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Multiple commutator estimates and resolvent smoothness in quantum scattering theory

by

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ABSTRACT. — We develop an abstract theory of multiple commutator estimates for a self-adjoint operator H and a suitable conjugate operator A which gives C^k smoothness of the resolvent as a function of the energy between suitable spaces. These estimates imply an abstract short-range scattering theory, local decay of scattering solutions for Schrödinger operators with smooth potentials, and asymptotic completeness for certain long-range potentials.

RÉSUMÉ. — On développe une théorie abstraite d'estimations de commutateurs multiples pour un opérateur auto-adjoint H et un opérateur conjugué convenable A. Cette théorie fournit la régularité C^k de la résolvante de H comme fonction de l'énergie, considérée comme opérateur entre des espaces convenables. Ces estimations entraînent une théorie abstraite de la diffusion pour des potentiels à courte portée, la décroissance locale des solutions diffusives pour des opérateurs de Schrödinger avec potentiels lisses, et la complétude asymptotique pour certains potentiels à longue portée.

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1. INTRODUCTION

A fundamental role is played in scattering theory for the Schrödinger equation by the boundary values of the resolvent $R(z) = (H - z)^{-1}$ of the Schrödinger operator H, as $z \to \lambda \pm i0$, for λ in the continuous spectrum of H. Existence of these boundary values in an appropriate topology implies absence of singular continuous spectrum for H. Smoothness of the boundary values $R(\lambda \pm i0)$ as a function of λ implies, via the Fourier transformation, local decay of solutions to the corresponding time-dependent Schrödinger equation. Smoothness of $R(\lambda \pm i0)$ also yields, in the context of stationary scattering theory, smoothness of the scattering matrix as a function of the energy.

In this paper we systematically expose and extend the commutator methods initiated by one of us (E. Mourre). The commutator methods have previously been used to prove existence of boundary values $R(\lambda \pm i0)$ [*24*, *30*], to study certain of their phase space localization properties [*23*, *25*, *26*, *27*], and to prove asymptotic completeness for two-body Schrödinger operators with short-range [*23*] and long-range [*29*] potentials, and for a large class of three-body Schrödinger operators [*27*]. All of these results rely on an abstract theory of resolvent estimates for a self-adjoint operator H in terms of a « conjugate operator » A which we discuss in Section 2 below. We note that this abstract theory has a long « prehistory » in the work of Kato [*16*, *17*], Lavine [*21*, *22*], Putnam [*31*] and others. Our extension consists in proving C^k-smoothness of the maps $\lambda \to R(\lambda \pm i0)$ in an appropriate topology and under appropriate hypotheses on H. Our results illuminate the connection between the commutator method and the dilation-analytic method of Aguilar, Balslev and Combes [*2*, *3*], and enables us to give a new proof of asymptotic completeness for long-range potentials along the lines of [*29*], but with considerably weaker hypotheses on the long-range potential. For other recent work on long-range potential, see [*10*, *28*].

Let us give an intuitive sketch of the theory we develop. Given a self-adjoint operator H on a Hilbert space $\mathscr{H}$, we suppose that there is another self-adjoint operator A, so that the unitary group $U(\theta) = \exp(i\theta A)$ preserves the domain $\mathscr{D}(H)$ of H. We suppose that the family of operators

$$H(\theta) = U(\theta)HU(\theta)^{-1}$$

varies smoothly with θ, i. e. derivatives of $H(\theta)$ exist up to some order $n \geq 1$ as bounded operators from $\mathscr{D}(H)$ (with the graph norm) to $\mathscr{H}$, and we suppose moreover that the first derivative is positive in a sense made precise below. We use smoothness of the map $\theta \to H(\theta)$ to prove smoothness of the resolvent as a function of the energy between suitable spaces defined by the operator A. If $(-i)^k B_k$, $1 \leq k \leq n$, denotes the k^{th} derivative of $H(\theta)$ at $\theta = 0$, our strategy is to study the resolvent of H by first studying the resolvent of the operator

$$H_n(\theta) = H + \sum_{k=1}^{n} \frac{(i\theta)^k}{k!} B_k$$

for *complex* θ. A special case ($n = 1$) of these results are those of Mourre [*24*], whose results coincide with many results of Aguilar, Balslev, and Combes [*2, 3*], although these latter authors assume *analyticity* of the map $\theta \to H(\theta)$. Thus our theory « interpolates » between Mourre's theory and the theory for dilation-analytic potentials.

A brief sketch of the contents of this paper follows. In Section 2 we present the main resolvent estimates that we prove in an abstract setting, and in Section 3 we give their proof. In Section 4 we use these results to give an abstract short-range scattering theory, and in Section 5 we present several applications to Schrödinger operators, including local decay of solutions to the time-dependent Schrödinger equation and asymptotic completeness for scattering with long-range potentials.

2. THE ABSTRACT RESULTS

In this section we introduce our notation, give some basic definitions, and state the abstract results. The proofs are given in the next section.

Let H be a self-adjoint operator in a separable Hilbert space $\mathscr{H}$ with domain $\mathscr{D}(H)$. Let E_H denote the spectral measure for H. Denote by $\mathscr{H}_n$ the completion of the vectors satisfying

$$\| \psi \|_n^2 = \int_{-\infty}^{\infty} (1 + \lambda^2)^{n/2} d \| E_H(\lambda)\psi \|^2 < \infty .$$

Then $\mathscr{H}_{+2}$ is the domain $\mathscr{D}(H)$ with the graph norm, and $\mathscr{H}_{-2}$ is the dual of $\mathscr{H}_{+2}$ obtained via the inner product on $\mathscr{H}$.

To motivate the definition below suppose that we are given another self-adjoint A on $\mathscr{H}$ such that the group $U(\theta) = \exp(i\theta A)$ maps $\mathscr{H}_{+2}$ into itself boundedly. Then $H(\theta) = U(\theta) H U(\theta)^{-1}$ belongs to $\mathscr{B}(\mathscr{H}_{+2}, \mathscr{H})$, the bounded operators from $\mathscr{H}_{+2}$ to $\mathscr{H}$. We say that H is *n-smooth with respect to* A, if the map $\theta \to H(\theta)$ is C^n as a map from $\mathbb{R}$ to $\mathscr{B}(\mathscr{H}_{+2}, \mathscr{H})$ with the operator norm. Let

$$B_k = (i)^k D_\theta^k H(0), \qquad 1 \leq k \leq n,$$

where the derivative is taken in the norm topology on $\mathscr{B}(\mathscr{H}_{+2}, \mathscr{H})$. Then $B_k \in \mathscr{B}(\mathscr{H}_{+2}, \mathscr{H})$, and *formally* we have

$$B_k = [B_{k-1}, A], \qquad 1 \leq k \leq n,$$

where $B_0 = H$. If $\mathscr{D}(A) \cap \mathscr{H}_{+2}$ is dense in $\mathscr{H}_{+2}$ in the graph norm, and the form $i^k[B_{k-1}, A]$ on $\mathscr{D}(A) \cap \mathscr{H}_{+2}$ is semibounded, we can identify B_k with $(i)^k$ times the operator obtained from the closure of this quadratic form. Conversely, suppose that $\mathscr{D}(A) \cap \mathscr{H}_{+2}$ is dense in $\mathscr{H}$ and that the commutators $\mathrm{Ad}^{(k)}(A)(H) = i[\mathrm{Ad}^{(k-1)}(A), H]$, $1 \leq k \leq n$, all define semibounded quadratic forms on $\mathscr{D}(A) \cap \mathscr{H}_{+2}$ which extend to bounded operators in $\mathscr{B}(\mathscr{H}_{+2}, \mathscr{H})$. It is not difficult to see that H is n-smooth with respect to A, with $B_k = [B_{k-1}, A]$ in the sense described above. We will focus on the following special class of n-smooth operators introduced by Mourre [*24*] in the case $n = 1$.

DEFINITION 2.1. — Let $n \geq 1$ be an integer. A self-adjoint operator A on $\mathscr{H}$ is said to be conjugate to H at the point $E \in \mathbb{R}$, and H is said to be n-smooth with respect to A, if the following conditions are satisfied:

a) $\mathscr{D}(A) \cap \mathscr{D}(H)$ is a core for H.

b) $e^{i\theta A}$ maps $\mathscr{D}(H)$ into $\mathscr{D}(H)$, and for each $\psi \in \mathscr{D}(H)$

$$\sup_{|\theta| \leq 1} \| H e^{i\theta A} \psi \| < \infty .$$

c_n) The form $i[H, A]$ defined on $\mathscr{D}(H) \cap \mathscr{D}(A)$, is bounded from below and closable. The self-adjoint operator associated with its closure is denoted iB_1. Assume $\mathscr{D}(B_1) \supset \mathscr{D}(H)$. If $n > 1$, assume for $j = 2, \ldots, n$ that the form $i[iB_{j-1}, A]$, defined on $\mathscr{D}(H) \cap \mathscr{D}(A)$, is bounded from below and closable. The associated self-adjoint operator is denoted iB_j, and it is assumed that $\mathscr{D}(B_j) \supset \mathscr{D}(H)$.

d_n) The form $[B_n, A]$, defined on $\mathscr{D}(H) \cap \mathscr{D}(A)$, extends to a bounded operator from $\mathscr{H}_{+2}$ to $\mathscr{H}_{-2}$.

e) There exist $\alpha > 0$, $\delta > 0$, and a compact operator K on $\mathscr{H}$ such that

$$E_H(J) iB_1 E_H(J) \geq \alpha E_H(J) + E_H(J) K E_H(J)$$

where $J = (E - \delta, E + \delta)$.

The interval J is called the interval of conjugacy. If H is n-smooth with respect to A for every integer $n \geq 1$, H is said to be ∞-smooth with respect to A.

Note that in [*30*], Perry, Sigal, and Simon showed that Mourre's theory could be carried through assuming only that $B_1 \in \mathscr{B}(\mathscr{H}_{+2}, \mathscr{H}_{-1})$. We could similarly extend our theory, but we do not do this.

It was shown in [*24*] that existence of a conjugate operator A to H at $E \in \mathbb{R}$ implies that the point spectrum of H is discrete in J. Furthermore, if $I \subset J \cap \sigma_c(H)$ ($\sigma_c(H)$ denotes the continuous spectrum of H) is a relatively compact interval, then for $s > 1/2$ the following *a priori* estimate holds:

$$\| (A^2 + 1)^{-s/2} (H - z)^{-1} (A^2 + 1)^{-s/2} \| \leq c$$

for all z with $\mathrm{Re}\, z \in I$, $\mathrm{Im}\, z \neq 0$. In particular, H has no singular continuous spectrum in J. To be precise, this result was proved for $s = 1$ in [*24*]. The proof for $s > 1/2$, due to Mourre, was given in [*30*].

For an operator A, P_A^+ (P_A^-) denotes the spectral projection corresponding to $(0, +\infty)$ $((-\infty, 0))$. The following result generalizes the results proved for $n = 1$ in [*24, 26*].

THEOREM 2.2. — Let H be a self-adjoint operator in $\mathscr{H}$ and $n \geqslant 1$ an integer. Let A be a conjugate operator to H at $E \in \mathbb{R}$. Assume H is n-smooth with respect to A. Let J be the interval of conjugacy, and $I \subset J \cap \sigma_c(H)$ a relatively compact interval. Let $s > n - (1/2)$.

i) For $\operatorname{Re} z \in I$, $\operatorname{Im} z \neq 0$, one has

$$\| (A^2 + 1)^{-s/2}(H - z)^{-n}(A^2 + 1)^{-s/2} \| \leq c\,. \tag{2.1}$$

ii) For $\operatorname{Re} z$, $\operatorname{Re} z' \in I$, $0 < |\operatorname{Im} z| \leq 1$, $0 < |\operatorname{Im} z'| \leq 1$, there exists a constant c, independent of z, z', such that

$$\| (A^2 + 1)^{-s/2}((H - z)^{-n} - (H - z')^{-n})(A^2 + 1)^{-s/2} \| \leq c\,|z - z'|^{\delta_1}$$

where

$$\delta_1 = \delta_1(s, n) = \frac{1}{1 + \dfrac{sn}{s - n + \dfrac{1}{2}}}.$$

iii) Let $\lambda \in I$. The norm limits

$$\lim_{\varepsilon \downarrow 0} (A^2 + 1)^{-s/2}(H - \lambda \pm i\varepsilon)^{-n}(A^2 + 1)^{-s/2}$$

exist and equal

$$\left(\frac{d}{d\lambda}\right)^{n-1} (A^2 + 1)^{-s/2}(H - \lambda \pm i0)^{-1}(A^2 + 1)^{-s/2}\,.$$

The norm limits are Hölder continuous with exponent $\delta_1(s, n)$ given above. (Here we use the notation

$$(A^2+1)^{-s/2}(H-\lambda \pm i0)^{-1}(A^2+1)^{-s/2} = \lim_{\varepsilon \downarrow 0} (A^2+1)^{-s/2}(H-\lambda \pm i\varepsilon)^{-1}(A^2+1)^{-s/2}.)$$

THEOREM 2.3. — Let H, $\mathscr{H}$, A, and I be as in Theorem 2.2. Let $s > n$.

i) For $\operatorname{Re} z \in I$, $\pm \operatorname{Im} z > 0$, one has

$$\| P_A^{\mp}(H - z)^{-n}(A^2 + 1)^{-s/2} \| \leq c\,. \tag{2.2}$$

ii) For $\operatorname{Re} z, \operatorname{Re} z' \in I, 0 < |\operatorname{Im} z| \leq 1, 0 < |\operatorname{Im} z'| \leq 1$, there exists $c > 0$, independent of z, z', such that for $\pm \operatorname{Im} z > 0$, $\pm \operatorname{Im} z' > 0$,

$$\| P_A^{\mp}((H - z)^{-n} - (H - z')^{-n})(A^2 + 1)^{-s/2} \| \leq c\,|z - z'|^{\delta_2}$$

where

$$\delta_2 = \delta_2(s, n) = \begin{cases} \dfrac{1}{2(n+1)}, & s \geq 2n \\ \dfrac{1}{1 + s \cdot \dfrac{n + \frac{1}{2}}{s - n}}, & n < s < 2n \end{cases}$$

iii) Let $\lambda \in \mathrm{I}$. The norm limits

$$\lim_{\varepsilon \downarrow 0} \mathrm{P}_\mathrm{A}^{\mp}(\mathrm{H} - \lambda \pm i\varepsilon)^{-n}(\mathrm{A}^2 + 1)^{-s/2}$$

exist and equal

$$\left(\frac{d}{d\lambda}\right)^{n-1} \mathrm{P}_\mathrm{A}^{\mp}(\mathrm{H} - \lambda \pm i0)^{-1}(\mathrm{A}^2 + 1)^{-s/2}.$$

The norm limits are Hölder continuous with exponent $\delta_2(s, n)$ given above.

THEOREM 2.4. — Let H, $\mathscr{H}$, A, and I be as in Theorem 2.2. Assume furthermore that H is $(n + 1)$-smooth w. r. t. A.

i) For $\operatorname{Re} z \in \mathrm{I}$, $\pm \operatorname{Im} z > 0$, one has

$$\| \mathrm{P}_\mathrm{A}^{\mp}(\mathrm{H} - z)^{-n}\mathrm{P}_\mathrm{A}^{\pm} \| \leq c. \tag{2.3}$$

ii) For $\operatorname{Re} z$, $\operatorname{Re} z' \in \mathrm{I}$, $0 < \pm \operatorname{Im} z \leq 1$, $0 < \pm \operatorname{Im} z' \leq 1$, there exists a constant $c > 0$, independent of z, z', such that

$$\| \mathrm{P}_\mathrm{A}^{\mp}((\mathrm{H} - z)^{-n} - (\mathrm{H} - z')^{-n})\mathrm{P}_\mathrm{A}^{\pm} \| \leq c \, | z - z' |.$$

iii) Let $\lambda \in \mathrm{I}$. The norm limits

$$\lim_{\varepsilon \downarrow 0} \mathrm{P}_\mathrm{A}^{\mp}(\mathrm{H} - \lambda \pm i\varepsilon)^{-n}\mathrm{P}_\mathrm{A}^{\pm}$$

exist and equal

$$\left(\frac{d}{d\lambda}\right)^{n-1} \mathrm{P}_\mathrm{A}^{\mp}(\mathrm{H} - \lambda \pm i0)^{-1}\mathrm{P}_\mathrm{A}^{\pm}.$$

The norm limits are Lipschitz-continuous.

3. PROOFS

As noted above the results are known for $n = 1$, so one can assume $n \geq 2$. Let A be a conjugate operator to H at $\mathrm{E} \in \mathbb{R}$ and assume H n-smooth w. r. t. A. Let J be the interval from Definition 2.1 (*e*). As shown in [*24*], any point $\mathrm{E}' \in \sigma_c(\mathrm{H}) \cap \mathrm{J}$ is contained in an interval $\mathrm{I} \subset \sigma_c(\mathrm{H}) \cap \mathrm{J}$ such that the following condition holds: Let ϕ be a smooth real-valued function which is identically one on I and has a sufficiently small compact support in $\sigma_c(\mathrm{H}) \cap \mathrm{J}$. Let $\mathrm{P_H} = \phi(\mathrm{H})$. Then for some $c > 0$

$$\mathrm{P_H} i \mathrm{B}_1 \mathrm{P_H} \geq c \mathrm{P}_\mathrm{H}^2 .$$

It is clear that it suffices to consider such intervals I in the proofs of Theorems 2.2-2.4. Throughout this section we fix one such interval I.

The estimates for powers of the resolvent are obtained using the auxiliary operator

$$\mathrm{C}_n(\varepsilon) = \sum_{j=1}^{n} \frac{\varepsilon^j}{j!} \mathrm{B}_j$$

which by assumption is H-bounded. The first step is to show existence of $(\mathrm{H} - z + \mathrm{C}_n(\varepsilon))^{-1}$ as a bounded operator under suitable restrictions on z and ε.

It is convenient to use the notation $\rho = (\mathrm{A}^2 + 1)^{-1/2}$. T^a denotes the closure of a closable operator T. In the sequel, c denotes various positive constants. *c is always independent of z and ε below.* This remark will not be repeated.

LEMMA 3.1. — There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$, $\operatorname{Re} z \in \mathrm{I}$, $\operatorname{Im} z \cdot \varepsilon > 0$, the following results hold:

i) The closed operator $\mathrm{H} - z + \mathrm{C}_n(\varepsilon)$ has a bounded inverse, denoted $\mathrm{G}_z(\varepsilon)$.

ii) $\mathrm{G}_z(\varepsilon)$ satisfies the estimates (c independent of z)

$$\| \mathrm{G}_z(\varepsilon) \| + \| \mathrm{H}\mathrm{G}_z(\varepsilon) \| \leq \frac{c}{|\varepsilon|}$$

$$\| \mathrm{G}_z(\varepsilon)\rho \| + \| \mathrm{H}\mathrm{G}_z(\varepsilon)\rho \| \leq \frac{c}{\sqrt{|\varepsilon|}} .$$

iii) $\mathrm{G}_z(\varepsilon)$ maps $\mathscr{D}(\mathrm{A})$ into $\mathscr{D}(\mathrm{A}) \cap \mathscr{D}(\mathrm{H})$.

iv) $\mathrm{G}_z(\varepsilon)$ is norm-differentiable w. r. t. ε, and on $\mathscr{D}(\mathrm{A})$

$$\frac{d}{d\varepsilon} \mathrm{G}_z(\varepsilon) = [\mathrm{G}_z(\varepsilon), \mathrm{A}] + \frac{\varepsilon^n}{n!} \mathrm{G}_z(\varepsilon) [\mathrm{B}_n, \mathrm{A}] \mathrm{G}_z(\varepsilon) .$$

REMARK 3.2. — Using the result $\mathrm{G}_z(\varepsilon)^* = \mathrm{G}_{\bar{z}}(-\varepsilon)$, one finds e. g. $\| \rho \mathrm{G}_z(\varepsilon) \| \leq c/\sqrt{|\varepsilon|}$. Such simple consequences of the lemma will be used without further comment.

Proof. — The idea of the proof is to apply a perturbation argument to a result in [*24*]. For the sake of clarity the proof is divided into several steps. The following result was proved in [*24*]: Let $\mathrm{P_H}$ be as above. Let $\operatorname{Re} z \in \mathrm{I}$

and $\varepsilon \cdot \operatorname{Im} z > 0$. Then $H - z + \varepsilon P_H B_1 P_H$ has a bounded inverse, denoted $G_z^M(\varepsilon)$ here. Let $P'_H = 1 - P_H$. Then one has [24]

$$\| P_H G_z^M(\varepsilon) \| + \| H P_H G_z^M(\varepsilon) \| \leq \frac{c}{|\varepsilon|} \tag{3.1}$$

$$\| P'_H G_z^M(\varepsilon) \| + \| H P'_H G_z^M(\varepsilon) \| \leq c \tag{3.2}$$

$$\| G_z^M(\varepsilon)\rho \| + \| H G_z^M(\varepsilon)\rho \| \leq \frac{c}{\sqrt{|\varepsilon|}}. \tag{3.3}$$

The above Remark 3.2 applies here, too. Thus (3.2) implies

$$\| \varepsilon B_1 P_H G_z^M(\varepsilon) P'_H \| \leq c\,|\varepsilon|\,.$$

Here and in the sequel $\operatorname{Re} z \in I$ and $\varepsilon \cdot \operatorname{Im} z > 0$ is assumed. There exists δ_1 such that for $|\varepsilon| < \delta_1$, one can define

$$G_z^0(\varepsilon) = G_z^M(\varepsilon) - G_z^M(\varepsilon) P'_H (1 + \varepsilon B_1 P_H G_z^M(\varepsilon) P'_H)^{-1} \varepsilon B_1 P_H G_z^M(\varepsilon)\,.$$

This approach is a standard technique for factored perturbations, see [16]. The computations are given in some detail in order to establish the estimates in (*ii*).

$G_z^0(\varepsilon)$ is bounded with range contained in $\mathscr{D}(H)$. A straightforward computation shows

$$(H - z + \varepsilon B_1 P_H) G_z^0(\varepsilon) = 1$$

on $\mathscr{H}$ and

$$G_z^0(\varepsilon)(H - z + \varepsilon B_1 P_H) = 1$$

on $\mathscr{D}(H)$.

Thus $H - z + \varepsilon B_1 P_H$ has $G_z^0(\varepsilon)$ as its bounded inverse. By construction and (3.1), (3.2), (3.3), $G_z^0(\varepsilon)$ satisfies the following estimates:

$$\| G_z^0(\varepsilon) \| + \| H G_z^0(\varepsilon) \| \leq \frac{c}{|\varepsilon|}$$

$$\| P'_H G_z^0(\varepsilon) \| + \| H P'_H G_z^0(\varepsilon) \| \leq c$$

$$\| G_z^0(\varepsilon)\rho \| + \| H G_z^0(\varepsilon)\rho \| \leq \frac{c}{\sqrt{|\varepsilon|}}.$$

The operator $P'_H G_z^0(\varepsilon)\varepsilon B_1$ is closable with a bounded closure which satisfies

$$\| (P'_H G_z^0(\varepsilon)\varepsilon B_1)^a \| \leq \| (P'_H G_z^0(\varepsilon)(H+i))^a \| \cdot \| ((H+i)^{-1} B_1)^a \| \cdot |\varepsilon| \leq c\,|\varepsilon|\,.$$

Thus there exists $\delta_2 \leq \delta_1$ such that for $|\varepsilon| < \delta_2$ one can define

$$G_z^1(\varepsilon) = G_z^0(\varepsilon) - (G_z^0(\varepsilon)\varepsilon B_1)^a (1 + (P'_H G_z^0(\varepsilon)\varepsilon B_1)^a)^{-1} P'_H G_z^0(\varepsilon)\,.$$

One shows as above that $G_z^1(\varepsilon)$ is a bounded inverse to $H - z + \varepsilon B_1$ and satisfies the estimates in (*ii*) of the lemma. This proves (*i*) and (*ii*) in case $n = 1$.

For $n \geq 2$ note that in this case $C_n(\varepsilon)$ is H-bounded and satisfies

$$\| (C_n(\varepsilon) - \varepsilon B_1)(H + i)^{-1} \| \leq c \,|\, \varepsilon \,|^2 .$$

Thus there exists $\varepsilon_0 \leq \delta_2$ such that for $|\, \varepsilon \,| < \varepsilon_0$ one can define

$$G_z(\varepsilon) = G_z^1(\varepsilon) - G_z^1(\varepsilon)(1 + (C_n(\varepsilon) - \varepsilon B_1)G_z^1(\varepsilon))^{-1} \cdot (C_n(\varepsilon) - \varepsilon B_1)G_z^1(\varepsilon) .$$

As above one verifies that $G_z(\varepsilon)$ satisfies (*i*) and (*ii*) of the lemma.

To prove (*iii*) note that the commutator

$$[A, G_z(\varepsilon)] = G_z(\varepsilon)[H + C_n(\varepsilon), A]G_z(\varepsilon)$$

extends to a bounded operator on $\mathscr{H}$ by (c_n) and (d_n) in Definition 2.1, cf. [*24*].

$$G_z(\varepsilon_1) - G_z(\varepsilon_2) = G_z(\varepsilon_1)(C_n(\varepsilon_2) - C_n(\varepsilon_1))G_z(\varepsilon_2)$$

shows that $G_z(\varepsilon)$ is norm-differentiable, and

$$\frac{d}{d\varepsilon} G_z(\varepsilon) = - G_z(\varepsilon)C_n'(\varepsilon)G_z(\varepsilon) .$$

Computing as a form on $\mathscr{D}(A) \cap \mathscr{D}(H)$, one finds

$$\begin{aligned} C_n'(\varepsilon) &= \sum_{j=1}^{n} \frac{\varepsilon^{j-1}}{(j-1)!} B_j = B_1 + \sum_{j=1}^{n-1} \frac{\varepsilon^j}{j!} [B_j, A] \\ &= [H + C_{n-1}(\varepsilon), A] \\ &= [H + C_n(\varepsilon), A] - \frac{\varepsilon^n}{n!} [B_n, A] \end{aligned}$$

from which (*iv*) of the lemma follows. □

The proofs of the theorems employ Mourre's differential inequality technique. The result needed is summarized in the following trivial lemma.

LEMMA 3.3. — Let X be a Banach space and $\varepsilon_0 > 0$. Let $f : (0, \varepsilon_0) \to X$ be a continuously norm-differentiable function. Assume there exists constants $\alpha, \beta, \gamma, c_1, c_2$, $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $-\infty < \gamma < \infty$, and $c_1, c_2 > 0$ such that

$$\| f'(\varepsilon) \| \leq c_1(\| f(\varepsilon) \|^{\alpha} \varepsilon^{-\beta} + 1)$$

and

$$\| f(\varepsilon) \| \leq c_2 \varepsilon^{-\gamma}, \qquad 0 < \varepsilon < \varepsilon_0 .$$

Then $\lim_{\varepsilon \downarrow 0} f(\varepsilon)$ exists in norm. Furthermore, there exists $c > 0$, c depending only on $c_1, c_2, \alpha, \beta, \gamma, \varepsilon_0$, such that $\| f(\varepsilon) \| \leq c$ for $0 \leq \varepsilon < \varepsilon_0$.

Proof. — Assume $\gamma \geq 0$. Otherwise the result is trivial. Fix ε_1, $0 < \varepsilon_1 < \varepsilon_0$. For $0 < \varepsilon < \varepsilon_1$, one has

$$f(\varepsilon_1) - f(\varepsilon) = \int_{\varepsilon}^{\varepsilon_1} f'(\mu) d\mu$$

and thus

$$
\begin{aligned}
\| f(\varepsilon) \| &\leq \| f(\varepsilon_1) \| + \int_{\varepsilon}^{\varepsilon_1} \| f'(\mu) \| \, d\mu \\
&\leq c_2 \varepsilon_1^{-\gamma} + \varepsilon_1 \cdot c_1 + c_1 \int_{\varepsilon}^{\varepsilon_1} \| f(\mu) \|^{\alpha} \mu^{-\beta} d\mu \\
&\leq c_2 \varepsilon_1^{-\gamma} + \varepsilon_1 c_1 + c_1 \,|\, \alpha\gamma + \beta - 1 \,|^{-1} \left(\varepsilon_1^{-\alpha\gamma-\beta+1} - \varepsilon^{-\alpha\gamma-\beta+1} \right) \cdot c_2^{\alpha} .
\end{aligned}
$$

Here one assumes $\alpha\gamma + \beta \neq 1$, since this can always be obtained by increasing γ slightly. If $-\alpha\gamma - \beta + 1 \geq 0$, the proof is obvious, so assume $-\alpha\gamma - \beta + 1 < 0$. Then there exists c_3 and ε_2, $0 < \varepsilon_2 \leq \varepsilon_1$, such that $\| f(\varepsilon) \| \leq c_3 \varepsilon^{-\alpha\gamma-\beta+1}$ for $0 < \varepsilon < \varepsilon_2$. This represents an improvement of $\gamma - \alpha\gamma - \beta + 1 \geq 1 - \beta > 0$, and thus in a finite number of iterations, one gets $\gamma \leq 0$ in which case the proof is trivial. Since only a finite number of iterations is needed, the last result follows. □

Proof of Theorem 2.2. — Assume $n \geq 2$, $s > n - (1/2)$, $\operatorname{Im} z > 0$, $\varepsilon > 0$, $\operatorname{Re} z \in \mathrm{I}$. Define for $0 < \varepsilon < \varepsilon_0$, $\mathrm{F}_z(\varepsilon) = \rho^s (\mathrm{G}_z(\varepsilon))^n \rho^s$. Lemma 3.1 (*iv*) implies

$$
\begin{aligned}
\frac{d}{d\varepsilon} \mathrm{F}_z(\varepsilon) &= \rho^s \frac{d}{d\varepsilon} (\mathrm{G}_z(\varepsilon))^n \rho^s \\
&= \rho^s \sum_{j=0}^{n-1} \mathrm{G}_z(\varepsilon)^j \left(\frac{d}{d\varepsilon} \mathrm{G}_z(\varepsilon) \right) \mathrm{G}_z(\varepsilon)^{n-j-1} \rho^s \\
&= \rho^s \sum_{j=0}^{n-1} \mathrm{G}_z(\varepsilon)^j [\mathrm{G}_z(\varepsilon), \mathrm{A}] \mathrm{G}_z(\varepsilon)^{n-j-1} \rho^s \\
&\quad + \frac{\varepsilon^n}{n!} \rho^s \sum_{j=0}^{n-1} \mathrm{G}_z(\varepsilon)^{j+1} [\mathrm{B}_n, \mathrm{A}] \mathrm{G}_z(\varepsilon)^{n-j} \rho^s \\
&= \rho^s [\mathrm{G}_z(\varepsilon)^n, \mathrm{A}] \rho^s + \frac{\varepsilon^n}{n!} \rho^s \sum_{j=0}^{n-1} \mathrm{G}_z(\varepsilon)^{j+1} [\mathrm{B}_n, \mathrm{A}] \mathrm{G}_z(\varepsilon)^{n-j} \rho^s \\
&= \mathrm{I}(\varepsilon) + \mathrm{II}(\varepsilon) .
\end{aligned}
$$

$s \geq 1$ implies

$$
\begin{aligned}
\| \mathrm{II}(\varepsilon) \| &\leq c\varepsilon^n \| \rho \mathrm{G}_z(\varepsilon) \| \cdot \sum_{j=0}^{n-1} \{ \| (\mathrm{G}_z(\varepsilon)^j (\mathrm{H} + i))^a \| \cdot \\
&\quad \cdot \| (\mathrm{H} + i)^{-1} [\mathrm{B}_n, \mathrm{A}] (\mathrm{H} + i)^{-1} \| \cdot \| (\mathrm{H} + i) \mathrm{G}_z(\varepsilon)^{n-j-1} \| \} \cdot \| \mathrm{G}_z(\varepsilon) \rho \| \\
&\leq c\varepsilon^n \varepsilon^{-1/2} \varepsilon^{-(n-1)} \varepsilon^{-1/2} \leq c .
\end{aligned}
$$

Using interpolation, one finds

$$\begin{aligned} \| \mathrm{I}(\varepsilon) \| &\leq 2 \| \rho^{s-1} \mathrm{G}_z(\varepsilon)^n \rho^s \| \\ &\leq 2 \| \rho^s \mathrm{G}_z(\varepsilon)^n \rho^s \|^{1-(1/s)} \| \mathrm{G}_z(\varepsilon)^n \rho^s \|^{1/s} \\ &\leq c \| \mathrm{F}_z(\varepsilon) \|^{1-(1/s)} (\varepsilon^{-(n-1)} \cdot \varepsilon^{-1/2})^{1/s} . \end{aligned}$$

Thus one gets

$$\left\| \frac{d}{d\varepsilon} \mathrm{F}_z(\varepsilon) \right\| \leq c_1 (\| \mathrm{F}_z(\varepsilon) \|^{1-(1/s)} \varepsilon^{-(n-(1/2))/s} + 1) . \tag{3.4}$$

Lemma 3.1 (*ii*) implies

$$\| \mathrm{F}_z(\varepsilon) \| \leq c_2 \varepsilon^{-(n-1)} .$$

Since c_1, c_2 above are independent of z (z restricted as stated above) and $(n - (1/2))/s < 1$, Lemma 3.3 gives (*i*) of the theorem.

To prove (*ii*), an extension of the argument given in [*30*] for $n=1$ is employed. (3.4) and $\| \mathrm{F}_z(\varepsilon) \| \leq c$ imply the existence of $\mathrm{F}_z(0)$ and the estimate

$$\| \mathrm{F}_z(0) - \mathrm{F}_z(\varepsilon) \| \leq c \varepsilon^{1-(n-(1/2))/s} .$$

One also has

$$\left\| \frac{d}{dz} \mathrm{F}_z(\varepsilon) \right\| = \| \rho^s \mathrm{G}_z(\varepsilon)^{n+1} \rho^s \| \leq c \varepsilon^{-n}$$

and thus

$$\| \mathrm{F}_z(\varepsilon) - \mathrm{F}_{z'}(\varepsilon) \| \leq c \, | z - z' | \cdot \varepsilon^{-n} .$$

Take

$$\varepsilon = | z - z' |^{\mu}, \qquad \mu = \frac{s}{s-n+sn+\dfrac{1}{2}},$$

and (*ii*), (*iii*) of the theorem follow easily. □

Proof of Theorem 2.3. — Assume $\mathrm{Re}\, z \in \mathrm{I}$, $\mathrm{Im}\, z > 0$, $\varepsilon > 0$, $s > n$. Define for $0 < \varepsilon < \varepsilon_0$, $\mathrm{F}_z(\varepsilon) = \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} \mathrm{G}_z(\varepsilon)^n \rho^s$. One then finds:

$$\begin{aligned} \frac{d}{d\varepsilon} \mathrm{F}_z(\varepsilon) &= \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} \mathrm{A} \mathrm{G}_z(\varepsilon)^n \rho^s + \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} \frac{d}{d\varepsilon} (\mathrm{G}_z(\varepsilon))^n \rho^s \\ &= \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} \mathrm{A} \mathrm{G}_z(\varepsilon)^n \rho^s + \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} [\mathrm{G}_z(\varepsilon)^n, \mathrm{A}] \rho^s \\ &\quad + \frac{\varepsilon^n}{n!} \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} \sum_{j=0}^{n-1} \mathrm{G}_z(\varepsilon)^{j+1} [\mathrm{B}_n, \mathrm{A}] \mathrm{G}_z(\varepsilon)^{n-j} \rho^s \\ &= \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} \mathrm{G}_z(\varepsilon)^n \mathrm{A} \rho^s + \frac{\varepsilon^n}{n!} \mathrm{P}_{\mathrm{A}}^- e^{\varepsilon \mathrm{A}} \sum_{j=0}^{n-1} \mathrm{G}_z(\varepsilon)^{j+1} [\mathrm{B}_n, \mathrm{A}] \mathrm{G}_z(\varepsilon)^{n-j} \rho^s \\ &= \mathrm{I}(\varepsilon) + \mathrm{II}(\varepsilon) . \end{aligned}$$

An interpolation argument yields

$$\begin{aligned} \| \mathrm{I}(\varepsilon) \| &\leq \| \mathrm{P}_{\mathrm{A}}^{-} e^{\varepsilon \mathrm{A}} \mathrm{G}_z(\varepsilon)^n \rho^{s-1} \| \\ &\leq \| \mathrm{P}_{\mathrm{A}}^{-} e^{\varepsilon \mathrm{A}} \mathrm{G}_z(\varepsilon)^n \rho^{s} \|^{1-(1/s)} \| \mathrm{P}_{\mathrm{A}}^{-} e^{\varepsilon \mathrm{A}} \mathrm{G}_z(\varepsilon)^n \|^{1/s} \\ &\leq c \| \mathrm{F}_z(\varepsilon) \|^{1-(1/s)} \varepsilon^{-n/s} . \end{aligned}$$

The second term is estimated using Lemma 3.1 (*ii*) and (d_n):

$$\| \mathrm{II}(\varepsilon) \| \leq c\varepsilon^{-1/2} .$$

Thus one finds

$$\left\| \frac{d}{d\varepsilon} \mathrm{F}_z(\varepsilon) \right\| \leq c(\| \mathrm{F}_z(\varepsilon) \|^{1-(1/s)} \varepsilon^{-n/s} + \varepsilon^{-1/2}) \tag{3.5}$$

and also

$$\| \mathrm{F}_z(\varepsilon) \| \leq c\varepsilon^{-(n-(1/2))} .$$

Since $n/s < 1$, a simple modification of Lemma 3.3, and arguments similar to those given above, will complete the proof. □

REMARK 3.4. — A differential inequality of the form (3.5) was first used in [*23*], and is one of the motivations for the present approach.

Proof of Theorem 2.4. — Assume H is $(n + 1)$-smooth w. r. t. A. For $\mathrm{Re}\, z \in \mathrm{I}$, $\mathrm{Im}\, z > 0$, $\varepsilon > 0$, define (using Lemma 3.1)

$$\begin{aligned} \mathrm{G}_z(\varepsilon) &= (\mathrm{H} - z + \mathrm{C}_{n+1}(\varepsilon))^{-1} \\ \mathrm{F}_z(\varepsilon) &= \mathrm{P}_{\mathrm{A}}^{-} e^{\varepsilon \mathrm{A}} (\mathrm{G}_z(\varepsilon))^n e^{-\varepsilon \mathrm{A}} \mathrm{P}_{\mathrm{A}}^{+} . \end{aligned}$$

$\mathrm{F}_z(\varepsilon)$ is weakly differentiable on $\mathscr{D}(\mathrm{A})$. Lemma 3.1 (*iv*) implies

$$\begin{aligned} \frac{d}{d\varepsilon} \mathrm{F}_z(\varepsilon) &= \mathrm{P}_{\mathrm{A}}^{-} e^{\varepsilon \mathrm{A}} [\mathrm{A}, \mathrm{G}_z(\varepsilon)^n] e^{-\varepsilon \mathrm{A}} \mathrm{P}_{\mathrm{A}}^{+} \\ &\quad + \mathrm{P}_{\mathrm{A}}^{-} e^{\varepsilon \mathrm{A}} \left(\frac{d}{d\varepsilon} \mathrm{G}_z(\varepsilon)^n \right) e^{-\varepsilon \mathrm{A}} \mathrm{P}_{\mathrm{A}}^{+} \\ &= \frac{\varepsilon^{n+1}}{(n+1)!} \mathrm{P}_{\mathrm{A}}^{-} e^{\varepsilon \mathrm{A}} \sum_{j=0}^{n-1} \mathrm{G}_z(\varepsilon)^{j+1} [\mathrm{B}_{n+1}, \mathrm{A}] \mathrm{G}_z(\varepsilon)^{n-j} e^{-\varepsilon \mathrm{A}} \mathrm{P}_{\mathrm{A}}^{+} . \end{aligned}$$

Since $[\mathrm{B}_{n+1}, \mathrm{A}]$ is bounded from $\mathscr{H}_{+2}$ to $\mathscr{H}_{-2}$, Lemma 3.1 (*ii*) implies $\left\| \frac{d}{d\varepsilon} \mathrm{F}_z(\varepsilon) \right\| \leq c$ from which the results follow. The details are omitted. □

4. SOME RESULTS ON ABSTRACT SCATTERING THEORY

The results in Section 2 have as one particular consequence an abstract scattering theory. In this section, H is assumed to be a semibounded self-adjoint operator in $\mathscr{H}$. It seems necessary to assume H semibounded to obtain a simple proof of the following lemma.

LEMMA 4.1. — Let H be a semibounded self-adjoint operator on $\mathscr{H}$. Assume that A satisfies (*a*), (*b*), (c_n), (d_n) in Definition 2.1 for some $n \geq 1$. Let $\phi \in C_0^\infty(\mathbb{R})$. Then $\phi(H)$ maps $\mathscr{D}(A^{n+1})$ into $\mathscr{D}(A^{n+1})$.

Proof. — This follows as in [*29*], if one notes that norm-differentiability can be replaced by strong differentiability. □

Theorems 2.3 and 2.4 have the following consequence which can be interpreted as a propagation property.

THEOREM 4.2. — Let H be a semibounded self-adjoint operator on Hilbert space $\mathscr{H}$. Assume that A is conjugate to H at $E \in \mathbb{R}$ and H is ∞-smooth with respect to A. Let J be the interval of conjugacy. Let $\phi \in C_0^\infty(J \backslash \sigma_p(H))$. Then for any s, s', $0 < s' < s$, there exists $c > 0$ such that the following estimates hold:

$$\| (A^2+1)^{-s/2} e^{-itH} \phi(H)(A^2+1)^{-s/2} \| \leq c(1+|t|)^{-s'}, \qquad t \in \mathbb{R} \quad (4.1)$$

$$\| (A^2+1)^{-s/2} e^{-itH} \phi(H) P_A^{\pm} \| \leq c(1+|t|)^{-s'}, \qquad \pm t > 0, \quad (4.2)$$

Proof. — Let $\rho = (A^2 + 1)^{-1/2}$. For $s > 1/2$ use Theorem 2.2 to write

$$\rho^s E'(\lambda)\rho^s = \frac{1}{2\pi i}(\rho^s(H - \lambda - i0)^{-1}\rho^s - \rho^s(H - \lambda + i0)^{-1}\rho^s).$$

Thus one has

$$\rho^s e^{-itH}\phi(H)\rho^s = \int_{-\infty}^{\infty} e^{-it\lambda}\phi(\lambda)\rho^s E'(\lambda)\rho^s d\lambda.$$

For $s > n + (1/2)$, Theorem 2.2 implies that $\phi(\lambda)\rho^s E'(\lambda)\rho^s$ is C^n in norm. Integration by parts then yields

$$\| \rho^s e^{-itH}\phi(H)\rho^s \| \leq c(1 + |t|)^{-n}, \qquad s > n + (1/2).$$

Since this result holds for all n, (4.1) follows by an interpolation argument, cf. [*23*].

The proof of (4.2) follows as in [*29*], using Lemma 4.1. □

The estimates (4.1) and (4.2) lead to an abstract scattering theory for H and H_1, where $H_1 - H$ is small in a suitable sense. One possible formulation is the following result.

THEOREM 4.3. — Let H be a semibounded self-adjoint operator on $\mathscr{H}$. Assume A is conjugate to H at $E \in \mathbb{R}$ and H is ∞-smooth w. r. t. A. Let J denote the interval of conjugacy. Let H_1 be another self-adjoint operator on $\mathscr{H}$ such that the following conditions are satisfied:

i) $(H_1 + i)^{-1} - (H + i)^{-1}$ is compact.

ii) There exists $s_0 > 1$ such that for every relatively compact interval

$I \subset J \cap \sigma_c(H)$ and some function ψ (ψ continuous, $\psi(x)$, $x\psi(x)$ bounded, $\psi(x) > 0$ for all $x \in I$), the operator

$$(E_{H_1}(I)H_1\psi(H) - E_{H_1}(I)H\psi(H))(A^2 + 1)^{s_0/2}$$

extends to a bounded operator on $\mathscr{H}$.

Then the following results hold:

a) $\sigma_{sc}(H_1) \cap J = \emptyset$.

b) $\sigma_p(H_1)$ is discrete in $J \backslash \sigma_p(H)$ (i. e. each eigenvalue of H_1 in $J \backslash \sigma_p(H)$ has finite multiplicity, and the only possible accumulation points are $\sigma_p(H) \cap J$ and the end points of J).

c) The wave operators $W_\pm = s - \lim_{t \to \pm\infty} e^{itH_1}e^{-itH}E_H(J \backslash \sigma_p(H))$ exist and are complete.

Proof. — The proof given in [*23*] based on estimates (4.1) and (4.2) will carry over without change to the present situation. To indicate the type of argument existence of W_+ will be shown. First one notes that assumption (*i*) and Lavine's argument (see e. g. [*32*; proof of Thm. XIII.31]) imply

$$s - \lim_{t \to \pm\infty} E_{H_1}(\mathbb{R} \backslash I)e^{itH_1}e^{-itH}E_H(I) = 0$$

where $I \subset J \cap \sigma_c(H)$ is a relatively compact interval. Let $f \in E_H(I)\mathscr{H}$ and assume $f = \phi(H)\psi(H)(A^2 + 1)^{-s_0/2}g$ where $\phi \in C_0^\infty(I)$. Vectors of this form are dense in $E_H(I)\mathscr{H}$. The usual Cook argument yields:

$$E_{H_1}(I)e^{it_2H_1}e^{-it_2H}f - E_{H_1}(I)e^{it_1H_1}e^{-it_1H}f = i\int_{t_1}^{t_2} e^{itH_1}(E_{H_1}(I)(H_1 - H))e^{-itH}f\,dt.$$

One has

$$\| E_{H_1}(I)(H_1 - H)e^{-itH}f \| = \| (E_{H_1}(I)H_1\psi(H) - E_{H_1}(I)H\psi(H))(A^2+1)^{s_0/2} \cdot (A^2+1)^{-s_0/2}e^{-itH}\phi(H)(A^2+1)^{-s_0/2}g \| \leq c(1+|t|)^{-s'} \| g \|$$

for $1 < s' < s_0$, and from this existence of $W_\pm$ follows. The rest of the proof follows the line of [*23*] except that the splitting

$$1 = F(|x| > c|t|) + F(|x| \leq c|t|)$$

is replaced by $1 = (1 + A^2)^{s/2}(1 + A^2)^{-s/2}$, as above. $\square$

The following theorem gives another version, where less is assumed on H and A, and more is assumed on the « interaction » $H_1 - H$.

THEOREM 4.4. — Let H be a semibounded self-adjoint operator on $\mathscr{H}$. Assume A is conjugate to H at $E \in \mathbb{R}$ and H is 2-smooth w. r. t. A. Let J denote the interval of conjugacy. Let H_1 be another self-adjoint operator on $\mathscr{H}$, such that the following conditions are satisfied:

i) $(H_1 + i)^{-1} - (H + i)^{-1}$ is compact.

ii) There exists $s_0 > 2$ such that for every relatively compact interval $I \subset J \cap \sigma_c(H)$ and some function ψ (ψ continuous, $\psi(x)$, $x\psi(x)$ bounded, $\psi(x) > 0$ for $x \in I$), the operator

$$(E_{H_1}(I)H_1\psi(H) - E_{H_1}(I)H\psi(H))(A^2 + 1)^{s_0/2}$$

extends to a bounded operator on $\mathscr{H}$.

Then the following results hold:

a) $\sigma_{sc}(H_1) \cap J = \emptyset$.

b) $\sigma_p(H_1)$ is discrete in $J \backslash \sigma_p(H)$.

c) The wave operators $W_\pm = s-\lim_{t \to \pm\infty} e^{itH_1}e^{-itH}E_H(J\backslash\sigma_p(H))$ exist and are complete.

Proof. — As in the proof of Theorem 4.3 the crucial step is to establish (4.1) and (4.2) for some $s' > 1$ with $s = s_0$. Consider first (4.1). Let $\rho = (A^2 + 1)^{-1/2}$, and $I \subset J \cap \sigma_c(H)$ a relatively compact interval. Let $\phi \in C_0^\infty(I)$. Then one has as in the proof of Theorem 4.2:

$$\rho^{s_0}e^{-itH}\phi(H)\rho^{s_0} = \int_{-\infty}^{\infty} e^{-it\lambda}\phi(\lambda)\rho^{s_0}E'(\lambda)\rho^{s_0}d\lambda\,.$$

Since $s_0 > 2 > 3/2$, Theorem 2.2 implies that the function

$$\Phi(\lambda) = \phi(\lambda)\cdot\rho^{s_0}E'(\lambda)\rho^{s_0},\ \Phi: \mathbb{R} \to \mathscr{B}(\mathscr{H})$$

(bounded operators on $\mathscr{H}$) is differentiable, and furthermore

$$\int_{-\infty}^{\infty} \|\Phi'(\lambda + h) - \Phi'(\lambda)\|\, d\lambda \le c\,|h|^{\delta_1}, \qquad \delta_1 > 0\,.$$

A well-known theorem on the Fourier transform then implies

$$\|\rho^{s_0}e^{-itH}\phi(H)\rho^{s_0}\| \le c(1 + |t|)^{-1-\delta_1}\,.$$

A similar argument shows

$$\|\rho^{s_0}e^{-itH}\phi(H)P_A^\pm\| \le c(1 + |t|)^{-1-\delta_2}, \qquad \pm t > 0\,.$$

The rest of the proof is identical to the proof of Theorem 4.3. □

REMARK 4.5. — It is clear that one can assume H is n-smooth w. r. t. A and by application of interpolation one can get a condition on the interaction with $1 < s_0 < 2$, for some s_0 depending on n.

A comparison of the above results with previous results of this type is difficult, partly due to the use of the conjugate operator, which was not used previously.

5. APPLICATIONS

Let us consider the case $H = -(1/2)\Delta + V(x)$ on $\mathscr{H} = L^2(\mathbb{R}^\nu)$ and $A = (1/2i)(x\cdot\nabla + \nabla\cdot x)$, the generator of dilations. If $V(x)$ is -Δ-bounded with relative bound less than one, $\mathcal{H}_{+2} = \mathcal{D}(-\Delta)$, and it is easy to see that hypotheses (*a*) and (*b*) of Section 2 hold, [*24*, *30*]. Moreover, $\mathcal{S}(\mathbb{R}^\nu)$ is a common core for H and A [*30*], so all commutators to be computed on $\mathcal{D}(A) \cap \mathcal{H}_{+2}$ may actually be computed on $\mathcal{S}(\mathbb{R}^\nu)$. If $B_0 = H$ and $B_k = [B_{k-1}, A]$ as forms on $\mathcal{S}(\mathbb{R}^\nu) \times \mathcal{S}(\mathbb{R}^\nu)$ all of the commutators are well-defined since A maps $\mathcal{S}(\mathbb{R}^\nu)$ into itself, and

$$i^k B_k = (-2)^k \left(-\frac{1}{2} \Delta \right) + (x \cdot \nabla)^k V(x)$$

where $(x \cdot \nabla)^k V(x)$ is interpreted in the sense of distributions. Hence, hypothesis (c_n) of Section 2 holds, if the distributional derivatives $(x \cdot \nabla)^k V(x)$, $1 \leq k \leq n$, all extend to bounded operators from $\mathcal{H}_{+2}$ to $\mathcal{H}$, and hypothesis (d_n) is satisfied, if $(x \cdot \nabla)^{n+1} V(x)$ extends to a bounded operator from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$. Hypothesis (*e*) holds provided V and $(x \cdot \nabla)V$ are both -Δ-compact, although these conditions can be weakened (cf. [*30*]).

Our first result concerns local decay of scattering solutions and is « input » to Theorem 5.2 on long-range scattering below.

THEOREM 5.1. — Let $H = -(1/2)\Delta + V(x)$ on $L^2(\mathbb{R}^\nu)$ where $V(x) \in C^{2m+1}(\mathbb{R}^\nu)$ and

$$| D^\alpha V(x) | \leq C_{|\alpha|} (1 + |x|)^{-(|\alpha| + \varepsilon)}$$

for some $\varepsilon > 0$ and all α with $|\alpha| \leq 2m + 1$. Let $\phi \in C_0^\infty((0, \infty) \backslash \sigma_p(H))$ and $\eta > 0$. Then for all $s \in (0, 2m - (1/2) + \eta)$ the estimate

$$\| (1 + x^2)^{-s/2} e^{-itH} \phi(H) (1 + x^2)^{-s/2} \| \leq c_s (1 + |t|)^{-\mu(s)}$$

holds, where $\mu(s) = s(2m - 1)/(2m - (1/2) + \eta)$.

REMARK 1. — Similar estimates can be proved for N-body Schrödinger operators with two-body potentials obeying the smoothness and decay hypothesis satisfied by V.

2. — Compare [*29*], where a similar result is proved under the more restrictive assumption that V is C^∞ and dilation-analytic.

3. — Muthuramalingam and Sinha have obtained slightly sharper estimates in [*28*].

Sketch of Proof. — The hypotheses on V guarantee that A is conjugate to H at any $E \in (0, \infty)$, and that H is $2m$-smooth with respect to A. Furthermore, they ensure $\mathcal{D}(H^k) = \mathcal{D}(H_0^k)$, $1 \leq k \leq m$, where $H_0 = -(1/2)\Delta$. It is not difficult to see that $(A^2 + 1)^{s/2} (H + c)^{-s/2} (1 + x^2)^{-s/2}$ is a bounded operator for $0 \leq s \leq 2m$ and suitable $c > 0$. Thus, Theorem 5.1 will follow, if we can show that

$$\| (A^2 + 1)^{-s/2} e^{-itH} \psi(H) (A^2 + 1)^{-s/2} \| \leq c_s (1 + |t|)^{-\mu(s)}$$

for any $\psi \in C_0^\infty(0, \infty)$ and $s \in (0, 2m - (1/2) + \eta)$. This follows by interpolation from the same estimate with $s = 2m-(1/2)+\eta$ and $\mu(s) = 2m-1$, since $e^{-itH}\psi(H)$ is bounded. To prove the estimate for $s=2m-(1/2)+\eta$ we use an argument with Fourier transforms together with the resolvent estimate of Theorem 2.2 (*i*) with $n=2m$, $s=2m-(1/2)+\eta$, cf. the proof of Theorem 4.2. □

We can use this result to prove

THEOREM 5.2. — Let $H = -(1/2)\Delta + V_s(x) + V_l(x)$ on $L^2(\mathbb{R}^\nu)$ where

i) $V_s(x)(1+x^2)^{(1+\varepsilon)/2}(-\Delta+1)^{-1}$ is a bounded operator on $L^2(\mathbb{R}^\nu)$ for some $\varepsilon > 0$, and

ii) $V_l(x) \in C^\infty(\mathbb{R}^\nu)$ and

$$| D^\alpha V_l(x) | \leq c_{|\alpha|}(1 + |x|)^{-(|\alpha|+\varepsilon_0)}$$

for some $\varepsilon_0 > 1/2$ and all α.

Then the modified wave operators ($p = -i\nabla$)

$$\Omega_D^\pm(H, H_0) = \underset{t \to \mp\infty}{s\text{-lim}}\, e^{itH} \exp\left(-i \int_0^t ds\left(\frac{1}{2}p^2 + V_l(ps)\right)\right) \tag{5.1}$$

exist and are strongly complete, i. e. H has no singular continuous spectrum, and Ran $\Omega_D^\pm$ $(H, H_0) = \mathscr{H}_{ac}(H)$, the subspace of absolute continuity for H. Moreover, eigenvalues of H can only accumulate at 0.

For existence of $\Omega_D^\pm(H, H_0)$ and discussion of modified wave operators, see [*32*], where references to the extensive literature on existence of modified wave operators may be found. Completeness under hypotheses similar to ours is a result of numerous authors, see [*1, 5, 7-11, 18-20, 33, 34*]. Note that most of these authors can treat arbitrary $\varepsilon_0 > 0$ (removing the restriction $\varepsilon_0 > 1/2$); we could also do this if we replaced the Dollard dynamics [*4*] with a more sophisticated modified free evolution [*6*]. The difference between [*29*] and the present theorem is that we remove the hypothesis of dilation analyticity. Note that Muthuramalingam and Sinha [*28*] have recently given a proof of asymptotic completeness similar in spirit to ours, but with less smoothness of V_l assumed.

Sketch of Proof. — We follow the outline of [*29*] except that Theorem 5.1 replaces Section 2 of [*29*] and the smoothing argument of Section 2 is skipped. We introduce an « intermediate » Hamiltonian $H' = -(1/2)\Delta + V_l(x)$, and prove completeness by showing that: (1) the modified wave operators $\Omega_D^\pm(H', H_0)$ exist and are complete, and (2) the ordinary wave operators $\Omega^\pm(H, H')$ exist and are strongly complete. We then appeal to the chain rule for wave operators to conclude the proof. Step (2) is an easy application of Theorem 4.4 and Remark 4.5 once we note that

$$(A^2 + 1)^{s/2}(1 + x^2)^{-s/2}(1 - \Delta)^{-s/2}$$

is bounded for all s (cf. e. g. [*23*]). Step (1) follows the outline of [*29*] except that we study the evolution of observables D, H, and x-tp using the dense of vectors in $\mathscr{H}_{ac}(\mathrm{H}')$

$$\mathscr{D} = \{ \phi(\mathrm{H}')\chi \mid \phi \in \mathrm{C}_0^\infty(0, \infty), \chi \in \mathrm{C}^\infty(\mathrm{A}) \}$$

where $\mathrm{C}^\infty(\mathrm{A})$ denotes the vectors $\chi \in \mathscr{D}(\mathrm{A}^n)$ for all positive integers n. For these vectors the basic estimate

$$\| (1 + x^2)^{-s/2} e^{-it\mathrm{H}'}\chi \| \leq c_{s,\eta}(1 + |t|)^{-(s-\eta)}$$

holds for all $s > 0$ and any $\eta > 0$, using Theorem 5.1 and interpolation. We can then study the evolution of observables as in Section 3 of [*29*], prove Theorem 3.1 of [*29*] on this new dense set $\mathscr{D}$, and conclude the proof of completeness of $\Omega_{\mathrm{D}}^{\pm}(\mathrm{H}', \mathrm{H}_0)$ as before. □

Let us conclude this section with some remarks on further applications. Smoothness of the boundary values of the resolvent is an important part of the results on time-decay given in [*13*]. The above Theorem 5.1 can be used in a discussion of time-decay in case $\mathrm{H} = -\Delta + \mathrm{V}$ with V a sum of a long-range and a sufficiently short-range potential. A complete discussion is rather involved and will be given elsewhere.

Smoothness of boundary values of the resolvent imply smoothness with respect to energy of various quantities in scattering theory. In particular, smoothness of the scattering matrix as a function of energy is obtained. Such results were given in [*14, 15*]. The above results provide the starting point for a similar discussion for the scattering matrix (and time-delay) for Schrödinger operators with long-range potentials. This discussion is complicated by the complexities of a stationary scattering theory for such operators [*10, 19, 34*] and by the non-uniqueness of the scattering matrix.

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Note added in proof: The multiple commutator technique has been used to study time-decay of scattering states for two-body Schrödinger operators with long-range potentials in: H. CYCON, P. PERRY, preprint, 1983.