ANNALES DE L'I. H. P., SECTION A

EMANUELA CALICETI

Perturbation theory for Schrödinger operators with complex potentials

Annales de l'I. H. P., section A, tome 42, nº 3 (1985), p. 235-251
<http://www.numdam.org/item?id=AIHPA_1985__42_3_235_0>

Ann. Inst. Henri Poincaré,
Vol. 42, n° 3, 1985, p. 235-251. *Physique théorique*

Perturbation theory for Schrödinger operators with complex potentials

by

Emanuela CALICETI

Dipartimento di Matematica, Università di Modena, 41100 Modena, Italy

ABSTRACT. — A complete analysis for the spectral and perturbation theory of the dilated Hamiltonian $H_g(\theta) = p^2 + ge^\theta V(e^{\theta/2}x)$, $g \in \mathbb{R}\backslash\{0\}$, $|Im\,\theta| < \theta_0 < \pi/2$, is provided. The potential V is dilation analytic and admits two different limits $a_+, a_- \in \mathbb{R}$ as $|z| \to \infty$, $|\arg z| < \theta_0$ and as $|z| \to \infty$, $|\arg z - \pi| < \theta_0$, respectively, to which it converges sufficiently fast. For a suitable range of the parameters g and θ, a spectral representation for the resolvent is obtained, in terms of a spectral family of projections, for which completeness can be proved. Furthermore, a functional calculus is developed for a restricted class of functions defined on $\sigma_{ess}(H_g(\theta))$, which decay sufficiently rapidly at infinity.

RÉSUMÉ. — On donne une analyse complète de la théorie spectrale et de la théorie des perturbations pour le Hamiltonien dilaté

$$H_g(\theta) = p^2 + ge^\theta V(e^{\theta/2}x), \qquad g \in \mathbb{R}\backslash\{0\}, \qquad |\operatorname{Im}\theta| < \theta_0 < \pi/2.$$

Le potentiel V est analytique par dilatation, admet deux limites différentes $a_+, a_- \in \mathbb{R}$ quand $|z| \to \infty$, $|\arg z| < \theta_0$ et quand $|z| \to \infty$, $|\arg z - \pi| < \theta_0$, respectivement, et converge suffisamment vite vers ces limites. Pour un domaine convenable des paramètres g et θ, on obtient une représentation spectrale de la résolvante en termes d'une famille spectrale de projecteurs pour lesquels on peut montrer la complétude. En outre, on développe un calcul fonctionnel pour une classe restreinte de fonctions définies sur $\sigma_{ess}(H_g(\theta))$ qui décroissent suffisamment vite à l'infini.

1. INTRODUCTION

We consider the two channel problem generated by the Schrödinger operator $H(g, 0) = p^2 + gV$ in $L^2(\mathbb{R})$, for $g \in \mathbb{R} \setminus \{0\}$, involving different spacial asymptotics as $x \to \pm\infty$. The potential V is assumed to satisfy the following.

HYPOTHESIS 1.1. — *i*) $V(x) \in \mathbb{R}$ for each $x \in \mathbb{R}$;

ii) there exists $\theta_0 \in (0, \pi/2)$ such that $V(x)$ is the restriction to $z \in \mathbb{R}$ of a function $V(z)$ holomorphic at least in the region

$$\{ z \in \mathbb{C} : |\arg z| < \theta_0 \} \cup \{ z \in \mathbb{C} : |\arg z - \pi| < \theta_0 \}$$

and bounded in a neighbourhood of the point $z = 0$. Moreover, for each fixed $\theta \in \mathbb{C}$ with $|\operatorname{Im} \theta| < \theta_0$, the function $x \in \mathbb{R} \to V(xe^{\theta})$ is of class $C^{\infty}(\mathbb{R})$;

iii) there exist real numbers a_- and a_+, with $a_- < 0 < a_+$, enjoying the following property:

for each pair of real numbers (β_1, β_2) with $|\beta_j| < \theta_0, j = 1, 2, \beta_1 < \beta_2$, there exists $\varepsilon = \varepsilon(\beta_1, \beta_2)$ such that

$$|V(z) - a_+| = 0(|z|^{-1-\varepsilon}) \quad \text{as} \quad |z| \to \infty, \qquad \beta_1 \leq \arg z \leq \beta_2$$

and

$$|V(z) - a_-| = 0(|z|^{-1-\varepsilon}) \quad \text{as} \quad |z| \to \infty, \qquad \pi+\beta_1 \leq \arg z \leq \beta_2 + \pi\,;$$

iv) for any fixed θ with $|\operatorname{Im} \theta| < \theta_0$

$$\int_{-\infty}^{0} |V(xe^{\theta/2}) - a_-|(1 + x^2)dx < \infty$$

and

$$\int_{0}^{+\infty} |V(xe^{\theta/2}) - a_+|(1 + x^2)dx < \infty\,.$$

Remark. — The above assumption $a_- < 0 < a_+$ is only made to fix ideas, but it is not restrictive, in the sense that the results obtained in Sections 2 and 3 can be extended to the more general case $a_- \neq a_+, a_-, a_+ \in \mathbb{R}$, with similar arguments.

Example. — A class of potentials satisfying Hypothesis 1.1 was first analyzed in [*4*]. It was proved that the Borel sum of the Rayleigh-Schrödinger perturbation expansion of any odd anharmonic oscillator $p^2 + x^2 + \beta x^{2k+1}$, $k = 1, 2, \ldots$, is the limit of a sequence of resonances in the standard sense of dilation analyticity. This was achieved by means of the following family of approximating potentials

$$V_\alpha(x) = (x^2 + \beta x^{2k+1})(\alpha^2 x^{4k+2} + 1)^{-1/2}, \qquad \text{as} \quad \alpha \to 0,$$

and the problem was reduced to a stability result for the eigenvalues of the dilated operator $e^{-\theta}p^2 + e^{\theta}x^2 + \beta e^{(2k+1)\theta/2}x^{2k+1}$, for $Im\ \theta > 0$, $\beta > 0$, with respect to the family $e^{-\theta}p^2 + V_\alpha(e^{\theta/2}x)$ as $\alpha \to 0$. The interest of this class of potentials lies in the fact that they exhibit the typical shape of a barrier and the result shows how in some cases the so called shape resonances [*5*] exist in the standard sense of dilation analyticity. It is easy to check that the potential V_α satisfies Hypothesis 1.1 for $k > 2$.

The purpose of the present paper is to give a complete spectral analysis of the dilated differential operators

$$(1.1) \quad H(g, \theta) = e^{-\theta}(-d^2/dx^2 + g e^{\theta} V(e^{\theta/2}x)) \equiv e^{-\theta}(-d^2/dx^2 + gV(x, \theta))$$

in $L^2(\mathbb{R})$, on the domain $D(H(g, \theta)) = H^2(\mathbb{R})$, the usual Sobolev space. For $\theta \in \mathbb{R}$, $H(g, \theta)$ is unitarily by equivalent to $H(g, 0)$, via the dilation operator $U(\theta)$ defined by $(U(\theta)f)(x) = e^{\theta/4}f(e^{\theta/2}x)$.

Since $V(x, \theta)$ is bounded and analytic in θ in the strip $|Im\,\theta| < \theta_0$, by standard arguments (see e. g. Kato [*9*]), $H(g, \theta)$ is a holomorphic family of type A in θ, for $|Im\,\theta| < \theta_0$. A complete description of $\sigma_{ess}(H(g, \theta))$ is given in the following theorem, whose proof is omitted since it is similar to the one provided in [*4*], Theorem 2.3, for the case

$$V(x) = (x^2 + \beta x^{2k+1})(\alpha^2 x^{4k+2} + 1)^{-1/2}, \qquad \beta > 0, \qquad \alpha > 0, \qquad k \in \mathbb{N}$$

THEOREM 1.2. — Let $|\operatorname{Im}\theta| < \theta_0$ and $g \in \mathbb{R}\setminus\{0\}$. Then

$$\sigma_{ess}(H(g, \theta)) = \{ e^{-\theta}x + ga_- : x \geq 0 \} \cup \{ e^{-\theta}x + ga_+ : x \geq 0 \}.$$

By a slight abuse of notation we set, for $g \in \mathbb{R}\setminus\{0\}$, $|\operatorname{Im}\theta| < \theta_0$;

$$(1.2) \quad H_g(\theta) = e^{\theta}H(g, \theta) = -d^2/dx^2 + gV(x, \theta), \qquad D(H_g(\theta)) = H^2(\mathbb{R}).$$

Since similar arguments apply for $g < 0$ and $-\theta_0 < \operatorname{Im}\theta < 0$, hereafter g and θ will be fixed so that $g > 0$ and $0 < \operatorname{Im}\theta < \theta_0$. A complete scattering theory for this kind of problem in the self-adjoint case has been developed in Davies-Simon [*6*]. Our purpose is to check whether the Schrödinger operators with complex potentials $H_g(\theta)$ fit in the category of spectral operators in the sense of Dunford-Schwartz [*7*].

It turns out that the operator $H_g(\theta)$ is not spectral and this is due to the fact that the norm of the resolvent $R(\lambda) \equiv R(\lambda; H_g(\theta)) = (\lambda - H_g(\theta))^{-1}$ it not uniformly bounded in λ as $|\lambda| \to \infty$, λ in the region R_2 between the two branches of $\sigma_{ess}(H_g(\theta))$,

$$\Sigma_+ = \{ x + ga_+e^{\theta} : x > 0 \} \qquad \text{and} \qquad \Sigma_- = \{ x + ga_-e^{\theta} : x > 0 \}.$$

On the other hand, at least for a spectral operator T of scalar type, T is similar to a normal operator, from which one can prove the uniform boundedness of $|R(\lambda)|$ in λ for dist $(\lambda, \sigma(T)) = $ const... In fact if T is any spectral operator, by definition it satisfies such an estimate on the norm of the resolvent. Nevertheless for the operator $H_g(\theta)$ we can construct a spectral family E(.) which is not uniformly bounded, but allows us to develop a spectral theory which accomodates a suitably restricted functional calculus.

In particular it is possible to prove, first of all, that, for a suitable choice of the parameters g and θ, $\sigma_d(H_g(\theta))$ is a finite subset of $\mathbb{C}$. Let $\sigma_d(H_g(\theta)) = \{\lambda_1, \ldots, \lambda_n\}$. Then we can define spectral projections $E(\Delta)$ for bounded Borel subsets Δ of $\mathbb{C}$, such that the resolvent $R(\lambda)$, $\lambda \in \rho(H_g(\theta))$. admits the following representation

$$(1.3)\quad R(\lambda) = \int_{\sigma_{ess}(H_g(\theta))} (z-\lambda)^{-1} dE(z) - (2\pi i)^{-1} \sum_{j=1}^{n} \oint_{\Gamma_j} R(z)(z-\lambda)^{-1} dz$$

where Γ_j is a closed curve about λ_j containing no other point of $\sigma(H_g(\theta))$.

We can also develop a functional calculus for a class of functions ψ defined on $\sigma_{ess}(H_g(\theta))$. For any such ψ we can define

$$(1.4)\quad \psi(H_g(\theta)) = \int_{\sigma_{ess}(H_g(\theta))} \psi(z) dE(z)$$

so that, if we neglect the contribution given by the eigenvalue part, Equation (1.4) reduces to (1.3) for the function $\psi(z) = (z-\lambda)^{-1}$ (see Theorem 3.3 and the following remark). To obtain these results we need to make asymptotic estimates for the solutions ϕ_+ and ϕ_- of the differential equation $H_g(\theta)\sigma = \lambda\sigma$, which, together with their derivatives, decay exponentially fast at $+\infty$ and $-\infty$ respectively. This in turn allows one to control the behavior of their Wronskian $W(\lambda)$ and of the resolvent $R(\lambda)$ when $|\lambda| \to \infty$. In particular it turns out that $|W(\lambda)| \to 0$ as $|\lambda| \to \infty$, λ in the region R_2 between Σ_+ and Σ_- and, since the projections $E(\Delta)$ are defined in terms of $W(\lambda)$ and of the functions ϕ_+, ϕ_-, this causes the non uniform boundedness of the projections $E(\Delta)$. However we can prove that $|\lambda^{1/2}W(\lambda)| \to d \neq 0$ as $|\lambda| \to \infty$, $\lambda \in R_2$, and this condition is sufficient to guarantee completeness for the projections in the sense of Equation (1.3) (see Theorems 2.8 and 3.2).

The problem of the spectral theory for Schrödinger operators has also been considered by a variety of authors [*1*] [*2*] [*7*] [*8*]. The complex-dilated two-body Schrödinger operator, which corresponds to a Schrödinger operator with complex potential on the half-line $[0, \infty)$, has been treated extensively (see [*7*] and [*8*]) and it gives rise to a spectral operator. A similar result is obtained in [*2*], where an analytic scattering theory is developed for a two-body Schrödinger operator $-\Delta + V$, with V a dilation analytic short-range interaction. For a more recent discussion of twoparticle systems, in the framework of spectral operators, when the potential is the boundary value of an analytic function, see also [*11*].

The situation is more complicated in the N-body case, with $N > 2$. The problem has been analyzed in Balslev [*1*]. He considers the complex-dilated many-body Schrödinger operator and decomposes it on invariant subspaces associated with the « cuts » $\Sigma_\mu = \mu + z^{-2}\mathbb{R}_+$, where μ is any threshold and z is the dilation parameter, and isolated spectral points. In particular a uniform bound on the norm of the resolvent $R(\lambda)$ is obtained, for λ varying in a strip properly contained in the region between two consecutive cuts Σ_{μ_1} and Σ_{μ_2}. Such estimate is used to construct spectral projections defined in terms of contour integrals of the resolvent along suitable curves surrounding each cut Σ_μ, but with empty intersection with the spectrum of the operator.

A stronger result is obtained in our case: completeness for the spectral projections is proved by virtue of a uniform estimate on the scalar products $\langle R(\lambda)f, h \rangle$ of the resolvent for $|\lambda| \to \infty$, $\lambda \in R_2$, where the bound depends on f and h in $C_0^\infty(\mathbb{R})$, but not on λ. Notice that here λ can be taken as close as we wish to any of the cuts Σ_+ or Σ_-. Moreover the spectral projections are defined in terms of contour integrals of the resolvent along (and not around) the essential spectrum of $H_g(\theta)$.

For the exposition of the paper we closely follow the author's doctoral thesis [*3*], in which the case $p^2 + V$ is considered, with

$$V(x) = (x^2 + \beta x^{2k+1})(\alpha^2 x^{4k+2} + 1)^{-1/2}, \quad \alpha > 0, \quad \beta > 0, \quad k \in \mathbb{N}.$$

Since most of the results can be proved with identical arguments, proofs will be omitted whenever the reader can be referred to the corresponding theorems in [*3*].

In Section 2 we obtain the above mentioned results about the discrete spectrum of $H_g(\theta)$ and the behavior of the Wronskian $W(\lambda)$ as a function of λ.

In Section 3 we complete the spectral analysis of the dilated Hamiltonian $H_g(\theta)$, by obtaining a spectral representation for the resolvent and the completeness of the projections $E(\Delta)$, in the sense of Equation (1.3). Finally we develop an operational calculus for functions defined on $\sigma_{ess}(H_g(\theta))$ which decay sufficiently fast at infinity.

2. SPECTRAL THEORY FOR THE DILATED HAMILTONIANS

Fist of all let us introduce some notation. For $g > 0$ and θ fixed with $0 < \operatorname{Im}\theta < \theta_0$, let $H_g(\theta)$ denote the operator defined by (1.2) with the potential V satisfying Hypothesis 1.1. Set

$$V^+(x, \theta) = V(x, \theta) - a_+ e^\theta = e^\theta(V(xe^{\theta/2}) - a_+)$$

and

$$V^-(x, \theta) = V(x, \theta) - a_- e^\theta = e^\theta(V(xe^{\theta/2}) - a_-).$$

For any function $u \in C^1(\mathbb{R})$, $u'(x)$ denotes the first derivative with respect to x.

Remark. — The relevance of the assumptions on the potential V will be stressed along the way as they are needed in the proofs of the theorems. In particular (*iv*) is required in Theorem 3.1 to prove the boundedness of the spectral operators $E(\Delta)$ (see also [7], Theorem XX.1.12, where the same assumption is made to prove a similar result for the half-line case $(0, \infty)$.).

The first part of our analytic work consists in making sufficiently fine asymptotic estimates of the solutions of the equation $H_g(\theta)\sigma = \lambda\sigma$, to be able to control the resolvent

$$R(\lambda; H_g(\theta)) = (\lambda - H_g(\theta))^{-1} \quad \text{when} \quad \lambda \to \hat{\lambda} \in \sigma_{ess}(H_g(\theta)) \quad \text{or} \quad |\lambda| \to \infty.$$

In this context the asymptotic relationship $f(x) \sim h(x)$ as $x \to \pm\infty$ is meant in the usual sense: $\lim_{x \to \pm\infty} |f(x)h(x)^{-1} - 1| = 0$.

The first results in this direction are obtained in the following lemma, which extends to the whole real line the half-line case considered in Lemma XX.1.1 of [7]. Its proof is analogous to that of Corollaries 3.3 and 3.6 of [*3*] and therefore will be omitted.

LEMMA 2.1. — Let $P = \{\mu : \operatorname{Im}\mu \geq 0\}$. Let $\lambda \in \mathbb{C}$ and

$$\mu_+ \equiv \mu_+(\lambda) = (\lambda - g a_+ e^\theta)^{1/2},$$

with $\operatorname{Im}\mu_+ \geq 0$, that is $\lambda = \mu_+^2 + g a_+ e^\theta$. Then the equation $H_g(\theta)\sigma = \lambda\sigma$ has a solution $\phi_+(x, \mu_+)$ defined for $(x, \mu_+) \in (-\infty, +\infty) \times P$, with the following properties:

i) $\phi_+(x, \mu_+)$ is C^∞ in x, for $\mu_+ \in P$;

ii) $\phi_+(x, \mu_+)$ and $\phi'_+(x, \mu_+)$ are analytic in μ_+ for μ_+ interior to P and continuous in μ_+ for μ_+ in P, $-\infty < x < +\infty$. Moreover, $\phi_+(x, \mu_+)$ satisfies the following asymptotic relationships

$$\phi_+(x, \mu_+) \sim \exp(i\mu_+ x), \qquad \phi'_+(x, \mu_+) \sim i\mu_+ \exp(i\mu_+ x)$$

as $x \to \infty$, uniformly for $\mu_+ \in P$, and also as $|\mu_+| \to \infty$, $\mu_+ \in P$, uniformly in $a \leq x < \infty$, $a \in \mathbb{R}$. Furthermore, for any $a \in \mathbb{R}$ there exists a constant K_a^+ such that

$$|\phi_+(x, \mu_+) - \exp(i\mu_+ x)| \leq K_a^+(1 + |\mu_+|)^{-1} \int_x^\infty (y - a + 1)|V^+(y, \theta)|\,dy$$

for $a \leq x < \infty$ and $\mu_+ \in P$.

Similarly if we let $\mu_- \equiv \mu_-(\lambda) = (\lambda - g a_- e^\theta)^{1/2}$, with $\operatorname{Im}\mu_- \geq 0$, then the equation $H_g(\theta)\sigma = \lambda\sigma$ has a solution $\phi_-(x, \mu_-)$, $(x, \mu_-) \in (-\infty, +\infty) \times P$, enjoying properties analogous to (*i*)-(*ii*), with the obvious modifications, replacing $+$ with $-$. In particular for any $a \in \mathbb{R}$ there exists a constant K_a^- such that

$$| \phi_-(x, \mu_-) - \exp(-i\mu_- x) | \leq K_a^-(1 + | \mu_- |)^{-1} \int_{-\infty}^{x} (a - y + 1) | V^-(y, \theta) | dy$$

for $-\infty < x \leq a$ and $\mu_- \in P$.

For $\lambda \in \mathbb{C}$ let $W(\lambda)$ denote the Wronskian of the two solutions ϕ_+ and ϕ_- of $H_g(\theta)\sigma = \lambda\sigma$, described in Lemma 2.1, i. e.

$$W(\lambda) = W(\phi_+, \phi_-) = \phi_+(x, \mu_+)\phi'_-(x, \mu_-) - \phi'_+(x, \mu_+)\phi_-(x, \mu_-)$$

for any $x \in (-\infty, +\infty)$.

Let $\Sigma_\pm = \{ z = x + ga_\pm e^\theta : 0 < x < \infty \}$, $\overline{\Sigma}_\pm = \Sigma_\pm \cup \{ ga_\pm e^\theta \}$. Let γ denote the straight line segment with endpoints $ga_- e^\theta$ and $ga_+ e^\theta$. Then the curve $\mathscr{C} = \Sigma_- \cup \gamma \cup \Sigma_+$ can be parametrized by a one-to-one function $z(t)$, defined for $-\infty < t < \infty$. $\mathscr{C}$ divides the complex sphere into two regions R_1 and R_2 surrounding the points of R_1 (resp. R_2) in the positive (resp. negative) sense of complex function theory.

The first results concerning the behaviour of $W(\lambda)$ as a function of λ are contained in the following

PROPOSITION 2.2. — $W(\lambda)$ is analytic in λ for λ in the complement of $\overline{\Sigma}_- \cup \overline{\Sigma}_+$ and approaches continuous limits $W^1_\pm(\lambda_\pm)$ and $W^2_\pm(\lambda_\pm)$ as $\lambda \to \lambda_\pm$, for any $\lambda_\pm \in \overline{\Sigma}_\pm$ from R_1 and R_2 respectively.

Moreover $W(\lambda)$ satisfies the following asymptotic relationships

$$(2.1) \qquad W(\lambda) = -2i\lambda^{1/2} + 0(1) \quad \text{as} \quad |\lambda| \to \infty, \quad \lambda \in R_1,$$

$$(2.2) \quad W(\lambda) = \frac{3}{4} ige^\theta \operatorname{sgn}(\operatorname{Im} \lambda)\lambda^{-1/2} \left\{ a_+ - a_- + \frac{g}{2}\left(\left(\int_0^\infty (V(x) - a_+)dx\right)^2 - \left(\int_{-\infty}^0 (V(x) - a_-)dx\right)^2\right)\right\} + 0(\lambda^{-1}), \quad \text{as} \quad |\lambda| \to \infty, \quad \lambda \in R_2,$$

where $\operatorname{sgn}(x) = +1$, if $x \geq 0$ and $\operatorname{sgn}(x) = -1$, if $x < 0$.

Sketch of the proof. — The first statement immediately follows from Lemma 2.1. In particular for $\hat{\lambda} \in \overline{\Sigma}_+$ we have

$$(2.3) \quad \begin{aligned} W^1_+(\hat{\lambda}) &= \phi_+(0, \mu_+(\hat{\lambda}))\phi'_-(0, \mu_-(\hat{\lambda})) - \phi'_+(0, \mu_+(\hat{\lambda}))\phi_-(0, \mu_-(\hat{\lambda})) \\ W^2_+(\hat{\lambda}) &= \phi_+(0, -\mu_+(\hat{\lambda}))\phi'_-(0, \mu_-(\hat{\lambda})) - \phi'_+(0, -\mu_+(\hat{\lambda}))\phi_-(0, \mu_-(\hat{\lambda})) \end{aligned}$$

where $\mu_+(\hat{\lambda}) = \lim_{\substack{\lambda \to \hat{\lambda} \\ \lambda \in R_1}} \mu_+(\lambda) \geq 0$. Similar expressions hold for $W^1_-(\hat{\lambda})$ and $W^2_-(\hat{\lambda})$, for $\hat{\lambda} \in \overline{\Sigma}_-$, with $\mu_-(\hat{\lambda}) = \lim_{\substack{\lambda \to \hat{\lambda} \\ \lambda \in R_1}} \mu_-(\lambda) \leq 0$.

The proof of (2.1) and (2.2) follows by retaining the terms of $0(\lambda^{-1})$ in the asymptotic expansion for $\phi_+(x, \mu_+(\lambda))$ and $\phi_-(x, \mu_-(\lambda))$, and the terms of $0(\lambda^{-1/2})$ in the asymptotic expansion for their derivatives $\phi'_+(x, \mu_+(\lambda))$ *and* $\phi'_-(x, \mu_-(\lambda))$. We first obtain the leading terms for these expansions and then substitute these expansions into the expression for $W(\lambda)$. Finally we consider $W(\lambda)$ in the regions R_1 and R_2. In particular (2.1) corresponds to Equation (3.20) in [*3*] and can be proved following the same steps. As for (2.2) we obtain

$$(2.4)\qquad \mu_-(\lambda) = \frac{3}{4} ige^{\theta}(a_+ - a_-) + \frac{3g^2}{8} i\left[\left(\int_0^{\infty} V^+(y, \theta)dy\right)^2 - \left(\int_{-\infty}^{0} V^-(y, \theta)dy\right)^2\right] + 0(\lambda^{-1/2}),$$

for $\lambda \in R_2$, $|\lambda|$ sufficiently large. Using Cauchy's theorem and Hypothesis 1.1 (*ii*)-(*iii*), the change of variables $x \to e^{\theta/2}x$ yields

$$\int_0^{\infty} V^+(y, \theta)dy = e^{\theta/2}\int_0^{\infty} (V(x) - a_+)dx$$

and

$$\int_{-\infty}^{0} V^-(y, \theta)dy = e^{\theta/2}\int_{-\infty}^{0} (V(x) - a_-)dx.$$

Now (2.2) immediately follows from (2.4) and the identity

$$\mu_-(\lambda) = (\lambda - ga_-e^{\theta})^{1/2}.$$

We now start the study of the discrete spectrum of $H_g(\theta)$. As remarked in the Introduction, $\sigma_{ess}(H_g(\theta)) = \overline{\Sigma}_- \cup \overline{\Sigma}_+$. Our goal is to show that $\sigma_d(H_g(\theta))$ consists of a finite number of eigenvalues; in particular we shall prove that, for suitable values of the parameters g and θ, eigenvalues can neither accumulate at ∞ nor at any point in $\overline{\Sigma}_- \cup \overline{\Sigma}_+$.

THEOREM 2.3. — With the possible exception of one value of $g > 0$, independent of θ, $\sigma_d(H_g(\theta))$ is a bounded subset of the complex plane, i. e. there is no sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \sigma_d(H_g(\theta))$ with $\lim_{n\to\infty} |\lambda_n| = \infty$.

Proof. — As in [*3*], Theorem 3.12, one can prove that if $\lambda_0 \notin \overline{\Sigma}_- \cup \overline{\Sigma}_+$, then $\lambda_0 \in \sigma(H_g(\theta))$ if and only if $W(\lambda_0) = 0$, in which case $\lambda_0 \in \sigma_d(H_g(\theta))$ and is a pole of the resolvent of $H_g(\theta)$. Moreover if $\lambda \notin \overline{\Sigma}_- \cup \overline{\Sigma}_+$ and $W(\lambda) \neq 0$, the resolvent $R(\lambda; H_g(\theta)) = (\lambda - H_g(\theta))^{-1}$ is an integral operator in $L^2(\mathbb{R})$ with integral kernel $R(x, y; \lambda)$ defined by

$$(2.5)\qquad \begin{aligned} R(x, y; \lambda) &= W(\lambda)^{-1}\phi_-(x, \mu_-)\phi_+(y, \mu_+), &&\text{for } x < y\\ &= W(\lambda)^{-1}\phi_-(y, \mu_-)\phi_+(x, \mu_+), &&\text{for } x \geq y. \end{aligned}$$

More precisely

$$(2.6)\qquad (R(\lambda; H_g(\theta))f)(x) = \int_{-\infty}^{\infty} R(x, y; \lambda)f(y)dy, \qquad f \in L^2(\mathbb{R}).$$

Thus, since $\sigma_d(H_g(\theta)) \subset \mathbb{C}\backslash(\bar{\Sigma}_- \cup \bar{\Sigma}_+)$, $W(\lambda) = 0$ for any $\lambda \in \sigma_d(H_g(\theta))$. Therefore it suffices to prove that there exists $g_0 > 0$, independent of θ, such that for all $g \neq g_0$, $g > 0$, and all θ with $0 < \operatorname{Im} \theta < \theta_0$, the operator $H_g(\theta)$ satisfies the condition: $W(\lambda) \neq 0$ for all λ with $|\lambda|$ sufficiently large.

This follows immediately from Proposition 2.2, since

$$\lim_{\substack{|\lambda| \to \infty \\ \lambda \in R_1}} | W(\lambda) | = \infty \qquad \text{and} \qquad \lim_{\substack{|\lambda| \to \infty \\ \lambda \in R_2}} \mu_-(\lambda) W(\lambda) = \frac{3}{4} i g e^{\theta} c(g)$$

where $c(g)$ does not depend on θ and vanishes for at most one value of $g > 0$.

Remark. — From (2.2) $W(\lambda) = \frac{3}{4} i g e^{\theta} \operatorname{sgn}(\operatorname{Im} \lambda) \lambda^{-1/2} c(g) + 0(\lambda^{-1})$ as $|\lambda| \to \infty$, $\lambda \in R_2$, where $c(g)$ vanishes for at most one value of $g > 0$. As remarked above, in the proof of Theorem 2.3, for any $g > 0$ such that $c(g) \neq 0$ we have $W(\lambda) \neq 0$ for all λ with $|\lambda|$ sufficiently large. Also notice that $W(\lambda) \to 0$ as $|\lambda| \to \infty$, $\lambda \in R_2$, and this prevents the operator from being spectral in the sense of Dunford-Schwartz [*7*]. In particular the resolvent $R(\lambda) \equiv R(\lambda; H_g(\theta))$ is not uniformly bounded as $|\lambda| \to \infty$, $\lambda \in R_2$, for dist $(\lambda, \sigma(H_g(\theta)) = \text{const.}$; in fact $\lim_{\substack{|\lambda| \to \infty \\ \lambda \in R_2}} | R(\lambda) | = \infty$.

This result is rather surprising, since, if operators $H_g^{\pm}(\theta)$ are defined by the same potentials $V^{\pm}(x, \theta)$ restricted to $(0, \infty)$ and $(-\infty, 0)$ respectively with essential spectrum $g a_{\pm} e^{\theta} + [0, \infty)$, their resolvents are bounded in norm on lines parallel to these half-lines. This situation is analogous to the limit case with $a_- = a_+$ on $(-\infty, \infty)$, when the region R_2 didappears. However as long as $a_- \neq a_+$ (non-limit case), the two separate branches Σ_- and Σ_+ of $\sigma_{ess}(H_g(\theta))$ cause a sort of pinching of eigenvalues, which becomes more and more appreciable as either g or $| a_+ - a_- |$ gets smaller and smaller: R_2 becomes more and more narrow, while $\sigma_d(H_g(\theta))$ is « compressed » in it. So the number of zeros of $W(\lambda)$, i. e. the number of discrete eigenvalues, stays finite, but they alter the behaviour of $W(\lambda)$ (in comparison with its behaviour in R_1, which is similar to the self-adjoint case), causing its decay at ∞.

Nevertheless we shall be able to construct spectral projections (not uniformly bounded) and to prove completeness, in a sense that will be made more precise below (see Theorem 3.2). In order to obtain these results it is crucial that $W(\lambda)$ decays sufficiently slowly, i. e.

$$\lim_{\substack{|\lambda| \to \infty \\ \lambda \in R_2}} | \lambda^{1/2} W(\lambda) | = \frac{3}{4} g | e^{\theta} c(g) | \neq 0$$

(see [*3*] Theorem 3.19, and Theorem 2.7 below). On the other hand if g

is a critical value, in the sense that $c(g) = 0$, then it is still possible that $W(\lambda) \neq 0$ for $|\lambda|$ sufficiently large. It suffices to find a positive integer n such that $\lim_{\substack{|\lambda| \to \infty \\ \lambda \in R_2}} |\lambda^n W(\lambda)| \neq 0$. However in this case we would obtain a weaker version of completeness for the spectral projections.

The worst case, of course, occurs if, for some value of $g > 0$, $W(\lambda)$ vanishes to all orders. Then not only does completeness fail, but also eigenvalues accumulate at infinity.

THEOREM 2.4. — There exists a sequence $\{ g_n \}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $\lim_{n \to \infty} g_n = \infty$ such that for all $g \neq g_n$, $n \in \mathbb{N}$, $\sigma_d(H_g(\theta))$ is a finite subset of $\mathbb{C}$ for all θ with $0 < \operatorname{Im} \theta < \theta_0$.

Proof. — By a standard dilation analyticity argument (see e. g. Reed-Simon [*10*]), $\sigma_d(e^{-\theta} H_g(\theta)) \subset \{ z \in \mathbb{C} : -\operatorname{Im} \theta < \arg(z - ga_-) < 0 \} \cup \mathbb{R}$ and its only possible limit points are ga_- and ga_+. The assertion is obtained following the steps of the proof of Theorem 3.14 in [*3*]. In particular we can find a sequence $g_n \xrightarrow{n \to \infty} \infty$ such that for all $g \neq g_n$, $n \in \mathbb{N}$, and all θ with $0 < \operatorname{Im} \theta < \theta_0$, for the operator $H_g(\theta)$ there is no accumulation of eigenvalues at the thresholds $ga_- e^\theta$ and $ga_+ e^\theta$. The proof is then complete if we combine this result with Theorem 2.3.

In order to construct spectral projections for $H_g(\theta)$ we need to study the behaviour of the Wronskian $W(\lambda)$ as $\lambda \to \hat{\lambda} \in \sigma_{ess}(H_g(\theta))$. In particular we shall prove that, for a suitable choice of the parameters g and θ, $W(\lambda)$ approaches non-zero limits as $\lambda \to \hat{\lambda} \in \overline{\Sigma}_- \cup \overline{\Sigma}_+$ from R_1 qnd R_2.

THEOREM 2.5. — The sequence $\{ g_n \}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ obtained in Theorem 2.4 can be chosen so as to satisfy also the following condition:

for every $g \neq g_n$, $\forall n \in \mathbb{N}$, there exists $\theta(g) \in (0, \theta_0)$ such that for all θ with $0 < \operatorname{Im} \theta < \theta(g)$ there is $r(\theta) > 0$ enjoying the following property:

$$W^1_\pm(\lambda) \neq 0 \quad \text{and} \quad W^2_\pm(\lambda) \neq 0 \quad \text{for all } \lambda \in \Sigma_\pm(\theta') = \{ x + ga_\pm e^{\theta'} : x > 0 \}$$

for all θ' with $\operatorname{Im} \theta < \operatorname{Im} \theta' < \operatorname{Im} \theta + r(\theta)$.

Proof. — See [*3*], Theorem 3.15.

Remark. — In view of the above results we can select a sequence $\{ g_n \}_{n \in \mathbb{N}} \subset \mathbb{R}_+$, with $\lim_{n \to \infty} g_n = \infty$, satisfying the following condition:

for every $g \neq g_n$, $\forall n \in \mathbb{N}$, there exists $\theta(g) \in (0, \theta_0)$ such that for every fixed θ_1 with $0 < \operatorname{Im} \theta_1 < \theta(g)$ there is $r(\theta_1) > 0$ enjoying the following properties:

a) $\sigma_d(H_g(\theta))$ is finite

b) $$\lim_{\substack{|\lambda| \to \infty \\ \lambda \in R_2}} |\lambda^{1/2} W(\lambda)| \neq 0$$

c) $W^1_{\pm}(\lambda) \neq 0, \quad W^2_{\pm}(\lambda) \neq 0 \quad \text{for all} \quad \lambda \in \{ x + ga_{\pm}e^{\theta} : x > 0 \}$

d) $$\lim_{\lambda \to ga_{\pm}e^{\theta}} W(\lambda) \neq 0$$

for all θ with $\operatorname{Im}\theta_1 < \operatorname{Im}\theta < \operatorname{Im}\theta_1 + r(\theta_1)$.

In what follows we shall work with the operator $H_g(\theta)$ where the parameters g and θ are chosen so that conditions (*a*)-(*d*) hold.

We shall now study the behaviour of the resolvent $R(\lambda; H_g(\theta))$ as λ approaches a point $\hat{\lambda} \in \overline{\Sigma}_- \cup \overline{\Sigma}_+$ from R_1 and R_2. We shall prove that $R(\lambda; H_g(\theta))$ admits continuous limits in the weak sense, in terms of which it will be possible to define spectral projections and to provide a spectral representation for the resolvent.

LEMMA 2.6. — Let $\lambda_+ \in \overline{\Sigma}_+$. Then

i) $W^1_+(\lambda_+)$ and $W^2_+(\lambda_+)$ are continuous for all $\lambda_+ \in \overline{\Sigma}_+$;

ii) $\lim_{|\lambda_+| \to \infty} |W^1_+(\lambda_+)| = \infty; \qquad \lim_{|\lambda_+| \to \infty} |W^2_+(\lambda_+)| = 0.$

Similar results hold for $W^1_-(\lambda_-)$ and $W^2_-(\lambda_-)$, $\lambda_- \in \overline{\Sigma}_-$. Moreover

iii) for each pair of functions $f, h \in C(-\infty, \infty)$ which vanish outside a bounded set, the limits

$$B^1_+(f, h, \lambda) = \lim_{\delta \to 0^+} \langle R(\lambda + i\delta; H_g(\theta))f, h \rangle$$

$$B^2_+(f, h, \lambda) = \lim_{\delta \to 0^-} \langle R(\lambda + i\delta; H_g(\theta))f, h \rangle$$

exist for $\lambda \in \Sigma_+$ and are expressible by the formulas

$$(2.7) \quad B^1_+(f, h, \lambda) = W^1_+(\lambda)^{-1} \iint_{-\infty < x < y < \infty} \phi_-(x, \mu_-)\phi_+(y, \mu_+)f(y)\overline{h(x)}dydx + W^1_+(\lambda)^{-1} \iint_{-\infty < y < x < \infty} \phi_-(y, \mu_-)\phi_+(x, \mu_+)f(y)\overline{h(x)}dydx$$

$$(2.8) \quad B^2_+(f, h, \lambda) = W^2_+(\lambda)^{-1} \iint_{-\infty < x < y < \infty} \phi_-(x, \mu_-)\phi_+(y, -\mu_+)f(y)\overline{h(x)}dydx + W^2_+(\lambda)^{-1} \iint_{-\infty < y < x < \infty} \phi_-(y, \mu_-)\phi_+(x, -\mu_+)f(y)\overline{h(x)}dydx$$

where $\mu_+ = \lim_{\substack{\lambda' \to \lambda \\ \lambda' \in R_1}} (\lambda' - ga_+e^{\theta})^{1/2} > 0$ and $\mu_- = (\lambda - ga_-e^{\theta})^{1/2}$, $\operatorname{Im}\mu_- \geq 0$.

Similarly, for $\lambda \in \Sigma_-$ the limits

$$B^1_-(f, h, \lambda) = \lim_{\delta \to 0^-} \langle R(\lambda + i\delta; H_g(\theta))f, h \rangle$$

$$B^2_-(f, h, \lambda) = \lim_{\delta \to 0^+} \langle R(\lambda + i\delta; H_g(\theta))f, h \rangle$$

exist and are expressible by formulas analogous to (2.7), (2.8) replacing $+$ with $-$.

Proof. — Statement (*i*) follows from Lemma 2.1 and Equation (2.3). (*ii*) is a consequence of Proposition 2.2, whereas (*iii*) follows immediately from (2.5) and (2.6).

In the next theorem we obtain a spectral representation for the resolvent in terms of integrals along $\overline{\Sigma}_- \cup \overline{\Sigma}_+$. Let $\sigma_d(H_g(\theta)) = \{\lambda_1, \ldots, \lambda_n\}$, where each λ_j is an eigenvalue of finite multiplicity and a pole of the resolvent; let Γ_j be a small circle centered at λ_j, which containes no other eigenvalue of $H_g(\theta)$ and such that $\Gamma_j \cap \sigma_{ess}(H_g(\theta)) = \phi$, $j = 1, \ldots, n$.

THEOREM 2.7. — Let $\lambda \in \mathbb{C}\backslash\sigma(H_g(\theta)) \equiv \rho(H_g(\theta))$ and let the circles Γ_j, $j = 1, \ldots, n$, be chosen so that they do not contain the point λ. Then for the resolvent $R(\lambda; H_g(\theta))$ we have the following representation

$$(2.9) \quad \langle R(\lambda; H_g(\theta))f, h \rangle = \sum_{j=1}^{n} \langle R_j(\lambda)f, h \rangle + (2\pi i)^{-1} \int_{\Sigma_+} (z-\lambda)^{-1}[B^1_+(f, h, z) - B^2_+(f, h, z)]dz + (2\pi i)^{-1} \int_{\Sigma_-} (z-\lambda)^{-1}[B^2_-(f, h, z) - B^1_-(f, h, z)]dz$$

for all $f, g \in C_0^\infty(\mathbb{R})$, where

$$R_j(\lambda) = -(2\pi i)^{-1} \oint_{\Gamma_j} R(z; H_g(\theta))(z-\lambda)^{-1}dz, \qquad j = 1, \ldots, n.$$

Remark. — Note that in the case that λ_j is a simple eigenvalue,

$$R_j(\lambda) = (\lambda - \lambda_j)^{-1}P_j,$$

where P_j is the projection corresponding to λ_j. If λ_j is not simple, then $R_j(\lambda)$ may in principle contain a nilpotent part.

The integrals along Σ_+ and Σ_- appearing in (2.9) are in the direction of increasing $|z|$ for $z \in \Sigma_+ \cup \Sigma_-$, and the integral along Γ_j defining $R_j(\lambda)$ is taken in the counterclockwise direction, $j = 1, \ldots, n$.

Proof. — See [3], Theorem 3.17.

3. THE SPECTRAL PROJECTIONS AND THEIR COMPLETENESS

The next goal is to define spectral projections $E(\Delta)$ on the bounded Borel subsets Δ of $\Sigma_+ \cup \Sigma_-$ in such a way that

$$(3.1) \quad \langle R(\lambda; H_g(\theta))f, h \rangle = \sum_{j=1}^{n} \langle R_j(\lambda)f, h \rangle + \int_{\Sigma_+ \cup \Sigma_-} (z-\lambda)^{-1} d \langle E(z)f, h \rangle$$

for all f, h.

THEOREM 3.1. — Let Δ_+ and Δ_- be bounded Borel subsets of Σ_+ and Σ_- respectively. Then there exist bounded operators $E(\Delta_+)$ and $E(\Delta_-)$ on $L^2(\mathbb{R})$ uniquely determined by the formulas

(3.2) $$(E(\Delta_+)f)(x) = \int_{\Delta_+} \mu_+(z)\{\pi W_+^1(z)W_+^2(z)\}^{-1}\phi_-(x,\mu_-(z))\int_{-\infty}^{\infty}\phi_-(y,\mu_-(z))f(y)dydz$$

and

(3.3) $$(E(\Delta_-)f)(x) = \int_{\Delta_-} \mu_-(z)[\pi W_-^1(z)W_-^2(z)]^{-1}\phi_+(x,\mu_+(z))\int_{-\infty}^{\infty}\phi_+(y,\mu_+(z))f(y)dydz$$

for any $f \in C_0^\infty(\mathbb{R})$, $\Delta_+ \subset \Sigma_+$, $\Delta_- \subset \Sigma_-$. Moreover the operator-valued function $E(.)$ defined on the bounded Borel subsets Δ of the complex plane by

(3.4) $$E(\Delta) = E(\Delta \cap \Sigma_+) + E(\Delta \cap \Sigma_-)$$

defines a spectral measure in the following sense:

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$$

for bounded Borel subsets Δ_1 and Δ_2 of $\mathbb{C}$, with $\Delta_1 \cap \Delta_2 = \phi$.

Furthermore the operator $E(\Delta)$ satisfies the following conditions:

i) $$E(\Delta)D(H_g(\theta)) \subset D(H_g(\theta));$$

ii) $$E(\Delta)H_g(\theta)f = H_g(\theta)E(\Delta)f, \quad \text{for all } f \in D(H_g(\theta));$$

iii) for any two bounded Borel subsets of $\mathbb{C}$, Δ_1 and Δ_2, we have $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$. In particular $E(\Delta)$ is a projection, i. e. $E(\Delta)^2 = E(\Delta)$.

Proof. — See [*3*]: the arguments of Theorems 3.18, 3.19 and 3.20 can be applied by making use of Hypothesis 1.1 (*iv*) to obtain similar results.

In particular the boundedness of the operator $E(\Delta)$ can be shown following the steps of the proof of Theorem 3.18 of [*3*]; it is based on the fact that for some constant $D > 0$

(3.5) $$\left(\int_0^\infty |(\Phi_+h)(\mu)|^2 d\mu\right)^{1/2} \leq D\,|h|_2, \qquad \forall h \in C_0^\infty(\mathbb{R})$$

where $(\Phi_+h)(\mu) = \int_0^\infty \phi_+(x,\mu)h(x)dx$, for $\mu > 0$. This in turn is achieved by comparing Φ_+ with Φ_0 defined by

$$(\Phi_0 h)(\mu) = \int_0^\infty e^{i\mu x}h(x)dx, \qquad h \in C_0^\infty(\mathbb{R}), \qquad \mu > 0.$$

By Plancharel's theorem $\left(\int_0^\infty |(\Phi_0 h)(\mu)|^2 d\mu\right)^{1/2} \leq (2\pi)^{1/2} |h|_2$ and by Lemma 2.1

$$|(\Phi_0 h)(\mu) - (\Phi_+ h)(\mu)| \leq K(1 + |\mu|)^{-1} \int_0^\infty \left\{ \int_x^\infty (y + 1) |V^+(y, \theta)| dy \right\} |h(x)| dx .$$

Now (3.5) immediately follows from Hypothesis 1.1 (*iv*), since

$$\int_0^\infty \left\{ \int_x^\infty (1 + y) |V^+(y, \theta)| dy \right\}^2 dx \leq \int_0^\infty (1 + y) |V^+(y, \theta)| dy \int_0^\infty (1 + y)^2 |V^+(y, \theta)| dy < \infty .$$

We can now prove the main result of this section, i. e. the completeness for the projections $E(\Delta)$, in the sense of (3.1).

THEOREM 3.2. — The operator $H_g(\theta)$ satisfies the following equation

$$(3.6) \quad \langle R(\lambda; H_g(\theta))f, h \rangle = \sum_{j=1}^{n} \langle R_j(\lambda)f, h \rangle + \int_{\sigma_{ess}(H_g(\theta))} (z - \lambda)^{-1} d \langle E(z)f, h \rangle$$

for all $\lambda \in \rho(H_g(\theta))$, and $f, h \in C_0^\infty(\mathbb{R})$, where $E(z) = E([ga_+e^\theta, z])$, if $z \in \Sigma_+$ and $[ga_+e^\theta, z] = \{ x + ga_+e^\theta : 0 \leq x \leq z - ga_+e^\theta \}$. Similarly for $z \in \Sigma_-$. $dE(z)$ is defined more precisely below.

Sketch of the proof. — Proceeding as in the proof of Theorem 3.21 in [*3*] we obtain

$$(2\pi i)^{-1} \int_{\Sigma_+} [B_+^1(f, h, z) - B_+^2(f, h, z)](z - \lambda)^{-1} dz = \int_{\Sigma_+} \mu_+(z)[\pi W_+^1(z) W_+^2(z)]^{-1}(z - \lambda)^{-1} \int_{-\infty}^\infty \int_{-\infty}^\infty \phi_-(x, \mu_-(z))\phi_-(y, \mu_-(z)) f(y)\overline{h(x)} dy dx dz .$$

Let $\{ \Delta_+^n \}_{n \in \mathbb{N}}$ be an increasing sequence of compact subsets of $\overline{\Sigma}_+$ such that $\bigcup_{n \in \mathbb{N}} \Delta_+^n = \overline{\Sigma}_+$ and $ga_+e^\theta \in \bigcap_{n \in \mathbb{N}} \Delta_+^n$. Since the function

$$\gamma_+^n(z) \equiv \langle E(z)f, h \rangle = E([ga_+e^\theta, z])f, h \rangle, \qquad z \in \Delta_+^n ,$$

is absolutely continuous on Δ_+^n, it defines a Borel measure γ_+^n on each Δ_+^n, such that

$$d\gamma_+^n(z)=\left\{ \mu_+(z)[\pi W_+^1(z)W_+^2(z)]^{-1}\int_{-\infty}^{\infty}\phi_-(x,\mu_-(z))\overline{h(x)}dx \int_{-\infty}^{\infty}\phi_-(y,\mu_-(z))f(y)dy\right\}dz\equiv d\langle E(z)f,h\rangle$$

which does not depend on $n \to \infty$. Hence,

$$(3.7)\quad (2\pi i)^{-1}\int_{\Sigma_+}(z-\lambda)^{-1}[B_+^1(f,h,z)-B_+^2(f,h,z)]\,dz = \lim_{n\to\infty}\int_{\Delta_+^n}(z-\lambda)^{-1}d\langle E(z)f,h\rangle \equiv \int_{\Sigma_+}(z-\lambda)^{-1}d\langle E(z)f,h\rangle$$

where the last equality simply defines the integral $\int_{\Sigma_+}(z-\lambda)^{-1}d\langle E(z)f,h\rangle$. Similarly we have

$$(3.8)\quad (2\pi i)^{-1}\int_{\Sigma_-}(z-\lambda)^{-1}[B_-^2(f,h,z)-B_-^1(f,h,z)]\,dz = \int_{\Sigma_-}(z-\lambda)^{-1}d\langle E(z)f,h\rangle .$$

Now the assertion follows if we combine (2.9) with (3.7) and (3.8).

We conclude this section by introducing a class of functions $\mathscr{F}$, defined on $\sigma_{ess}(H_g(\theta))$, for which we can develop a functional calculus relative to the operator $H_g(\theta)$. The functional calculus for functions defined on the discrete part of the spectrum for $H_g(\theta)$ reduces to that for a finite dimensional matrix, since each discrete eigenvalue corresponds to a finite dimensional projection (see Kato [9]). Therefore we shall not consider this part further. In order to generalize the spectral theorem for self-adjoint operators, we shall show that with each $\psi \in \mathscr{F}$ we can associate a bounded operator $\psi(H_g(\theta))$ on $L^2(\mathbb{R})$, defined in terms of the spectral projections $E(\Delta)$. In particular $\psi(H_g(\theta))$ will be uniquely determined by the formula

$$(3.9)\qquad \langle \psi(H_g(\theta))f_1,f_2\rangle = \int_{\sigma_{ess}(H_g(\theta))}\psi(z)d\langle E(z)f_1,f_2\rangle$$

with $E(z)$ the projection-valued spectral family of Theorems 3.1 and 3.2.

Notice that this representation for the operator $\psi(H_g(\theta))$ is of scalar type, in the sense of Dunford-Schwartz [7].

The class $\mathscr{F}$ will consist of functions for which the right hand side of (3.9) defines a bounded bilinear form. More precisely

$$\mathscr{F} = \{ \psi \in L^{\infty}_{loc}(\sigma_{ess}(H_g(\theta))) : \sup_{z\in\sigma_{ess}(H_g(\theta))} |z\psi(z)| < \infty \} .$$

THEOREM 3.3. — For each $\psi \in \mathscr{F}$ there exists a bounded operator $\psi(H_g(\theta))$ in $L^2(\mathbb{R})$, uniquely determined by the Equation (3.9) for all f_1, $f_2 \in L^2(\mathbb{R})$. Moreover, if $\{ \psi_n \}_{n \in \mathbb{N}} \subset \mathscr{F}$, $\psi \in \mathscr{F}$, and $\| \psi_n - \psi \|_{\mathscr{F}} \overset{n \to \infty}{\longrightarrow} 0$, then $\psi_n(H_g(\theta)) \overset{n \to \infty}{\longrightarrow} \psi(H_g(\theta))$ in norm. Here $\| \psi \|_{\mathscr{F}} = \sup_{z \in \sigma_{ess}(H_g(\theta))} | z\psi(z) |$ for $\psi \in \mathscr{F}$.

Proof. — See [*3*], Theorem 3.22.

Remark. — For fixed $\lambda \in \rho(H_g(\theta))$ the function $\psi(z) = (z-\lambda)^{-1}$ belongs to $\mathscr{F}$ and the corresponding operator $\psi(H_g(\theta))$ satisfies the relationship

$$R(\lambda; H_g(\theta)) = \psi(H_g(\theta)) + \sum_{j=1}^{n} R_j(\lambda)$$

where $R_j(\lambda) = -(2\pi i)^{-1} \oint_{\Gamma_j} (z-\lambda)^{-1} R(z; H_g(\theta)) dz$, as in Theorem 2.1.

Moreover, by linearity, by differentiating with respect to λ and by the resolvent equation, we have that, if $P(x_1, \ldots, x_p)$ is a polynomial in $x_1, \ldots, x_p$ equal to zero at $x_1 = x_2 = \ldots = x_p = 0$, then

$$P(R(\beta_1; H_g(\theta)), \ldots, R(\beta_p; H_g(\theta))) = \psi(H_g(\theta)) + \text{eigenvalue part},$$

with $\psi(x) = P((x-\beta_1)^{-1}, (x-\beta_2)^{-1}, \ldots, (x-\beta_p)^{-1})$, $\beta_i \in \rho(H_g(\theta))$, $\forall i = 1, \ldots, p$. Furthermore, if ψ is the $\mathscr{F}$-norm limit of a sequence $\{ P_n \}_{n \in \mathbb{N}}$ of such polynomials then $P_n(H_g(\theta)) \overset{n \to \infty}{\longrightarrow} \psi(H_g(\theta))$ in norm. Hence it is natural to define $\psi(H_g(\theta))$ by the spectral integral.

ACKNOWLEDGMENTS

This paper is based on the author's doctoral thesis written at the University of Virginia. The author wishes to express her gratitude to Professor L. E. Thomas for his guidance and encouragement. It is a pleasure to thank Prof. E. Balslev for critical remarks and fruitful discussions.

REFERENCES

[*1*] E. BALSLEV, *Comm. Math. Phys.*, t. **52**, 1977, p. 127-146.

[*2*] E. BALSLEV, *J. Funct. Anal.*, t. **29**, 1978, p. 375-396.

[*3*] E. CALICETI, *Perturbation Theory for Schrödinger Operators with Complex Potentials.* Doctoral dissertation presented at the University of Virginia, Department of Mathematics. August 1983.

[*4*] E. CALICETI, M. MAIOLI, *Ann. Inst. H. Poincaré, Section A (Physique Théorique)*, t. **XXXVIII**, no. 2, 1983, p. 175-186.

[*5*] S. COLEMAN, *The Uses of Instantons.* Lectures delivered at the 1977 International School of Subnuclear Physics Ettore Majorana.

[6] E. B. DAVIES, B. SIMON, *Comm. Math. Phys.*, t. **63**, 1978, p. 277-301.
[7] N. DUNFORD, J. T. SCHWARTZ, *Linear Operators* Part III. New York, Wiley-Interscience, 1971.
[8] T. KATO, *Math. Annal.*, t. **162**, 1966, p. 258-279.
[9] T. KATO, *Perturbation Theory for Linear Operators*. Berlin, Heidelberg, New York, Springer 1966.
[10] M. REED, B. SIMON, *Methods of Modern Mathematical Physics*, vol. IV, New York, Academic Press, 1978.
[11] C. VAN WINTER, *J. Math. Anal. Appl.*, t. **94**, 1983, p. 406-434.

(Manuscrit reçu le 15 février 1984)
(Version révisée reçue le 1er octobre 1984)