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by

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ABSTRACT. — The compatibility relations are applied in order to study the evolution of shock waves in relativistic continuum mechanics. General results are presented on the damping of shock waves in relativistic fluids when the shock velocity tends to the light speed.

RÉSUMÉ. — On utilise les relations de compatibilité pour étudier l'évolution des ondes de choc en mécanique relativiste des milieux continus. On présente des résultats généraux sur l'amortissement des ondes de choc dans les fluides relativistes lorsque la vitesse du choc tend vers la vitesse de la lumière.

1. INTRODUCTION

The study of shock waves presents several problems in continuum mechanics. One of these is that, once a shock wave is formed (say by the steepening of an acceleration wave in a solid or of an acoustic wave in a fluid), it is difficult to follow its subsequent evolution short of finding an exact solution to the general dynamical equations. In some particular cases heuristic methods can be used [1] but their mathematical validity is doubtful.

In general, however, the only way of tackling the evolution of a shock

wave is by resorting to a numerical integration scheme. This situation is to be contrasted with that arising in the case of weak discontinuity waves (as acceleration waves, thermal waves, etc.) where the evolution of the wave can be studied independently of the full solution. In fact, if the state ahead of the wavefront is known, it is possible to separate the evolution of the wave from the solution behind the wavefront. In fact the wave decouples from the field behind the wavefront and its evolution can be determined, once the state ahead of the wavefront has been specified, by the knowledge of the initial data for the wave's amplitude [2] [3] [4].

This is not possible in general in the case of a shock wave because the wavefront is subsonic with respect to the flow behind it and therefore the evolution of the shock wave can be influenced by acoustic signals coming from behind the wavefront.

However, although the general program of separating the wave's evolution from the solution behind the wavefront is not feasible for a shock wave, a limited progress can be achieved by using Thomas' method of iterated discontinuities [5]. A general method can be devised which leads to a sort of transport equation for the shock amplitude which, however, is not sufficient to determine completely its subsequent evolution, but still allows some qualitative results to be obtained.

This method has been applied to various situations in classical continuum mechanics, as in non linear elasticity [6] and viscoelasticity [7], and has been proved to be extremely useful in deriving exact qualitative results on the evolution of shock waves.

Relativistic shock waves can occur in various physical situations. Relativistic blast waves can be produced in laboratory plasmas by strong laser beams [8] [9]. Also relativistic electromagnetic shocks could perhaps propagate in polarizable media [10]. Finally in many areas of astrophysics such as supernovae, extragalatic radio sources, and galaxy formation, relativistic shocks seem to be a basic ingredient [11] [12] [13].

For these reasons it is desirable to extend Thomas' method of iterated discontinuities to relativistic shock waves.

A first attempt in this direction was made in [14] [15] [16] where the case of one-dimensional shock waves in a relativistic fluid was treated in detail. The aim of this paper is to develop Thomas' method of iterated discontinuities for relativistic shock waves of arbitrary geometry.

The plan of the paper is the following.

In Sec. 2 the basic formalism of the relativistic compatibility relations is briefly recalled.

In Sec. 3 the main compatibility equations are derived for relativistic shock waves in an arbitrary continuum, drawing only on the conservation equations.

In Sec. 4 a foliation of space-time consisting of space-like hypersurfaces

is explicitely introduced and the previous equations are rewritten in terms
of data on one of these hypersurfaces.

In Sec. 5 the compatibility equations are specialized to the case of a
relativistic fluid.

In Sec. 6 some general qualitative results are obtained and the limiting
case of an extremely relativistic shock is investigated.

2. THE COMPATIBILITY RELATIONS

Let $\mathcal{M}$ be a space-time, i. e. a 4-dimensional differentiable manifold,
oriented, paracompact, endowed with a Lorentz metric g (of signature
+ 2) and time-oriented.

Let Σ be an orientable hypersurface of $\mathcal{M}$. It is easy to prove [17] that
there exist:

i) a differentiable 1-form l such that $l_{|\Sigma} \neq 0$, which is orthogonal to
all tangent vectors of Σ.

ii) three differentiable submanifolds Ω, Ω_+, Ω_- of $\mathcal{M}$, Ω_+ and Ω_-
having boundaries, such that:

a) $\Omega = \Omega_+ \cup \Omega_-$
b) Ω is an open neighbourhood of Σ
c) $\Omega_\pm \cap \Sigma = \partial \Omega_\pm$.

A tensor field T *of type* (r, s), *smooth on* $\Omega - \Sigma$, *is said regularly disconti-*
nuous across Σ *if there exist two tensor fields* $\tilde{T}^+$, $\tilde{T}^-$, *of type* (r, s), *smooth*
on Ω_+ *and* Ω_- *respectively, such that*

$$T_{|\Omega_\pm - \Omega_\pm \cap \Sigma} = \tilde{T}^\pm_{|\Omega_\pm - \Omega_\pm \cap \Sigma}.$$

In this case the jump of T across Σ is defined by

(1) $$[T] = \tilde{T}^-_{|\Sigma} - \tilde{T}^+_{|\Sigma}.$$

Let ∇ denote the riemannian connection of $\mathcal{M}$ and T be regularly dis-
continuous across Σ. Then ∇T is a tensor field of type (r, s + 1) which is
regularly discontinuous across Σ. In fact $\nabla \tilde{T}^\pm$ are well defined and smooth
on $\Omega_\pm \cup \Sigma$ and the jump of ∇T is then given by

$$[\nabla T] = \nabla \tilde{T}^-_{|\Sigma} - \nabla \tilde{T}^+_{|\Sigma}.$$

Similar remarks apply to the higher covariant derivatives of T.

From the previous definition it is apparent that [T] can be extended
(in an obviously non unique way) to a tensor field smooth on a neigh-
bourhood of Σ.

Let now Σ be a time-like hypersurface and let $\underline{n}$ denotes its unit normal. In local coordinates $\underline{n}$ is given by

$$n^\mu = \frac{g^{\mu\nu}l_\nu}{(g^{\alpha\beta}l_\alpha l_\beta)^{\frac{1}{2}}}$$

where l_μ are the components of the 1-form l.

The *inner covariant derivative* $\tilde{\nabla}$ is defined on the smooth tensor fields of Ω defined on Σ as follows [18], in local coordinates,

(2) $\tilde{\nabla}_\mu T^{\ldots}_{\ldots} = h_\mu{}^\nu \nabla_\nu T^{\ldots}_{\ldots}$

where $h_\mu{}^\nu = \delta_\mu{}^\nu - n_\mu n^\nu$ is the projection tensor onto Σ. $\tilde{\nabla}_\mu T^{\ldots}_{\ldots}$ is a smooth tensor field of Ω defined on Σ, which depends only on the restriction of $T^{\ldots}_{\ldots}$ to Σ.

Except when acting on scalar functions, $\tilde{\nabla}$ is different from the induced riemannian connection of Σ because, in general, $\tilde{\nabla}_\mu T^{\ldots}_{\ldots}$ is not tangent to Σ even when $T^{\ldots}_{\ldots}$ is tangent to Σ.

The definition of the inner covariant derivative given by (2) is *local* and *holds only for non-null hypersurfaces*. A global definition which holds also for null hypersurfaces is given in [19].

The compatibility relations restrict the jump $[\nabla T]$ of the covariant derivative of a tensor field T regularly discontinuous across Σ and are given by [18] [19]

(3) $[\nabla_\mu T^{\ldots}_{\ldots}] = \tilde{\nabla}_\mu [T^{\ldots}_{\ldots}] + [\nabla_{\underline{n}} T^{\ldots}_{\ldots}] \otimes n_\mu$.

These relations are the natural extension of the classical Hadamard ones [5] to general relativity. A different approach to the compatibility relations is given in [20] where use is made of the theory of tensor distributions. For the aim of this paper, however, the present approach is adequate.

3. THE GENERAL COMPATIBILITY EQUATIONS

In this section general compatibility equations are derived on the basis of the conservation equations alone. Therefore they hold for an arbitrary relativistic continuum. Further compatibility equations can be obtained from the constitutive equations once a particular class of media has been selected.

Let $T^{\mu\nu}$ be the energy-momentum tensor and j^μ the mass-flux vector of the continuum. Both $T^{\mu\nu}$ and j^μ are assumed to be regularly discontinuous across Σ. Then, as a consequence of the conservation laws

(4) $\nabla_\mu T^{\mu\nu} = 0$

(5) $\nabla_\mu j^\mu = 0$

the following junction conditions hold across Σ, [20] [21],

(6) $$n_\mu [T^{\mu\nu}] = 0$$

(7) $$n_\mu [j^\mu] = 0$$

which state that both $[T^{\mu\nu}]$ and $[j^\mu]$ are tangent to Σ.

Let
$$\tilde{T}^{\mu\nu} = h^\mu{}_\alpha h^\nu{}_\beta T^{\alpha\beta}$$
$$\tilde{j}^\mu = h^\mu{}_\alpha j^\beta$$

then the junction conditions can be rewritten in the form

$$[T^{\mu\nu}] = [\tilde{T}^{\mu\nu}]$$
$$[j^\mu] = [\tilde{j}^\mu] .$$

From (4), (5), the compatibility relations (3) and the junction conditions, it follows easily

(8) $$\tilde{\nabla}_\mu [T^{\mu\nu}] - n_\mu [n^\alpha \nabla_\alpha T^{\mu\nu}] = 0$$

(9) $$\tilde{\nabla}_\mu [\tilde{j}^\mu] + n_\mu [n^\alpha \nabla_\alpha j^\mu] = 0 .$$

Let $x \in \Sigma$ and U be a coordinate neighbourhood around x in $\mathscr{M}$ such that $\overline{U} = U \cap \Sigma$ is a coordinate neighbourhood of x in Σ. If (y^0, y^i) are local coordinates in U, then $y^0 = 0$ is the local equation of Σ in U and (y^i) are local coordinates in $\overline{U}$.

In the following all considerations will be restricted to U and $\overline{U}$ or subsets thereof.

It is easy to see that in $\overline{U}$ there exists an orthonormal basis field $\{\underline{k}, \underline{e}_A\}$, $A = 2, 3$, such that $\underline{k}$ is time-like and future-directed and $\underline{e}_A$ is space-like. A way to construct it is the following. $\left\{ \dfrac{\partial}{\partial y^i} \right\}$ is a basis field in $\overline{U}$ and $h_\mu{}^\nu = \delta_\mu{}^\nu - n_\mu n^\nu$ defines a metric of signature $+1$ on $\overline{U}$. Therefore it is possible to orthonormalize the basis vectors $\dfrac{\partial}{\partial y^i}$ with respect to the metric $h_\mu{}^\nu$ thereby obtaining the orthonormal basis $\{\underline{k}, \underline{e}_A\}$. Hence in $\overline{U}$ one has

$$k_\mu k^\mu = -1, \qquad e_A{}^\mu e_{A\mu} = 1, \quad k_\mu e_A{}^\mu = 0 ,$$
$$e_{A\mu} e_B{}^\mu = \delta_{AB} .$$

With respect to the basis $\{\underline{k}, \underline{e}_A\}$ one has the following decomposition

(10) $$[j^\mu] = I^1 k^\mu + I^A e_A{}^\mu$$

(11) $$[\tilde{T}^{\mu\nu}] = \theta^{11} k^\mu k^\nu + \theta^{1A}(k^\mu e_A{}^\nu + e_A{}^\mu k^\nu) + \theta^{AB} e_A{}^\mu e_B{}^\nu .$$

A simple substitution of (10) and (11) into (8) and (9) yields

$$(12) \qquad k^\mu \tilde{\nabla}_\mu I^1 + I^1 \tilde{\nabla}_\mu k^\mu + e_A{}^\mu \tilde{\nabla}_\mu I^A + I^A \tilde{\nabla}_\mu e_A{}^\mu + n_\mu [n^\alpha \nabla_\alpha j^\mu] = 0$$

$$
\begin{aligned}
(13) \quad & k^\nu k^\mu \tilde{\nabla}_\mu \theta^{11} + k^\nu \theta^{11} \tilde{\nabla}_\mu k^\mu + k^\mu \theta^{11} \tilde{\nabla}_\mu k^\nu + \left(k^\mu e_A{}^\nu + e_A{}^\mu k^\nu \right) \tilde{\nabla}_\mu \theta^{1A} \\
& + \theta^{1A} \left(e_A{}^\nu \tilde{\nabla}_\mu k^\mu + k^\mu \tilde{\nabla}_\mu e_A{}^\nu + k^\nu \tilde{\nabla}_\mu e_A{}^\mu + e_A{}^\mu \tilde{\nabla}_\mu k^\nu \right) + e_A{}^\mu e_B{}^\nu \tilde{\nabla}_\mu \theta^{AB} \\
& + e_B{}^\nu \theta^{AB} \tilde{\nabla}_\mu e_A{}^\mu + \theta^{AB} e_A{}^\mu \tilde{\nabla}_\mu e_B{}^\nu + n_\mu [n^\alpha \nabla_\alpha T^{\mu\nu}] = 0 .
\end{aligned}
$$

At this stage it is convenient to introduce the induced riemannian connection of Σ, $\overline{\overline{\nabla}}$, defined on smooth tensor fields $T^{\mu\ldots}_{\ \nu\ldots}$ tangent to Σ by

$$(14) \qquad \overline{\overline{\nabla}}_\mu T^{\sigma\ldots}_{\ \nu\ldots} = h^\alpha{}_\mu h^\sigma{}_\beta h_\nu{}^\gamma \ldots \nabla_\alpha T^{\beta\ldots}_{\ \gamma\ldots}$$

Obviously, on scalars, $\tilde{\nabla}$ coincides with $\overline{\overline{\nabla}}$.

Also, from

$$\overline{\overline{\nabla}}_\mu k^\nu = h^\alpha{}_\mu h^\nu{}_\beta \nabla_\alpha k^\beta$$

it follows

$$\overline{\overline{\nabla}}_\mu k^\mu = h^\alpha{}_\beta \nabla_\alpha k^\beta = \tilde{\nabla}_\mu k^\mu$$

and analogous results hold for $\tilde{\nabla}_\mu e_A{}^\mu$.

The second fundamental form of Σ is defined by

$$(15) \qquad \chi_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu \nabla_\alpha n_\beta$$

and, for any couple of vector fields v^μ, z^μ, tangent to Σ, one has

$$(16) \qquad v^\mu \tilde{\nabla}_\mu z^\nu = v^\mu \nabla_\mu z^\nu = v^\mu \overline{\overline{\nabla}}_\mu z^\nu - n^\nu \chi_{\alpha\beta} v^\alpha z^\beta .$$

Eq. (12) can be rewritten in the form

$$(17) \qquad k^\mu \overline{\overline{\nabla}}_\mu I^1 + I^1 \overline{\overline{\nabla}}_\mu k^\mu + e_A{}^\mu \overline{\overline{\nabla}}_\mu I^A + I^A \overline{\overline{\nabla}}_\mu e_A{}^\mu + n_\mu [n^\alpha \nabla_\alpha j^\mu] = 0$$

After some manipulations eq. (13), contracting with n_ν, yields

$$(18) \quad \theta^{11} \chi_{\alpha\beta} k^\alpha k^\beta + 2\theta^{1A} \chi_{\alpha\beta} k^\alpha e_A{}^\beta + \theta^{AB} \chi_{\alpha\beta} e_A{}^\alpha e_B{}^\beta - n_\mu n_\nu [n^\alpha \nabla_\alpha T^{\mu\nu}] = 0 .$$

From (18) it follows that, if x is a flat point of Σ (i. e. $\chi_{\alpha\beta}(x) = 0$), then

$$[n^\alpha \nabla_\alpha (n_\mu n_\nu T^{\mu\nu})]_x = 0 .$$

By further contracting eq. (13) with k_ν and $e_{C\nu}$ respectively one obtains

$$
\begin{aligned}
(19) \quad & k^\mu \overline{\overline{\nabla}}_\mu \theta^{11} + \theta^{11} \overline{\overline{\nabla}}_\mu k^\mu + e_A{}^\mu \overline{\overline{\nabla}}_\mu \theta^{1A} + \theta^{1A} k^\mu e_{A\tau} \overline{\overline{\nabla}}_\mu k^\tau + \theta^{1A} \overline{\overline{\nabla}}_\mu e_A{}^\mu \\
& + \theta^{AB} e_A{}^\mu e_{B\tau} \overline{\overline{\nabla}}_\mu k^\tau - k_\nu n_\mu [n^\alpha \nabla_\alpha T^{\mu\nu}] = 0
\end{aligned}
$$

$$
\begin{aligned}
(20) \quad & \theta^{11} e_{C\tau} k^\mu \overline{\overline{\nabla}}_\mu k^\tau + k^\mu \overline{\overline{\nabla}}_\mu \theta^{1C} + \theta^{1C} \overline{\overline{\nabla}}_\mu k^\mu + \theta^{1A} \left(e_{C\tau} k^\mu \overline{\overline{\nabla}}_\mu e^\tau{}_A + e_{C\tau} e_A{}^\mu \overline{\overline{\nabla}}_\mu k^\tau \right) \\
& + e_A{}^\mu \overline{\overline{\nabla}}_\mu \theta^{AC} + \theta^{AC} \overline{\overline{\nabla}}_\mu e_A{}^\mu + \theta^{AB} e_{C\tau} e_A{}^\mu \overline{\overline{\nabla}}_\mu e_B{}^\tau + e_{C\nu} n_\mu [n^\alpha \nabla_\alpha T^{\mu\nu}] = 0 .
\end{aligned}
$$

It is interesting to notice that in eqs. (18), (19), (20), the operators

$$k^\mu \overline{\overline{\nabla}}_\mu + \overline{\overline{\nabla}}_\mu k^\mu$$

$$e_A{}^\mu \overline{\overline{\nabla}}_\mu + \overline{\overline{\nabla}}_\mu e_A{}^\mu$$

appear which are suggestive of conservation laws along the lines tangent

to k^μ and $e_A{}^\mu$ respectively. The extra terms appearing in these equations are then suggestive of diffusive terms while the jumps of the derivatives would represent the contribution from the flow behind the wavefront.

In eqs. (17-20) the normal jumps of $\nabla_\alpha T^{\mu\nu}$ and $\nabla_\alpha j^\mu$ appear, which, in general, are unknown and therefore these equations cannot be called « transport equations » in a proper sense. However they would become transport equations after some knowledge of $[n^\alpha \nabla_\alpha T^{\mu\nu}]$ and $[n^\alpha \nabla_\alpha^n j^\mu]$ has been obtained.

4. THE INITIAL DATA

Let the open subset U be endowed with a foliation of space-like hyper-surfaces $\mathscr{F}_t$, $t \in [a, b] \subset R$ and let $\underline{N}$ be the future-directed time-like vector field of unit normals to $\mathscr{F}_t$. The existence of such a foliation is entailed, for instance, by global hyperbolicity [22].

Denote by H the projection tensor onto $\mathscr{F}_t$, given in local coordinates by

$$H^\mu{}_\nu = \delta^\mu{}_\nu + N^\mu N_\nu .$$

The propagation speed V_Σ of the hypersurface Σ with respect to the family of observers identified with the time-like vector field $\underline{N}$ is given by

(21) $$\Gamma_\Sigma^2 = \frac{1}{1 - V_\Sigma^2} = 1 + (n_\mu N^\mu)^2$$

where Γ_Σ is the Lorentz factor of Σ.

Here the condition

$$n_\mu N^\mu > 0$$

will be assumed, in order to have progressive waves ($V_\Sigma > 0$).

By suitably restricting U it is possible to characterize $\mathscr{F}_t$ by a smooth function $\psi(y^\mu)$, such that

$$\mathscr{F}_t : \psi(y^\mu) = t .$$

On $\overline{U}$ then one has

$$\psi(0, y^i) = t$$

which represents a 2-surface σ_t.

On $\overline{U}$ one has a foliation by the 2-surfaces σ_t.

Let $\underline{k}$ be the unit normal vector to σ_t in U,

$$k_\mu = \frac{h_{\mu\nu} N^\nu}{|h_{\alpha\beta} N^\alpha N^\beta|^{\frac{1}{2}}}, \qquad k_\mu k^\mu = -1 ,$$

and $\underline{e}_A$ be two orthogonal unit vectors tangent to σ_t in $\overline{U}$. Then $\{ \underline{k}, \underline{e}_A \}$ is a basis field in $\overline{U}$. Explicitly one has

(22) $$k^\mu = \frac{1}{\Gamma_\Sigma} (N^\mu - \Gamma_\Sigma V_\Sigma n^\mu) .$$

In the sequel, instead of using the anholonomic frame $\{\underline{k}, \underline{e}_A\}$, one could as well use the holonomic basis $\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial z^A}$ where z^A are two coordinates on σ_t.

Let one of the hypersurfaces $\mathscr{F}_t$, say $\mathscr{F}_0$, be considered as *the initial hypersurface*, i. e. the data for $T^{\mu\nu}$ and j^μ are given on $\mathscr{F}_0$.

On $\mathscr{F}_0$ one can define an orthonormal basis field as follows.

In U $\underline{n}$ is a vector field (in fact dy^0 is a 1-form in U) and therefore $\underline{n}$ is defined on $\mathscr{F}_0$. One can then construct the basis field on $\mathscr{F}_0$,

$$\{\,\underline{b}, \underline{\varepsilon}_A\,\}$$

where

$$b^\mu = \frac{H^\mu{}_\nu n^\nu}{|\,H_{\alpha\beta} n^\alpha n^\beta\,|^{\frac{1}{2}}}$$

and $\underline{\varepsilon}_A$ are two orthogonal unit vectors such that

$$\underline{\varepsilon}_A \cdot \underline{b} = 0 \qquad \text{and} \qquad \underline{\varepsilon}_A = \underline{e}_A \qquad \text{on} \quad \sigma_0\,.$$

Explicitly

$$\text{(23)} \qquad\qquad b^\mu = \frac{1}{\Gamma_\Sigma}(n^\mu + v_\Sigma \Gamma_\Sigma N^\mu)\,.$$

Notice that b^μ, as a vector field, is defined on U.

Now let f be any quantity which is regularly discontinuous across Σ. Then f has a jump $[f]_0$ across σ_0, which can be considered as an initial datum for $[f]$. Also, if f_0 is the initial datum for f on $\mathscr{F}_0$, one has, across σ_0

$$[f]_0 = \lfloor f_0 \rfloor\,.$$

Moreover $[\nabla_b f]_0$ can be computed from f_0, being equal to $[\nabla_b f_0]$.

Therefore it is convenient to express $[\nabla_n f]_0$ (which appears in the transport equations of the previous section) in terms of $[\nabla_b f]_0$ (which can be computed from the initial datum).

This can be done by expressing $\underline{n}$ in terms of $\underline{b}$ and $\underline{k}$, as follows

$$\text{(24)} \qquad\qquad n^\mu = \frac{1}{\Gamma_\Sigma}(b^\mu - \Gamma_\Sigma V_\Sigma k^\mu)\,.$$

Then one obtains

$$[\nabla_n f] = \frac{1}{\Gamma_\Sigma}[\nabla_b f] - V_\Sigma[\nabla_k f]\,,$$

hence

$$\text{(25)} \qquad\qquad [\nabla_n f]_0 = \frac{1}{\Gamma_\Sigma}[\nabla_b f_0] - V_\Sigma \nabla_k[f_0]\,.$$

After some manipulations eqs. (17)-(20) rewrite, on σ_0,

$$(26) \quad k^\mu \bar{\bar{\nabla}}_\mu I^1 + e_A{}^\mu \bar{\bar{\nabla}} I^A + I^1(\bar{\bar{\nabla}}_\mu k^\mu + V_\Sigma \chi_{\alpha\beta} k^\alpha k^\beta)$$
$$+ I^A(\bar{\bar{\nabla}}_\mu e_A{}^\mu + V_\Sigma \chi_{\alpha\beta} k^\alpha e_A{}^\beta) + \frac{1}{\Gamma_\Sigma} n_\mu [b^\alpha \nabla_\alpha j^\mu] = 0$$

$$(27) \quad \theta^{11} \chi_{\alpha\beta} k^\alpha k^\beta + 2\theta^{1A} \chi_{\alpha\beta} k^\alpha e_A{}^\beta + \theta^{AB} \chi_{\alpha\beta} e_A{}^\alpha e_B{}^\beta - \frac{1}{\Gamma_\Sigma} n_\mu n_\nu [b^\alpha \nabla_\alpha T^{\mu\nu}] = 0$$

$$(28) \quad k^\mu \bar{\bar{\nabla}}_\mu \theta^{11} + \theta^{11}(\bar{\bar{\nabla}}_\mu k^\mu + V_\Sigma \chi_{\alpha\beta} k^\alpha k^\beta) + e_A{}^\mu \bar{\bar{\nabla}}_\mu \theta^{1A} - \theta^{1A}(k^\mu k_\tau \bar{\bar{\nabla}}_\mu e_A{}^\tau$$
$$+ \bar{\bar{\nabla}}_\mu e_A{}^\mu + V_\Sigma \chi_{\alpha\beta} k^\alpha e_A{}^\beta) - \theta^{AB} e_A{}^\mu k_\tau \bar{\bar{\nabla}}_\mu e_B{}^\tau - \frac{1}{\Gamma_\Sigma} k_\nu n_\mu [b^\alpha \nabla_\alpha T^{\mu\nu}] = 0$$

$$(29) \quad k^\mu \bar{\bar{\nabla}}_\mu \theta^{1C} + \theta^{11} k^\mu e_{C^\tau} \bar{\bar{\nabla}}_\mu k^\tau$$
$$+ \theta^{1A}(\delta_{AC} \bar{\bar{\nabla}}_\mu k^\mu + k^\mu e_{C^\tau} \bar{\bar{\nabla}}_\mu e_A{}^\tau + e_A{}^\mu e_{C^\tau} \bar{\bar{\nabla}}_\mu k^\tau + \delta_{AC} V_\Sigma \chi_{\alpha\beta} k^\alpha k^\beta)$$
$$+ e_A{}^\mu \bar{\bar{\nabla}}_\mu \theta^{AC} + \theta^{AB}(\delta_{BC} \bar{\bar{\nabla}}_\mu e_A{}^\mu + e_A{}^\mu e_{C^\tau} \bar{\bar{\nabla}}_\mu e_B{}^\tau + \delta_{BC} V_\Sigma \chi_{\alpha\beta} k^\alpha e_A{}^\beta)$$
$$+ \frac{1}{\Gamma_\Sigma} e_{C^\nu} n_\mu [b^\alpha \nabla_\alpha T^{\mu\nu}] = 0.$$

Obviously these equations hold also on any σ_t considered as an initial surface.

It is convenient to view σ_0 as a 2-surface of the pseudo-riemannian manifold $\mathscr{F}_0$ and introduce the induced metric on σ_0,

$$(30) \qquad\qquad S^\mu{}_\nu = h^\mu{}_\nu + k^\mu k_\nu = H^\mu{}_\nu - b^\mu b_\nu$$

and its second fundamental form

$$(31) \qquad\qquad A_{\mu\nu} = S_\mu{}^\alpha S_\nu{}^\beta \nabla_\alpha b_\beta.$$

Let $B_{\mu\nu}$ be the second fundamental form of $\mathscr{F}_0$,

$$(32) \qquad\qquad B_{\mu\nu} = H_\mu{}^\alpha H_\nu{}^\beta \nabla_\alpha N_\beta,$$

then it is easy to prove that

$$(33) \qquad\qquad \bar{\bar{\nabla}}_\mu k^\mu = -\Gamma_\Sigma V_\Sigma A^\mu{}_\mu + \Gamma_\Sigma B^\mu{}_\mu - \Gamma_\Sigma b^\alpha b_\mu \nabla_\alpha N^\mu.$$

In fact

$$\bar{\bar{\nabla}}_\mu k^\mu = h_\mu{}^\alpha \nabla_\alpha k^\mu = h_\mu{}^\alpha \{ N^\mu \nabla_\alpha \Gamma_\Sigma + \Gamma_\Sigma \nabla_\alpha N^\mu - b^\mu \nabla_\alpha(\Gamma_\Sigma V_\Sigma) - \Gamma_\Sigma V_\Sigma \nabla_\alpha b^\mu \}.$$

Also

$$h^\alpha{}_\mu N^\mu = \Gamma_\Sigma k^\alpha, \qquad h^\alpha{}_\mu b^\mu = \Gamma_\Sigma V_\Sigma k^\alpha$$
$$h^\alpha{}_\mu \nabla_\alpha b^\mu = A^\mu{}_\mu + \Gamma_\Sigma b_\mu k^\alpha \nabla_\alpha N^\mu$$
$$h^\alpha{}_\mu \nabla_\alpha N^\mu = B^\mu{}_\mu - b^\alpha b_\mu \nabla_\alpha N^\mu + \Gamma_\Sigma V_\Sigma b_\mu k^\alpha \nabla_\alpha N^\mu$$

whence eq. (33) follows.

With some simple manipulations it is easy to prove the following formulae:

$$(34) \qquad \chi_{\alpha\beta}k^\alpha k^\beta = \Gamma_\Sigma^2 k^\mu \nabla_\mu V_\Sigma + \frac{1}{V_\Sigma \Gamma_\Sigma} k^\nu k^\mu \nabla_\mu N_\nu$$

$$(35) \qquad \chi_{\alpha\beta}k^\alpha e_A{}^\beta = \frac{1}{\Gamma_\Sigma V_\Sigma} e_{A\nu} k^\mu \nabla_\mu N^\nu - \frac{1}{V_\Sigma} e_{A\nu} k^\mu \nabla_\mu k^\nu$$

$$(36) \qquad \overline{\overline{\nabla}}_\mu e_A{}^\mu = \nabla_\mu e_A{}^\mu - k_\mu k^\alpha \nabla_\alpha e_A{}^\mu$$

$$(37) \qquad e_A{}^\mu e_{C\tau} \overline{\overline{\nabla}}_\mu e_B{}^\tau = e_{C\tau} e_A{}^\mu \check{\nabla}_\mu e_B{}^\tau ,$$

where $\check{\nabla}$ denotes the induced riemannian connection on σ_0,

$$(38) \qquad \check{\nabla}_\mu z^\beta = S_\mu{}^\alpha S^\beta{}_\nu \nabla_\alpha z^\nu ,$$

for any vector $\underline{z}$ tangent to σ_0,

$$(39) \qquad \chi_{\alpha\beta}e_A{}^\alpha e_B{}^\beta = \frac{1}{\Gamma_\Sigma V_\Sigma} e_A{}^\alpha e_B{}^\beta \nabla_\alpha N_\beta - \frac{1}{V_\Sigma} e_A{}^\alpha e_B{}^\beta \overline{\overline{\nabla}}_\alpha k_\beta .$$

Therefore eqs. (26-29) can be rewritten in the following form

$$(40) \qquad DI^1 + e_A{}^\mu \check{\nabla}_\mu I^A + EI^1 + I^A F_A + \frac{1}{\Gamma_\Sigma} n_\mu [b^\alpha \nabla_\alpha j^\mu] = 0$$

$$(41) \quad \theta^{11}\left(\Gamma^2 DV_\Sigma + \frac{1}{V_\Sigma \Gamma_\Sigma} k^\nu DN_\nu \right) + 2\theta^{1A}\left(\frac{1}{\Gamma_\Sigma V_\Sigma} e_{A\nu} DN^\nu - \frac{1}{V_\Sigma} e_{A\nu} Dk^\nu \right)$$
$$+ \theta^{AB}\left(\frac{1}{\Gamma_\Sigma V_\Sigma} e_A{}^\alpha e_B{}^\beta \nabla_\alpha N_\beta - \frac{1}{V_\Sigma} e_A{}^\alpha e_B{}^\beta \overline{\overline{\nabla}}_\alpha k_\beta \right) - \frac{1}{\Gamma_\Sigma} n_\mu n_\nu [b^\alpha \nabla_\alpha T^{\mu\nu}] = 0$$

$$(42) \quad D\theta^{11} + \theta^{11}E + e_A{}^\mu \check{\nabla}_\mu \theta^{1A} - \theta^{1A}\left(F_A - e_{A\mu} Dk^\mu \right)$$
$$+ \theta^{AB} e_A{}^\mu e_B{}^\tau \nabla_\mu k_\tau - \frac{1}{\Gamma_\Sigma} k_\nu n_\mu [b^\alpha \nabla_\alpha T^{\mu\nu}] = 0$$

$$(43) \quad D\theta^{1C} + \theta^{11} e_{C\tau} \overline{\overline{D}} k^\tau + \theta^{1C} E + \theta^{1A}\left(e_{C\tau} e_A{}^\mu \overline{\overline{\nabla}}_\mu k^\tau + e_{C\tau} De_A{}^\tau \right)$$
$$+ e_A{}^\mu \check{\nabla}_\mu \theta^{AC} + \theta^{AC} F_A + \theta^{AB} e_A{}^\mu e_{C\tau} \nabla_\mu e_B{}^\tau + \frac{1}{\Gamma_\Sigma} e_{C\nu} n_\mu [b^\alpha \nabla_\alpha T^{\mu\nu}] = 0$$

where $D \equiv k^\mu \nabla_\mu$ is the derivative along $\underline{k}$, $\overline{\overline{D}} \equiv k^\mu \overline{\overline{\nabla}}_\mu$ (on scalars D and $\overline{\overline{D}}$ coincide),

$$(44) \quad E = -\Gamma_\Sigma V_\Sigma A^\mu{}_\mu + \Gamma_\Sigma B^\mu{}_\mu - \Gamma_\Sigma b^\alpha b_\mu \nabla_\alpha N^\mu + V_\Sigma \Gamma_\Sigma^2 DV_\Sigma + \frac{1}{\Gamma_\Sigma} k^\nu DN_\nu$$

$$(45) \qquad F_A = \check{\nabla}_\mu e_A{}^\mu + \frac{1}{\Gamma_\Sigma} e_{A\mu} DN^\mu .$$

Obviously these equations too hold on any σ_t considered as an initial surface.

5. THE RELATIVISTIC FLUID

In this section the equations derived in the previous section are specialized to the important case of the relativistic perfect fluid. The energy-momentum tensor is [21]

$$(46) \qquad T^{\mu\nu} = wu^\mu u^\nu + pg^{\mu\nu}$$

where u^μ is the fluid 4-velocity, w and p are the enthalpy and pressure respectively, both measured in the local rest frame.

The mass-density current is

$$(47) \qquad j^\mu = \rho u^\mu$$

where ρ is the proper rest-mass density.

It is convenient to introduce the index f of the fluid, defined by [18]

$$f = \frac{w}{\rho}.$$

The Rankine-Hugoniot relations express the invariance across Σ of the scalar [18]

$$(48) \qquad m = \rho u^\mu n_\mu$$

and of the vector

$$(49) \qquad Z^\mu = mf u^\mu + pn^\mu.$$

Let $\tau = \dfrac{f}{\rho}$ be the dynamical volume [20]. Then it is easy to show that [20]

$$(50) \quad m^2 [T^{\mu\nu}] = \left[\frac{1}{\tau}\right] Z^\mu Z^\nu - \left[\frac{p}{\tau}\right](Z^\mu n^\nu + n^\mu Z^\nu) + \left[\frac{p^2}{\tau}\right] n^\mu n^\nu + m^2 [p]g^{\mu\nu}$$

$$(51) \qquad m[j^\mu] = \left[\frac{1}{\tau}\right] Z^\mu - \left[\frac{p}{\tau}\right] n^\mu.$$

For the sake of simplicity let the fluid ahead of the shock front be at rest with respect to the family of observers defined by $\underline{N}$. Then one has, on Σ,

$$u_+^\mu = N^\mu$$

and u^μ coincides with N^μ on $U^+ = \Omega^+ \cap \Sigma$.

It follows easily that

$$Z^\mu = mf_+ N^\mu + p_+ n^\mu$$

whence

$$(52) \qquad I^1 = -\Gamma_\Sigma f_+ \left[\frac{1}{\tau}\right], \qquad I^A = 0$$

$$(53) \qquad \theta^{11} = \left[\frac{1}{\tau}\right] f_+^2 \Gamma_\Sigma^2 - [p], \qquad \theta^{AB} = [p]\delta^{AB}, \qquad \theta^{1A} = 0.$$

Therefore eqs. (40-43) can be rewritten as follows

$$(54) \qquad D\left(\Gamma_\Sigma f_+\left[\frac{1}{\tau}\right]\right) + E\Gamma_\Sigma f_+\left[\frac{1}{\tau}\right] - \frac{1}{\Gamma_\Sigma} n_\mu[b^\alpha\nabla_\alpha j^\mu] = 0$$

$$(55) \qquad \left(\left[\frac{1}{\tau}\right]f_+^2\Gamma_\Sigma^2 - [p]\right)\left(\Gamma_\Sigma^2 DV_\Sigma + \frac{1}{V_\Sigma\Gamma_\Sigma}k^\nu DN_\nu\right)$$
$$+ [p]\left(\frac{1}{\Gamma_\Sigma V_\Sigma}S^{\alpha\beta}\nabla_\alpha N_\beta - \frac{1}{V_\Sigma}\bar{\bar{\nabla}}_\mu k^\mu\right) - \frac{1}{\Gamma_\Sigma}n_\mu n_\nu[b^\alpha\nabla_\alpha T^{\mu\nu}] = 0$$

$$(56) \qquad D\left(\left[\frac{1}{\tau}\right]f_+^2\Gamma_\Sigma^2\right) - D[p] + E\left[\frac{1}{\tau}\right]f_+^2\Gamma_\Sigma^2$$
$$- [p]\left(V_\Sigma\Gamma_\Sigma^2 DV_\Sigma + \frac{1}{\Gamma_\Sigma}k^\nu DN_\nu\right) - \frac{1}{\Gamma_\Sigma}k_\nu n_\mu[b^\alpha\nabla_\alpha T^{\mu\nu}] = 0$$

$$(57) \qquad \left(\left[\frac{1}{\tau}\right]f_+^2\Gamma_\Sigma^2 - [p]\right)e_{C\tau}\bar{\bar{D}}k^\tau + e_C{}^\mu\check{\nabla}_\mu[p]$$
$$+ \frac{1}{\Gamma_\Sigma}[p]e_{C\mu}DN^\mu + \frac{1}{\Gamma_\Sigma}e_{C\nu}n_\mu[b^\alpha\nabla_\alpha T^{\mu\nu}] = 0$$

Also, it is easy to see that

$$(58) \qquad n_\mu[b^\alpha\nabla_\alpha j^\mu] = \Gamma_\Sigma V_\Sigma(b^\alpha\nabla_\alpha\rho)_+ - \frac{\rho_+}{\rho_-}\Gamma_\Sigma V_\Sigma(b^\alpha\nabla_\alpha\rho)_-$$
$$+ \rho_+ n_\mu b^\alpha\nabla_\alpha N^\mu - \rho_- n_\mu(b^\alpha\nabla_\alpha u^\mu)_-$$

$$(59) \qquad n_\mu n_\nu[b^\alpha\nabla_\alpha T^{\mu\nu}] = \Gamma_\Sigma^2 V_\Sigma^2(b^\alpha\nabla_\alpha w)_+ - \left(\frac{\rho_+}{\rho_-}\Gamma_\Sigma V_\Sigma\right)^2(b^\alpha\nabla_\alpha w)_-$$
$$+ w_+\Gamma_\Sigma V_\Sigma n_\mu b^\alpha\nabla_\alpha N^\mu - 2w_-\frac{\rho_+}{\rho_-}\Gamma_\Sigma V_\Sigma n_\mu(b^\alpha\nabla_\alpha u^\mu)_-$$
$$+ w_+\Gamma_\Sigma V_\Sigma n_\nu b^\alpha\nabla_\alpha N^\nu + [b^\alpha\nabla_\alpha p]$$

$$(60) \qquad k_\nu n_\mu[b^\alpha\nabla_\alpha T^{\mu\nu}] = -\Gamma_\Sigma^2 V_\Sigma(b^\alpha\nabla_\alpha w)_+ + \frac{\rho_+}{\rho_-}\Gamma_\Sigma^2 V_\Sigma\frac{f_+}{f_-}(b^\alpha\nabla_\alpha w)$$
$$- w_+\Gamma_\Sigma n_\mu b^\alpha\nabla_\alpha N^\mu + \frac{w_-\Gamma_\Sigma f_+}{f_-}n_\mu(b^\alpha\nabla_\alpha u^\mu)_- + w_+\Gamma_\Sigma V_\Sigma k_\nu b^\alpha\nabla_\alpha N^\nu$$
$$- \frac{w_-\rho_+}{\rho_-}\Gamma_\Sigma V_\Sigma k_\nu(b^\alpha\nabla_\alpha u^\nu)_-$$

$$(61) \qquad e_{C\nu}n_\mu[b^\alpha\nabla_\alpha T^{\mu\nu}] = w_+\Gamma_\Sigma V_\Sigma e_{C\nu}b^\alpha\nabla_\alpha N^\nu - \frac{w_-\rho_+}{\rho_-}\Gamma_\Sigma V_\Sigma e_{C\nu}(b^\alpha\nabla_\alpha u^\nu)_-$$

A better understanding of these equations is obtained in the case of special relativity.

Let $\mathcal{M}$ be the Minkowski space-time, (t, x^a) global inertial cartesian coordinates, where
$$g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$

The hypersurfaces $\mathscr{F}_t$ can be taken to be

$$t = \text{cost}.$$

Then

$$N^\mu = (1, 0, 0, 0)$$

$$n_\mu = \Gamma_\Sigma(V_\Sigma, -\underline{v})$$

where $\underline{v}$ is a unit 3-vector,

$$k^\mu = \Gamma_\Sigma(1, V_\Sigma\underline{v})$$

$$b^\mu = (0, -\underline{v})$$

$$e_A{}^\mu = (0, \underline{i}_A)$$

where $\underline{i}_A$ are two unit 3-vectors orthogonal to $\underline{v}$,

$$\underline{i}_A \cdot \underline{v} = 0.$$

The fluid 4-velocity is in the inertial frame,

$$u^\mu = \Gamma(1, \underline{v})$$

where $\underline{v}$ is the velocity 3-vector and Γ is the Lorentz factor.

From the invariance of $Z^\mu = mfu^\mu + pn^\mu$, it follows that

$$(\underline{V})_- = V_-\underline{v}.$$

Let $\dfrac{\delta}{\delta s} = \dfrac{1}{\Gamma_\Sigma}\overline{\overline{D}}\left(\text{which on scalars coincides with } \dfrac{1}{\Gamma_\Sigma}D\right)$, then eqs. (54-57)

yield, with $\partial_{\underline{v}} = v^a\partial_a$,

(62) $\quad \dfrac{\delta}{\delta s}\left(f_+\left[\dfrac{1}{\tau}\right]\right) + 2\Gamma_\Sigma^2 V_\Sigma f_+\left[\dfrac{1}{\tau}\right]\dfrac{\delta V_\Sigma}{\delta s} - V_\Sigma f_+\left[\dfrac{1}{\tau}\right]A^\mu{}_\mu + \dfrac{V_\Sigma}{\Gamma_\Sigma^2}(\partial_{\underline{v}}\rho)_+$

$$-\dfrac{\rho_+}{\rho_-}\dfrac{V_\Sigma}{\Gamma_\Sigma^2}(\partial_{\underline{v}}\rho)_- - \dfrac{\rho_-\Gamma_\Sigma^3}{\Gamma_\Sigma^2}(V_- V_\Sigma - 1)\underline{v}\cdot(\partial_{\underline{v}}\underline{V})_- = 0$$

(63) $\quad \left(\left[\dfrac{1}{\tau}\right]f_+^2\Gamma_\Sigma^2 - [p]\right)\Gamma_\Sigma^3\dfrac{\delta V_\Sigma}{\delta s} + [p]\Gamma_\Sigma A^\mu{}_\mu + \Gamma_\Sigma V_\Sigma^2(\partial_{\underline{v}}w)_+$

$$-\dfrac{\rho_+^2}{\rho_-^2}\Gamma_\Sigma V_\Sigma(\partial_{\underline{v}}w)_- + \dfrac{1}{\Gamma_\Sigma}[\partial_{\underline{v}}p] - 2w_-\dfrac{\rho_+}{\rho_-}\Gamma_\Sigma V_\Sigma\Gamma^3(V_\Sigma V_- - 1)\underline{v}\cdot(\partial_{\underline{v}}\underline{V}) = 0$$

(64) $\quad \dfrac{\delta}{\delta s}\left(\left[\dfrac{1}{\tau}\right]f_+^2\right) + \left(3\Gamma_\Sigma^2 V_\Sigma f_+^2\left[\dfrac{1}{\tau}\right] - V_\Sigma[p]\right)\dfrac{\delta V_\Sigma}{\delta s} - \dfrac{V_\Sigma}{\Gamma_\Sigma^2}(\partial_{\underline{v}}w)_+$

$$+\dfrac{V_\Sigma}{\Gamma_\Sigma^2}\dfrac{w_+}{w_-}(\partial_{\underline{v}}w)_- + \dfrac{w_-\Gamma^3}{\Gamma_\Sigma^2}\left\{\dfrac{f_+}{f_-}(V_\Sigma V_- - 1) + \dfrac{\rho_+}{\rho_-}V_\Sigma(V_- - V_\Sigma)\right\}\underline{v}\cdot(\partial_{\underline{v}}\underline{V})_- = 0$$

(65) $\quad \left(\left[\dfrac{1}{\tau}\right]f_+^2\Gamma_\Sigma^2 - [p]\right)e_{C^\tau}\dfrac{\delta k^\tau}{\delta s} + \dfrac{1}{\Gamma_\Sigma}e_C{}^\mu\check{V}'_\mu[p] + \dfrac{w_-\rho_+}{\rho_-}\dfrac{\Gamma_-}{\Gamma_\Sigma}V_\Sigma\underline{i}_C\cdot(\partial_{\underline{v}}\underline{V})_- = 0.$

These equations look rather formidable. However some qualitative information can be obtained from them, as will be seen in the next section.

6. QUALITATIVE RESULTS AND CONCLUSION

From eq. (65) one can obtain at once the following important result. Let $[p]$ be uniform all over the shock wavefront σ_t,

$$e_C{}^\mu \nabla_\mu [p] = 0$$

and $(\partial_{\underline{v}} \underline{V})_-$ be normal to the wavefront,

$$\underline{i}_C \cdot (\partial_{\underline{v}} \underline{V})_- = 0,$$

then it follows

$$e_{C\tau} \frac{\delta k^\tau}{\delta s} = 0.$$

Now, since $\dfrac{\delta k^\tau}{\delta s}$ lies in the two-plane spanned by $e_{2\tau}$, $e_{3\tau}$, it follows

(66)
$$\frac{\delta k^\tau}{\delta s} = 0$$

which states that in this case k^μ is a geodesic vector field on Σ with respect to the induced connection, and s is an affine parameter on these geodesics.

One might enquire under which conditions k^μ is a geodesic vector field in the full space-time $\mathcal{M}$. The answer is that, beyond (66), k^μ must satisfy

$$\chi_{\alpha\beta} k^\alpha k^\beta = 0$$

which, from eq. (34), implies

$$\frac{\delta V_\Sigma}{\delta s} = 0,$$

which means a constant amplitude shock.

In the general case one obtains from the condition $e_C{}^\tau \dfrac{\delta k^\tau}{\delta s} = 0$, being $k^\mu = \Gamma_\Sigma(1, V_\Sigma \underline{v})$,

(67)
$$\frac{\delta v}{\delta s} = 0$$

which states that the shockfront 2-surfaces σ_t are parallel surfaces.

This result is analogous to a well known theorem on wave propagation in classical continuum mechanics [7].

Another important result can be obtained from eqs. (63-64). Consider the case of a radiation fluid, for which the equation of state is

$$w = 4p.$$

A simple analysis of the Rankine-Hugoniot relations (48), (49), shows

that, in the limit of an extremely relativistic shock, $V_\Sigma \to 1$, one has the asymptotic relations, to the order Γ_Σ^{-1},

$$p_- = \frac{2}{3}\Gamma_\Sigma^2 w_+ \quad , \qquad \rho_- = 2\sqrt{2}\Gamma_\Sigma\rho_+ \, ,$$

$$f_- = \frac{4}{3\sqrt{2}}\Gamma_\Sigma f_+ \, , \qquad \Gamma_-^2 = \frac{1}{2}\Gamma^2 \, .$$

It follows that $\left[\dfrac{1}{\tau}\right]$ remains finite in the limit $V_\Sigma \to 1$, and in particular one has

$$\left[\frac{1}{\tau}\right] = -\frac{\rho_+}{f_+}\left(2 + \frac{9}{4}\frac{\sqrt{2}}{\Gamma_\Sigma}\right).$$

Therefore, asymptotically, one has,

$$\frac{\delta}{\delta s}\left[\frac{1}{\tau}\right] = -2\frac{\delta}{\delta s}\left(\frac{\rho_+}{f_+}\right) + \frac{9\sqrt{2}}{4}\frac{\rho_+}{f_+}\Gamma_\Sigma\frac{\delta V_\Sigma}{\delta s}$$

to the order Γ_Σ^{-1}.

Hence from eq. (64) it follows, asymptotically,

(68)
$$\frac{16}{3}\rho_+ f_+ \frac{\delta V_\Sigma}{\delta s} = \frac{3}{8\Gamma_\Sigma^6}(\partial_\nu w)_- - \frac{5}{3}\frac{\rho_+ f_+}{\Gamma_\Sigma^2}\underline{\nu}\cdot(\partial_\nu\underline{V})_- \, .$$

Similarly, from eq. (63) one obtains, asymptotically,

(69)
$$2\rho_+ f_+ \frac{\delta V_\Sigma}{\delta s} = \frac{1}{8\Gamma_\Sigma^6}(\partial_\nu w)_- + \frac{\rho_+ f_+}{\Gamma_\Sigma^2}\underline{\nu}\cdot(\partial_\nu\underline{V})_- \, .$$

Now one can assume that $\underline{\nu}\cdot(\partial_\nu\underline{V})_-$ remains finite in the limit $V_\Sigma \to 1$. Then, from eqs. (68)-(69) it is easy to show that, in the limit $V_\Sigma \to 1$,

(70
$$\frac{\delta V_\Sigma}{\delta s} = 0(\Gamma_\Sigma^{-2}).$$

This result which is at striking variance with the behaviour of shock waves in classical fluiddynamics had already been conjectured by Liang and Baker [23]. The first rigorous proof was given by Anile, Miller and Motta [14] in the case of a plane shock wave propagating into a constant state in a relativistic barotropic fluid and confirmed in more general cases by numerical calculations [13].

In this section the asymptotic behaviour (70) has been proved for a shock wave of arbitrary geometry propagating into an arbitrary state in a radiation fluid. By using a similar method it should be possible to prove the asymptotic behaviour (70) without any restriction on the equation of state.

A. M. ANILE

In this section the first order compatibility relations (62-65) have been investigated and some qualitative results have been obtained. Further work could be done on the following two lines. The first is to envisage an approximation method for weak shocks which would close the system (62-65) thereby obtaining a system of propagation equations. The second is to postulate special relationships among $(\partial_v \rho)_-$, $\underline{v} \cdot (\partial_v \underline{V})_-$ and $(\partial_v p)_-$, drawn from the requirements of self-similarity, which would also permit to obtain proper propagation equations for the shock amplitude.

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