta\delta}^{\lambda} g_{\mu\nu} / 2 \\ g^{\beta\delta} \partial C_{\beta\delta}^{\alpha} / \partial g^{\mu\nu};_{\lambda} &= g^{\alpha\lambda} g_{\mu\nu} / 2 - \delta_{(\mu}^{\alpha} \delta_{\nu)}^{\lambda} \\ C_{\alpha\delta}^{\gamma} \partial C_{\beta\gamma}^{\alpha} / \partial g^{\mu\nu};_{\lambda} &= (g_{\beta(\mu} C_{\nu)}^{\lambda} - g_{\beta(\mu} C_{\nu)}^{\lambda} - C_{(\mu\nu)\delta} \delta_{\beta}^{\lambda}) / 2 \end{aligned} \quad (\text{A.2})$$

From (A.2) and (1.9) one easily obtains equations (1.13).

The field equations (1.12) may be written in tensor (and not tensor density) form:

$$\nabla_{\lambda} P_{\mu\nu}^{\lambda} = \partial^* L_1 / \partial g^{\mu\nu} \quad (\text{A.3})$$

From (1.9), (1.13) and (A.3) one obtains

$$\nabla_{\lambda} D_{\mu\nu}^{\lambda} + g_{\mu\nu;\lambda} Z^{\lambda} / 2 + e \partial^* A / \partial g^{\mu\nu} = e R g_{\mu\nu} / 2 \quad (\text{A.4})$$

From the identities

$$\begin{aligned} \partial^* C_{\beta\gamma}^{\alpha} / \partial g^{\mu\nu} &= (e_{\beta\gamma;(\mu} \delta_{\nu)}^{\alpha} + g_{\beta(\mu} e_{\nu)}^{\alpha};_{\gamma} - g_{\beta(\mu} e_{\nu)\gamma}^{\alpha} + g_{\gamma(\mu} e_{\nu)}^{\alpha};_{\beta} - g_{\gamma(\mu} e_{\nu)\beta}^{\alpha}) / 2 \\ &= \{ (C_{\gamma}^{\alpha}{}_{(\mu} - C_{\gamma(\mu}^{\alpha} g_{\nu)\beta}) + (C_{\beta}^{\alpha}{}_{(\mu} - C_{\beta(\mu}^{\alpha} g_{\nu)\gamma}) - C_{\beta\gamma(\mu} \delta_{\nu)}^{\alpha} - C_{\gamma\beta(\mu} \delta_{\nu)}^{\alpha}) \} / 2, \end{aligned} \quad (\text{A.5})$$

one obtains

$$\begin{aligned} C^{\gamma\delta}{}_{\alpha} \partial^* C_{\gamma\delta}^{\alpha} / \partial g^{\mu\nu} &= -C_{\beta\mu}^{\alpha} C_{\alpha\nu}^{\beta}, \\ \partial^* C_{\beta\alpha}^{\alpha} / \partial g^{\mu\nu} &= -C_{(\mu\nu)\beta}, \\ g^{\beta\gamma} \partial^* C_{\beta\gamma}^{\alpha} / \partial g^{\mu\nu} &= C_{(\mu\nu)}^{\alpha} - C_{\mu\nu}^{\alpha} - C_{\beta(\mu}^{\beta} \delta_{\nu)}^{\alpha}. \end{aligned} \quad (\text{A.6})$$

From (A.6) one obtains

$$\partial^* A / \partial g^{\mu\nu} = \Omega_{\mu\nu} - C_{\beta\mu}^{\alpha} C_{\alpha\nu}^{\beta} - C_{(\mu\nu)\lambda} Z^{\lambda} / e + C_{\alpha\mu}^{\alpha} C_{\beta\nu}^{\beta} \quad (\text{A.7})$$

From

$$R_{\mu\nu} = R_{\nu\mu} \quad \Omega_{\mu\nu} = \Omega_{\nu\mu}$$

one obtains

$$C_{\alpha\mu;\nu}^{\alpha} = C_{\alpha\nu;\mu}^{\alpha} \quad (\text{A.8})$$

(in fact $C_{\alpha\mu}^{\alpha}$ is a total differential). Equation (A.8) gives

$$D_{\mu\nu}^{\lambda};_{\lambda} = e(R_{\mu\nu} - \Omega_{\mu\nu} + C_{\beta\mu}^{\alpha} C_{\alpha\nu}^{\beta} - C_{\alpha\nu}^{\alpha} C_{\beta\mu}^{\beta}) \quad (\text{A.9})$$

Inserting (A.7) and (A.9) into (A.4) gives, as expected, the Einstein equations for $g^{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu} / 2 = 0.$$

APPENDIX B

First it will be shown, that if the field equations are satisfied by the physical metric $g_{\mu\nu}$, then, for X^α covariantly constant with respect to the background

$$t^\alpha_{\ \kappa} X^\kappa = - U_\kappa^{\alpha\beta}{}_{;\beta} X^\kappa = (- U_\kappa^{\alpha\beta} X^\kappa)_{;\beta} \tag{B.1}$$

where $U_\kappa^{\alpha\beta}$ is given by (3.8). From (3.8), (1.6) and (1.7) one obtains

$$U_\kappa^{\alpha\beta} = 2(eC^{[\beta\alpha]}_\kappa + \delta_\kappa^{[\alpha} Z^{\beta]}). \tag{B.2}$$

This implies

$$U_\kappa^{\alpha\beta}{}_{;\beta} = \delta_\kappa^\alpha Z^\beta{}_{;\beta} - Z^\alpha{}_{;\kappa} - 2e \{ C_{\sigma\beta}^\alpha C^{[\alpha\beta]}_\kappa + (C^{[\alpha\beta]}_\kappa)_{;\beta} \} \tag{B.3}$$

(remember that raising and lowering of indices does not commute with covariant differentiation). From

$$(C^{[\alpha\beta]}_\kappa)_{;\beta} = C^{[\alpha\beta]}_{\ \kappa;\beta} - C^{(\beta\gamma)}_\beta C^\alpha_{\ \gamma\kappa} + C^{(\alpha\gamma)}_\beta C^\beta_{\ \gamma\kappa}, \tag{B.4}$$

and from

$$Z^\alpha{}_{;\kappa} = C^\beta_{\ \beta\kappa} Z^\alpha + 2e(C^{[\beta\alpha]}_{\ \beta;\kappa} + C^{\beta\gamma}{}_\kappa C^\alpha_{\ \beta\gamma} - C^{(\alpha\beta)}_\kappa C^\gamma_{\ \beta\gamma}) = 2eC^{[\beta\alpha]}_{\ \beta;\kappa} + t^\alpha_\kappa + \delta^\alpha_\kappa L_1 \tag{B.5}$$

one obtains

$$U_\kappa^{\alpha\beta}{}_{;\beta} = -t^\alpha_\kappa - \delta^\alpha_\kappa L + e \{ 2C^{\beta\alpha}{}_{[\kappa;\beta]} + 2C^{\alpha\beta}{}_{\ \beta;\kappa]} + C^\alpha_{\ \sigma\beta} C^{\beta\alpha}{}_\kappa + C^{\beta\sigma}{}_\alpha C^\alpha_{\ \beta\kappa} - 2C^{(\alpha\gamma)}_\beta C^\beta_{\ \gamma\kappa} \}. \tag{B.6}$$

Since

$$2C^{\alpha\beta}{}_{[\kappa;\beta]} = -R^\alpha_\kappa - g^{\beta\gamma} \Omega_{\beta\gamma\kappa}^\alpha + C^\alpha_{\ \sigma\kappa} C^{\sigma\beta}{}_\beta - C^\alpha_{\ \gamma}{}^\beta C^\gamma_{\ \beta\kappa}, \tag{B.7}$$

$$2C^{\beta\alpha}{}_{\ \beta;\kappa]} = -R^\alpha_\kappa + g^{\alpha\beta} \Omega_{\beta\kappa}^\alpha + C^\beta_{\ \gamma\beta} C^\gamma_{\ \alpha\kappa} - C^\beta_{\ \gamma\kappa} C^\gamma_{\ \alpha\beta},$$

one finally obtains

$$U_\kappa^{\alpha\beta}{}_{;\beta} = -t^\alpha_\kappa + 2eG^\alpha_\kappa + eS^\alpha_\kappa, \tag{B.8}$$

where

$$S^\alpha_\kappa = g^{\beta\gamma} \Omega_{\beta\gamma\kappa}^\alpha - g^{\alpha\beta} \Omega_{\beta\kappa}^\alpha. \tag{B.9}$$

Since X^α is a background Killing vector, it satisfies the equation

$$X^\kappa{}_{;\alpha\beta} = \Omega^\kappa_{\ \alpha\beta\sigma} X^\sigma. \tag{B.10}$$

(B.10) implies, for $X^\alpha{}_{;\beta} = 0$,

$$S^\alpha_\kappa X^\kappa = 0, \tag{B.11}$$

which proves (B.1) if $G_{\mu\nu} = 0$.

Let us now turn our attention to the case of a general Killing vector X . The divergence of (3.10) gives

$$-16\pi E^{\nu\lambda}{}_{;\nu} = U_\kappa^{\lambda\nu}{}_{;\nu} X^\kappa + \{ U_\kappa^{\lambda\nu} + (2eg^{\nu[\lambda} \delta_\kappa^{\beta]})_{;\beta} \} X^\kappa{}_{;\nu} + 2eg^{\alpha[\lambda} \delta_\beta^{\nu]} X^\beta{}_{;\alpha\nu} \tag{B.12}$$

From (B.10) one obtains

$$2eg^{\alpha[\lambda} \delta_\beta^{\nu]} X^\beta{}_{;\alpha\nu} = -eS^\lambda_\kappa X^\kappa. \tag{B.13}$$

One also has

$$(2eg^{\nu[\lambda} \delta_\kappa^{\beta]})_{;\beta} = e(g^{\nu\lambda} C^\beta_{\ \beta\kappa} - 2C^{(\nu\lambda)}_{\ \kappa} + \delta_\kappa^{[\nu} C^{\lambda\beta]}_{\ \beta}) = 2P_{\kappa\alpha}{}^\lambda g^{\alpha\nu} - U_\kappa^{\lambda\nu}. \tag{B.14}$$

(B.8) combined with (B.13) and (B.14) allow us to write (B.12) in the following form:

$$| 16\pi E^{\nu\lambda}{}_{;\nu} = t^\lambda_\kappa X^\kappa - 2P_{\kappa\alpha}{}^\lambda g^{\alpha\nu} X^\kappa{}_{;\nu} = 16\pi E^\lambda(X)$$

which shows, that $E^{\lambda\nu}$ given by (3.10) is indeed the superpotential for the energy momentum tensors of the gravitational field with respect to a background metric.

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