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Uncertainty principle and joint distributions of observables

by

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ABSTRACT. — Connections between Heisenberg uncertainty relations and the existence of joint probability distributions of corresponding observables in the frame of quantum logics are investigated. It is shown that, provided the set of physical states is sufficient, uncertainly relations imply the nonexistence of joint probability distributions in any state.

RÉSUMÉ. — On étudie la connexion entre les relations d'incertitude de Heisenberg et l'existence de distributions de probabilités multiples des observables correspondants dans le cadre des logiques quantiques. On montre que, si l'ensemble des états physiques est suffisant, les relations d'incertitude impliquent la nonexistence des distributions de probabilité multiples dans quelque état que ce soit.

1. INTRODUCTION

The logic L of a physical system, i. e. the set of all experimentally verifiable propositions of that system, is usually supposed to be an orthomodular σ-poset [*1*] [*2*]. We shall suppose, in agreement with Varadarajan [*3*], that $(L, \leqslant, \perp, 0, 1)$ is an orthomodular σ-lattice.

Two elements a, b of L are orthogonal ($a \perp b$) if $a \leqslant b^{\perp}$, and they are compatible ($a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b^{\perp})$, $b = (a \wedge b) \vee (a^{\perp} \wedge b)$ (one of the last equalities is enough). A subset K of L is compatible if $a \leftrightarrow b$ for any $a, b \in K$. Any compatible subset of L is contained in a Boolean sub-σ-algebra of L [3].

Physical states are identified with the states (i. e. probability measures) on L, i. e. a state on L is the map $m: L \to \langle 0, 1 \rangle$ such that (i) $m(1) = 1$, (ii) $m(\vee a_i) = \Sigma m(a_i)$ for any sequence $\{ a_i \}_{i=1}^{\infty}$ of pairwise orthogonal elements of L.

Physical quantities are identified with the observables on L. If (X, S) is a measurable space (where S is a σ-algebra of subsets of the space X), the (X, S)-observable x on L is a σ-homomorphism from S to L, i. e. the map $x: S \to L$ such that (i) $x(X) = 1$, (ii) $x(E^c) = x(E)^{\perp}$, $E \in S$, $E^c = X \backslash E$, (iii) $x(\cup E_i) = \vee x(E_i)$ for any sequence $\{ E_i \}_{i=1}^{\infty}$ of S. If X is a topological space, we denote by B(X) the σ-algebra of Borel subsets of X, i.e. the σ-algebra generated by all open subsets of X. The physically most important case are (R, B(R))-observables, where R is the real line. We shall call the (R, B(R))-observables the real observables. The range $R(x)$ of an (X, S)-observable x is the set $R(x) = \{ x(E): E \in S \}$, which is a Boolean sub-σ-algebra of L. A set $\{ x_\alpha \}_\alpha$ of (X, S)-observables is compatible if $\bigcup_\alpha R(x_\alpha)$ is a compatible subset of L. If x is an (X, S)-observable, and m is a state on L, then the map $m_x: E \leftrightarrow m(x(E))$, $E \in S$, is a probability measure on S, which is called the probability distribution of the observable x in the state m. If x is a real observable, the expectation of x in the state m is

$$m(x) = \int_R \lambda m_x(d\lambda), \tag{1}$$

if the integral exists. The variance of x in m is

$$\mathrm{var}_m(x) = \int_R [\lambda - m(x)]^2 m_x(d\lambda), \tag{2}$$

if the integral exists.

2. JOINT DISTRIBUTIONS

The real observables $x_1, x_2, \ldots, x_n$ are said to have a joint distribution in the state m if there is a measure μ on $B(R^n)$ such that

$$\mu(E_1 \times E_2 \times \ldots \times E_n) = m(x_1(E_1) \wedge x_2(E_2) \wedge \ldots \wedge x_n(E_n)) \tag{3}$$

for any rectangle $E_1 x E_2 x \ldots x E_n \in B(R^n)$.

This notion has been introduced by Gudder [4] and by Jauch [5]. The notion of joint probability distribution can be generalized to any set

$\{ x_\alpha : \alpha \in A \}$ of real observables in a natural way: we say that the observables $\{ x_\alpha : \alpha \in A \}$ have a joint probability distribution in a state m if any finite subset of $\{ x_\alpha : \alpha \in A \}$ has one. The generalization of this notion to (X, B(X))-observables is also straightforward. The existence of joint distributions has been studied in [6] [7] [8] [9]. It is not *a priori* clear if the criteria of the existence used in [7] and [8] are the same as that used in [9]. Now we shall unify them. To this aim we need some definitions.

We say that a subset M of L is partially compatible with respect to an element a of L (abbreviated: M is p. c. a) if (*i*) $M \leftrightarrow a$ i. e. $b \leftrightarrow a$ for any $b \in M$, and (*ii*) $M \wedge a = \{ b \wedge a : b \in M \}$ is a compatible subset of L. It is a fact that $L_{[0,a]} = \{ b \in L : b \leqslant a \}$ is a logic with the orthocomplementation $b^* = b^\perp \wedge a$. The set $M \wedge a$ is compatible in L iff it is compatible in $L_{[0,a]}$. Put $D = \{ 0, 1 \}$ and $a^1 = a$, $a^0 = a^\perp$, $a \in L$. Let $F = \{ a_1, a_2, \ldots, a_n \}$ be any finite subset of L. The element

$$\text{com } F = \bigvee_{d \in D^n} a_1^{d_1} \wedge \ldots \wedge a_n^{d_n} \tag{4}$$

is called the commutator of F. It holds that F is p. c. com F (see [9]).

Now we shall introduce the following convention. Let $a = \Lambda \{ a_\alpha : \alpha \in A \}$, where A is any set of indices. We shall say that a is countably obtainable (over $\{ a_\alpha : \alpha \in A \}$) if there is an at most countable subset $N \subset A$ such that $a = \Lambda \{ a_{\alpha_i} : \alpha_i \in N \}$.

Let M be a subset of L and let $\wedge \{ \text{com } F : F \text{ is a finite subset of } M \}$ exist. Then we put

$$\text{com } M = \bigwedge_{F \subset M} \text{com } F \tag{5}$$

where the infimum is taken over all finite subsets F of M and we call com M the commutator of M. Clearly, com M = 1 iff M is compatible. Similarly as in [10], we can prove that M is p. c. com M if com M exists.

For the (X, S)-observables $x_1, x_2, \ldots, x_n$ on L put

$$\text{com } (x_1, \ldots, x_n) = \text{com } \left(\bigcup_{i=1}^{n} R(x_i) \right) \tag{6}$$

We shall show further that for the real observables $x_1, \ldots, x_n$ the commutator always exists and is countably obtainable.

Let us denote by K (K^σ) any finite (countable) measurable partition of (X, S), i. e. $K = \{ E_1, E_2, \ldots, E_n \}$ ($K^\sigma = \{ E_i \}_{i=1}^\infty$), $E_i \in S$, $E_i \cap E_j = \phi$ for $i \neq j$ and $\cup E_i = X$. Let us denote by K^s the s-partition of X, *i. e.* $K^s = \{ E_1, E_2, \ldots, E_s \}$, $s = 2, 3, \ldots$. For a given set of observables

$\{x_1, x_2, \ldots, x_n\}$ and partitions $\{K_1, K_2, \ldots, K_n\}$ $(\{K_1^\sigma, K_2^\sigma, \ldots, K_n^\sigma\})$ let us denote

$$a(K_1, K_2, \ldots, K_n) = \bigvee_{\{j_i : E_i^{j_i} \in K_i\}} \bigwedge_{i=1}^{n} x_i(E_i^{j_i}) \tag{7}$$

$$a(K_1^s, K_2^s, \ldots, K_n^s) = \bigvee_{\{j_i : E_i^{j_i} \in K_i^s\}} \bigwedge_{i=1}^{n} x_i(E_i^{j_i}) \tag{8}$$

$s = 2, 3, \ldots, \sigma$.

Especially, if $K_i = K_i^2 = \{E_i, E_i^c\}$, $i = 1, 2, \ldots, n$, put

$$a(E_1, E_2, \ldots, E_n) = \bigvee_{d \in D^n} x_1(E_1)^{d_1} \wedge \ldots \wedge x_n(E_n)^{d_n}$$
$$= \mathrm{com}\,(x_1(E_1), x_2(E_2), \ldots, x_n(E_n)) \tag{9}$$

Clearly, if K_i' is a refinement of K_i, then

$$a(K_1, \ldots, K_i', \ldots, K_n) \leqslant a(K_1, \ldots, K_i, \ldots, K_n),$$

and the same holds for the σ-partitions.

Let us further denote, provided the right sides exist,

$$a^0 \equiv a^0(x_1, \ldots, x_n) = \bigwedge_{(E_1, \ldots, E_n)} a(E_1, \ldots, E_n) \tag{10 i}$$

where the infimum is taken over all n-tuples $(E_1, \ldots, E_n)$ of the elements of S,

$$a^s \equiv a^s(x_1, \ldots, x_n) = \bigwedge_{(K_1^s, \ldots, K_n^s)} a(K_1^s, \ldots, K_n^s) \tag{10 ii}$$

where the infimum is taken over all s-partitions $(K_1^s, \ldots, K_n^s)$ of X, $s = 2, 3, \ldots$,

$$a^f \equiv a^f(x_1, \ldots, x_n) = \bigwedge_{(K_1, \ldots, K_n)} a(K_1, \ldots, K_n) \tag{10 iii}$$

where the infimum is taken over all finite partitions $(K_1, \ldots, K_n)$,

$$a^\sigma \equiv a^\sigma(x_1, \ldots, x_n) = \bigwedge_{(K_1^\sigma, \ldots, K_n^\sigma)} a(K_1^\sigma, \ldots, K_n^\sigma) \tag{10 iv}$$

where the infimum is taken over all σ-partitions $(K_1^\sigma, \ldots, K_n^\sigma)$.

It is clear that $a^0 = a^2$, and as any s-partition can be considered as an $s+1$-partition by adding the empty set, $a^s \geq a^{s+1} \geq a^f \geq a^\sigma$, $s = 2, 3, \ldots$. It is also easy to see that, provided it exists, $a^f = \mathrm{com}\,(x_1, \ldots, x_n)$.

THEOREM 2.1. — *i*) The real observables $x_1, \ldots, x_n$ have a joint distribution in the state m iff one of (*a*), (*b*) or (*c*) holds, where

a) $m(a(K_1, \ldots, K_n)) = 1$ for all $(K_1, \ldots, K_n)$

b) $m(a(K_1^\sigma, \ldots, K_n^\sigma)) = 1$ for all $(K_1^\sigma, \ldots, K_n^\sigma)$

c) $m(a(E_1, \ldots, E_n)) = 1$ for all $(E_1, \ldots, E_n)$

ii) Let b be any of the elements defined in (10) which exists and is countably obtainable. Then the joint distribution in the state m exists iff $m(b) = 1$.

The statement of Theorem 2.1 has been proved in [*7*] for $a(E_1, \ldots, E_n)$, resp. a^0. The generalization to the other cases is straightforward. The elements $a(K_1, \ldots, K_n)$ resp. a^f have been used in the criterion of the existence of the joint distributions in [*9*].

We note that the statement of Theorem 2.1 holds not only for real observables, but also for $(X, B(X))$-observables if the topology on X is tight, e. g. for complete separable metric spaces (see [*6*] [*7*]). (We recall that a Hausdorff topology is tight if each open set is σ-compact, i. e. is a countable union of compact sets).

PROPOSITION 2.1. — The element a^f exists iff a^σ exists and $a^f = a^\sigma$. Moreover, if one of them is countably obtainable, the other is also countably obtainable.

Proof. — For a given set of partitions $(K_1^\sigma, \ldots, K_n^\sigma)$ put

$$A = A(K_1^\sigma, \ldots, K_n^\sigma) = \{ x_i(E_i^{j_i}) : E_i^{j_i} \in K_i,\ i = 1, \ldots, n \}.$$

The set A is at most countable, so that

$$\mathrm{com}\,A = \wedge \{ \mathrm{com}\,F : F \subset A,\ F \text{ is finite} \}$$

exists and the set A is p. c. com A. This implies that $A \wedge \mathrm{com}\,A$ is a compatible subset in $L_{[0,\mathrm{com}\,A]}$, from which we obtain $a(K_1^\sigma, \ldots, K_n^\sigma) \wedge \mathrm{com}\,A = \mathrm{com}\,A$. On the other hand, for any finite subset F of A we have $a(K_1^\sigma, \ldots, K_n^\sigma) \leqslant \mathrm{com}\,F$, because $(K_1^\sigma, \ldots, K_n^\sigma)$ is a refinement of the partitions corresponding to the set F. Hence,

$$\mathrm{com}\,(A(K_1^\sigma, \ldots, K_n^\sigma)) = a(K_1^\sigma, \ldots, K_n^\sigma).$$

Now we show that the set $\bigcup_{i=1}^{n} R(x_i)$ is p. c. a^σ. Indeed, let $x_i(E)$, $x_j(G)$ be given. The two-element partitions $K_i = \{ E, E^c \}$, $K_j = \{ G, G^c \}$ define the refinements $K_i \cap K_i^\sigma = \{ B \cap C : B \in K_i, C \in K_i^\sigma \}$, analogically we define $K_j \cap K_j^\sigma$. It can be easily checked that if a^σ exists, then

$$a^\sigma = \bigwedge_{(K_1^\sigma, \ldots, K_n^\sigma)} a(K_1^\sigma, \ldots, K_n^\sigma) = \bigwedge_{(K_1^\sigma, \ldots, K_n^\sigma)} \operatorname{com}(A(K_1^\sigma, \ldots, K_n^\sigma))$$

$$\geq \bigwedge_{(K_1^\sigma, \ldots, K_n^\sigma)} \operatorname{com}(A(K_1^\sigma, \ldots, K_n^\sigma) \cup \{ x_i(E), x_j(G) \})$$

$$= \bigwedge_{(K_1^\sigma, \ldots, K_n^\sigma)} \operatorname{com}(A(K_1^\sigma, \ldots, K_i \cap K_i^\sigma, \ldots, K_j \cap K_j^\sigma, \ldots, K_n^\sigma)) \geq a^\sigma .$$

Put $\overline{A}(K_1^\sigma, \ldots, K_n^\sigma) \equiv \overline{A} = A(K_1^\sigma, \ldots, K_i \cap K_i^\sigma, \ldots, K_j \cap K_j^\sigma, \ldots, K_n^\sigma)$. Since $x_i(E) \leftrightarrow \operatorname{com} \overline{A}$ for any $(K_1^\sigma, \ldots, K_n^\sigma)$, we have $x_i(E) \leftrightarrow a^\sigma$. Similarly $x_j(G) \leftrightarrow a^\sigma$. Then the equality $\operatorname{com} \overline{A} = \operatorname{com} \overline{A} \wedge \operatorname{com} \{ x_i(E), x_j(G) \}$ implies that

$$a^\sigma = a^\sigma \wedge \operatorname{com} \{ x_i(E), x_j(G) \} = a^\sigma \wedge \operatorname{com} \{ x_i(E) \wedge a^\sigma, x_j(G) \wedge a^\sigma \} .$$

The last expression is the commutator of $x_i(E) \wedge a^\sigma$ and $x_j(G) \wedge a^\sigma$ in the logic $L_{[0,a^\sigma]}$, and as it equals to a^σ, we obtain that $x_i(E) \wedge a^\sigma \leftrightarrow x_j(G) \wedge a^\sigma$. This implies that $\bigcup_{i=1}^{n} R(x_i)$ is p. c. a^σ. From [*10*] we have that $a^\sigma \leqslant \operatorname{com} F$ for any finite subset of $\bigcup_{i=1}^{n} R(x_i)$. Let $c \in L$ be such that $c \leqslant \operatorname{com} F$ for all F.

Then from above we conclude that for all countable partitions we have $c \leqslant a(K_1^\sigma, \ldots, K_n^\sigma)$, so that $a^\sigma = a^f$. We have shown that if a^σ exists then a^f exists and they are equal.

Let now a^f exist. Since $\bigcup_{i=1}^{n} R(x_i)$ is p. c. a^f, we have $a(K_1^\sigma, \ldots, K_n^\sigma) \wedge a^f = a^f$. Hence a^σ exists and $a^\sigma = a^f$.

Now suppose that a^σ is countably obtainable. Then there is a sequence of countable partitions $\{ K_{i,k}^\sigma : i = 1, \ldots, n \}_{k=1}^{\infty}$ such that

$$a^\sigma = \bigwedge_{k=1}^{\infty} a(K_{1,k}^\sigma, \ldots, K_{n,k}^\sigma) .$$

The set $M = \bigcup_{k=1}^{\infty} A(K_{1,k}^\sigma, \ldots, K_{n,k}^\sigma)$ is at most countable, so that $\operatorname{com} M$ is countably obtainable. As M is p. c. a^σ, by [*10*] is $a^\sigma \leqslant \operatorname{com} M$. On the other hand, $a(K_{1,k}^\sigma, \ldots, K_{n,k}^\sigma) = \operatorname{com}(A(K_{1,k}^\sigma, \ldots, K_{n,k}^\sigma)) \geqslant \operatorname{com} M$, which implies $a^\sigma = \operatorname{com} M = a^f$.

LEMMA 2.1. — Let $x_1, x_2, \ldots, x_n$ be (X, S)-observables on L and let $\{ E_1^1, \ldots, E_{k_1}^1 \}, \ldots, \{ E_1^s, \ldots, E_{k_s}^s \}$ be systems of disjoint sets from S, $1 \leqslant s \leqslant n$. Then for any $G_{s+1}, \ldots, G_n \in S$ we have

$$\bigwedge_{j_1=1}^{k_1} \cdots \bigwedge_{j_s=1}^{k_s} \operatorname{com} \{ x_1(E_{j_1}^1), \ldots, x_s(E_{j_s}^s), x_{s+1}(G_{s+1}), \ldots, x_n(G_n) \} = \operatorname{com} \{ \mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_s, \mathbb{G} \},$$

where $\mathbb{E}_i = \{ x_i(E_1^i), \ldots, x_i(E_{k_i}^i) \}$, $i = 1, 2, \ldots, s$, $\mathbb{G} = \{ x_{s+1}(G_{s+1}), \ldots, x_n(G_n) \}$.

For the proof see [*10*] [*18*].

COROLLARY 2.1. — Let $K_1, K_2, \ldots, K_n$ be finite partitions of the set S and let $x_1, \ldots, x_n$ be (X, S)-observables on L. Then

$$\operatorname{com} \left(\bigcup_{i=1}^{n} \{ x_i(E) : E \in K_i \} \right) = \bigwedge_{\substack{E_1 \in K_1 \\ \vdots \\ E_n \in K_n}} \operatorname{com} \{ x_1(E_1), \ldots, x_n(E_n) \}$$

PROPOSITION 2.2. — Let $x_1, \ldots, x_n$ be (X, S)-observables on L. Then a^f exists iff a^0 exists and $a^0 = a^f$. Moreover, if one of them is countably obtainable, so is the other.

Proof. — For a given set of finite partitions $K_1, \ldots, K_n$ we put

$$A \equiv A(K_1, \ldots, K_n) = \bigcup_{i=1}^{n} \{ x_i(E) : E \in K_i \} .$$

Similarly as in the case of the σ-partitions we show that

$$a(K_1, \ldots, K_n) = \operatorname{com} (A(K_1, \ldots, K_n))$$

Let $a^f = \bigwedge_{(K_1, \ldots, K_n)} a(K_1, \ldots, K_n)$ exist. Clearly, $a^f \leqslant \operatorname{com} \{ x_1(E_1), \ldots, x_n(E_n) \}$ for all $(E_1, \ldots, E_n)$. Let $c \leqslant \operatorname{com} \{ x_1(E_1), \ldots, x_n(E_n) \}$ for all $(E_1, \ldots, E_n)$. Then

$$a(K_1, \ldots, K_n) = \operatorname{com} (A(K_1, \ldots, K_n)) = \bigwedge_{\substack{E_1 \in K_1 \\ \vdots \\ E_n \in K_n}} \operatorname{com} \{ x_1(E_1), \ldots, x_n(E_n) \} \geqslant c$$

Therefore a^0 exists and $a^f = a^0$.

Now let $a^0 = \bigwedge_{(E_1, \ldots, E_n)} \operatorname{com} \{ x_1(E_1), \ldots, x_n(E_n) \}$ exist. From Corollary 2.1 it follows that $a^0 \leqslant a(K_1, \ldots, K_n)$ for any $(K_1, \ldots, K_n)$. On the other hand, if $c \leqslant a(K_1, \ldots, K_n)$ for all $(K_1, \ldots, K_n)$, then $c \leqslant a^0$. This implies that a^f exists and $a^f = a^0$. The remaining part of the theorem follows also from Corollary 2.1 (*).

COROLLARY 2.2. — Let the (X, S)-observables $x_1, \ldots, x_n$ be such that $x_i(\{\lambda_1^i, \lambda_2^i, \ldots\}) = 1$, $i = 1, 2, \ldots, n$, $\{\lambda_j^i\} \in S$ $j = 1, 2, \ldots$. Then com $(x_1, \ldots, x_n)$ exists and

$$\operatorname{com}(x_1, \ldots, x_n) = \bigvee_{j_1, \ldots, j_n} x_1\{\lambda_{j_1}^1\} \wedge \ldots \wedge x_n\{\lambda_{j_n}^n\} \tag{11}$$

Proof. — Observe that the element on the right in (11) is $a^\sigma(x_1, \ldots, x_n)$ and see Proposition 2.1. (See also [*10*] for another proof of this statement for real observables.)

Let $\{x_\alpha : \alpha \in A\}$ be any set of (X, S)-observables on L. We put

$$\operatorname{com}\{x_\alpha : \alpha \in A\} = \operatorname{com}(\cup\{R(x_\alpha) : \alpha \in A\}),$$

if the element on the right exists. Now we shall investigate the existence of the commutators.

PROPOSITION 2.3. — Let $\{x_\alpha : \alpha \in A\}$ be a set of observables on a logic L such that there is an at most countable subset $\mathscr{A} \subset \cup\{R(x_\alpha) : \alpha \in A\}$, where $\mathscr{A}$ generates the minimal sublogic L_0 of L containing the set $\cup\{R(x_\alpha) : \alpha \in A\}$. Then com $\{x_\alpha : \alpha \in A\}$ exists and is countably obtainable (in fact, it is equal to com $\mathscr{A}$).

Proof. — Since the set $\mathscr{A}$ is at most countable, the set of all finite subsets of $\mathscr{A}$ is at most countable, too, so that

$$\operatorname{com}\mathscr{A} = \wedge\{\operatorname{com}F : F \text{ is a finite subset of } \mathscr{A}\}$$

exists. Put $a = \operatorname{com}\mathscr{A}$ and let Q(a) be the maximal subset of L which is partially compatible with respect to a and contains $\mathscr{A}$. By [*9*], Q(a) is a sublogic of L. We have

$$\cup\{R(x_\alpha) : \alpha \in A\} \subset L_0 \subset Q(a).$$

Let $F \subset \cup\{R(x_\alpha) : \alpha \in A\}$ be finite. Then F is p. c. a, so that $a \leqslant \operatorname{com}F$ (*see* [*10*]). Now let $c \in L$ be such that $c \leqslant \operatorname{com}F$ for all finite subsets F of $\cup\{R(x_\alpha) : \alpha \in A\}$. Then, especially, $c \leqslant \operatorname{com}G$ for any finite subset G of $\mathscr{A}$, hence $c \leqslant \operatorname{com}\mathscr{A} \equiv a$. We have shown that

$$\begin{aligned} a \equiv \operatorname{com}\mathscr{A} &= \wedge\{\operatorname{com}F : F \text{ finite subset of } \cup\{R(x_\alpha) : \alpha \in A\}\} \\ &= \operatorname{com}\{x_\alpha : \alpha \in A\}. \end{aligned}$$

COROLLARY 2.3. — If $\{x_\alpha : \alpha \in A\}$ is a sequence (i. e. A is countable) of real observables, then com $\{x_\alpha : \alpha \in A\}$ is countably obtainable.

(¹) Equivalence of these two elements has been stated in [*9*], but it is not seen immediately. (See [*10*] for the detailed proof).

Proof. — It follows from the preceding proposition and the fact that any $R(x_\alpha)\,;\alpha \in A$ is countably generated. In fact, put $\mathscr{A} = \cup \{ \mathscr{A}_\alpha : \alpha \in A \}$, where $\mathscr{A}_\alpha$ is the generator of $R(x_\alpha)$. Then $\mathscr{A}$ is countable and the sublogic L_0 generated by $\mathscr{A}$ contains the set $\cup \{ R(x_\alpha) : \alpha \in A \}$ owing to the fact that the minimal sublogic of L containing $\mathscr{A}_\alpha$ is $R(x_\alpha)$.

COROLLARY 2.4. — For any $s = 2, 3, \ldots, \sigma$ the elements a^f and a^s for the real observables $x_1, \ldots, x_n$ always exist, are countably obtainable and are equal to the commutator.

Proof. — The equality of a^f and a^s, $s = 2$, to $\mathrm{com}(x_1, \ldots, x_n)$ follows from Proposition 2.3 and Proposition 2.2. Since any s-element partition may be considered as an $(s+1)$-partition adding the empty set, it follows that the elements a^s exist and are equal for $s = 2, 3, \ldots, \sigma$. The countable obtainability of them follows from Corollary 2.1 (for $2 \leqslant s < \sigma$) and from Proposition 2.1 (for $s = \sigma$).

REMARK 2.1. — The statement of Corollary 2.4 remains true also for a set $\{ x_\alpha : \alpha \in A \}$ of observables if we define

$$a^s \{ x_\alpha : \alpha \in A \} = \bigwedge_{(\alpha_{i_1}, \ldots, \alpha_{i_n})} a^s(x_{\alpha_{i_1}}, \ldots, x_{\alpha_{i_n}}) \tag{12}$$

and similarly for a^f, provided the set $\{ x_\alpha : \alpha \in A \}$ fulfils the conditions of Proposition 2.3.

REMARK 2.2. — Let $x_1, \ldots, x_n$ be real observables and let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ be the generators of $R(x_1), R(x_2), \ldots, R(x_n)$, respectively, composed of mutually orthogonal elements. Proposition 2.3 and Corollary 2.1 imply that

$$\mathrm{com}(x_1, \ldots, x_n) = \bigwedge_{(a_1, \ldots, a_n) \in \mathscr{A}_1 \times \mathscr{A}_2 \times \ldots \times \mathscr{A}_n} \mathrm{com}(a_1, \ldots, a_n) \tag{13}$$

COROLLARY 2.5. — Let the system of real observables $\{ x_\alpha : \alpha \in A \}$ satisfy the conditions of Proposition 2.3. Then the joint distribution of them in a state m exists iff $m(\mathrm{com}\{ x_\alpha : \alpha \in A \}) = 1$.

Proof. — Follows from Proposition 2.3, Remark 2.1 and Theorem 2.1.

As easy consequence of Corollary 2.3 and Corollary 2.5 we obtain the following statement, that we need in the sequel.

THEOREM 2.2. — The real observables $x_1, x_2, \ldots, x_n$ have a joint distribution in the state m iff

$$m(\mathrm{com}(x_1, \ldots, x_n)) = 1 .$$

A set M of states on L is sufficient if for any $a \in L$, $a \neq 0$, there is a state $m \in M$ such that $m(a) = 1$.

We shall say that the (real) observables $x_1, \ldots, x_n$ are (*i*) compatible if $\operatorname{com}(x_1, \ldots, x_n) = 1$, (*ii*) partially compatible if $0 < \operatorname{com}(x_1, \ldots, x_n) < 1$, (*iii*) totally noncompatible if $\operatorname{com}(x_1, \ldots, x_n) = 0$.

THEOREM 2.3. — Real observables $x_1, \ldots, x_n$ on a logic L with a sufficient set of states M are (*i*) compatible, iff the joint distributions exist in any state $m \in M$, (*ii*) partially compatible, iff there is a state $m_1 \in M$ such that the joint distribution exists in m_1, and there is a state $m_2 \in M$ such that the joint distribution does not exist in m_2, (*iii*) totally noncompatible, iff the joint distribution does not exist in any state $m \in M$.

Proof. — (*i*) If the joint distribution exist in any state $m \in M$, then by Theorem 2.2 $m(\operatorname{com}(x_1, \ldots, x_n)) = 1$ for any $m \in M$. If $\operatorname{com}(x_1, \ldots, x_n) < 1$, then, owing to the sufficiency of M, there is $m \in M$ such that $m(\operatorname{com}(x_1, \ldots, x_n)^{\perp}) = 1$, a contradiction. (*ii*) Let $0 < \operatorname{com}(x_1, \ldots, x_n) < 1$. The sufficiency of M implies that there is $m_1 \in M$ such that $m_1(\operatorname{com}(x_1, \ldots, x_n)) = 1$ and $m_2 \in M$ such that $m_2(\operatorname{com}(x_1, \ldots, x_n)^{\perp}) = 1$. (*iii*) Let the joint distribution do not exist in any state $m \in M$ and let $m(\operatorname{com}(x_1, \ldots, x_n)) \neq 0$. Then there is $m \in M$ such that $m(\operatorname{com}(x_1, \ldots, x_n)) = 1$, so that the joint distribution exists in m, a contradiction.

3. UNCERTAINTY PRINCIPLE

In this section, we shall suppose that M is a sufficient set of states for the logic L. We shall consider only the real observables.

For the observables $x_1, \ldots, x_n$ let us denote by $V(x_1, \ldots, x_n)$ the set of states of M in which the variances of $x_1, \ldots, x_n$ exist and are finite, i. e.

$$V(x_1, \ldots, x_n) = \{ m \in M : \operatorname{var}_m(x_i) < \infty,\ i = 1, 2, \ldots, n \} \tag{14}$$

Following two cases can occure:

$$\begin{aligned} &(1) \quad (\forall \varepsilon > 0)(\exists m \in V(x_1, \ldots, x_n))(\operatorname{var}_m(x_1) \ldots \operatorname{var}_m(x_n) < \varepsilon) \\ &(2) \quad (\exists \varepsilon > 0)(\forall m \in V(x_1, \ldots, x_n))(\operatorname{var}_m(x_1) \ldots \operatorname{var}_m(x_n) \geqslant \varepsilon) \end{aligned} \tag{15}$$

If (2) occurs, we say that the uncertainly relation holds for the observables $x_1, \ldots, x_n$ (*see* [*11*] [*12*] [*13*]).

The following lemma has been proved in [*13*]. We give a simpler proof of it

LEMMA 3.1. — If the observables $x_1, \ldots, x_n$ are compatible, then (1) in (15) holds.

Proof. — By [*3*], there is a joint observable $\tau : B(R^n) \to L$ such that

$$\tau(E_1 \times E_2 \times \ldots \times E_n) = x_1(E_1) \wedge x_2(E_2) \wedge \ldots \wedge x_n(E_n).$$

for any rectangle $E_1 \times E_2 \times \ldots \times E_n \in B(R^n)$. By [*15*], $(\omega_1, \omega_2, \ldots, \omega_n) \in \sigma(\tau)$, where $\sigma(\tau)$ is the spectrum of τ, iff for any $r > 0$,

$$\tau((\omega_1 - r, \omega_1 + r) \times \ldots \times (\omega_n - r, \omega_n + r)) \neq 0.$$

As M is sufficient, there is $m \in M$ such that

$$m[\tau((\omega_1 - r, \omega_1 + r) \times \ldots \times (\omega_n - r, \omega_n + r))] = 1.$$

Then also $m(x_i(\omega_i - r, \omega_i + r)) = 1$, $i = 1, 2, \ldots, n$ so that m_{x_i} is concentrated on $(\omega_i - r, \omega_i + r)$. If we choose r sufficiently small, we obtain that (1) holds.

Now we are ready to prove the main theorem of this section.

THEOREM 3.1. — If for the observables $x_1, x_2, \ldots, x_n$ (2) of (15) holds, then $\mathrm{com}(x_1, \ldots, x_n) = 0$, i. e. the observables $x_1, \ldots, x_n$ are totally noncompatible.

Proof. — Put $a = \mathrm{com}(x_1, \ldots, x_n)$ and suppose that $a \neq 0$. Then the set $M_a = \{ m \in M : m(a) = 1 \}$ is nonempty. It is easy to see that M_a is sufficient for the logic $L_{[0,a]}$. Indeed, let $b \in L_{[0,a]}$, $b \neq 0$. There is a state $m \in M$ such that $m(b) = 1$. But $b \leqslant a$ implies that $m(a) = 1$, i. e. $m \in M_a$. Let us consider the maps $x_i \wedge a : E \mapsto x_i(E) \wedge a$, $E \in B(R)$. From the fact that $\bigcup_{i=1}^{n} R(x_i)$ is p. c. a, we obtain that $x_i \wedge a$, $i = 1, 2, \ldots, n$ are mutually compatible observables on $L_{[0,a]}$. Moreover, it can be easily seen that for any $m \in M_a$, $\mathrm{var}_m(x_i \wedge a) = \mathrm{var}_m(x_i)$, $i = 1, 2, \ldots, n$. By Lemma 3.1, (1) holds for the observables $x_i \wedge a$, $i = 1, 2, \ldots, n$, which contradicts to the supposition that (2) holds for x_i, $i = 1, 2, \ldots, n$.

By Theorem 2.3 (*iii*), total noncompatibility of the observables $x_1, x_2, \ldots, x_n$ implies the nonexistence of joint distributions for the observables in any state $m \in M$ (2). The absence of joint distributions is, in the probabilistic sense, the expression of simultaneous nonmeasurability of the observables. For example, by Suppes [*16*] « the conclusion that momentum and position are not simultaneously measurable at all does not follow from the Heisenberg relations but from the more fundamental result about the absence of genuine joint distributions ». Now we see that, provided the set of states is sufficient, the Heisenberg relation implies the nonexistence of joint distributions and hence the simultaneous nonmeasurability of corresponding observables.

(2) The joint distributions defined on p. 254 is called « joint distribution of type 1 ». There is also another type of joint distributions, so called « type 2 » (see [*4*]). The last, weaker form of joint distributions exists even for the complementary momentum and position observables in some states.

At the end of this section, we shall compare the notion of total noncompatibility with the notion of complementarity introduced in [12] [13]: two elements a, b of L are complementary if $a \wedge b = 0$; and two observables x and y are complementary if $x(E) \wedge y(F) = 0$ for any bounded Borel sets E, F such that $\sigma(x) \not\subset E$ and $\sigma(y) \not\subset F$. It has been shown in [13] that compatible observables are complementary iff at least one of them is a constant.

By our opinion, the above definition of complementarity is a little misleading. There is a one-to-one correspondence between the elements of L and the simple observables: to any $a \in L$ there is a unique observable x_a such that $\sigma(x_a) \subset \{0, 1\}$ and $x_a\{1\} = a$. If, for example, $a \perp b$, then a and b are complementary, but x_a and x_b are not complementary. But it is usual to identify the elements of L with the corresponding simple observables, e. g. the closed linear subspaces of the Hilbert space with corresponding projections.

We suggest to define the complementarity of the elements of L as follows: a and b of L are complementary if x_a and x_b are complementary.

It is easy to see that two simple observables x_a and x_b are complementary iff $0 = (a \wedge b) \vee (a^{\perp} \wedge b) \vee (a \wedge b^{\perp}) \vee (a^{\perp} \wedge b^{\perp}) = \text{com}\{x_a, x_b\}$, so that complementarity is equivalent to total noncompatibility.

In general, it is not the case. Proposition 2.1 implies that if x and y are complementary and noncompatible. then they are totally noncompatible. The converse implication does not hold. To see this, let us introduce following simple example. Let us consider the logic $L(R^3)$ of the tree dimensional Hilbert space R^3. By [7] and [14], there is no pair of nontrivial complementary observables. Let (e_1, e_2, e_3) and (f_1, f_2, f_3) be two different (and disjoint) bases. Choose real numbers α_1, α_2, α_3 and define the observables x and y as follows: $x\{\alpha_i\} = [e_i]$, $y\{\alpha_i\} = [f_i]$, $i = 1, 2, 3$, where $[e]$ is the one-dimensional subspace generated by the vector $e \in R^3$. The set of all common eigenvectors of x and y is empty, so that by Corollary 2.2, $\text{com}(x, y) = 0$. Hence, the observables x and y are totally noncompatible, but they are not complementary.

4. CONCLUDING REMARKS

In [17], next problems are introduced:

V. Uncertainty principle. Problem: Is there a generalization of the Heisenberg uncertainty principle for quantum logics?

VII. Joint distribution. Problem: Can joint distribution be defined for noncompatible observables?

The problem of uncertainty principle has been solved by Lahti in [12], resp. [13]. In the present paper a completion of this solution is obtained.

It is shown that the uncertainty relation implies the simultaneous non-measurability of corresponding observables.

The answer to the problem of joint distributions has been obtained in the present paper together with the papers [6]-[10]. For noncompatible observables x and y on a logic the joint distribution in a state m exists iff $m(\mathrm{com}(x, y)) = 1$. If the joint distribution exists, then with respect to the state m, we can instead of the observables x and y consider the observables $x \wedge \mathrm{com}(x, y)$ and $y \wedge \mathrm{com}(x, y)$ on the logic $L_{[0, \mathrm{com}(x,y)]}$, and these observables are compatible.

REFERENCES

[1] G. W. MACKEY, *Mathematical foundations of quantum mechanics*, W. A. Benjamin, New York, 1963.

[2] M. J. MACZYŃSKI, A remark on Mackey's axiom system for quantum mechanics, *Bull. Acad. Polon. Sci.*, t. **15**, 1967, p. 583-587.

[3] V. S. VARADARAJAN, *Geometry of quantum theory*, Van Nostrand, Princeton, N. Y., 1968.

[4] S. P. GUDDER, Joint distributions of observables, *Journ. of Math. and Mech.*, t. **18**, 1968, p. 325-335.

[5] J. M. JAUCH, *The quantum probability calculus, in: Logic and probability in quantum mechanics*, P. Suppes ed., p. 123-146, Reidel, Dordrecht, 1976.

[6] S. PULMANNOVÁ, Relative compatibility and joint distributions of observables, *Found. Phys.*, t. **10**, 1980, p. 641-653.

[7] A. DVUREČENSKIJ, S. PULMANNOVÁ, Connection between joint distributions and compatibility of observables. *Rep. Math. Phys.*, t. **19**, 1984, p. 349-359.

[8] A. DVUREČENSKIJ, S. PULMANNOVÁ, On joint distributions of observables, *Math. Slovaca*, t. **32**, 1982, p. 155-166.

[9] S. PULMANNOVÁ, Compatibility and partial compatibility in quantum logics, *Ann. Inst. Henri Poincaré*, t. **34**, 1981, p. 391-403.

[10] S. PULMANNOVA, *Commutators in orthomodular lattices*, submitted for publication.

[11] S. BUGAJSKI, P. LAHTI, Fundamental principles of quantum theory, *Int. Journ. Theoret. Phys.*, t. **19**, 1980, p. 499-514.

[12] P. LAHTI, Uncertainty principle and complementarity in axiomatic quantum mechanics, *Rep. Math. Phys.*, t. **17**, 1980, p. 99-110.

[13] P. LAHTI, Uncertainty and complementarity in axiomatic quantum mechanics, *Int. Journ. Theor. Phys.*, t. **19**, 1980, p. 789-842.

[14] P. LAHTI, Characterization of quantum logics, *Int. Journ. Theor. Phys.*, t. **19**, 1980, p. 905-923.

[15] S. P. GUDDER, Spectral methods for a generalized probability theory, *Trans. Amer. Math. Soc.*, t. **119**, 1965, p. 428-442.

[16] P. SUPPES, Probability concepts in quantum mechanics, *Philosophy of Science*, t. **22**, 1961, p. 378-389.

[17] S. P. GUDDER, *Some unsolved problems in quantum logics, in: Mathematical foundations of quantum theory*, A. R. Marlow ed., p. 87-103, Academic Press, New York, 1978.

[18] A. DVUREČENSKIJ, On two problems of quantum logics, *Math. Slovaca*, in press.

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