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C. TRESSER

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About some theorems by L. P. Šil'nikov

by

C. TRESSER
Équipe de Mécanique Statistique, L. A. 190,
Parc Valrose, 06034 Nice Cedex, France

ABSTRACT. — Some theorems by L. P. Šil'nikov, which describe the dynamics in the neighbourhood of homoclinic orbits, bi-asymptotic to a saddle-focus, and initially proved for real analytic vector fields, are collected here. Recent results in dynamical systems theory allow us to precise some of the conclusions, and to generalize these theorems to the $C^{1,1}$ class. Certain heteroclinic loops involving a saddle-focus are also considered.

RÉSUMÉ. — On rassemble ici quelques théorèmes de L. P. Šil'nikov qui décrivent la dynamique dans le voisinage d'une orbite homocline, bi-asymptotique à une selle-foyer. Des résultats récents en théorie des systèmes dynamiques nous permettent de préciser les conclusions, et de généraliser à la classe $C^{1,1}$, des théorèmes initialement démontrés pour des champs de vecteur analytiques réels. On considère aussi le cas de certaines boucles hétéroclines faisant intervenir une selle-foyer.

I. INTRODUCTION

A local C^1 linearization theorem by P. Hartman [*21*] (improved notably in [*8*]) and hyperbolicity criteria [*23*] [*34*] which apply in particular to the non-wandering set of a horseshoe [*47*] allowed us to improve (at least in dimension 3) a theorem [*43*] [*46*] obtained by L. P. Šil'nikov for certain real analytic vector fields. More precisely, we obtained the following [*52*]:

THEOREM A. — *Consider the system:*

$$\begin{cases} \dot{x} = \rho x - \omega y + P(x, y, z), \\ \dot{y} = \omega x + \rho y + Q(x, y, z), \\ \dot{z} = \lambda z + R(x, y, z), \end{cases} \tag{1}$$

where P, Q, R *are* $C^{1,1}$ *functions which vanish, as well as their first order partial derivatives, at* O. *Suppose that there exists a homoclinic orbit* Γ_0, *biasymptotic to* O, *which remains at finite distance from any other singularity. Suppose at least that:*

$$\lambda > -\rho > O. \tag{2}$$

Then, if θ *is a first return map correctly defined on a well chosen piece of surface* π_0, *one gets the following conclusions:*

a) *for each positive integer m, there exists a map:*

$$h_m : \Sigma^m \to \pi_0 ,$$

which is a homeomorphism of Σ^m *onto* $O_m = h_m(\Sigma^m)$ *such that:*

$$\theta |_{O_m} = h_m \circ \sigma \circ h_m^{-1} ,$$

b) *the set* O_m *is hyperbolic,*

c) *for each real* α *with* $1 \leqslant \alpha \leqslant \lambda/\rho$, *there exists a map:*

$$h_{*,\alpha} : \Sigma^{*,\alpha} \to \pi_0 ,$$

which is a homeomorphism of $\Sigma^{*,\alpha}$ *onto* $O_{*,\alpha} = h_{*,\alpha}(\Sigma^{*,\alpha})$ *such that:*

$$\theta |_{O_{*,\alpha}} = h_{*,\alpha} \circ \sigma \circ h_{*,\alpha}^{-1} .$$

We have used the notations:

$$\Sigma^m = \{ 1, \ldots, m \}^{\mathbb{Z}} ,$$

$$\Sigma^{*,\alpha} = \left\{ \underline{s} = \{ s_i \}_{i=-\infty}^{+\infty} \mid s_i \in \mathbb{Z} - \{ 0 \} \text{ and } | s_{i+1} | \geqslant \frac{| s_i |}{\alpha} \right\}.$$

These sets are natural domain for a shift map, denoted σ in both cases, and defined as usual by:

$$\sigma(\{ s_i \}_{i=-\infty}^{+\infty}) = \{ s_i' \}_{i=-\infty}^{+\infty} ; \; s_i' = s_{i+1} .$$

REMARK 1. — In this theorem, *c*) implies *a*); we isolated *a*) since *a*) and *b*) together describe well known hyperbolic sets.

REMARK. — One of the consequences of *a*) or *c*) is that any neighbourhood of Γ_0 contains infinitely many periodic orbits of saddle type: this was the main conclusion of [*43*].

In section II of the present paper, we recall the proof of theorem A, since all other results can be obtained by small modifications of this proof.

Section III is devoted to other kinds of vectors fields involving homoclinic orbits biasymptotic to saddle-foci.

In section IV, we describe consequences of small perturbations on the flows examined in sections I to III. In section V, we give some results about certain heteroclinic loops involving at least one saddle-focus.

A large part of an unpublished version of this paper was devoted to applications. Since many references are now available on this matter, we shall only make brief comments, mainly bibliographical ones, in the last section.

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II. PROOF OF THEOREM A [*52*]

The proof will consist in four steps. The geometric construction in steps 2 and 3 follows [*32*] [*42*]-[*46*]. Most of the proof is indeed quite standard after a glance at Figure 1. Let us remark that in three dimensions, we do not need the genericity hypothesis formulated in [*46*] for the case of a dimension at least equal to 3.

STEP 1.

By [*21*] (or [*8*]), the flow described by (1) is C^1-conjugate to its linear part in some neighbourhood U of O. Extending the change of variables which allows this conjugacy, one gets a flow which is globally C^1 and linear in U. We shall continue to denote the coordinates by (x, y, z).

STEP 2.

In U, the local stable manifold W^s_{loc} of O is given by $z=0$, and the local unstable manifold W^u_{loc}, by $x=y=0$; we will suppose (for definiteness) that it is the $z \geqslant 0$ part of W^u_{loc} which belongs to the homoclinic orbit Γ_0. We shall denote by Π_0 the intersection of U with the half-plane $\{ y = 0, x > 0 \}$, and by $\Omega_0 = (x_0, O, O)$ one of the points of $\Gamma_0 \cap W^s_{loc} \cap \Pi_0$.

The linear flow in U induces a first return map P_0 on Π_0, at least for initial conditions such that $|z|$ is small enough. One then chooses a point A_1 in Π_0, with $z_1 > 0$, such that with $A'_1 = P_0(A_1)$, Ω_0 is inside the orthogonal projection of $\overline{A_1A'_1}$ on W^s_{loc}. On the vertical line through A_1, we define a sequence $\{A_i\}_{i=0}^{\infty}$ of points with $z = z_1 e^{-2\pi(i-1)\lambda/\omega}$. This gives a sequence $\{A'_i\}_{i=1}^{\infty}$ with $A'_i = P_0(A_i)$ on the vertical line through A'_i. We shall denote by π_0 the interior of the union (over $i \geqslant 1$) of the rectangles $R_i \equiv A_i A'_{i-1} A_i A_{i-1}$ in Π_0. For a correctly chosen $h > 0$, if $\Omega_1 = (O, O, h)$ is a point of W^u_{loc} in U, and Π_1 the plane orthogonal to W^u_{loc} at Ω_1, one can define a mapping $\theta_0 : \pi \to \Pi_1$, which associates to $M \in \pi_0$, the first intersection $\overline{M}$ of Π_1 with the orbit issued from M. We then define $\pi_1 = \theta_0(\pi_0)$. The expression of θ_0 reads:

$$\theta_0 : \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \mapsto \begin{pmatrix} x(h/z)^{\rho/a} \cos\left(\left(\dfrac{\omega}{\lambda}\right) . \operatorname{Log}\left(\dfrac{h}{z}\right)\right) \\ x(h/z)^{\rho/a} \sin\left(\left(\dfrac{\omega}{\lambda}\right) . \operatorname{Log}\left(\dfrac{h}{z}\right)\right) \\ h \end{pmatrix}$$

Let us remark that π_1 is the complementary set of a logarithmic spiral in a snail-shaped quasi-disk (Figure 1) and that the particular choice we have made for π_0 allows θ_0 to be a bijection.

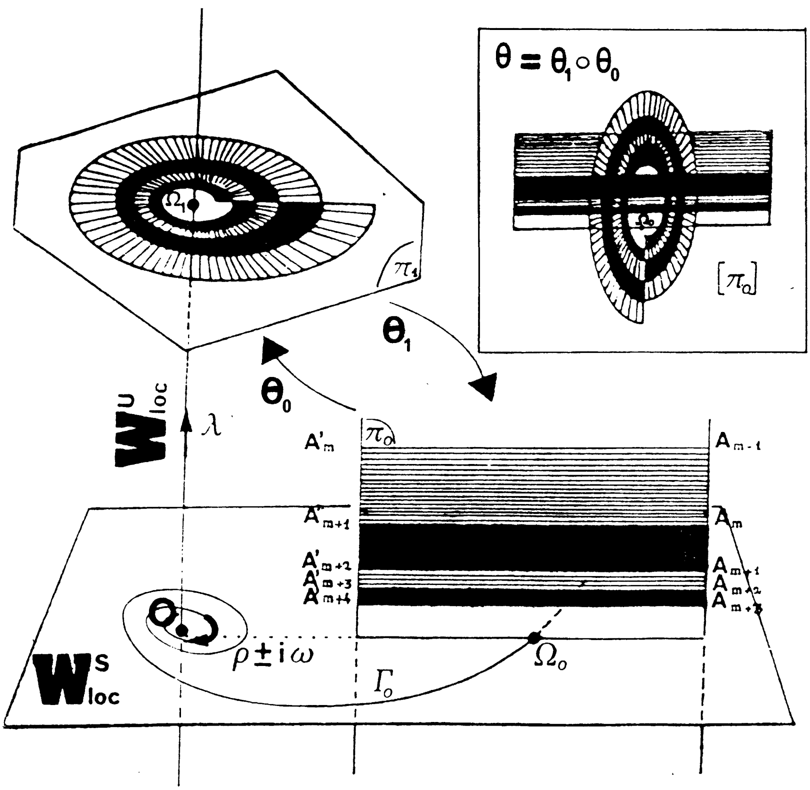

FIG. 1.

STEP 3.

The global part of the flow (out of U), induces a first intersection map $\theta_1 : \pi_1 \to \Pi_0$. If one identifies Π_1 and Π_0 so that Ω_1 and Ω_0 coincide, this map appears as a small perturbation of a linear non singular map θ_{1L}, since Γ_0 is supposed to be bounded away from any singularity out of U (see for instance [*33*]). One can force θ_1 to be as close as one wants from θ_{1L} in the C^1-topology by choosing U small enough. The product $\theta = \theta_1 \circ \theta_0 : \pi_0 \to \Pi_0$ is the first return map we wanted to construct: let us denote that the very existence of a singularity at O prevents the possibility of defining a global section for the flow. It is however usual, specially when one wants to exhibit a horseshoe effect, to consider a mapping from a rectangle to a larger surface which contains it.

STEP 4.

To get statements *a*) and *b*) of theorem A, we shall invoque the horseshoe theorem, as it is presented for instance in [*31*], and the remark that theorem (3.1) of [*34*] allows us to avoid the particular condition of theorem (3.3) of [*31*], and thus gives the hyperbolicity of the non-wandering set of the horseshoe map, and its well known dynamics, under the sole hypotheses of theorem (3.2) in [*31*] (essentially the existence of contracting and expanding invariant sector bundles). Statement *c*) of theorem A needs a straightforward rewording of the horseshoe theorem: the coding by $\Sigma^{*.\alpha}$ comes evidently from the spiral structure of $\theta(\pi_0)$. More precisely, one associates the symbols $+i$ and $-i$ to the two horizontal strips of the rectangle R_i, whose image under θ cuts vertically the rectangles R_k for $k \geqslant \frac{i}{\alpha}$; these vertical images are as usual also associated to the symbols $\pm i$. To get the coding by Σ^m for statements *a*) and *b*), one isolates m contiguous rectangles R_i, $i = k, \dots, k + m$, close enough to W^s_{loc}, and one chooses one strip per rectangle $\left(\text{or one isolates } E\left(\frac{m+1}{2}\right) \text{ rectangles and}\right.$ chooses two strips per rectangle, and only one for one of them if m is odd).

To check the hypotheses of the horseshoe theorems and conclude the proof, we use the fact that θ is arbitrarily C^1-close to $\theta_L = \theta_{1L} \circ \theta_0$. θ_L is given by:

$$\theta_L : \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \mapsto \begin{pmatrix} x(h/z)^{\rho/\lambda}\left(a \cos\left(\frac{\omega}{\lambda} \operatorname{Log}\left(\frac{h}{z}\right)\right) + c \sin\left(\frac{\omega}{\lambda} \operatorname{Log}\left(\frac{h}{z}\right)\right)\right) + x_0 \\ 0 \\ x(h/z)^{\rho/\lambda}\left(b \cos\left(\frac{\omega}{\lambda} \operatorname{Log}\left(\frac{h}{z}\right)\right) + d \sin\left(\frac{\omega}{\lambda} \operatorname{Log}\left(\frac{h}{z}\right)\right)\right) \end{pmatrix},$$

where $ad - bc \neq 0$, since θ_{1L} is non singular. One then remarks that one can find (x, z) pairs such that $T\theta_L$ has zero-trace: this however cannot occur for « interesting » points. Indeed, if U is small enough, the intersection condition along two vertical strips, of $\theta(R_k)$ with R_n $\left(n \geqslant \frac{k}{\alpha}\right)$, warrants the good behavior under direct (resp. inverse) iteration of contracting (resp. expanding) angular sectors. ∎

REMARK. — This proof can be readily adapted to piecewise $C^{1,1}$ vector fields which are only Lipschitz-continuous on smooth enough surfaces transversal to Γ_0. Examples of this kind are given by the piecewise linear vector fields considered in [*5*] and [*7*]. To adapt the proof it is enough to decompose θ_1 in $p + 1$ mappings: $\theta_1 = \theta_{1,p+1} \circ \theta_{1,p} \circ \ldots \circ \theta_{1,1}$, where p is the number of bad surfaces crossed by Γ_0. Jack Hale informed me, before publication, of an independent adaptation of Sil'nikov theorem to piecewise linear cases [*16*].

III. OTHER HOMOCLINIC CASES INVOLVING A SADDLE-FOCUS

The most direct consequence of theorem A is that it applies as well for equations of type (1) but with:

$$-\lambda > \rho > 0, \tag{3}$$

since one has only to reverse time to get the hypotheses of theorem A. On the contrary, when:

$$-\rho > \lambda > 0 \qquad (\text{or} \quad \rho > -\lambda > 0) \tag{4}$$

one has:

THEOREM B. — *Under the hypotheses of Theorem* A, *except that* (4) *replaces* (2), *there is no periodic orbit in a neighbourhood of* Γ_0 *if it is chosen small enough.*

Remark. — For real analytic vector fields, a generalization of this result can be found in [*44*].

Proof. — One constructs a map θ like in steps 1 to 3 of the proof theorem A. It then remains only to check that the image of a point of R_m, m large enough, is either in $\Pi_0 \backslash \pi_0$, or in some $R_{m'}$ with $m' > m$. ∎

Figure 2 represents π_0 and $\theta(\pi_0)$ under the hypotheses of theorems A and B.

We shall now consider differential equations like (1), under the supplementary hypothesis that they are invariant under the change of variables

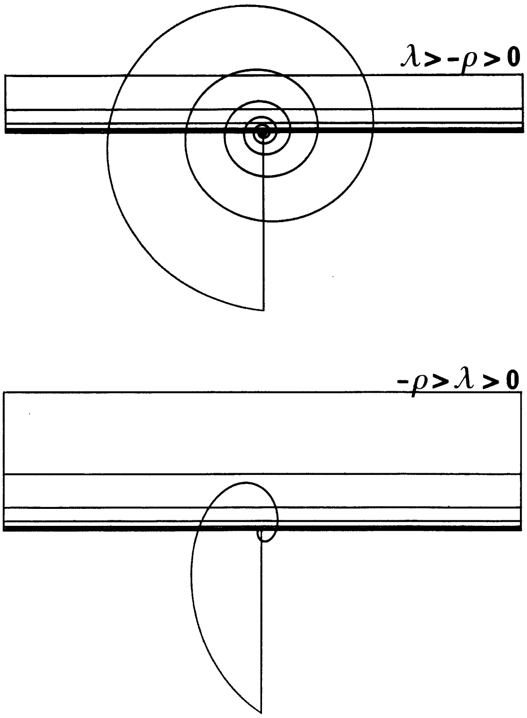

FIG. 2.

$(x, y, z) \mapsto (-x, -y, -z)$. In other words, P, Q, R, are supposed to be odd functions, or there is a central symmetry. Then, for each orbit, there are only two possibilities:

— either the orbit is invariant under the central symmetry,
— or there exists a pair of orbits which are exchanged by the central symmetry.

In particular, homoclinic orbits biasymptotic to O can only occur by pairs $\Gamma_0^{\pm} = W_0^{u\pm}$. Clearly, such symmetric systems can fulfil the hypotheses of theorems A or B and the conclusions of these theorems remain relevant if one is only interested in the neighbourhood of the orbit Γ_0^+ or Γ_0^-. It is however interesting to consider what occurs in the neighbourhood of the « figure eight » $\Gamma_0^+ \cup \Gamma_0^-$. This yields theorems AS and BS below. The formulation of these theorems and of theorem C in section V will be easier by the following notations.

Let $$\mathbb{Z}_n^* = \mathbb{Z}^* \times \{1, \ldots, n\},$$
and $$\|(a, i)\| = |a| \quad \text{for} \quad (a, i) \in \mathbb{Z}_n^*.$$

We shall denote by $\Sigma_n^{*,\alpha}$ the subset of $\mathbb{Z}_n^{*\mathbb{Z}}$ constituted by those sequences $\underline{s}$ such that, for each i:

$$\| s_{i+1} \| \geqslant \frac{\| s_i \|}{\alpha},$$

where $\alpha > 1$ is a real number.

Also, $\Sigma_n^{:,\beta^+}$ will be the subset of $\mathbb{Z}_n^{*\mathbb{Z}^+}$ constituted by those sequences $\underline{s} = \{ s_i \}_{i=0}^{+\infty}$ such that, for each $i \geqslant 0$:

$$\| s_{i+1} \| \geqslant \frac{\| s_i \|}{\beta},$$

where $\beta < 1$ is a positive real number.

The main advantage of introducing these notations is that the coding will be somehow « natural »: for instance, all symbols beginning with a same $|j|$, j in $\mathbb{Z}$, will correspond to a single horseshoe with $2n$ branches in theorems AS and C.

THEOREM AS. — *Under the hypotheses of theorem* A, *and if* P, Q, R *are odd functions, one gets, beside the conclusions of theorem* A, *a map:*

$$h_{*,\alpha,2} : \Sigma_2^{*,\alpha} \rightarrow \pi_2$$

which is defined for each α *with* $1 \leqslant \alpha \leqslant -\lambda/\rho$ *and which is a homeomorphism onto its image* $O_{*,\alpha,2} = h_{*,\alpha,2}(\Sigma_2^{*,\alpha})$ *such that:*

$$\theta_2 |_{O_{*,\alpha,2}} = h_{*,\alpha,2} \circ \sigma \circ h_{*,\alpha,2}^{-1},$$

where θ_2 *stands for a first return map on* π_2.

θ_2 and π_2 will be defined in the proof: they are natural adaptations of θ and π_0 in theorem A.

THEOREM BS. — *Under the hypotheses of theorem* B, *and if* P, Q, R *are odd functions, one gets, beside the conclusions of theorem* B, *a map:*

$$h_{:,\beta,2} : \Sigma_2^{:,\beta^+} \rightarrow \pi_2$$

which is defined for each β *with* $0 < \beta < -\lambda/\rho < 1$ *and which is a bijection onto its image* $O_{:,\beta,2} = h_{:,\beta,2}(\Sigma_2^{:,\beta^+})$ such that:

$$\theta_2 |_{O_{:,\beta,2}} = h_{:,\beta,2} \circ \sigma \circ h_{:,\beta,2}^{-1}.$$

Furthermore, $\Gamma_0^+ \cup \Gamma_0^-$ *is an attractor.*

One can simplify the coding in theorem BS to get the:

COROLLARY [*24*]. — *Under the hypothese of theorem* BS, *in each neighbourhood of* $\Gamma_0^+ \cup \Gamma_0^-$ *there exist sets of orbits in one to one correspondance with* $\{ 1, 2 \}^{\mathbb{Z}^+}$. *This correspondance can be described as follows: to each sequence* $\underline{s} = \{ s_i \}_{i=0}^{+\infty}$, $s_i \in \{ 1, 2 \}$ *one associates an orbit which starts, at an arbitrary initial time, close to* $W_{loc}^{u^+}$ (resp. $W_{loc}^{u^-}$) *if* $s_i = 1$ (resp. *if* $s_i = 2$)

and which starts each new loop in the neighbourhood of Γ_0^+ *or* Γ_0^- *according to the same law.*

Proof. for theorems AS *and* BS. — The main modification which needs to be made in the proofs of theorems A and B is a geometrical construction which takes into account the central symmetry. This construction is more easily explained in two steps.

STEP 1.

One renames, by adding a superscript « + », all sets and maps used in the proof of theorem A, and one uses the same symbols, except that a « – » replaces the « + », for the sets and maps induced by the symmetry. In particular, one gets two maps:

$$\begin{cases} \theta^+ : \pi_0^+ \to \Pi_0^+ , \\ \theta^- : \pi_0^- \to \Pi_0^- . \end{cases}$$

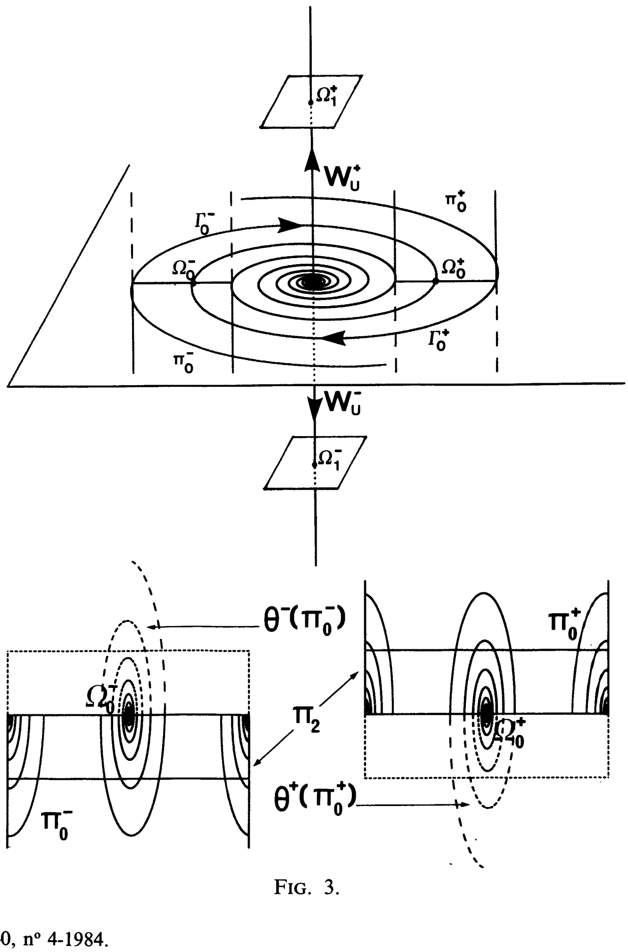

FIG. 3.

STEP 2.

Denoting by ϕ^t the linear flow in U, one remarks that, for each M in Π_0^+ (resp. Π_0^-), one and only one of the points $\phi^{\pi/\omega}(M)$ and $\phi^{-\pi/\omega}(M)$ belongs to Π_0^- (resp. Π_0^+) : we shall denote this point by P(M). Setting then:

$$\Pi_2 = \Pi_0^+ \cup \Pi_0^- \; ; \; \pi_2 = \pi_0^+ \cup \pi_0^- ,$$

one defines a map:

$$\theta_2 : \pi_2 \to \Pi_2 ,$$

by:

$$\begin{cases} \theta_2(M) = \theta^{\pm}(M) & \text{if} \quad z(M) \cdot z(\theta^{\pm}(M)) > 0 , \\ \theta_2(M) = P(\theta^{\pm}(M)) & \text{if} \quad z(M) \cdot z(\theta^{\pm}(M)) < 0 . \end{cases}$$

where $\theta^{\pm}$ means that θ^+ (resp. θ^-) has been applied if $M \in \pi_0^+$ (resp. π_0^-). The action of θ_2 has been schematized in Figure 3.

Remark. — P. Holmes gives another geometrical construction for the symmetric case in [*24*], corresponding to the one given in [*21*] for the non symmetric case. The choices made in the present paper were motivated by our attempt to be as close as possible to the original proof in [*43*], at least for theorem A.

IV. PERTURBATION OF SYSTEMS HAVING A HOMOCLINIC CURVE, BIASYMPTOTIC TO A SADDLE FOCUS

In this section, we shall consider small C^1 perturbations of flow verifying the hypotheses of the previous theorems. We shall denote by Z the z-coordinate of the point $\theta_1(\Omega_1)$ in the general cases, and by $Z^{\pm}$ the z-coordinates of the points $\theta_1^{\pm}(\Omega_1^{\pm})$ in the symmetric cases. We use the same symbols to refer to sets and maps related to unperturbed and perturbed systems. The results below are determined by the sign of Z (or $Z^{\pm}$), assuming the perturbations are small enough in the C^1 topology. In each case, we shall be interested in the dynamical behavior in a small neighbourhood V of Γ_0 (or $\Gamma_0^+ \cup \Gamma_0^-$). In order to avoid repetitions, most of the hypotheses of the theorems we shall formulate will be contained in the name of the theorem: for instance theorem AP describes C^1 perturbations to systems satisfying the hypotheses of theorem A. We shall begin with theorems AP and ASP which are immediate consequences of the proofs of theorems A and AS (see also [*24*] [*32*] [*46*]).

THEOREM AP. — *For* $|Z|$ *small enough, one can find* $M > 1$, *and for each* m *with* $1 < m \leqslant M$, *a map:*

$$h_m : \Sigma^m \to \pi_0$$

which is a homeomorphism onto its image $O_m = h_m(\Sigma^m)$ *such that:*

$$\theta |^{O}_{m} = h_m{}^1 \sigma{}^1 h_m^{-1} .$$

Furthermore, O_m *is hyperbolic.*

Remark. — If $Z < 0$, there cannot be any homoclinic curve in V. On the contrary, for $Z_0 > 0$, one can find Z in $]0, Z_0[$ such that there exists a homoclinic curve in V; for each of these values of Z one gets the hypotheses of theorem A, but in order to find the corresponding $O_{*,\alpha}$, one has to redefine $\pi_0, \pi_1, \ldots$ with respect to the new homoclinic orbit.

THEOREM ASP. — *Beside the conclusions of theorems* AS, *which work for a neighbourhood* V^+ (resp. V^-) *of* Γ_0^+ (resp. Γ_0^-), *there can be zero, one or two homoclinic orbits in* V *for a general* C^1 *perturbation of the flow, zero or one pair of symmetric homoclinic orbits for symmetric perturbations in either case* $Z^+ = Z^- > 0$ *and* $Z^+ = - Z^- < 0$.

Remark. — Theorem AP can be interpreted as follows: among the infinity of nested horseshoes that occur when there is a homoclinic curve, finitely many persist under a small C^1 perturbation.

This remark was already reported in [*46*] (see also [*32*]). The two following theorems are proved (with different smoothness assumptions), the first one in [*32*] [*45*] the second in [*24*]. They will be given here without further comment, and we leave to the reader the task of listing the possible consequences of a general perturbation of a system verifying the hypotheses of theorem BS.

THEOREM BP.— *If* $Z > 0$, *there is a single periodic orbit in* V, *and this orbit is stable. If* $Z < 0$, 0 *is the unique non wandering point in* V.

THEOREM BPS. — *For a symmetric perturbation, if* $Z^+ = - Z^- > 0$, *there is a pair of stable and symmetric periodic orbits. If* $Z^+ = - Z^- < 0$ *there is a single periodic orbit which is stable and invariant by the central symmetry.*

V. ABOUT CERTAIN HETEROCLINIC LOOPS

The aim of this section is to extend some of the previous results to some heteroclinic loops. One is then faced with the fact that the problem of local C^1 linearization is more delicate for three real eigenvalues than for saddle-foci (see the example of P. Hartman in [*21*]) As a consequence of a lemma by S. Sternberg in [*49*] (p. 812), one can use a theorem by G.R. Belitskii [*8*]. This theorem generalizes the theorem of P. Hartman [*21*], by insuring the existence of a local C^1 linearization for $C^{1,1}$ maps (and thus, by [*49*] for $C^{1,1}$ flows), under the mildest non resonance-type conditions. These non-resonance hypotheses can be formulated for maps as:

$$\forall k: \qquad |\rho_k| \neq |\rho_i| \cdot |\rho_j|; \; |\rho_i| \leqslant 1 \leqslant |\rho_j|, \tag{5}$$

and for flows, the ρ_i's represent the exponentials of the eigenvalues λ_i's.

This chapter is organized as follows: after some general considerations in (*a*), we define what we call « simple type » heteroclinic loops in (*b*): another class (semi-simple type) will be brievely examined in the Appendix. The theorem C of (*c*) apply to the simple type loops.

a) *General considerations*

Let X be a $C^{1,1}$ vector field in $\mathbb{R}^3$ and $\{ O_i \}_{i=1,\dots,n}$, n hyperbolic critical points of X. All O_i's are supposed to be codimension -1 unstable with $W^u_{O_i} \cap W^s_{O_{i+1}} \neq \emptyset$ where we have used the notation $O_{i+1} \equiv O_i$.

The intersections:

$$\Gamma_i = W^u_{O_i} \cap W^s_{O_{i+1}}, \quad i \in \{1, \dots, n\}$$

are heteroclinic connections and the union:

$$\Gamma_0 = \bigcup_{i=1}^{n} (\Gamma_i \cup O_i),$$

will be called « heteroclinic loop » (if $n = 1$, the heteroclinic loop $\Gamma_0 = W^u_{O_1} \cap W^s_{O_1}$ appears as a particular case of heteroclinic loop since $O_1 \in \Gamma_0$ in this case).

Remark. — Unlike homoclinic orbits, homoclinic loops are generally not non-wandering sets. This is already true in two dimensions: Figure 4(*a*) gives an example of a non wandering case, and Figure 4(*b*) an example of a wandering case.

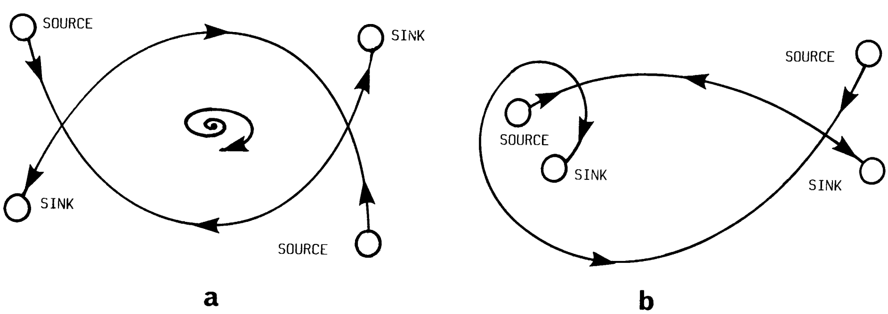

FIG. 4.

b) *Simple type heteroclinic loops*

We shall use the following notations and hypotheses:

— $O_{n+1} \equiv O_1$, $O_0 \equiv O_n$ and more generally: $O_{n+k} \equiv O_k$,
— $W_i^{u^+} = \Gamma_i \cup O_i$, the other part of $W_{O_i}^u$ being denoted by $W_i^{u^+}$,
— each Γ_i is bounded away from all singularities except O_i and O_{i+1},
— for each O_i of saddle type, the eigenvalues of the linearized flow at O_i are:

$$\lambda_{1,i} > 0\,;\ \lambda_{3,i} < \lambda_{2,i} < 0\,.$$

— one has m saddles with $O \leqslant m < n$,
— for each saddle focus O_i, the eigenvalues of the linearized flow at O_i are:

$$\lambda_i > 0,\ \rho_i \pm i\omega_i \qquad \text{with} \qquad \rho_i < 0\,.$$

DÉFINITION. — Let Γ_0 be a heteroclinic loop joining the critical points O_i, $1 \leqslant i \leqslant n$ and let O_j, for some $j \in \{1, n\}$, be a saddle. We shall say that O_j is of simple type if $W_{O_{j+1}}^s \cup O_j$ contains a disk which in turns contains $W_{O;\mathrm{loc}}^u$ and the local part of the strong stable manifold corresponding to $\lambda_{3,j}$. In the sequel, such a disk, say D_j, will be supposed to contain O_j and O_{j+1} in its interior. Each D_j generates two half-tubes $\mathscr{C}_j^{\pm}$ and $\mathscr{C}_j^{+}$ will refer to the one which contains the local part of Γ_{j-1} near O_j, which is tangent to the eigenvector corresponding to $\lambda_{2,j}$.

DÉFINITION. — We shall say that a heteroclinic loop involving at least one saddle focus, is of simple type if:

i) all saddles are of simple type,
ii) for each of them, $\mathscr{C}_j^{+}$ contains $W_{j+1}^{u^+}$,
iii) Γ_0 always follows the leading direction in $W_{O_j}^s$ near saddles.

PROPOSITION. — *A heteroclinic loop of simple type is a non wandering set.*

The proof is left to the reader.

c) *Dynamics in the neighbourhood of a heteroclinic loop of simple type.*

THEOREM C. — *Let* X *be a* $C^{1,1}$ *vector field in* $\mathbb{R}^3$, Γ_0 *a heteroclinic loop of simple type such that all saddles in* Γ_0 *verify the non resonance conditions* (5). *Let*

$$p = \prod_{\substack{O_i \text{ is a} \\ \text{saddle focus}}} -\frac{\lambda^i}{\rho_i} \cdot \prod_{\substack{O_i \text{ is a} \\ \text{saddle}}} -\frac{\lambda_{1,i}}{\lambda_{2,i}},$$

Then:

a) *if* $p > 1$, *one has the conclusions of theorem* A, *with* $\Sigma^{*,\alpha}$ *replaced by* $\Sigma^{*,\alpha}_{2(n-m-1)}$ *and* $1 \leqslant \alpha \leqslant p$.

b) *if* $p < 1$, *one has the conclusion of theorem* B.

Proof. — With section II in mind, we first illustrate the role of a saddle in a heteroclinic loop of simple type. To this end, let us consider the heteroclinic in Fig. 5. The two main phenomena which have been represented are:

1) the splitting of the spiral produced by the saddle focus (see section II), due to the stable manifold of the saddle,
2) the pinching effect characteristic of hyperbolic equilibria of saddle type; further saddles would not bring any new qualitative effect and would only contribute to the quantity p.

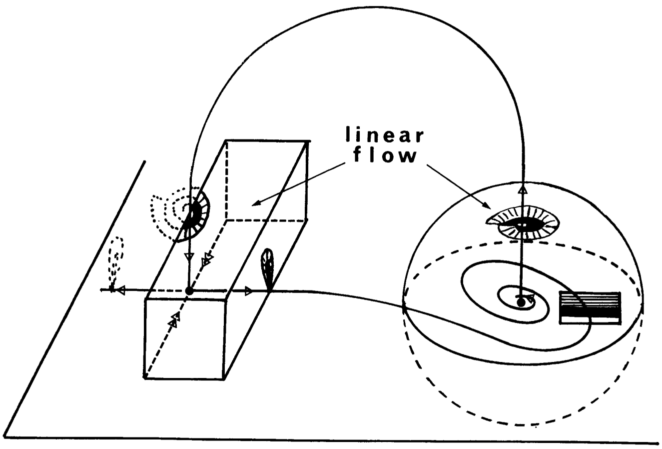

FIG. 5.

If there is only one saddle focus and at least one saddle in Γ_0, the action of the first return map defined similarly to what has been done in section II is like what is represented in Fig. 6, where *a*) and *b*) refer to corresponding parts in theorem C. The other important characteristic of the heteroclinic

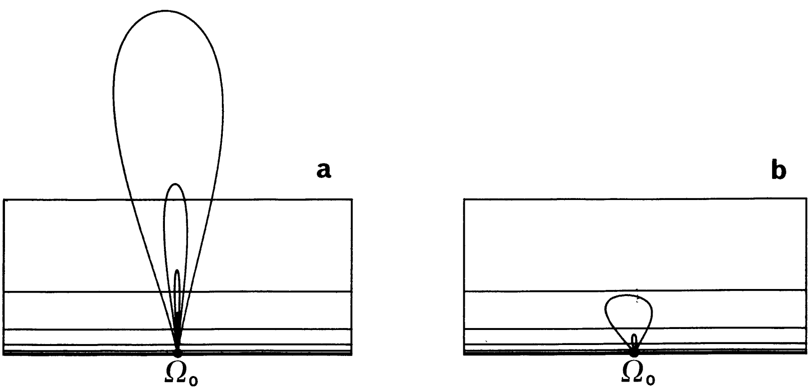

FIG. 6.

case is that the $\Sigma^{*,\alpha}$ of theorem A is replaced by $\Sigma^{*,\alpha}_{2(n-m-1)}$. This comes from iterated deformations (indeed $n - m - 1$ of them) consisting in:

1° stretching (if $\lambda > -\rho$) or compressing (if $\lambda < -\rho$) a pinched rainbow as those in Fig 6,

2° roling it in a logarithmic spiral. ■

VI. APPLICATIONS AND FINAL REMARKS

Beside works already mentioned, there have been many articles devoted to theorems « à la Šil'nikov » *[9]* *[11]* *[13]* *[32]* *[42]* *[44]* *[45]* (1). The considerations of more general homoclinic loops leads to the rough classification of volume contracting flows in $\mathbb{R}^3$, proposed in *[51]* *[53]* and still in process. For the relevance of Šil'nikov's results on saddle foci in the context of the problem of the onset of turbulence (*[41]*), see *[6]* for an explicit example of piecewise linear differential equation satisfying the hypotheses of theorem A, and *[4]* for an example where central symmetry allows the numerical observation of an attractor with spiral structure when a pair of homoclinic orbits exists.

Remark. — Condition (3) instead of (2) on the eigenvalues of (1) yields another kind of attractor simultaneous to a homoclinic curve, but with less « spectacular » Poincaré maps *[5]* *[17]* *[51]*.

Part of the importance of Šil'nikov's theorem in applications to Physics is put forward in the following two remarks (see also the end of this section):

REMARK 1. — Many « classical » systems displaying numerically observed chaotic behavior, such as the ones presented in *[11]* *[28]* *[30]* *[35]* *[37]* *[38]* *[39]* *[57]* have been successfully analyzed at the light of Šil'nikov's theorems (see for instance *[5]* *[7]* *[17]* *[18]* *[36]* *[51]*). The equations in *[30]* and *[38]* are rather special in this respect since one has to extend the parameters space in order to find heteroclinic loops joining saddle foci, and get a comprehensive interpretation of the origin and structure of the « attractors » observed numerically *[51]*.

REMARK 2. — The z-projection of the action of the map θ (θ_2 in symmetric cases) yields a one dimensional map. Such map allow a good qualitative understanding of the behavior of one parameter families of flows which meet homoclinic conditions, with saddle foci *[3]* *[18]* *[51]*.

To complete these remarks, one should mention that systems like those presented in *[6]* *[53]*, or those presented in *[28]* *[35]* *[38]*, (in different

(1) For return maps near critical points in $\mathbb{R}^2$, see e. g. *[1]* or *[19]*.

parameter ranges than those discussed in the present paper: more precisely near the onset of chaos) should be analyzed using different approaches: see [*53*] for some references.

Most authors decided to reconstruct their own proof of Theorem A, often because the geometry is not quite transparent in [*43*] or [*46*]. All these proofs start with a local linearization near O. One recognized in [*52*] that good (optimal $^{\rho}$) smoothness conditions ($C^{1,1}$) for the flows were possible, thanks to a theorem of P. Hartman [*21*]. The generalization of [*21*] by G. R. Belitskii [*8*] allowed the study of heteroclinic cases involving saddles in the present paper.

Center manifold type arguments are presented in [*3*] [*20*] [*24*] (see also [*10*]) to suggest the relevance of Šil'nikov's theorem in the description of certain macroscopic states (corresponding to P. D. E.'s). In this context, the interest of a $C^{1,1}$ version of bifurcation theorems lies in the fact that in general, a center manifold is not C^{∞} [*50*], but always C^k (for $k \geqslant$ regularity of the initial problem) and the size of the center manifold can only increase if one requires less smoothness.

Let us end this section by the observation that Šil'nikov's results seem to be relevant for the interpretation of the data of some experiments on the onset of turbulence (see for instance [*26*] [*40*] [*55*]).

APPENDIX

Lotka-Volterra equations and semi-simple loops.

Lotka-Volterra equations read [*29*] [*56*]:

$$\dot{X}_i = X_i(v_i + \gamma_{ij}X_j),\ i \in \{1, \ldots, n\} \tag{A.1}$$

and only the case $n = 3$ will be considered here.

One is generally interested in the region R such that, $\forall i$, $X_i \geqslant 0$ in phase space: let us remark that coordinate planes and axis are globally invariant. Beside O, (A.1) admits:

— at most one critical point per coordinate axis
— at most one critical point per coordinate plane, out of the axis
— at most one critical point of the coordinate planes.

If we restrict ourselves to the cases with the last critical point belonging to R, (A.1) can be rewritten as:

$$\dot{x}_i = x_i(\alpha_{ij}(1 - x_j)) \tag{A.2}$$

The fact that chaotic behavior can be numerically observed with (A.2) was announced in [*54*]. In [*4*] examples were given and interpreted using theorem AP (perturbation of homoclinic orbits). We shall see that Theorem C could not be invoqued, despite the fact that indeed, one has perturbations to heteroclinic loops. However, the claim in [*4*] that no chaotic behavior can occur near heteroclinic loop involving a saddle is incorrect.

The examples in [*4*] (these examples do not belong to Smale's class in [*48*]: see [*2*]) involved a heteroclinic loop joining one saddle focus and two saddles (the loop was observed numerically). This loop is not of simple type and I do not know any example of loop of simple type arising with equations (A.2), despite the fact that many examples of chaotic dynamics can be generated using the computer program mentioned in [*4*]. Instead of looking at all possible cases of heteroclinic loops generated by (A.2), we shall analyse three heteroclinic loops, represented in Figure 7. These examples correspond to the simplest configurations, which can be numerically observed with Lotka-Volterra equations. More precisely:

— in Figure 7(*a*), neither O_2 nor O_3 are of simple type,
— in Figure 7(*b*), O_2 is of simple type, O_3 is not,
— in Figure 7(*c*), O_3 is of simple type, O_2 is not.

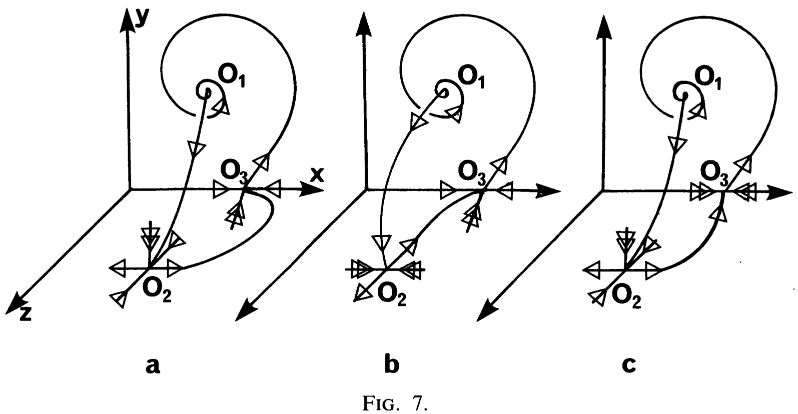

FIG. 7.

Nevertheless, in all these cases, the violation of the simple type of the loop is of the simplest kind: the disks D_j are still defined, but they contain the local part of the invariant manifold corresponding to $\lambda_{2,j}$, instead of $\lambda_{3,j}$. This motivates the following definitions:

DEFINITION 1. — A saddle is of semi-simple type if it satisfies the same properties as a simple type one, with the roles of $\lambda_{2,j}$ and $\lambda_{3,j}$ permuted.

DEFINITION 2. — A heteroclinic loop involving at least one saddle focus, is semi-simple if all saddles it contains are simple or semi-simple, one at least being semi-simple.

Remark. — The proposition in section V remains true if one replaces « simple type » by « simple type or semi-simple type ».

If one wants to study the dynamics in the neighbourhood of a semi-simple heteroclinic loop, one has first to replace the quantity p computed in theorem C by the quantity

$$p' = \prod_{\substack{O_i \text{ is a} \\ \text{saddle focus}}} -\frac{\lambda_i}{\rho_i} \prod_{\substack{O_i \text{ is a} \\ \text{simple} \\ \text{saddle}}} -\frac{\lambda_{1,i}}{\lambda_{2,i}} \prod_{\substack{O_i \text{ is a} \\ \text{semi-simple} \\ \text{saddle}}} -\frac{\lambda_{1,i}}{\lambda_{3,i}}.$$

We have represented in Figures 8 and 9 how the spiral issued from the saddle focus O_1 are transformed in the neighbourhoods of O_2 and O_3 for the configurations in Figure 7. In Figure 8 (neighbourhood of O_2), *a*) corresponds to the figures 7*a*) and 7*c*).

In a case with a semi-simple configuration, and near a semi-simple saddle, the images of the rectangles R_i (as defined in section II) under the flow come closer and closer to the stable manifold of the saddle when i becomes larger and larger. As a consequence, the computation of the quantity p' does not allow any more to conclude that there exist horseshoes in the case $p' > 1$: the result depends on the details of the non-linear dynamics. The only significant result corresponds to the cases when $p' < 1$:

THEOREM D. — *If one replaces « simple » by « semi-simple » in the hypotheses of theorem* C, *and if* $p' < 1$, *the conclusion remains unchanged.*

Consequently, the point of view adopted in [*4*] to interpret the numerical results presented there seems reasonable: one just remarked that the perturbations of a heteroclinic loop resembles the perturbation of a homoclinic orbit and invoked the stability of horseshoes that occur under the hypotheses of theorem A.

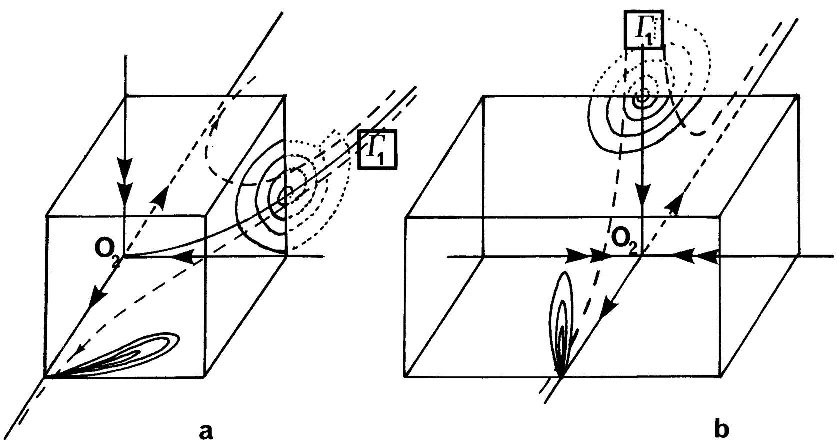

FIG. 8.

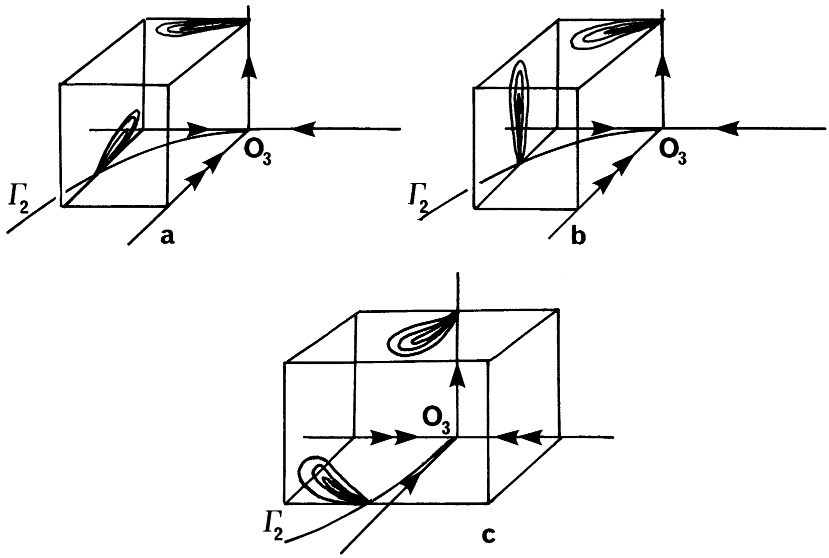

FIG. 9.

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