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Classical particle limit of non-relativistic quantum mechanics

by

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ABSTRACT. — We study the Classical Particle Limit of Non-Relativistic Quantum Mechanics. We show that the unitary group describing the evolution of the quantum fluctuation around any classical phase orbit has a classical limit as $\hbar \to 0$ in the strong operator topology for a very large class of time independent scalar and vector potentials, which in practice covers all physically interesting cases. We also show that the mean values of the quantum mechanical position and velocity operators on suitable states, obtained by time evolution of the product of a Weyl operator centred around the large coordinates and momenta $\hbar^{-\frac{1}{2}}\xi$ and $\hbar^{-\frac{1}{2}}\pi$ and a fixed $\hbar$-independent wave function, converge to the solution of the classical equations with initial data ξ, π as $\hbar \to 0$ for a broad class of repulsive interactions.

RÉSUMÉ. — On étudie la limite classique des particules de la mécanique quantique non relativiste. On démontre que le groupe unitaire qui décrit l'évolution des fluctuations quantiques autour d'une orbite classique quelconque a une limite classique quand $\hbar \to 0$ dans la topologie forte des opérateurs, pour une vaste classe de potentiels qui couvre en pratique tous les cas physiquement intéressants. On démontre aussi que les valeurs moyennes des opérateurs quantiques position et vitesse, dans des états convenables obtenus par l'évolution temporelle du produit d'un opérateur de Weyl centré autour de grandes coordonnées et de grandes impulsions $\hbar^{-\frac{1}{2}}\xi$, $\hbar^{-\frac{1}{2}}\pi$ et d'une fonction d'onde indépendante de $\hbar$, tendent vers la solution des équations classiques avec données initiales ξ, π quand $\hbar \to 0$ pour une grande classe d'interactions répulsives.

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1. INTRODUCTION

The Classical Particle Limit is a problem as old as Quantum Mechanics [1] [2] [3]. There are several approaches to this subject. The oldest of these, and maybe one of the most known, is founded on Ehrenfest's equations [4].

$$\begin{aligned}\frac{dq_\hbar}{dt}(t) &= \frac{\partial \mathscr{H}}{\partial \pi}(q_\hbar(t), p_\hbar(t)) \\ \frac{dp_\hbar}{dt}(t) &= -\frac{\partial \mathscr{H}}{\partial \xi}(q_\hbar(t), p_\hbar(t))\end{aligned} \tag{1.1}$$

Taking the mean values of (1.1) on coherent states centred around the "large" coordinates and momenta $\hbar^{-\frac{1}{2}}\xi$, $\hbar^{-\frac{1}{2}}\pi$, we obtain the classical equations of motion when $\hbar \to 0$ [5] [6] [7]. In the WKB method the wave function $\psi(x)$ is written as $\exp\left(\frac{i}{\hbar} S_\hbar(x)\right)$ and then $S_\hbar(x)$ is expanded in a power series of $\hbar$.

When the number of degrees of freedom is greater than one, this becomes complicated [8]. Feynman's path integral approach requires the of a functional measure, and it is difficult to put it on a firm mathematical basis except for a very small number of cases [9] [10] [12].

By now K. Hepp's paper [11] is a classic work on the subject. It is about correlation functions rather than the operators $q_\hbar$ and $p_\hbar$, which are unbounded (see also [5]). It would be interesting to deal with this more general problem. This is exactly the aim of this paper.

Consider a quantum mechanical system with ν degrees of freedom and with an analogous classical system. In Quantum Mechanics the states of the system are represented by rays of some Hilbert space $\mathscr{L}$ and the observables by selfadjoint operators in $\mathscr{L}$. Conversely, in Classical Mechanics the states are points of a real 2ν-dimensional vector space $\mathscr{E}$, called phase space, and the observables real measurable functions defined on $\mathscr{E}$. Suppose that the quantum mechanical $\hbar$-dependent operator $A_\hbar$ and the phase function $a(\xi, \pi)$ represent the same observable. There is only one physically meaningful method of obtaining $a(\xi, \pi)$ from $A_\hbar$, when the classical limit $\hbar \to 0$ is performed. This consists in selecting states $|\hbar, \xi, \pi\rangle$ such that

$$\langle A_\hbar \rangle_{|\hbar,\xi,\pi\rangle} \to a(\xi, \pi), \qquad \hbar \to 0 \tag{1.2}$$

$$(\Delta A_\hbar)_{|\hbar,\xi,\pi\rangle} \to 0 \qquad \hbar \to 0 \tag{1.3}$$

where for any selfadjoint operator S and any quantum state $| \Psi \rangle$ such that $\langle \Psi | \Psi \rangle = 1$

$$\langle S \rangle_{|\Psi\rangle} = \langle \Psi | S | \Psi \rangle \tag{1.4}$$

$$(\Delta S)_{|\Psi\rangle} = (\langle (S - \langle S \rangle_{|\Psi\rangle})^2 \rangle_{|\Psi\rangle})^{\frac{1}{2}} \tag{1.5}$$

The problem of finding the explicit form of the states $| \hbar, \xi, \pi \rangle$ can be simplified if $\Delta_\hbar$ is one of the operators $q_\hbar$ and $p_\hbar$. To this end, we choose any 2ν selfadjoint operators $q_1, \ldots, q_\nu, p_1, \ldots, p_\nu$ such that

$$[q_j, q_k] = [p_j, p_k] = 0, \qquad [q_j, p_k] = i\delta_{jk} \tag{1.6}$$

They form the vector operators

$$q = (q_1, \ldots, q_\nu), \qquad p = (p_1, \ldots, p_\nu) \tag{1.7}$$

The Weyl's operators are defined by

$$C(\xi, \pi) = \exp\,[i(\pi \cdot q - \xi \cdot p)] \tag{1.8}$$

for any $\xi, \pi \in \mathbb{R}^\nu$. It is convenient to use the following shortened notation

$$z = (q, p) \qquad \zeta = (\xi, \pi) \tag{1.9}$$

For any $\zeta \in \mathbb{R}^{2\nu}$, $C(\zeta)$ is unitary and

$$C(\zeta)^+ z C(\zeta) = z + \zeta \tag{1.10}$$

When dealing with the Classical Particle Limit, the only appropriate representation of the canonical commutation rules is given by:

$$z_\hbar = \hbar^{\frac{1}{2}} z \tag{1.11}$$

Let $| \Omega \rangle$ be any quantum state such that $\langle \Omega | \Omega \rangle = 1$. From (1.10), it follows that

$$\langle z_\hbar \rangle_{C(\zeta/\hbar^{\frac{1}{2}})|\Omega\rangle} = \zeta + \hbar^{\frac{1}{2}} \langle z \rangle_{|\Omega\rangle} \tag{1.12}$$

$$(\Delta z_\hbar)_{C(\zeta/\hbar^{\frac{1}{2}})|\Omega\rangle} = \hbar^{\frac{1}{2}} (\Delta z)_{|\Omega\rangle} \tag{1.13}$$

Therefore, it seems sensible to set

$$| \hbar, \zeta \rangle = C(\zeta/\hbar^{\frac{1}{2}}) | \Omega \rangle \tag{1.14}$$

The problem becomes more difficult when the time evolution of the system is taken into account. The quantum evolution operator is

$$U_\hbar(t, s) = \exp\left(-\frac{i}{\hbar}(t - s) H_\hbar \right) \tag{1.15}$$

Since the quantum system has an analogous classical system, the quantum hamiltonian $H_\hbar$ must have the form

$$H_\hbar = \mathscr{H}(z_\hbar) \tag{1.16}$$

where

$$\mathscr{H}(\zeta) = \frac{1}{2}(\pi - a(\xi))^2 + v(\xi), \qquad \zeta = (\xi, \pi) \tag{1.17}$$

is the classical Hamiltonian. The Heisenberg's operators corresponding to the $z_\hbar$'s are

$$z_\hbar(t) = \mathrm{U}_\hbar(t, s)^+ z_\hbar \mathrm{U}_\hbar(t, s) \tag{1.18}$$

Probably, when $\hbar$ is small, $\langle z_\hbar(t) \rangle_{|\hbar,\zeta\rangle}$ satisfies the canonical equations with initial data ζ at $t = s$. In order to demonstrate this, we take any solution $\zeta(t)$ of the canonical equations. Then it is easy to see that

$$\langle z_\hbar(t) \rangle_{|\hbar,\zeta(s)\rangle} = \zeta(t) + \langle \mathrm{W}_\hbar(t, s)^+ z_\hbar \mathrm{W}_\hbar(t, s) \rangle_{|\Omega\rangle} \tag{1.19}$$

$$(\Delta z_\hbar(t))_{|\hbar,\zeta(s)\rangle} = \langle \mathrm{W}_\hbar(t, s)^+ (z_\hbar + \zeta(t) - \langle z_\hbar(t) \rangle_{|\hbar,\zeta(s)\rangle})^2 \mathrm{W}_\hbar(t, s) \rangle^{\frac{1}{2}}_{|\Omega\rangle} \tag{1.20}$$

where

$$\mathrm{W}_\hbar(t, s) = \mathrm{C}(\zeta(t)/\hbar^{\frac{1}{2}})^+ \mathrm{U}_\hbar(t, s) \mathrm{C}(\zeta(s)/\hbar^{1/2}) \cdot \exp\left\{ \frac{i}{\hbar} \int_s^t \left[\mathscr{H}(\zeta(r)) - \frac{1}{2} \nabla_\zeta \mathscr{H}(\zeta(r)) \cdot \zeta(r) \right] dr \right\} \tag{1.21}$$

A formal calculation shows that

$$\begin{aligned} \frac{\partial}{\partial s} \mathrm{W}_\hbar(t, s) &= \frac{i}{\hbar} \mathrm{W}_\hbar(t, s) \mathrm{K}_\hbar(s) \\ \frac{\partial}{\partial t} \mathrm{W}_\hbar(t, s) &= -\frac{i}{\hbar} \mathrm{K}_\hbar(t) \mathrm{W}_\hbar(t, s) \end{aligned} \tag{1.22}$$

where

$$\mathrm{K}_\hbar(s) = \mathscr{H}(z_\hbar + \zeta(s)) - \mathscr{H}(\zeta(s)) - \nabla_\zeta \mathscr{H}(\zeta(s)) \cdot z_\hbar \tag{1.23}$$

Therefore,

$$\mathrm{W}_\hbar(t, s) = \mathrm{T} \exp\left(-\frac{i}{\hbar} \int_s^t \mathrm{K}_\hbar(r) dr \right) \tag{1.24}$$

If the potentials are smooth enough,

$$\hbar^{-1} \mathrm{K}_\hbar(r) \to \mathrm{H}_2(r) = \frac{1}{2} z \cdot \nabla_\zeta \nabla_\zeta \mathscr{H}(\zeta(r)) \cdot z, \qquad \hbar \to 0 \tag{1.25}$$

Let

$$\mathrm{U}_2(t, s) = \mathrm{T} \exp\left(-i \int_s^t \mathrm{H}_2(r) dr \right) \tag{1.26}$$

Then we can expect that

$$\mathrm{W}_\hbar(t, s) \to \mathrm{U}_2(t, s), \qquad \hbar \to 0 \tag{1.27}$$

so that

$$\langle z_\hbar(t) \rangle_{|\hbar,\zeta(s)\rangle} \to \zeta(t), \qquad \hbar \to 0 \tag{1.28}$$

$$(\Delta z_\hbar(t))_{|\hbar,\zeta(s)\rangle} \to 0, \qquad \hbar \to 0 \tag{1.29}$$

The operators $W_\hbar(t, s)$ and $U_2(t, s)$ were first introduced by K. Hepp [11], although Hepp's definition of $W_\hbar(t, s)$ was slightly different from ours. Hepp proved that (1.27) holds in the strong operator topology, provided the potential does not increase too fast at infinity. We shall prove the same result without this restriction (for a rigorous statement of the problem see sections 3, 4 and 5). Operators formally analogous to $W_\hbar$ and U_2 were introduced by J. Ginibre and G. Velo [13] in the context of the Classical Field Limit for non-Relativistic Bosons. In this case, (1.27) still holds. We also prove (1.28) and (1.29) (see section 6). The major difficulty consists in dealing with unbounded operators instead of bounded ones (as Hepp did). This will force us to reduce the class of quantum potentials for which (1.27) holds. The most severe restriction concerns the scalar potential v, which cannot have negative singularities. Thus, our results fit only repulsive interactions (for instance, the repulsive Coulomb interaction). Furthermore, we shall find a classical limit for the velocity $p_\hbar - a(q_\hbar)$, instead of the momentum $p_\hbar$, when a vector potential a is present. Further investigation is required on these points.

2. NOTATIONS

In general, for any Hilbert space $\mathscr{H}$, we denote by $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and $\| \cdot \|_{\mathscr{H}}$ the inner product and the norm in $\mathscr{H}$ respectively. Assume that A is any linear operator in $\mathscr{H}$. We denote by D(A) the domain of A. If A is closable, $\bar{A}$ denotes the operator closure of A. If A is densely defined, A^+ denotes the adjoint operator of A. For any linear manifold M in $\mathscr{H}$, we denote by $A \restriction_M$ the restriction of A to M. If A is selfadjoint, we can associate A with a scale of Hilbert spaces $(\mathscr{H}_\lambda(A))_{\lambda \in \mathbb{R}}$ defined in the following manner [15] [16]:

$$\mathscr{H}_\lambda(A) = \text{completion of } \left(D((1+|A|)^{\frac{\lambda}{2}}), \langle \cdot, \cdot \rangle_{A,\lambda}\right) \tag{2.1}$$

where

$$\langle \varphi, \psi \rangle_{A,\lambda} = \langle (1+|A|)^{\frac{\lambda}{2}}\varphi, (1+|A|)^{\frac{\lambda}{2}}\psi \rangle, \ \varphi, \psi \in D((1+|A|)^{\frac{\lambda}{2}}) \tag{2.2}$$

As usual, $Q(A) = \mathscr{H}_1(A)$ and $D(A) = \mathscr{H}_2(A)$.

For any two Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$, we denote by $B(\mathscr{H}, \mathscr{H}')$ the set of all continuous linear mappings from $\mathscr{H}$ to $\mathscr{H}'$ and by $\| \cdot \|_{B(\mathscr{H},\mathscr{H}')}$ the usual operator norm.

In this paper we denote by ν the number of degrees of freedom of the mechanical system under consideration. $L^2(\mathbb{R}^\nu)$ (or simply L^2) is the Hilbert space of the wave functions. The norm and the inner product in L^2 are simply denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively.

The operators q and p are formally defined by

$$q_j\psi(x) = x_j\psi(x), \qquad p_j\psi(x) = -i\nabla_j\psi(x), \qquad j = 1, \ldots, \nu. \tag{2.3}$$

It is well known that these turn out to be selfadjoint on appropriate domains and satisfy the commutation rules.

$$[q_j, q_k] = [p_j, p_k] = 0, \qquad [q_j, p_k] = i\delta_{jk}, \qquad j, k = 1, \ldots, \nu \tag{2.4}$$

Moreover, they form the operator ν-vectors

$$q = (q_1, \ldots, q_\nu), \qquad p = (p_1, \ldots, p_\nu) \tag{2.5}$$

By means of the q's and p's we can define the following positive self-adjoint operators.

$$q^2 = \sum_1^\nu {}_j q_j^2, \qquad p^2 = \sum_1^\nu {}_j p_j^2 \tag{2.6}$$

$$\mathrm{N} = \frac{1}{2}(q^2 + p^2) - \frac{\nu}{2} \tag{2.7}$$

We denote by $(\mathrm{Q}_\lambda)_{\lambda\in\mathbb{R}}$, $(\mathrm{P}_\lambda)_{\lambda\in\mathbb{R}}$ and $(\mathrm{X}_\lambda)_{\lambda\in\mathbb{R}}$ the scales of spaces associated with q^2, p^2 and N respectively. However, in place of the inner product $\langle \cdot, \cdot \rangle_{\mathrm{N},\lambda}$, defined by (2.2), it is convenient to use the following one, which is equivalent to the former:

$$\langle \varphi, \psi \rangle_{\mathrm{X}_\lambda} = \langle f(\lambda, \mathrm{N})\varphi, f(\lambda, \mathrm{N})\psi \rangle, \ \varphi, \psi \in \mathrm{D}((1 + \mathrm{N})^{\frac{\lambda}{2}}) \tag{2.8}$$

where

$$f(\lambda, \mathrm{N}) = \left[\frac{\Gamma(3/2 + \mathrm{N} + |\lambda|)}{\Gamma(3/2 + \mathrm{N})} \right]^{\frac{1}{2}\operatorname{sgn}\lambda} \tag{2.9}$$

Sometimes the space $\mathrm{X}_\infty = \cap_\lambda \mathrm{X}_\lambda = \mathscr{S}(\mathbb{R}^\nu)$ is useful.

The symbol $\langle \cdot, \cdot \rangle$ also denotes the duality on the scales of spaces defined above [*15*] [*16*].

For any two scalar or operator ν-vectors a and b and any scalar $\nu \times \nu$ matrix α we set

$$a . b = \sum_1^\nu {}_j a_j b_j, \qquad a . \alpha . b = \sum_1^\nu {}_{j,k} a_j \alpha_{jk} b_k \tag{2.10}$$

As usual, $a^2 = a . a$.

The Weyl's operators [*20*] [*21*] are defined by

$$\mathrm{C}(\eta, \zeta) = \exp(i\zeta \cdot q) \exp(-i\eta \cdot p) \exp\left(-\frac{i}{2}\eta \cdot \zeta\right) \tag{2.11}$$

Since these play an important role in what follows we give a short list of their main properties which we shall frequentely refer to.

For any $\tau \in \mathbb{R}$ and all $\eta, \zeta \in \mathbb{R}^\nu$ we have

$$\mathrm{C}(\eta, \zeta) \in \mathrm{B}(\mathrm{P}_\tau, \mathrm{P}_\tau), \qquad \mathrm{C}(\eta, \zeta) \in \mathrm{B}(\mathrm{Q}_\tau, \mathrm{Q}_\tau) \tag{2.12}$$

$$\mathrm{C}(\eta, \zeta) \in \mathrm{B}(\mathrm{X}_\tau, \mathrm{X}_\tau) \tag{2.13}$$

uniformly for bounded η, ζ. If f is a polynomial on $\mathbb{R}^\nu$ and $\tau \in \mathbb{R}$,

$$\begin{aligned} C(\eta, \zeta)^+ f(q) C(\eta, \zeta) &= f(q + \eta) \quad \text{on} \quad Q_\tau \\ C(\eta, \zeta)^+ f(p) C(\eta, \zeta) &= f(p + \zeta) \quad \text{on} \quad P_\tau \end{aligned} \tag{2.14}$$

More generally, for any measurable function f, we have

$$\begin{aligned} C(\eta, \zeta)^+ f(q) C(\eta, \zeta) &= f(q + \eta) \\ C(\eta, \zeta)^+ f(p) C(\eta, \zeta) &= f(p + \zeta) \end{aligned} \tag{2.15}$$

as operators in L^2. The following formula is useful

$$C(\eta, \zeta)\psi(x) = \exp\left(i\zeta \cdot x - \frac{i}{2}\eta \cdot \zeta \right) \psi(x - \eta) \tag{2.16}$$

3. THE ADMISSIBLE PHASE ORBITS

Theorems 5.3 and 6.3 hold only if the potentials are C^2 functions in a neighborhood of the chosen classic configuration orbit and if the quantum Hamiltonian has a suitable expression on the wave functions whose support is contained in this neighborhood. The following notations are useful for simplifying the statement of the problem.

DEFINITION 3.1. — Let $k \in \mathbb{N}$, $v : \mathbb{R}^\nu \to \mathbb{R}$ and $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$. We denote by $\mathscr{A}_k(v, a)$ the set of all $(\xi(\cdot), \pi(\cdot))$ satisfying the following assumptions :

1) $\xi(\cdot)$ and $\pi(\cdot)$ are differentiable functions from some open interval I in $\mathbb{R}$ to $\mathbb{R}^\nu$.

2) There is an open set O in $\mathbb{R}^\nu$ such that $\{ \xi(t) \mid t \in I \} \subseteq O$ and v and a C^k-functions on O.

3) For all $t \in I$,

$$\begin{aligned} d\xi/d\tau(t) &= \partial\mathscr{H}/\partial\pi(\xi(t), \pi(t) ; v, a) \\ d\pi/d\tau(t) &= -\partial\mathscr{H}/\partial\xi(\xi(t), \pi(t) ; v, a) \end{aligned} \tag{3.1}$$

where, for any $x, y \in \mathbb{R}^\nu$,

$$\mathscr{H}(x, y ; v, a) = \frac{1}{2}(y - a(x))^2 + v(x) \tag{3.2}$$

I is said to be the domain of $(\xi(\cdot), \pi(\cdot))$. We denote by $\Omega_k(\xi, \pi)$ the family of all open sets O for which condition 2 holds.

DEFINITION 3.2. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$ and $(\xi, \pi) \in \mathscr{A}_1(v, a)$. We denote by $\mathscr{E}(\xi, \pi)$ the set of all $(H_\hbar)_{\hbar > 0}$ such that:

1) For all $\hbar > 0$, $H_\hbar$ is a linear selfadjoint operator in $L^2(\mathbb{R}^\nu)$.

2) There is $O \in \Omega_1(\xi, \pi)$ such that for each $\hbar > 0$ and $\psi \in P_2$ with supp ψ a compact set contained in $\hbar^{-1/2}O$ we have $\psi \in D(H_\hbar)$ and

$$H_\hbar\psi(x) = -\frac{\hbar}{2}\Delta\psi(x) + i\hbar^{1/2}a(\hbar^{1/2}x)\cdot\nabla\psi(x) + \frac{i\hbar}{2}\nabla\cdot a(\hbar^{1/2}x)\psi(x) + \frac{1}{2}a^2(\hbar^{1/2}x)\psi(x) + v(\hbar^{1/2}x)\psi(x) \quad (3.3)$$

for almost all $x \in \hbar^{-1/2}O$.

DEFINITION 3.3. — For any $\hbar > 0$, we set (cfr. (2.3))

$$q_{j,\hbar} = \hbar^{1/2}q_j \qquad p_{j,\hbar} = \hbar^{1/2}p_j \qquad j = 1, 2, \ldots, \nu \quad (3.4)$$

$$q_\hbar = \hbar^{1/2}q \qquad p_\hbar = \hbar^{1/2}p \quad (3.5)$$

Remark. — There might be several families $(H_\hbar)_{\hbar>0}$ belonging to $\mathscr{E}(\xi, \pi)$. The use of a particular family is a matter of physics, and the validity of theorem 5.3 does not depend on it.

Remark. — The $\hbar$'s are put in (3.3) according to (3.4-5). In general it is not possible to write $H_\hbar$ by means of $p_\hbar$ and $q_\hbar$ because no assumption has been made about the behavior of v and a on $\complement O$. However, in many interesting cases some explicit expressions can be obtained.

Remark. — The natural question arises of whether $\mathscr{E}(\xi, \pi)$ is non empty or not. The following theorem gives some conditions for which $\mathscr{E}(\xi, \pi) \neq \emptyset$.

THEOREM 3.4. — Assume that $v \in L^2_{loc}(\mathbb{R}^\nu, \mathbb{R})$ $a \in L^4_{loc}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ and

$$\sum_1^\nu {}_j\partial_j a_j \in L^2_{loc}(\mathbb{R}^\nu, \mathbb{R}).$$

Suppose that for each $\hbar > 0$

$$H^0_\hbar = \frac{1}{2}p^2_\hbar - \frac{1}{2}a(q_\hbar)\cdot p_\hbar - \frac{1}{2}p_\hbar\cdot a(q_\hbar) + \frac{1}{2}a(q_\hbar)^2 + v(q_\hbar)\restriction_{C^\infty_0(\mathbb{R}^\nu)} \quad (3.6)$$

is essentially selfadjoint. Then, if $(\xi, \pi) \in \mathscr{A}_1(v, a)$, $(\overline{H^0_\hbar})_{\hbar>0} \in \mathscr{E}(\xi, \pi)$.

Proof. — Clearly, the first condition stated in definition (3.3) is satisfied. Assume that $\hbar > 0$, $\psi \in P_2$ and that supp ψ is a compact set contained in $\hbar^{-\frac{1}{2}}O$, where $O \in \Omega_1(\xi, \pi)$. Then, there are a sequence $(\psi_j)_{j\in\mathbb{N}} \subseteq C^\infty_0(\mathbb{R}^\nu)$ and a compact set $K \subseteq \hbar^{-1/2}O$ such that $\psi_j \overset{P_2}{\to} \psi, j \to \infty$ and supp $\psi_j \subseteq K$ for each j. From the assumptions on (ξ, π), it follows easily that

$$\| H^0_\hbar(\psi_j - \psi_l) \| \to 0, \qquad j, l \to \infty.$$

Thus $\psi \in D(\overline{H_\hbar^0})$. Furthermore, for any $j \in \mathbb{N}$, $H_\hbar^0 \psi_j = \overline{H_\hbar^0}\psi_j$ is given by (3.3). We conclude that $\overline{H_\hbar^0}\psi$, also, must be given by (3.3) almost everywhere on $\hbar^{-1/2}O$. Q. E. D.

Among the numerous corollaries of this theorem we choose:

COROLLARY 3.5. — Assume that

1) $v_1 \in L^2_{loc}(\mathbb{R}^\nu, \mathbb{R})$, $v_1(x) \geqslant -cx^2 - d$ for some $c, d > 0$.

2) $v_2 \in L^2_{loc}(\mathbb{R}^\nu, \mathbb{R})$, $v_2 \leqslant 0$,

v_2 is Δ-bounded with relative bound smaller than 1.

3) $a \in L^4_{loc}(\mathbb{R}^\nu, \mathbb{R}^\nu)$, $\sum_1^\nu {}_j\partial_j a_j \in L^2_{loc}(\mathbb{R}^\nu, \mathbb{R})$.

Let $H_\hbar^0$ be defined as in (3.6). Set $H_\hbar = \overline{H_\hbar^0}$. Then, if $(\xi, \pi) \in \mathscr{A}_1(v, a)$, $(H_\hbar)_{\hbar > 0} \in \mathscr{E}(\xi, \pi)$.

Proof. — It follows easily from [*19*] and the previous theorem. Q. E. D.

4. THE UNITARY GROUP $U_2(t, s\,;\, \xi, \pi)$

In the introduction we showed that the unitary group $W_\hbar(t, s)$ had a classical limit $U_2(t, s)$. In this section we shall study the main properties of the group $U_2(t, s)$. Most of the results listed below are due to J. Ginibre and G. Velo [*23*].

DEFINITION 4.1. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$ and $(\xi, \pi) \in \mathscr{A}_2(v, a)$ with domain I. We set for any $t \in I$ (cfr. (2.3))

$$H_2(t\,;\,\xi, \pi) = \frac{1}{2}p^2 - \frac{1}{2}p \cdot \widetilde{\nabla a(\xi(t))} \cdot q - \frac{1}{2}q \cdot \nabla a(\xi(t)) \cdot p + \frac{1}{2}q \cdot \nabla\nabla(-\pi(t)\cdot a(\cdot) + \frac{a^2(\cdot)}{2} + v(\cdot))(\xi(t)) \cdot q \quad (4.1)$$

The symbol $\tilde{}$ denotes the operation of transposition.

Whenever no confusion arises we shall write $H_2(t)$ instead of $H_2(t\,;\,\xi, \pi)$.

THEOREM 4.2. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, and $(\xi, \pi) \in \mathscr{A}_2(v, a)$ with domain I. Then, for all $\tau \in [2, +\infty]$ and $t \in I$, $H_2(t)$ is essentially selfadjoint on X_τ (cfr. sec. 2)

Proof. — This follows easily from the estimate (4.13) given below and from W. G. Faris and R. Lavine's theorem [*17*]. Q. E. D.

THEOREM 4.3. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $(\xi, \pi) \in \mathscr{A}_2(v, a)$ with domain I. Then there is a unique L^2-strongly continuous L^2-unitary group $(U_2(t, s))_{t,s\in I}$ having the following L^2-strong derivatives for all $\psi \in X_2$ and $t, s \in I$

$$\begin{aligned} \frac{\partial}{\partial t} U_2(t, s)\psi &= - iH_2(t)U_2(t, s)\psi \\ \frac{\partial}{\partial s} U_2(t, s)\psi &= iU_2(t, s)H_2(s)\psi \end{aligned} \tag{4.2}$$

Furthermore, for any $\tau \in \mathbb{R}$, $U_2(t, s)$ is bounded and jointly strongly continuous in t, s in X_τ and

$$\| U_2(t, s) \|_{B(X_\tau, X_\tau)} \leqslant \exp\left(\left| \tau \int_s^t \theta(r)dr \right|\right) \tag{4.3}$$

where, for any $r \in I$,

$$\theta(r) = \frac{1}{2}\left(\sum_{m,n}{}_1^{\nu} \mid \mu_{m,n}(r) \mid^2\right)^{1/2} \tag{4.4}$$

$$\begin{aligned} \mu_{j,l} = &- \delta_{jl} + i\nabla_j a_l(\xi(r)) + i\nabla_l a_j(\xi(r)) \\ &+ \nabla_j\nabla_l(- \pi(r)\cdot a(\cdot) + \frac{1}{2}a(\cdot)^2 + v(\cdot))(\xi(r)), \qquad j, l = 1, \ldots, \nu \end{aligned} \tag{4.5}$$

Proof. — The proof is founded on a general result of T. Kato [*18*].
Let $\alpha, \beta \in I$, $\alpha < \beta$ and $u \in [0, 1]$. Set

$$A_{\alpha,\beta}(u) = (\beta - \alpha)H_2(\alpha + (\beta - \alpha)u) \tag{4.6}$$

$$\theta_{\alpha,\beta}(u) = (\beta - \alpha)\theta(\alpha + (\beta - \alpha)u) \tag{4.7}$$

$$T_{\alpha,\beta}(s\,; u) = \exp(- isA_{\alpha,\beta}(u)), \qquad s \in \mathbb{R} \tag{4.8}$$

Let us prove the following lemma,

LEMMA 4.3 *a.* — For any $\tau \in \mathbb{R}$
1) $T_{\alpha,\beta}(s\,; u) \in B(X_\tau, X_\tau)$, $\quad \| T_{\alpha,\beta}(s\,; u) \|_{B(X_\tau, X_\tau)} \leqslant \exp(\mid \tau s \mid \theta_{\alpha,\beta}(u))$ (4.9)
2) $s \to T_{\alpha,\beta}(s\,; u)$ is X_τ-strongly continuous.

Proof. — Let $t \in I$. Write $H_2(t)$ by means of the creation and anihilation operators

$$b = \frac{q + ip}{2^{1/2}} \qquad b^+ = \frac{q - ip}{2^{1/2}} \tag{4.10}$$

It is easy to check that for $\tau > 0$ (cfr. (2.9))

$$\begin{aligned} [H_2(t), f(\tau, N)^2] = &\, b\cdot\mu(t)\cdot b(f(\tau, N)^2 - f(\tau, N - 2)^2) \\ &- (f(\tau, N)^2 - f(\tau, N - 2)^2)b^+\cdot\overline{\mu(t)}\cdot b^+ \end{aligned} \tag{4.11}$$

as form on X_∞. It can be proved that for any complex function f_1 and any real positive function f_2 defined on $\mathbb{Z}_1$ [13].

$$\overline{f_1(N)b^+ \cdot \mu(t)} \cdot b^+ + b \cdot \mu(t) \cdot b f_1(N) \leqslant \frac{1}{2}\theta(t)\{ f_2(N-2)\,|\,f_1(N)\,|^2 N(N-1) + f_2(N)\} \quad (4.12)$$

Setting $f_1(N) = \pm i(f(\tau, N)^2 - f(\tau, N-2)^2)$, and $f_2(N) = 2\tau f(\tau, N)^2$ in (4.12) and using (4.11), we obtain

$$\pm\, i[H_2(t), f(\tau, N)^2] \leqslant \tau\theta(t) f(\tau, N)^2 \left\{ 1 + \frac{N(N-1)}{N^2 - 1/4} \cdot \frac{(N + \tau/2)^2}{(N+\tau)^2 - 1/4} \right\} \leqslant 2\tau\theta(\tau) f(\tau, N)^2 \quad (4.13)$$

Therefore, by (4.6-7),

$$\pm\, i\,[A_{\alpha,\beta}(u), f(\tau, N)^2] \leqslant 2\tau\theta_{\alpha,\beta}(u) f(\tau, N)^2 \quad (4.14)$$

as form on X_∞. Define for $\tau \geqslant 0$, $\varepsilon > 0$, $\psi \in L^2$

$$F_\varepsilon(s\,;\psi) = \left\langle T_{\alpha,\beta}(s\,;u)\psi, \frac{f(\tau, N)^2}{1 + \varepsilon f(\tau, N)^2} T_{\alpha,\beta}(s\,;u)\psi \right\rangle \quad (4.15)$$

Using (4.14) it is easy to prove that, for any $\psi \in X_2$

$$\left| \frac{d}{ds} F_\varepsilon(s\,;\psi) \right| \leqslant 2\tau\theta_{\alpha,\beta}(u) F_\varepsilon(s\,;\psi) \quad (4.16)$$

so that

$$F_\varepsilon(s\,;\psi) \leqslant \exp\,(2\tau\,|\,s\,|\,\theta_{\alpha,\beta}(u)) F_\varepsilon(0\,;\psi) \quad (4.17)$$

Since X_2 is dense in L^2, (4.17) also holds for any $\psi \in L^2$. Choosing $\psi \in X_\tau$ and taking the limit $\varepsilon \to 0+$ of both sides of (4.17), we see that (4.9) holds in the case where $\tau \geqslant 0$. A duality argument allows us to extend the validity of this result to negative τ's.

Assume that $\tau \in \mathbb{R}$ and $\psi \in X_\infty$. Using the asymptotic properties of the Γ function, it is easy to prove that

$$\| T_{\alpha,\beta}(s\,;u)\psi - T_{\alpha,\beta}(s_0\,;u)\psi \|_{X_\tau}^2 \leqslant C_\tau \| T_{\alpha,\beta}(s\,;u)\psi - T_{\alpha,\beta}(s_0\,;u)\psi \| \cdot \| T_{\alpha,\beta}(s\,;u)\psi - T_{\alpha,\beta}(s_0\,;u)\psi \|_{X_{2\tau}} \quad (4.18)$$

for some $C_\tau > 0$ depending only on τ. By (4.8-9), the right hand side of (4.18) vanishes when $s \to s_0$. Since X_∞ is dense in X_τ and (4.9) holds, point 2 holds. Q. E. D.

End of the proof of theorem 4.3. — Let $\tau \in \mathbb{R}$. From lemma 4.3 *a*,

$(T_{\alpha,\beta}(s\,;u))_{s\in\mathbb{R}}$ is a X_τ-strongly continuous group of operators in $B(X_\tau, X_\tau)$, whose infinitesimal generator is a suitable extension or restriction of $-iA_{\alpha,\beta}(u)$, which we shall still denote by $-iA_{\alpha,\beta}(u)$.

Let $\sigma \in \mathbb{R}$, $\sigma - \tau \geqslant 2$. By lemma 4.3 *a*, it follows that

$$-iA_{\alpha,\beta}(u) \in G(X_\rho, 1, |\rho|\,\theta_{\alpha,\beta}(u)).$$

and that $-iA_{\alpha,\beta}(u)$ is stable in X_ρ, for $\rho = \tau, \sigma$ [*18*] and that X_σ is $-iA_{\alpha,\beta}(u)$ – admissible for each $u \in [0,1]$ [*18*]. Furthermore, it can be straightforwardly verified that $A_{\alpha,\beta}(u) \in B(X_\sigma, X_\tau)$ and that $u \to A_{\alpha,\beta}(u)$ is continuous from $[0,1]$ to $B(X_\sigma, X_\tau)$ (norm topology). Now we can use theorem 5.2 and remarks 5.3-4 of Kato's paper with $\|\cdot\|_u = \|\cdot\|_{X_\sigma}$ $X = X_\tau$, $Y = X_\sigma$. Therefore, there is a unique X_τ-strongly continuous group $(\tilde{U}^{(\tau)}_{\alpha,\beta}(u, w))_{u,w\in[0,1]}$ of operators in $B(X_\tau, X_\tau)$, having for each $\psi \in X_\sigma$, the the X_τ-strong derivatives

$$\frac{\partial}{\partial u}\tilde{U}^{(\tau)}_{\alpha,\beta}(u, w)\psi = -iA_{\alpha,\beta}(u)\tilde{U}^{(\tau)}_{\alpha,\beta}(u, w)\psi$$

$$\frac{\partial}{\partial w}\tilde{U}^{(\tau)}_{\alpha,\beta}(u, w)\psi = i\tilde{U}^{(\tau)}_{\alpha,\beta}(u, w)A_{\alpha,\beta}(w)\psi \tag{4.20}$$

Futhermore $U^{(\tau)}_{\alpha,\beta}(u, w) \in B(X_\sigma, X_\sigma)$ and is jointly X_σ-strongly continuous in u, w. Using the Yosida's approximants (like Kato) and lemma 4.3 *a* we find that

$$\|\tilde{U}^{(\tau)}_{\alpha,\beta}(u, w)\|_{B(X_\tau,X_\tau)} \leqslant \exp\left(\left|\tau\int_u^w \theta_{\alpha,\beta}(z)dz\right|\right) \tag{4.21}$$

and that $\tilde{U}^{(0)}_{\alpha,\beta}(u, w)$ is L^2-unitary. From the uniqueness property,

$$\tilde{U}^{(\tau)}_{\alpha,\beta}(u, w) \subseteq \tilde{U}^{(\tau')}_{\alpha,\beta}(u, w)$$

for $\tau' \leqslant \tau$ so that the index τ can be suppressed. Set for $t, s \in [\alpha, \beta]$

$$U_{2,\alpha,\beta}(t, s) = \tilde{U}_{\alpha,\beta}\left(\frac{t-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) \tag{4.22}$$

It is a matter of a straightforward verification to show that $U_{2,\alpha,\beta}(t, s)$ has the properties listed in the statement of the theorem for $t, s \in [\alpha, \beta]$. Since α, β are arbitrary and $U_{2,\alpha,\beta}(t, s)$ is unique on $[\alpha, \beta]$, the theorem is completely proved. Q. E. D.

DEFINITION 4.4. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $(\xi, \pi) \in \mathscr{A}_2(v, a)$ with domain I. We denote by $(U_2(t, s\,;\,\xi, \pi))_{t,s\in I}$ the unitary group whose existence and uniqueness was proved in theorem 4.3.

Whenever no confusion arises, we shall write $U_2(t, s)$ instead of $U_2(t, s\,;\,\xi, \pi)$.

5. THE CLASSICAL LIMIT FOR THE GROUP $W_\hbar(t, s\,; (H_\hbar)_{\hbar>0}\,; \xi, \pi)$ AND THE OBSERVABLES REPRESENTED BY UNBOUNDED OPERATORS

In this section a rigorous proof of (1.26-1.28) will be provided.

DEFINITION 5.1. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $(\xi, \pi) \in \mathscr{A}_1(v, a)$ with domain I, $(H_\hbar)_{\hbar>0} \in \mathscr{E}(\xi, \pi)$. We set for $t, s \in I$

$$W_\hbar(t, s; (H_\hbar)_{\hbar>0}\,; \xi, \pi) = \exp\left\{ i\hbar^{-1} \int_s^t \left[\mathscr{H}(\xi(r), \pi(r)\,; v, a) - \frac{1}{2}\frac{d\xi}{d\tau}(r)\cdot\pi(r) + \frac{1}{2}\xi(r)\cdot\frac{d\pi}{d\tau}(r) \right] dr \right\} \times C(\hbar^{-\frac{1}{2}}\xi(t), \hbar^{-\frac{1}{2}}\pi(t)) + \exp(-i\hbar^{-1}(t-s)H_\hbar)C(\hbar^{-\frac{1}{2}}\xi(s), \hbar^{-\frac{1}{2}}\pi(s)) \quad (5.1)$$

where $C(\cdot\,,\cdot)$ denotes the Weyl's operators defined by (2.11).

Whenever no confusion arises, we shall write simply $W_\hbar(t, s)$.

THEOREM 5.2. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a\,; \mathbb{R}^\nu \to \mathbb{R}^\nu$, $(\xi, \pi) \in \mathscr{A}_1(v, a)$ with domain I, $(H_\hbar)_{\hbar>0} \in \mathscr{E}(\xi, \pi)$. Then $(W_\hbar(t, s))_{t,s\in I}$ is a L^2-strongly continuous L^2-unitary group.

Proof. — This follows easily from a simple algebraic calculation and from (2.11). Q. E. D.

DÉFINITION 5.3. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $(\xi, \pi) \in \mathscr{A}_1(v, a)$ with domain I. Assume that $A_\hbar$ is any $\hbar$-dependent linear operator in L^2. We set for $t \in I$,

$$A_\hbar(t\,; \xi, \pi) = C(\hbar^{-\frac{1}{2}}\xi(t), \hbar^{-\frac{1}{2}}\pi(t))^+ A_\hbar C(\hbar^{-\frac{1}{2}}\xi(t), \hbar^{-\frac{1}{2}}\pi(t)) \quad (5.2)$$

Whenever no confusion arises, we shall write simply $A_\hbar(t)$.

THEOREM 5.4. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $(\xi, \pi) \in \mathscr{A}_1(v, a)$ with domain I. Assume that $A_\hbar$ is any $\hbar$-dependent selfadjoint operator. Then for any $t \in I$, $A_\hbar(t)$ is selfadjoint. Furthermore, if $A_\hbar$ is positive, $A_\hbar(t)$, also, is positive.

Proof. — Trivial. Q. E. D.

The following theorem is the main result of this paper.

THEOREM 5.5. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $k \in \mathbb{N} \cup \{0\}$, $\bar{k} = \max(2, k)$, $(\xi, \pi) \in \mathscr{A}_{\bar{k}}(v, a)$ with domain I, $(H_\hbar)_{\hbar>0} \in \mathscr{E}(\xi, \pi)$. Let $S_\hbar$ be any $\hbar$-dependent operator such that:

1) For any $\hbar > 0$, $S_\hbar = S_\hbar^+$.

2) There is $O \in \Omega_{\bar{k}}(\xi, \pi)$ such that for any $\hbar > 0$ and $\psi \in P_k$ with supp ψ a compact set contained in $\hbar^{-\frac{1}{2}} O$, $\psi \in D(S_\hbar)$ and

$$S_\hbar \psi(x) = \sum_{\alpha, |\alpha| \leqslant k} \eta_\alpha(\hbar^{\frac{1}{2}} x)(-i\hbar^{\frac{1}{2}} D)^\alpha \psi(x) \tag{5.3}$$

Here the η_α's are continuous functions on O.

Furthermore, assume that there is an $\hbar$-dependent operator $M_\hbar$ such that:

3) For any $\hbar > 0$, $M_\hbar = M_\hbar^+ \geqslant 0$.

4) There is a non negative constant κ such that for any $\varepsilon, \hbar > 0$,

$$(1 + \varepsilon M_\hbar)^{-1} D(H_\hbar) \subseteq D(H_\hbar)$$

and

$$\pm i\hbar^{-1} [H_\hbar, M_\hbar] \leqslant \kappa M_\hbar \tag{5.4}$$

as form on $(1 + \varepsilon M_\hbar)^{-1} D(H_\hbar)$ [*17*].

5) For any $\hbar > 0$ and any $\psi \in P_k$ with supp ψ a compact set contained in $\hbar^{-\frac{1}{2}} O$, $\psi \in Q(M_\hbar)$ and

$$M_\hbar \psi(x) = \sum_{\alpha, |\alpha| \leqslant 2k} \zeta_\alpha(\hbar^{\frac{1}{2}} x)(-i\hbar^{\frac{1}{2}} D)^\alpha \psi(x) \tag{5.5}$$

where each ζ_α is a C^l-function on O with $l = \max(0, |\alpha| - k)$.

6) For any $\hbar > 0$, $Q(M_\hbar) \subseteq Q(S_\hbar^2)$ and $S_\hbar^2 \leqslant M_\hbar$ as form on $Q(M_\hbar)$.

Suppose that $s \in I$, $\psi \in X_k$. Assume that $\chi \in C^\infty([0, +\infty[, [0, +\infty[)$, $\chi(r) = 1$ for $r \in [0, 1]$, $\chi(r) = 0$ for $r \in [2, +\infty[$, $\chi(.)$ is decreasing in $[1, 2]$ and

$$\lim_{\rho \to 0+} \limsup_{\hbar \to 0} \| M_\hbar(s)^{\frac{1}{2}}(1 - \chi)(2 | q_\hbar | / \rho)\psi \| = 0 \tag{5.6}$$

Then we have uniformly for bounded $t \in I$.

7) For $\hbar$ small enough, $W_\hbar(t, s)\psi \in D(S_\hbar(t))$

8) $$\lim_{\hbar \to 0} \| S_\hbar(t) W_\hbar(t, s)\psi - \sum_{\alpha, |\alpha| \leqslant k} \eta_\alpha(\xi(t))\pi(t)^\alpha U_2(t, s)\psi \| = 0 \tag{5.7}$$

Furthermore, if the limit (5.6) holds uniformly in s varying in any compact subset of I, then points 7-8, also, hold uniformly in s varying in the same subset.

Proof. — The proof of theorem 5.5 is quite technical and a brief explanation of the general strategy is necessary. The major difficulty consists in checking the action of the operator $W_\hbar(t, s)$ when $\hbar \to 0$. This can be achieved by using estimates which do not contain $W_\hbar(t, s)$. To this purpose, assumptions 3-4 and Gronwall's lemma are vital.

LEMMA 5.5 *a*. — Let $\hbar > 0$ and $t, s \in I$. If $\psi \in Q(M_\hbar(s))$, then

$$W_\hbar(t, s)\psi \in Q(M_\hbar(t))$$

and

$$\| M_\hbar(t)^{\frac{1}{2}} W_\hbar(t, s)\psi \| \leqslant e^{\kappa|t-s|/2} \| M_\hbar(s)^{\frac{1}{2}}\psi \| \tag{5.8}$$

Proof. — By assumptions 3-4 and a result of W. G. Faris and R. Lavine [*17*] we find that for all $t, s \in \mathbb{R}$ and all $\psi \in Q(M_\hbar)$, $\exp(-i\hbar^{-1}(t-s)H_\hbar)\psi \in Q(M_\hbar)$ and

$$\| M_\hbar^{\frac{1}{2}} \exp(-i\hbar^{-1}(t-s)H_\hbar)\psi \| \leqslant e^{\kappa|t-s|/2} \| M_\hbar^{\frac{1}{2}}\psi \| \tag{5.9}$$

Moreover, since the Weyl's operators are L^2-unitary, we see that for all $r \in I$ $C(\hbar^{-\frac{1}{2}}\xi(r), \hbar^{-\frac{1}{2}}\pi(r))$ is unitary from $Q(M_\hbar(r))$ onto $Q(M_\hbar)$. The lemma follows easily from this remark and (5.9). Q. E. D.

LEMMA 5.5 *b*. — For any $\hbar > 0$ and any $r \in I$, $Q(M_\hbar(r)) \subseteq Q(S_\hbar^2(r))$ and $S_\hbar^2(r) \leqslant M_\hbar(r)$ as form on $Q(M_\hbar(r))$.

Proof. — It is a trivial consequence of assumption 6, the L^2-unitarity of the Weyl's operators, and (5.2). Q. E. D.

LEMMA 5.5 *c*. — Let $\hbar > 0$, $r \in I$, $\psi \in P_k$ with supp ψ a compact set contained in $\hbar^{-\frac{1}{2}}(O - \xi(r))$. Then $\psi \in Q(M_\hbar(r))$ and

$$M_\hbar(r)\psi(x) = \sum_{\alpha, |\alpha| \leqslant 2k} \zeta_\alpha(\hbar^{\frac{1}{2}}x + \xi(r))(-i\hbar^{1/2}D + \pi(r))^\alpha \psi(x) \tag{5.10}$$

Proof. — This follows easily from assumption 5, (5.2) and (2.12, 15-16). Q. E. D.

End of the proof of theorem 5.5. — Let K be any compact subinterval of I. Set

$$\rho_K = \max_{r \in K} \frac{1}{2} d(\xi(r), \complement O) \tag{5.11}$$

$$\mathcal{U}_K = \overline{\cup_{r \in K} S(\xi(r), \rho_K)} \tag{5.12}$$

$$\chi_\rho(x) = \chi(2 | x | / \rho), \qquad x \in \mathbb{R}^\nu, \qquad \rho > 0 \tag{5.13}$$

Let $s \in K$ and $\psi \in X_k$ such that (5.6) holds. Then $(1-\chi_\rho)(q_\hbar)\psi \in Q(M_\hbar(s))$ for $\hbar$, ρ small enough. Choose $\rho \in]0, \rho_K]$. Then $\chi_\rho(q_\hbar)\psi \in X_k \subseteq P_k$ and supp ψ is a compact set in $\hbar^{-1/2}(O - \xi(s))$, because $\chi_\rho \in C_0^\infty(\mathbb{R}^\nu)$ and supp $\chi_\rho = \overline{S(0, \rho)}$. By lemma 5.5 *c* $\chi_\rho(q_\hbar)\psi \in Q(M_\hbar(s))$. As a consequence,

$\psi \in Q(M_\hbar(s))$ for $\hbar$ small enough. Using lemma 5.5 *a* we find that $W_\hbar(t, s)\psi$, $W_\hbar(t, s)(1 - \chi_\rho)(q_\hbar)\psi$, $W_\hbar(t, s)\chi_\rho(q_\hbar)\psi \in Q(M_\hbar(t))$ for $\hbar$ and ρ small enough, uniformly in $t \in K$. By theorem 4.3 and lemma 5.5 *c*, $\chi_\rho(q_\hbar)U_2(t, s)\psi \in Q(M_\hbar(t))$ for $\hbar$ and ρ small enough, uniformly in $t \in K$. Moreover, from lemma 5.5 *b*, we have that $Q(M_\hbar(t)) \subseteq D(S_\hbar(t))$. Therefore,

$$\begin{aligned} \Big\| S_\hbar(t)W_\hbar(t, s)\psi - \sum_{\alpha, |\alpha| \leqslant k} \eta_\alpha(\xi(t))\pi(t)^\alpha U_2(t, s)\psi \Big\| &\leqslant \| S_\hbar(t)W_\hbar(t, s)(1 - \chi_\rho)(q_\hbar)\psi \| + \\ &+ \| S_\hbar(t)(W_\hbar(t, s)\chi_\rho(q_\hbar) - \chi_\rho(q_\hbar)U_2(t, s))\psi \| + \\ &\Big\| \Big(S_\hbar(t)\chi_\rho(q_\hbar) - \sum_{\alpha, |\alpha| \leqslant k} \eta_\alpha(\xi(t))\pi(t)^\alpha \Big) U_2(t, s)\psi \Big\| \end{aligned} \quad (5.14)$$

Now we have to estimate the three terms of the RHS of (5.14).

Estimate of the first term of the RHS *of* (5.14). — Using lemmas 5.5 *a* – *b*, we find that

$$\| S_\hbar(t)W_\hbar(t, s)(1 - \chi_\rho)(q_\hbar)\psi \| \leqslant e^{\frac{\kappa|t-s|}{2}} \| M_\hbar(s)^{\frac{1}{2}}(1 - \chi_\rho)(q_\hbar)\psi \| \quad (5.15)$$

Then, by (5.6), we get

$$\lim_{\rho \to 0+} \limsup_{\hbar \to 0} \| S_\hbar(t)W_\hbar(t, s)(1 - \chi_\rho)(q_\hbar)\psi \| = 0 \quad (5.16)$$

uniformly in $t \in K$. Moreover, if the limit (5.6) is uniform in $s \in K$, the limit (5.16), also, is uniform in $s \in K$.

Estimate of the second term of the RHS *of* (5.14). — The general strategy is still that outlined at the beginning of the proof. Define

$$\begin{aligned} \tilde{U}_\hbar(t, s) = {} & C(\hbar^{-\frac{1}{2}}\xi(t), \hbar^{-\frac{1}{2}}\pi(t))U_2(t, s)C(\hbar^{-\frac{1}{2}}\xi(s), \hbar^{-\frac{1}{2}}\pi(s))^+ \times \\ & \times \exp\left\{ -i\hbar^{-1} \int_s^t \left[\mathscr{H}(\xi(r), \pi(r); v, a) - \frac{1}{2}\frac{d\xi}{d\tau}(r)\cdot\pi(r) + \frac{1}{2}\xi(r)\cdot\frac{d\pi}{d\tau}(r) \right] dr \right\} \cdot \end{aligned} \quad (5.17)$$

$$\begin{aligned} \tilde{H}_\hbar(r) = {} & \frac{1}{2}p_\hbar^2 - \frac{1}{2}(p_\hbar - \pi(r))\cdot(a(\xi(r)) + \overline{\nabla a(\xi(r))\cdot(q_\hbar - \xi(r))}) \\ & - \frac{1}{2}(a(\xi(r)) + \overline{\nabla a(\xi(r))\cdot(q_\hbar - \xi(r))})\cdot(p_\hbar - \pi(r)) \\ & + F(\xi(r); r) + \nabla F(\xi(r); r)\cdot(q_\hbar - \xi(r)) \\ & + \frac{1}{2}(q_\hbar - \xi(r))\cdot\nabla\nabla F(\xi(r); r)\cdot(q_\hbar - \xi(r)) \end{aligned} \quad (5.18)$$

where

$$F(x;r) = -\pi(r).a(x) + a^2(x)/2 + v(x) \tag{5.19}$$

and the notation $\overline{|\quad\quad|}$ means that the vector indices of the vector quantities at the ends have to be saturated. Using (2.11), it can be directly proved that for any $\varphi \in X_1$, $r \in I$

$$\frac{d}{dt} C(\hbar^{-\frac{1}{2}}\xi(r), \hbar^{-\frac{1}{2}}\pi(r))\varphi = i\hbar^{-1} C(\hbar^{-\frac{1}{2}}\xi(r), \hbar^{-\frac{1}{2}}\pi(r)) \cdot \left\{ \left(q_\hbar + \frac{\xi(r)}{2}\right) \cdot \frac{d\pi}{d\tau}(r) - \left(p_\hbar + \frac{\pi(r)}{2}\right) \cdot \frac{d\xi}{d\tau}(r) \right\} \varphi \tag{5.20}$$

Using (2.13-2.14), (4.2) and (5.20), we find that for any $\psi \in X_2$

$$\frac{\partial}{\partial t} \tilde{U}_\hbar(t, s)\varphi = -i\hbar^{-1}\tilde{H}_\hbar(t)\tilde{U}_\hbar(t, s)\varphi \tag{5.21}$$

Define

$$\tilde{\Xi}_{\hbar,\rho}(t, s) = \exp(-i\hbar^{-1}(t-s)H_\hbar)\chi_\rho(q_\hbar - \xi(s)) - \chi_\rho(q_\hbar - \xi(t))\tilde{U}_\hbar(t, s) \tag{5.22}$$

Let $\varphi \in X_\infty$. Without any loss of generality, we can suppose that point 2 of definition 3.2 holds for functions of P_2 whose support is a compact set contained in $\hbar^{-\frac{1}{2}}O$. Since $\chi_\rho(q_\hbar - \xi(s))\varphi$ is one of those functions, we have

$$\frac{\partial}{\partial t} \tilde{\Xi}_{\hbar,\rho}(t, s)\varphi = i\hbar\chi_\rho(q_\hbar - \xi(t))\tilde{H}_\hbar(t)\tilde{U}_\hbar(t, s)\varphi - i\hbar^{-1}H_\hbar \exp(-i\hbar^{-1}(t-s)H_\hbar)\chi_\rho(q_\hbar - \xi(s))\varphi + \nabla\chi_\rho(q_\hbar - \xi(t)) \cdot \frac{d\xi}{d\tau}(t)\tilde{U}_\hbar(t, s)\varphi \tag{5.23}$$

From (2.13), theorem 4.3 and (5.17), it follows that

$$\chi_\rho(q_\hbar - \xi(t))\tilde{U}_\hbar(t, s)\varphi \in X_\infty$$

with support contained in $\hbar^{-\frac{1}{2}}O$. Using point 2 of definition 3.2 again, we can rewrite the RHS of (5.23) in the following manner

$$-i\hbar^{-1}H_\hbar \tilde{\Xi}_{\hbar,\rho}(t, s)\varphi + \tilde{R}_{\hbar,\rho}(t)\tilde{U}_\hbar(t, s)\varphi \tag{5.24}$$

where

$$\tilde{R}_{\hbar,\rho}(t) = -i\hbar^{-1}(H_\hbar - \tilde{H}_\hbar(t))\chi_\rho(q_\hbar - \xi(t)) + \nabla\chi_\rho(q_\hbar - \xi(t)) \cdot \frac{d\xi}{d\tau}(t) + i\hbar^{-1}[\chi_\rho(q_\hbar - \xi(t)), \tilde{H}_\hbar(t)] \tag{5.25}$$

Set for $\varepsilon > 0$ and $\omega \in L^2$

$$(M_\hbar)_\varepsilon = M_\hbar(1 + \varepsilon M_\hbar)^{-1} \tag{5.26}$$

$$\tilde{G}_{\varepsilon,\rho,\hbar}(t, s; \omega) = \langle \tilde{\Xi}_{\hbar,\rho}(t, s)\omega, (M_\hbar)_\varepsilon \tilde{\Xi}_{\hbar,\rho}(t, s)\omega \rangle \tag{5.27}$$

Using (5.23-5.27), we obtain

$$\frac{\partial}{\partial t}\tilde{G}_{\varepsilon,\rho,\hbar}(t, s; \varphi) = \langle \tilde{\Xi}_{\hbar,\rho}(t, s)\varphi, i\hbar^{-1}[H_\hbar, (M_\hbar)_\varepsilon\tilde{\Xi}_{\hbar,\rho}(t, s)\varphi \rangle + [\langle \tilde{R}_{\hbar,\rho}(t)\tilde{U}_\hbar(t, s)\varphi, (M_\hbar)_\varepsilon\tilde{\Xi}_{\hbar,\rho}(t, s)\varphi \rangle + \text{complex conjugate}] \quad (5.28)$$

From assumption 4, it follows that

$$\left|\frac{\partial}{\partial t}\tilde{G}_{\varepsilon,\rho,\hbar}(t, s; \varphi)\right| \leqslant (\kappa + 1)\tilde{G}_{\varepsilon,\rho,\hbar}(t, s; \varphi) + \| (M_\hbar)_\varepsilon^{1/2}\tilde{R}_{\hbar,\rho}(t)\tilde{U}_\hbar(t, s)\varphi \|^2 \quad (5.29)$$

Substituting φ with $C(\hbar^{-\frac{1}{2}}\xi(s), \hbar^{-\frac{1}{2}}\pi(s))\varphi$ in (5.27) and using (5.17, 22, 25, 27), we find that

$$\tilde{G}_{\varepsilon,\rho,\hbar}(t, s; C(\hbar^{-\frac{1}{2}}\xi(s), \hbar^{-\frac{1}{2}}\pi(s))\varphi) \equiv G_{\varepsilon,\rho,\hbar}(t, s; \varphi) = \langle \Xi_{\hbar,\rho}(t, s)\varphi, (M_\hbar)_\varepsilon(t)\Xi_{\hbar,\rho}(t, s)\varphi \rangle \quad (5.30)$$

$$\| (M_\hbar)_\varepsilon^{1/2}\tilde{R}_{\hbar,\rho}(t)\tilde{U}_\hbar(t, s)C(\hbar^{-\frac{1}{2}}\xi(s), \hbar^{-\frac{1}{2}}\pi(s))\varphi \| = \| (M_\hbar)_\varepsilon^{1/2}(t)R_{\hbar,\rho}(t)U_2(t, s)\varphi \| \quad (5.31)$$

where

$$\Xi_{\hbar,\rho}(t, s) = W_\hbar(t, s)\chi_\rho(q_\hbar) - \chi_\rho(q_\hbar)U_2(t, s) \quad (5.32)$$

$$\begin{aligned} R_{\hbar,\rho}(t) = {} & \frac{i\hbar}{2}\Delta\chi_\rho(q_\hbar) + (\delta^1 a(.\,; t)\cdot\nabla\chi_\rho(\cdot))(q_\hbar) \\ & + i\hbar^{-1}(\delta^1 a(.\,; t)\chi_\rho(\cdot))(q_\hbar)\cdot p_\hbar - i\hbar^{-1}(\delta^2 F(.\,; t)\chi_\rho(\cdot))(q_\hbar) \\ & - \nabla\chi_\rho(q_\hbar)\cdot p_\hbar + \nabla\chi_\rho(q_\hbar)\cdot \overline{\nabla a(\xi(t))\cdot q_\hbar} \\ & + \frac{1}{2}((\nabla\cdot a(. + \xi(t)) - \nabla\cdot a(\xi(t)))\chi_\rho(\cdot))(q_\hbar) \end{aligned} \quad (5.33)$$

with

$$\delta^1 a(x; t) = a(x + \xi(t)) - a(\xi(t)) - x\cdot\nabla a(\xi(t)) \quad (5.34)$$

$$\delta^2 F(x; t) = F(x + \xi(t); t) - F(\xi(t); t) - x\cdot\nabla F(\xi(t); t) - \frac{1}{2}x\cdot\nabla\nabla F(\xi(t); t)\cdot x \quad (5.35)$$

Since v and a are C^k-functions on $\mathcal{U}_K$ and $\varphi \in X_\infty$, by theorem 4.3 and (5.35), $R_{\hbar,\rho}(t)U_2(t, s)\varphi \in X_k \subseteq P_k$ and its support is a compact set contained in $\hbar^{-\frac{1}{2}}(O - \xi(t))$. Thus, we can use lemma 5.5 *c*. From (5.10), we see that the RHS of (5.31) is majorized by

$$C_K \sum_{|\alpha| \leqslant k} \| p_\hbar^\alpha R_{\hbar,\rho}(t)U_2(t, s)\varphi \| \quad (5.36)$$

where C_K is a non negative constant depending only on the compact set K. In order to estimate (5.36), we have to use (4.3) and (5.33) and the relation

$$p_\hbar^\alpha f(q_\hbar) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} ((-i\hbar D)^{\alpha-\beta} f)(q_\hbar) p_\hbar^\beta \tag{5.37}$$

which is valid for $|\alpha| \leqslant k$ and $f \in C_0^{|\alpha|}(\mathbb{R}^\nu)$. After performing these operations a divergent factor $\hbar^{-1}$ remains in some terms. In order to compensate it, we have to express $(\delta^1 a(.\,;t)\chi_\rho(\cdot))(q_\hbar)$ and $(\delta^2 F(.\,;t)\chi_\rho(\cdot))(q_\hbar)$ by means of Taylor's formula. In this way, we find that (5.36) is majorized by

$$g_K(\hbar, \rho) \| \varphi \|_{X_m} \qquad m = \max(2, k+1) \tag{5.38}$$

with $g_K(\hbar, \rho)$ a non negative function satisfying

$$\lim_{\rho \to 0+} \limsup_{\hbar \to 0} g_K(\hbar, \rho) = 0 \tag{5.39}$$

By (5.29-31, 36, 39) and Gronwall's lemma

$$G_{\varepsilon,\rho,\hbar}(t, s; \varphi) \leqslant |t-s| e^{(\kappa+1)|t-s|} (g_K(\hbar, \rho) \| \varphi \|_{X_m})^2 \tag{5.40}$$

Let $\omega \in X_k$. Then $\chi_\rho(q_\hbar)\omega \in Q(M_\hbar(s))$, and $\chi_\rho(q_\hbar)U_2(t, s)\omega \in Q(M_\hbar(t))$ by lemma 5.5 *c* and theorem 4.3. From lemma 5.5 *a* and (5.10, 32-37), the following estimate holds

$$\| M_\hbar(t)^{1/2} \Xi_{\hbar,\rho}(t, s)\omega \| \leqslant \tilde{g}_K(\hbar, \rho) \| \omega \|_{X_k} \tag{5.41}$$

where $\tilde{g}_K(\hbar, \rho)$ is a non negative function such that

$$\lim_{\rho \to 0+} \limsup_{\hbar \to 0} \tilde{g}_K(\hbar, \rho) < +\infty \tag{5.42}$$

Since X_∞ is dense in X_k and (5.40-42) hold on X_∞, we obtain

$$\lim_{\rho \to 0+} \limsup_{\hbar \to 0} \| M_\hbar(t)^{1/2} \Xi_{\hbar,\rho}(t, s)\psi \| = 0 \tag{5.43}$$

uniformly in $t, s \in K$. By lemma 5.5 *a*, we have

$$\| S_\hbar(t) \Xi_{\hbar,\rho}(t, s)\psi \| \leqslant \| M_\hbar(t)^{1/2} \Xi_{\hbar,\rho}(t, s)\psi \| \tag{5.44}$$

Therefore, uniformly in $t, s \in K$

$$\lim_{\rho \to 0+} \limsup_{\hbar \to 0} \| S_\hbar(t) \Xi_{\hbar,\rho}(t, s)\psi \| = 0 \tag{5.45}$$

Estimate of the third term of the RHS *of* (5.14). — From assumption 2, (5.2) and (2.12, 15-16), we find that $\chi_\rho(q_\hbar)\psi \in D(S_\hbar(t))$ and

$$S_\hbar(t)\psi(x) - \sum_{\alpha, |\alpha| \leqslant k} \eta_\alpha(\hbar^{\frac{1}{2}} x + \xi(t))(p_\hbar + \pi(t))^\alpha \psi(x) \tag{5.46}$$

Using (5.46) and theorem 4.3, it is not difficult to prove that

$$\lim_{\rho\to 0} \limsup_{\hbar\to 0} \left\| (S_\hbar(t)\chi_\rho(q_\hbar) - \sum_{\alpha,|\alpha|\leq k} \eta_\alpha(\xi(t))\pi(t)^\alpha)U_2(t,s)\psi \right\| = 0 \quad (5.47)$$

uniformly in $s \in K$. The limit is also uniform in $t \in K$, because of the uniform continuity of the functions η_α on the compact set $\mathcal{U}_K$.

This concludes the proof of the theorem. Q. E. D.

The previous theorem has an immediate important application.

THEOREM 5.6. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$, $(\xi, \pi) \in \mathcal{A}_2(v, a)$ with domain I, $(H_\hbar)_{\hbar>0} \in \mathcal{E}(\xi, \pi)$. Then

$$L^2 - s - \lim_{\hbar\to 0} W_\hbar(t, s) = U_2(t, s) \quad (5.48)$$

uniformly for bounded $t, s \in I$.

Proof. — Choose $k = 0$, $\kappa = 1$, $M_\hbar = S_\hbar = 1$, $\eta_\alpha = \zeta_\alpha = 1$ for $|\alpha| = 0$, $\eta_\alpha = \zeta_\alpha = 0$ otherwise. It is easily verified that all the assumptions of theorem 5.5 are fulfilled. Q. E. D.

COROLLARY 5.7. — Let v and a be any two potentials satisfying the assumptions of corollary 3.5. Let $(H_\hbar)_{\hbar>0}$ be the family of selfadjoint operators defined in the same corollary. Let $(\xi, \pi) \in \mathcal{A}_2(v, a)$ with domain I. Then (5.48) holds uniformly for bounded $t, s \in I$.

Proof. — This follows directly from the previous theorem. Q. E. D.

The method of proof of this paper allows us to obtain a result for the mean value of the operator $p_\hbar - a(q_\hbar)$ instead of $p_\hbar$. The following theorem selects a class of vector potentials for which $p_\hbar - a(q_\hbar)$ turns out to be selfadjoint.

THEOREM 5.8. — Let $g \in L^2_{loc}(\mathbb{R}^\nu, \mathbb{R})$. Then $p_{j,\hbar} - g(q_\hbar)$ is essentially selfadjoint on C_0^∞ for each $j = 1, 2, \ldots, \nu$.

Proof. — It is possible to set $\hbar = 1$, without any loss of generality. If it can be shown that the deficiency indices of $p_j - g(q) \restriction_{C_0^\infty}$ both vanish, the theorem will be proved [*15*]. Let $\psi \in L^2$ such that for any $\varphi \in C_0^\infty(\mathbb{R}^\nu)$

$$\langle \psi, (\pm i + p_j - g(q))\varphi \rangle = 0 \quad (5.49)$$

Since $\bar\psi$ and $g\bar\psi$ both belong to $L^1_{loc}(\mathbb{R}^\nu)$, $\bar\psi$ satisfies the following distributional differential equation [*14*] in $\mathcal{D}'(\mathbb{R}^\nu)$

$$\partial_j \bar\psi = (\mp 1 - ig)\bar\psi \quad (5.50)$$

so that also $\partial_j \bar\psi \in L^1_{loc}(\mathbb{R}^\nu)$. Thus, it is possible to apply Nykodim's theorem [22]: for almost all $\tilde{x} \in \mathbb{R}^{\nu-1}$, $\overline{\psi}(\tilde{x}, .) \in AC_{loc}(\mathbb{R})$ and for almost all $x_j \in \mathbb{R}$,

$$\frac{\partial}{\partial x_j} \overline{\psi}(\tilde{x}, x_j) = \partial_j \psi(\tilde{x}, x_j) \tag{5.51}$$

where we denote x by $(\tilde{x}, x_j)$ with $\tilde{x} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_\nu)$. Moreover, for almost all $\tilde{X} \in \mathbb{R}^{\nu-1}$, $g(\tilde{X}, .) \in L^1_{loc}(\mathbb{R})$. Using this remark, it is easy to see that any function which satisfies (5.50-51) must have the form

$$\overline{\psi}(\tilde{x}, x_j) = \overline{\psi}(\tilde{x}, 0) \exp\left(\mp x_j - i \int_0^{x_j} g(\tilde{x}, \xi_j) d\xi_j \right) \tag{5.52}$$

for almost all $\tilde{x} \in \mathbb{R}^{\nu-1}$. This belongs to $L^2(\mathbb{R}^\nu)$ only if $\overline{\psi}(\tilde{x}, 0) \equiv 0$ almost everywhere. Q. E. D.

DEFINITION 5.9. — Let $a \in L^2_{loc}(\mathbb{R}^\nu, \mathbb{R}^\nu)$. We define

$$P_{j,\hbar}(a) = \overline{p_{j,\hbar} - a_j(q_\hbar) \restriction C_0^\infty} \qquad j = 1, 2, \ldots, \nu \tag{5.53}$$

We shall write $P_{j,\hbar}$ instead of $P_{j,\hbar}(a)$ in order to simplify the notation.

The following theorem gives a rigorous statement of Bohr's correspondence principle.

THEOREM 5.10. — Let $v : \mathbb{R}^\nu \to \mathbb{R}$, $a : \mathbb{R}^\nu \to \mathbb{R}^\nu$ such that

1) $a \in L^4(\mathbb{R}^\nu, \mathbb{R}^\nu) + L^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu)$, $\sum_{1}^{\nu}{}_j \, \partial_j a_j \in L^2(\mathbb{R}^\nu, \mathbb{R}) + L^\infty(\mathbb{R}^\nu, \mathbb{R})$.

2) $v = v' + v''$, $\quad v' \in L^2(\mathbb{R}^\nu, \mathbb{R})$, $\quad v' \geqslant 0$,

v'' is real and measurable and

$$\alpha |x|^\gamma + \beta \geqslant v''(x) \geqslant -cx^2 - d \tag{5.54}$$

for some α, β, c, $d > 0$ and $\gamma \geqslant 2$.

Suppose that $(\xi, \pi) \in \mathscr{A}_2(v, a)$ with domain I. Let $H^0_\hbar$ be defined as in (3.6) and $H_\hbar = \overline{H^0_\hbar}$. Then $(H_\hbar)_{\hbar > 0} \in \mathscr{E}(\xi, \pi)$. Define

$$D = \bigcup_{\lambda > \nu/2} P_{\max\left(\frac{\lambda}{2}, 1\right)} \cap Q_{\max\left(\gamma/2, \frac{\nu/4}{1 - \nu/2\lambda}\right)} \tag{5.55}$$

Let $\psi \in D$ with $\| \psi \| = 1$. Then for $\hbar$ small enough and any $t, s \in I$.

$$\psi_\hbar(t, s; \psi) = \exp(-i\hbar^{-1}(t-s)H_\hbar) C(\hbar^{-\frac{1}{2}}\xi(s), \hbar^{-\frac{1}{2}}\pi(s))\psi \in \bigcap_{1}^{\nu}{}_j \, D(P_{j,\hbar}) \cap D(q_{j,\hbar}) \tag{5.56}$$

so that the following quantities are well defined

$$\xi_{j,\hbar}{}^*(t, s\,;\psi) = \langle\, \psi_\hbar(t, s\,;\psi),\, q_{j,\hbar}\psi_\hbar(t, s\,;\psi)\,\rangle \tag{5.57}$$

$$\Delta\xi_{j,\hbar}{}^*(t, s\,;\psi) = \langle\, \psi_\hbar(t, s\,;\psi),\, (q_{j,\hbar} - \xi_{j,\hbar}{}^*(t, s\,;\psi))^2\psi_\hbar(t, s\,;\psi)\,\rangle^{\frac{1}{2}} \tag{5.58}$$

$$\Pi_{j,\hbar}{}^*(t, s\,;\psi) = \langle\, \psi_\hbar(t, s\,;\psi),\, \mathrm{P}_{j,\hbar}\psi_\hbar(t, s\,;\psi)\,\rangle \tag{5.59}$$

$$\Delta\Pi_{j,\hbar}{}^*(t, s;\psi) = \langle\, \psi_\hbar(t, s\,;\psi),\, (\mathrm{P}_{j,\hbar} - \Pi_{j,\hbar}{}^*(t, s\,;\psi))^2\psi_\hbar(t, s\,;\psi)\,\rangle^{\frac{1}{2}} \tag{5.60}$$

for each $j = 1, 2, \ldots, v$. Furthermore the following limits exist for each $j = 1, 2, \ldots, v$ uniformly for bounded $t, s \in \mathrm{I}$:

$$\lim_{\hbar\to 0} \xi_{j,\hbar}{}^*(t, s\,;\psi) = \xi_j(t) \tag{5.61}$$

$$\lim_{\hbar\to 0} \Delta\xi_{j,\hbar}{}^*(t, s\,;\psi) = 0 \tag{5.62}$$

$$\lim_{\hbar\to 0} \Pi_{j,\hbar}{}^*(t, s\,;\psi) = \pi_j(t) - a_j(\xi(t)) = \frac{d\xi_j}{d\tau}(t) \tag{5.63}$$

$$\lim_{\hbar\to 0} \Delta\Pi_{j,\hbar}{}^*(t, s\,;\psi) = 0 \tag{5.64}$$

Proof. — The proof is founded on theorem 5.5. Define for $\hbar > 0$ [*17*]

$$\mathrm{M}_\hbar = \mathrm{H}_\hbar + 2cq_\hbar^2 + d \tag{5.65}$$

From a result of H. Leinfelder and C. G. Simader [*19*] and assumptions 1-2, it follows that both $\mathrm{H}_\hbar$ and $\mathrm{M}_\hbar$ are essentially selfadjoint on $C_0^\infty(\mathbb{R}^v)$ and $\mathrm{M}_\hbar \geqslant 0$, for any $\hbar > 0$. By corollary 3.5, $(\mathrm{H}_\hbar)_{\hbar>0} \in \mathscr{E}(\xi, \pi)$, because $(\xi, \pi) \in \mathscr{A}_2(v, a)$. Furthermore, $\mathrm{M}_\hbar$ satisfies assumption 3 of theorem 5.5. A straightforward calculation shows that

$$(\mathrm{H}_\hbar + d)^2 \leqslant \mathrm{M}_\hbar^2 + 2cv\hbar^2 \tag{5.66}$$

$$\pm\, i\hbar^{-1}[\mathrm{H}_\hbar, \mathrm{M}_\hbar] \leqslant \max(4c, 2)\mathrm{M}_\hbar \tag{5.67}$$

as form on $C_0^\infty(\mathbb{R}^v)$ [*17*]. Thus, assumption 4 of theorem 5.5 is fulfilled. Assumption 5 of theorem 5.5 is easily verified. Let $\mathrm{S}_\hbar$ be one of the operator $c^{\frac{1}{2}}q_{j,\hbar}$, $2^{-\frac{1}{2}}\mathrm{P}_{j,\hbar}$ with $j = 1, 2, \ldots, v$. It is a matter of an easy verification that assumptions 1-2, 6 of theorem 5.5 are fulfilled. Now we have to show that if $\psi \in \mathrm{D}$, (5.6) is satisfied uniformly for bounded $s \in \mathrm{I}$. To this purpose, we need the following lemma:

LEMMA 5.10 *a.* — Let $g \in \mathrm{L}^2(\mathbb{R}^v, \mathbb{R}) + \mathrm{L}^\infty(\mathbb{R}^v, \mathbb{R})$. Then, there is $\mathrm{R} > 0$ such that for all $\lambda > v/2$, $z \in \mathbb{R}^v$, $\hbar \in \,]0, 1[$,

$$\mathrm{P}_{\lambda/2} \subseteq \mathrm{Q}(g(q_\hbar + z)) \tag{5.68}$$

$$\pm\, g(q_\hbar + z) \leqslant |\,p\,|^\lambda + \mathrm{R}\hbar^{-\frac{v/4}{1 - v/2}} \tag{5.69}$$

as form on $\mathrm{P}_{\lambda/2}$.

Proof. — The proof is founded on Fourier representation and Hausdorff Young theorem. Q. E. D.

End of the proof of theorem 5.10. — It is easy to check that the functions $a^2(\cdot)$, $\sum_{1}^{\nu} \partial_j a_j$ and v' satisfy the assumptions of lemma 5.10 *a* so that (5.68-69) hold. Furthermore, by (5.54)

$$\pm\, v''(\hbar^{1/2}x + z) \leqslant 2^{\nu-1} \max(\alpha, c)\hbar^{\frac{\gamma}{2}} \,|\, x \,|^{\gamma} + b(z) \tag{5.70}$$

where $b(\cdot)$ is a continuous function. Using (5.68-70), we find that

$$\| \mathrm{M}_\hbar(s)^{\frac{1}{2}}(1 - \chi_\rho)(q_\hbar)\psi \| \leqslant \mathrm{C}[\, \| \,|\, p \,|^{\frac{\lambda}{2}}(1 - \chi_\rho)(q_\hbar)\psi \|^2$$
$$+ \| \,|\, q \,|^{\frac{\gamma}{2}}(1 - \chi_\rho)(q_\hbar)\psi \|^2 + (\hbar^{-\frac{\nu/4}{1-\nu/2\lambda}} + b(\xi(s))) \, \| (1 - \chi_\rho)(q_\hbar)\psi \|^2] \tag{5.71}$$

for $\psi \in \mathrm{P}_{\lambda/2} \cap \mathrm{Q}_{\gamma/2}$, $\lambda > \nu/2$ and $s \in \mathrm{I}$, where C is a positive constant and χ_ρ is defined by (5.13). By interpolation, it follows that for any $\tau \geqslant 0$, $\rho > 0$, $(1 - \chi_\rho)(q_\hbar)$ belongs both to $\mathrm{B}(\mathrm{Q}_\tau, \mathrm{Q}_\tau)$ and $\mathrm{B}(\mathrm{P}_\tau, \mathrm{P}_\tau)$ and $(1-\chi_\rho)(q_\hbar) \to 0$, $\hbar \to 0$ strongly in Q_τ and P_τ. Moreover, for any, $\delta > 0$ and $\psi \in \mathrm{Q}_\delta$,

$$\lim_{\hbar \to 0} \hbar^{-\delta} \| (1 - \chi_\rho)(q_\hbar)\psi \|^2 = 0 \tag{5.72}$$

Using this remark and (5.72), we see that the LHS of (5.71) vanishes when $\hbar \to 0$ uniformly for bounded $s \in \mathrm{I}$, provided $\psi \in \mathrm{P}_{\frac{\lambda}{2}} \cap \mathrm{Q}_{\frac{\gamma}{2}} \cap \mathrm{Q}_{\frac{\nu/4}{1-\nu/2\lambda}}$. Therefore, (5.6) is verified when $\psi \in \mathrm{D}$ uniformly for bounded $s \in \mathrm{I}$. Applying theorem 5.5, we obtain

$$\lim_{\hbar \to 0} \| (q_{j,\hbar} + \xi_j(t))\mathrm{W}_\hbar(t, s)\psi - \xi_j(t)\mathrm{U}_2(t, s)\psi \| = 0 \tag{5.73}$$

$$\lim_{\hbar \to 0} \| \mathrm{P}_{j,\hbar}(t)\mathrm{W}_\hbar(t, s)\psi - (\pi_j(t) - a_j(\xi(t)))\mathrm{U}_2(t, s)\psi \| = 0 \tag{5.74}$$

uniformly for bounded t, $s \in \mathrm{I}$. From these relations, the theorem follows easily. Q. E. D.

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