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Φ^4 field theory in dimension 4 : a modern introduction to its unsolved problems

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ABSTRACT. — We introduce the notion of renormalization scheme for regularized φ_4^4 theories in order to investigate the continuum limit. We point out simple phenomena which suggest the interest of generalized cut-off actions containing « antiferromagnetic » terms for the construction of non trivial (non-gaussian) limits. Parts of this paper are expository and may serve as an introduction to φ_4^4.

RÉSUMÉ. — On introduit la notion de schéma de renormalisation pour des théories ϕ_4^4 régularisées, afin d'étudier la limite continue. On signale des phénomènes simples qui suggèrent l'intérêt d'actions généralisées avec cut-off contenant des termes antiferromagnétiques, pour la construction de limites non triviales (non gaussiennes). Certaines parties de l'article sont didactiques et peuvent servir d'introduction à la théorie ϕ_4^4.

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1. INTRODUCTION

Old arguments by Landau and al. [*1*] [*2*], followed by renormalization group [*3*] and numerical analysis [*4*], suggest that the ultra-violet limit of a regularized φ_4^4 or QED_4 field theory should be « trivial », i. e. a gaussian field or a generalized free field, or more vaguely just a theory whose dependence on the renormalized couping constant has nothing to do with the one foreseen by supposing that the renormalized perturbation series is asymptotic to its Schwinger functions.

Although such arguments have generated a strongly held common belief among physicists on the triviality of φ_4^4 and QED_4, there are still a few scientists who feel that there should be something behind perturbative renormalization theory and advocate the importance of further study on this issue.

Recently considerable progress has been made towards the clarification of the nature of the problem. In the framework of lattice regularized φ_4^4-theories with a nearest neighbor realization of the laplacian it has been shown [*5*] [*6*], that a large class of interesting choices of the bare constants leads to continuum limits which are trivial theories. Such results are more complete in dimension of space-time $d \geq 5$ but even in $d = 4$ they cover a rather wide class of bare coupling constants. Most of the specialists are convinced that, sooner or later, the $d = 4$ case will be settled for the triviality in the same generality in which the $d \geq 5$ case has been already settled, and we shall not discuss the possibility that they might be wrong. In both cases, however, the issue of existence of non-trivial continuum limits has not been completely settled, to the best of our understanding, by the above major contributions, in spite of the flood of light they throw on problems hitherto reputed out of range of today's mathematically rigorous techniques. It is the aim of the present paper to explain what are in our opinion some remaining important open questions and to hint some possible ways to attack them. We shall first explore a naive approach based on perturbative renormalization theory and built with the unpopular hope of obtaining a non-trivial continuum limit asymptotic to the renormalized perturbative series. We shall see that this approach leads naturally to consider a class of cut-off theories wider than the ones studied in [*5*] [*6*]. In particular we shall produce examples of cut-off φ_4^4 theories which are « antiferromagnetic » (i. e. with negative wave function coupling constant) but which have a perturbative expansion for the Schwinger functions which upon removal of the cut-off converges term by term to the usual renormalized series. In our opinion this remark (see [*7*]) compels one to consider, in the triviality proofs, the antiferromagnetic case on equal footing with the ferromagnetic one. The consideration of antiferromagnetic couplings leads us also quite naturally to still more general regularized theories in which the realization of the laplacian is not necessarily between nearest neighbors and contains therefore several length scales. By displaying the greater richness and flexibility of these theories we shall finally argue that they may be better suited than the standard ones for the analysis of the ultraviolet limit and the possibility of non triviality.

We shall also point out that the sign of the renormalized coupling constant is not fixed by renormalization theory: this will be done by providing examples of cut-off theories (having positive bare coupling constant) giving rise to renormalized coupling constant of arbitrary sign. This might be interesting since the theories with negative renormalized coupling constant are asymptotically free (see [8] for a discussion of these theories).

In this paper we limit ourselves to two regularizations: the « Pauli-Villars » because of its simplicity, and the lattice regularization, because of its conceptual beauty, and mainly to make use of the results in [5] [6] and show how they are compatible with our ideas. Sect. 2 is devoted to some notations and is an introduction to φ_4^4 theories with « no prerequisites »; it also contains a discussion of the notion of triviality. We introduce what we call « renormalization schemes » and formulated some of our simple results, which are proved in an Appendix. In Sect. 3, 4 we provide some studies of the simplest « renormalization schemes », which unfortunately do not lead to interesting theories. In Sect. 5, 6, 7 we discuss further what the idea of antiferromagnetic field theories suggests towards the possible construction of a non trivial φ_4^4 theory: in particular we show that the presence of antiferromagnetic couplings and the « spreading » of the laplacian on many sites can dramatically alter the features of the theory without preventing it from having eventually a non trivial ultraviolet limit. Finally in Sect. 8 we discuss briefly, and mainly for completeness, some other non-standard approaches to the construction of φ_4^4 and the corresponding triviality conjectures.

2. Φ_4^4 FROM A CONSTRUCTIVE POINT OF VIEW

We discuss the case $d = 4$ only (though many of our comments apply also to $d \geq 5$ when, however, the theory is not renormalizable and Prop. 1 below is false). We shall also confine our attention to finite volume theories choosing for simplicity this volume to be a torus $\mathbf{T}^4 = [0, 2\pi]^4$.

Let $\mathbf{Z}^4$ denote the lattice of the points $\mathbf{n} \in \mathbf{R}^4$ with integer coordinates and let $a\mathbf{Z}^4 = \{ \mathbf{x} \mid \mathbf{x} = a\mathbf{n}, \mathbf{n} \in \mathbf{Z}^4 \}$. We define the free field as the gaussian measure P_{free} on $S'(\mathrm{T}^4)$ (the distributions on $\mathbf{T}^4$) whose covariance (free propagator) is:

$$C(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{n} \in \mathbf{Z}^4} \mathrm{C}(\mathbf{x}, \mathbf{y} + 2\pi\mathbf{n}) \tag{2.1}$$

where

$$\mathbf{C}(\mathbf{x}, \mathbf{y}) = \int \frac{d^4\mathbf{p}}{(2\pi)^4} \exp \frac{i.\mathbf{p}(\mathbf{x}-\mathbf{y})}{1 + \mathbf{p}^2} \tag{2.2}$$

The cut-off fields with cut-off of length $a = 2^{-N}2\pi$, $N = 0, 1, \ldots$, will be of two types denoted P_N^α with $\alpha = PV$ or $\alpha = l$ (for « Pauli-Villars » or « lattice »). The Pauli-Villars free field regularized at length $2\pi 2^{-N}$ will be the gaussian measure on $S'(\mathbf{T}^4)$ whose covariance is given by (2.1) with $\mathbf{C}$ replaced by:

$$\mathbf{C}_N^{PV}(\mathbf{x}, \mathbf{y}) = \int \frac{d^4\mathbf{p}}{(2\pi)^4} \exp i.\mathbf{p}(\mathbf{x}-\mathbf{y})\left[\frac{1}{1 + p^2} - \frac{1}{2^{2N} + p^2}\right] \tag{2.3}$$

The l-free field regularized at length $a = 2^{-N}2\pi$ will also be a gaussian probability measure on $S'(\mathbf{T}^4)$: it will give probability 1 to simple distributions φ, described by functions which are constant on the cubes of the pavement of $\mathbf{T}^4$ consisting of the cubes with centers at the points $\mathbf{x} \in a\mathbf{Z}^4$ and side size a. Of course the field φ is completely described by the values of φ at the centers of these cubes, hence we describe it completely by giving the covariance function $C(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in a\mathbf{Z}^4$ only. It will be given again by (2.1) with $\mathbf{C}$ replaced by:

$$\mathbf{C}_N^l(\mathbf{x}, \mathbf{y}) = \int_{-2^N}^{2^N} \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{i.\mathbf{p}(\mathbf{x}-\mathbf{y})}{1 + 2\sum_{j=1}^{4}(1 - \cos ap_j)/a^2} \tag{2.4}$$

In general if P^α is a gaussian measure on some space of distributions we shall denote: $:_\alpha$ the Wick ordering operator acting on the polynomials of the field.

Then a regularized φ_4^4-theory with bare couplings $G_N > 0$, M_N, A_N is defined as the normalized measure on $S'(\mathbf{T}^4)$:

$$P_{int,N}^\alpha(d\varphi) = Z^{-1} \exp - \int_{\mathbf{T}^4} d\mathbf{x}\,[G_N\!: \varphi_\mathbf{x}^4 :_\alpha + (1/2)a^{-2}M_N\!: \varphi_\mathbf{x}^2 :_\alpha + (1/2)A_N\!: (\partial\varphi_\mathbf{x})^2 :_\alpha].\,P_N^\alpha(d\varphi) \tag{2.5}$$

the ∂ denoting the gradient if $\alpha = PV$ and denoting if $\alpha = l$:

$$(\partial_i\varphi)_\mathbf{x} = (1/a)\,[\varphi_{\mathbf{x}+ae_i} - \varphi_\mathbf{x}] \qquad e_i = i\text{-th unit lattice vector} \tag{2.6}$$

and the constant Z in (2.5) being a normalization constant.

Clearly if $\alpha = l$ the field is defined by its values at a finite number of points and (2.5) can be written as a probability distribution for the values $(\varphi_\mathbf{x})$, $\mathbf{x} \in a\mathbf{Z}^4 \cap \mathbf{T}^4$; a simple computation, recalling the definition of Wick ordering, allows to write such a distribution more explicitly as:

$$Z'^{-1}\left(\exp\left[\sum_\mathbf{x}\left(-\frac{\lambda}{4}\varphi_\mathbf{x}^4 + \frac{\mu}{2}\varphi_\mathbf{x}^2\right) + \sum_{\mathbf{x},\mathbf{y}} \beta\varphi_\mathbf{x}\varphi_\mathbf{y}\right]\right)\prod_\mathbf{x} d\varphi_\mathbf{x} \tag{2.7}$$

where the sum $\sum_{\mathbf{x}}$ and the product $\prod_{\mathbf{x}}$ are taken over $a\mathbf{Z}^4 \cap \mathbf{T}^4$ and $\sum_{\mathbf{x},\mathbf{y}}$ is performed over the pairs of nearest neighbors in $a\mathbf{Z}^4 \cap \mathbf{T}^4$, hence such that $|\mathbf{x} - \mathbf{y}| = a$. Furthermore:

$$\frac{\lambda}{4} = a^4 G_N, \quad \frac{\mu}{2} = [6G_N \mathbf{C}_N^l - (1/2)(1 + a^{-2}M_N) - 2a^{-2}A_N(1 + A_N)]a^4,$$
$$\beta = a^2(1 + A_N) \qquad (2.8)$$

and we have the following correspondence:

$$\mathbf{C}_N^l \equiv \mathbf{C}_N^l(\mathbf{x}, \mathbf{x})$$
$$Z'^{-1} \equiv Z^{-1} \exp\left[-(2\pi)^4 3 G_N \mathbf{C}_N^{l\,2} + \frac{(2\pi)^4}{2} \mathbf{C}_N^l M_N\right] \exp\left[\frac{(2\pi)^4}{2} A_N \mathbf{D}_N^l\right]$$
$$\mathbf{D}_N^l \equiv \sum_{i=1}^{4} a^{-2}[\mathbf{C}^l(\mathbf{x}, \mathbf{x}) - \mathbf{C}^l(\mathbf{x}, \mathbf{x} - e_i . a)]$$

In (2.7) we adopt the notations of [*6*] for the purpose of later comparison. The assumption

$$G_N > 0 \qquad (2.9)$$

which we shall *always* make, and the fact that the product in (2.7) is finite show that for $a = l$ there is no problem in giving a meaning to (2.5).

If $\alpha = PV$, the field $\varphi_{\mathbf{x}}$ at the point $\mathbf{x} \in \mathbf{T}^4$ is not a good random variable and even less so is $\partial\varphi_{\mathbf{x}}$, because $\int \varphi_{\mathbf{x}}^2 P_N^{PV}(d\varphi) = +\infty$: therefore a discussion of the meaning of (2.5) is necessary.

We shall give a meaning to (2.5) with $\alpha = PV$ via a sequence of formal identities transforming it into a meaningful expression which is then taken as the rigorous definition of the right hand side of (2.5). In the procedure which follows there are some arbitrary choices which we make with the aim of obtaining a simple final form for $P_{N,int}^{PV}(d\varphi)$.

Let $\rho_N = (2^{2N} - 1)^{-1}$ and note that the function in (2.3) can be written as:

$$\mathbf{C}_N^{PV}(\mathbf{x}, \mathbf{y}) = \int \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{\exp i\mathbf{p}.(\mathbf{x} - \mathbf{y})}{\rho_N \mathbf{p}^4 + (1 + 2\rho_N)\mathbf{p}^2 + (1 + \rho_N)} \qquad (2.10)$$

We introduce also the function:

$$\mathbf{C}_N^l(\mathbf{x}, \mathbf{y}) = \int \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{\exp i\mathbf{p}.(\mathbf{x} - \mathbf{y})}{\rho_N[(\mathbf{p}^2 - \xi_N)^2 + \xi_N^2\omega^2]} \qquad (2.11)$$

where $\omega^2 > 0$ is a free parameter, which we shall take equal to 1, and

$$\xi_N = -(1 + A_N + 2\rho_N)/2\rho_N \qquad (2.12)$$

Then a simple calculation of formal nature aiming at including

$$-(1/2)A_N:(\partial\varphi_{\mathbf{x}})^2:_{PV}$$

into the gaussian measure, together with a term $-(1/2)a_N^{-2}M_N^*:\varphi_{\mathbf{x}}^2:$ added to balance it if $A_N < -1$, leads to expressing (2.5) in terms of the gaussian measure $P_N^1(d\varphi)$ whose covariance is given by (2.1) with $\mathbf{C}$ replaced by (2.11). The result is:

$$P_{N,int}^{PV}(d\varphi)=Z^{*-1}\left[\exp-\int_{\mathbf{T}^4}(G_N:\varphi_{\mathbf{x}.1}^4+(1/2)M^*{}_Na^{-2}:\varphi_{\mathbf{x}.1}^2)d\mathbf{x}\right]P_N^1(d\varphi) \tag{2.13}$$

where Z^* is a suitable normalization constant and, using the « Wick reordering » relations, one finds:

$$a^{-2}M^*{}_N\equiv a^{-2}M_N-12G_N(\mathbf{C}_N^{PV}-\mathbf{C}_N^1)+(1+\rho_N)-(1+\omega^2)\xi_N^2\rho_N \tag{2.14}$$

$$\equiv a^{-2}M_N-12G_N(C_N^{PV}-C_N^1)+(1+\rho_N)+(1+2\rho_N+A_N)^2(1+\omega^2)/4\rho_N \tag{2.14}$$

$$\mathbf{C}_N^{PV}-\mathbf{C}_N^1\equiv\sum_{\mathbf{n}\in\mathbf{Z}^4}\int_{\mathbf{T}^4}\frac{d^4\mathbf{p}}{(2\pi)^4}\exp\,[2\pi i\mathbf{n}.\mathbf{p}].$$

$$.\left\{\frac{1}{\rho_N\mathbf{p}^4+(1+2\rho_N)\mathbf{p}^2+(1+\rho_N)}-\frac{1}{\rho_N[(\mathbf{p}^2-\xi_N)^2+\xi_N^2\omega^2]}\right\} \tag{2.14}$$

It is now possible to see that the exponential function in (2.13) is, if (2.9) holds, in $L_p(P_N^1)$ for all $p > 0$. This follows from the logarithmic nature of the singularity that the distribution (2.11) as well as $\mathbf{C}_N^1$ itself have at $\mathbf{x}=\mathbf{y}$. Therefore it is possible to apply Nelson's theorem. It shows that in any even dimension d the function $\exp-\int_{\mathbf{T}^d}:P(\varphi_{\mathbf{x}}):d\mathbf{x}$, where P is a polynomial of even order and positive leading coefficient, is in $L_p(\mu)$, for any $p > 0$ and any gaussian measure μ generated by a positive polynomial of the laplace operator of degree $d/2$. Actually Nelson's theorem is formulated usually for $d = 2$ [*9*], but its proof clearly covers the above mentioned slight generalization. So (2.13) makes sense.

With the above notations and conventions for $P_{N,int}^{\alpha}$ we introduce the « Schwinger functions »:

$$S(f_1,f_2,\ldots,f_n;N,\alpha)=\int_{S'(\mathbf{T}^4)}\varphi(f_1)\ldots\varphi(f_n)P_{N,int}^{\alpha}(d\varphi) \tag{2.15}$$

and we say:

DEFINITION 1. — An (ordinary) φ_4^4 theory (on $\mathbf{T}^4$) of type α is a probability measure P on $S'(\mathbf{T}^4)$ which is the weak limit of a sequence of $P_{N,int}^{\alpha}$, hence whose Schwinger functions are the limits as $N \to \infty$ of sequences (2.15) when G_N, M_N, A_N are chosen such that these limits exist (α is fixed to be PV, l or a third value el (for « extended lattice ») introduced in (2.16) below.

So the φ_4^4 theories may depend on α and the choice of the sequences (G_N, M_N, A_N), $N = 1, \ldots \infty$. The (ordinary) triviality conjecture (or T. C.), which we have the impression that many believe but which does not seem to have been ever explicitly written down, is that all (ordinary) φ_4^4 theories are gaussian processes on $S'(\mathbf{T}^4)$. In Sect. 8 we introduce an other possible definition of extended φ_4^4 theories, based on perturbation theory, and discuss briefly the corresponding triviality conjecture, which we call the « strong triviality conjecture » or S. T. C. We do not think that to write down these different definitions and conjectures is just an unnecessary complication; we hope that it might help to understand what are the mathematical issues to clarify, and what rigorous meaning one might give to the often-heard sentence « φ_4^4 is trivial ». Apart from Sect. 8, however, we will only deal with the ordinary φ_4^4 theories as defined in Def. 1.

It should be stressed that the physical interpretation of any φ_4^4 field requires other properties on P besides the ones following from Def. 1: one wants the Schwinger functions associated to P to be the imaginary time values of a family of Wightman functions, so that they really correspond to a relativistic quantum field in Minkowski space. It is not very clear in the literature whether the triviality is conjectured among the φ_4^4 fields which also admit this quantum field interpretation (for instance, which satisfy some sufficient set of euclidean axioms, like Osterwalder-Schrader's). We can only remark that the various heuristic or rigorous papers on triviality, besides some restrictive assumptions, do not discuss this point in detail. In every case at least one of the properties coming from the Wightman axioms is violated *a priori* by the regularization and it is unclear how this affects the physical relevance of the results obtained afterwards.

In our opinion, however, any triviality conjecture should mention this point: we shall therefore speak of « physical triviality » if one restricts the above triviality conjectures to the theories which admit a quantum field interpretation.

In the context of our cutoffs the possible limits P fall short of verifying the O. S. axioms because the euclidean invariance is *a priori* broken by the lattice regularization, and the reflection positivity is broken in the P. V. regularization. Actually it is also partly broken in the lattice regularization if $1 + A_N < 0$; in this case and for a cubic lattice, only the reflection positivity through the planes containing sites holds and the other one, through the planes between the sites, is lost (to see this, use transformation (4.1)). Since the first set of planes becomes dense as the lattice spacing goes to zero there is however no reason to rule out the possibility that a continuum limit of such antiferromagnetic theories verifies O. S. positivity.

Before formulating the basic results of renormalization theory we wish to go more deeply into a question which naturally arises from the above discussion. How are the l and PV theories related? One would like to say that in the limit $N \to \infty$ they yield the same class of theories. Unfortunately this is unknown.

One can remark that there seems to be a basic difference between the two regularizations: if one « discretizes » the free field P_N^{PV} one obtains a lattice field with, at least, nearest and next-nearest neighbor couplings, which also have different signs; the wave function counter term then adds, after discretization, other contributions to the nearest neighbor coupling. Therefore it might be interesting to extend the lattice regularization by introducing instead of (2.5) the more general field:

$$P_{N,int}^{el}(d\varphi) = Z^{-1} \exp\left\{ - a^4 \sum_{\mathbf{x}\in a\mathbf{Z}^4\cap\mathbf{T}^4} (G_N : \varphi_{\mathbf{x}}^4 :_l + (1/2)a^{-2}M_N : \varphi_{\mathbf{x}}^2 :_l) \right.$$
$$\left. - (1/4)a^4 \sum_{\mathbf{x},\mathbf{n}} A_N(\mathbf{n}) : \frac{(\varphi_{\mathbf{x}+\mathbf{n}a} - \varphi_{\mathbf{x}})^2}{a^2} : \right\} P_N^l(d\varphi) \quad (2.16)$$

with $\mathbf{n} \in \mathbf{Z}^4$, $|\mathbf{n}| \equiv \sum_{i=1}^{4} |n_i|$ and:

$$A_N(\mathbf{n}) = 0 \qquad \text{for} \qquad a|\mathbf{n}| \geq b_N \quad (2.17)$$

where $A_N(\mathbf{n})$ is rotation symmetric (on the lattice) and $b_N > 0$ is arbitrary. A natural restriction on b_N could be $b_N \to 0$ as $N \to \infty$: this condition on b_N can be imposed to require that the field φ be formally local as $N \to \infty$. The definition 1 can be immediately extended to allow the index α to take the value *el*, letting G_N, M_N, A_N arbitrary as $N \to \infty$ in a way compatible with (2.17).

This completes our description of the bare φ_4^4 theories and their different cutoffs. We now turn to the perturbative theory of renormalization. Its basic result is formulated in term of a free parameter g called by universal agreement the « renormalized coupling constant » and opposed to the « bare » constants G_N, M_N, A_N.

A formulation of the results of renormalization theory requires understanding that a certain number of formal operations can be made on the measure $P_{N,int}^{\alpha}$. In particular $S(f_1, \dots, f_n; N, \alpha)$ can be written formally as

$$S(f_1, \dots, f_n; N, \alpha) = S^0(f_1, \dots, f_n; N, \alpha)$$
$$+ \sum_{m_1+m_2+m_3\geq 1} G_N^{m_1} M_N^{m_2} A_N^{m_3} S^{(m_1,m_2,m_3)}(f_1, \dots, f_n; N, \alpha) \quad (2.18)$$

where $S^{(m_1,m_2,m_3)}(f_1, \dots, f_n; N, \alpha)$ are well defined integrals of products of Wick ordered polynomials, $S^0(f_1, \ldots, f_n; N, \alpha)$ being the Schwinger functions of the P_N^α-field. They are obtained by simply expanding the exponential in (2.13) or (2.16) in powers of G_N, M_N, A_N (in the case of (2.16) one has to interpret m_3 as a vector as well as A_N and $A_N^{m_3}$ means $\prod_{\mathbf{n}} A_N^{m_3(\mathbf{n})}(\mathbf{n})$ with the product running on the $\mathbf{n}$'s such that $a \,|\, \mathbf{n} \,| \leq b_N$).

If we now suppose that G_N, M_N, A_N are formal power series in a parameter g:

$$G_N = \sum_{k=1}^{\infty} \gamma_k(N) g^k, \qquad M_N = \sum_{k=2}^{\infty} M_k(N) g^k, \qquad A_N = \sum_{k=2}^{\infty} \alpha_k(N) g^k \tag{2.19}$$

it is clear that by substituting (2.19) into (2.18) and by collecting equal powers in g, we obtain a formal series:

$$S(f_1, \ldots, f_n; N, \alpha) = \sum_{k=0}^{\infty} g^k S_k(f_1, \ldots, f_n; N, \alpha) \tag{2.20}$$

where each of the coefficient of (2.20) is a *finite* combination of coefficients of (2.18).

This construction permits the following definition:

DEFINITION 2. — Given the three formal power series (2.19), we call the formal power series (2.20) constructed above the « formal perturbation theory » for φ_4^4 (of type α), with formal couplings (2.19). For fixed α, we say that a family of functions $S_k(f_1, \ldots, f_n)$, $k = 1, 2, \ldots, f_1, \ldots, f_n \in S(\mathbf{T}^4)$, $n = 1, 2, \ldots$ is a « formal perturbative φ_4^4 theory » (of type α) if there is a sequence of formal power series of the form (2.19) whose formal power series for the Schwinger functions has the property that the limits:

$$S_k(f_1, \ldots, f_n) = \lim_{N \to \infty} S_k(f_1, \ldots, f_n; N, \alpha) \tag{2.21}$$

exist for all $k, f_1, \ldots, f_n$.

Note that in order to define a formal perturbative φ_4^4 theory one does not even need that the bare couplings (2.19) are given by convergent series.

The striking result of renormalization theory (see e. g. [*10*] and references therein) is then:

PROPOSITION 1. — For any $\alpha = PV$, l, or el, there exist three formal power series

$$G_N^\infty = g + \sum_{k=2}^{\infty} \gamma_k^\alpha(N) g^k, \qquad M_N^\infty = \sum_{k=2}^{\infty} m_k^\alpha(N) g^k, \qquad A_N^\infty = \sum_{k=2}^{\infty} a_k^\alpha(N) g^k \tag{2.22}$$

for $N = 0, 1, 2, \ldots$, such that the limits (2.21) exist. In other words there exist formal φ_4^4 perturbative theories. In the case el the A_N^∞ and $\alpha_k^{el}(N)$

are vector valued functions indexed by $\mathbf{n} \in \mathbf{Z}^4$, $a \mid \mathbf{n} \mid \leq b_N$ and $\lim_{N \to \infty} b_N = 0$.

Furthermore the above *el* theories yield formally the same results as the *l* theories with same G_N^∞, M_N^∞ and:

$$\alpha_k^l(N) = (1/4) \sum_{\mathbf{n}} \mathbf{n}^2 \alpha_k^{el}(\mathbf{n}; N) \tag{2.23}$$

in the limit $N \to \infty$.

If two formal φ_4^4 perturbative theories of *el* or of *l* type are described by bare constants G_N^∞, M_N^∞ and by A_N^∞ constants such that (2.23) holds we say that they are « corresponding » theories.

The above proposition is by no means obvious since its analogue in dimension $d > 4$ is simply false (this is the statement that φ_4^4 is renormalizable, but φ_d^4 is not renormalizable for $d > 4$). The remarkable part of Proposition 1 is that $\gamma_k^\alpha(N)$, $M_k^\alpha(N)$, $\alpha_k^\alpha(N)$ called respectively, for $k \geq 2$, « coupling », « mass », and « wave function » counterterms) are independent of $f_1, \ldots, f_n$, n.

There are many formal power series like (2.22) enjoying the property of producing a perturbative φ_4^4 field theory, for the following trivial reason: suppose that there is one choice of the series (2.22) which produces a formal φ_4^4 theory, then let

$$g = g^* + \sum_{k=2}^{\infty} \Gamma_k(N) g^{*k} \tag{2.24}$$

and substitute this formal series in (2.22). Working out the algebra one ends up with three new formal series $G_N^{*\infty}$, $M_N^{*\infty}$, $A_N^{*\infty}$ which of course still verify the necessary properties to define a formal φ_4^4 theory (of the same type).

One can say that the group of the formal power series with lowest order g « acts on the set of the formal perturbative φ_4^4 theories » in a natural way. This just described action is called a « renormalization group action » on the bare couplings of φ_4^4. There are other transformations of the bare couplings which leave invariant the set of the formal perturbative φ_4^4 theories: they can be obtained from other operations on the measures defining the fields like rescalings of the fields ($\varphi_\xi \to \zeta \varphi_\xi$) or of the length scale on which the space coordinates $\mathbf{x}$ are measured ($\varphi_{\mathbf{x}} \to \varphi_{\lambda \mathbf{x}}$). It is hard to imagine other classes of transformations of the bare couplings which could be added to the ones just described and presumably there are no other. Anyway, the full set of these transformations forms the « renormalization group » of φ_4^4.

As we see the above ambiguities on the choice of the bare constants leading to a formal φ_4^4 theory are somewhat trivial. Very few explicit choices have been in fact investigated in the literature and only one in real detail: it is the B. P. H. Z. subtraction scheme in momentum space (see e. g. *[10]*

[*11*], and references therein) for which there are not only qualitative, but also some quantitative information available [*11*]. Among the possible choices to define a formal φ_4^4 theory we decide therefore to select always this B. P. H. Z. choice. We are aware that this choice is somewhat arbitrary and that it would be interesting to analyze how much it can affect our study; but this would be hard and we must therefore restrict ourselves to the only case for which some detailed work has already been done.

Another remark on Proposition 1 is that it shows the formal equivalence between the field theories of type *l* and *el*. It does not seem possible for the time being that one could go beyond this formal equivalence and show that the set of all the possible continuum limits of *l* type coincide with that of all the possible limits of *el* type. The reason why we have introduced the *el* theories is that they may be easier to treat from a technical point of view. In particular they allow much more flexibility in the study of cases in which the convergence of the measures $P^{\alpha}_{N,int}$ to a limit P is so singular that $S(f_1, f_2; N, \alpha)$ converges to a limit $S(f_1, f_2)$ only in the sense of the distributions but not pointwise (i. e. the kernel $S(x, y; N, \alpha)$ of $S(f_1, f_2; N, \alpha)$ does not converges as $N \to \infty$ for fixed x and y but only as a distribution; this is precisely the situation that we can expect to meet when $1 + A_N < 0$. We shall later provide an explicit example of how a φ_4^4 theory of *el* type can be constructed in this way (without verification of the physical axioms, however)).

We now proceed to define in a more precise way the problem of the « construction of a φ_4^4 theory by perturbation theory ». By this we mean the following: fix a formal φ_4^4 perturbation theory, i. e. a sequence of three formal power series with the properties described in Definition 2 above; then attempt to construct a one parameter family of random fields depending on a parameter g and with Schwinger functions admitting an asymptotic series at g = 0, defined in a right or a left neighborhood of the origin, with coefficients given by the formal φ_4^4 perturbative theory.

The construction problem can be formulated in terms of the new notion of *renormalization schemes* which we define as follows:

DEFINITION 3. — Fix $\alpha = PV$, l, or el, $\varepsilon > 0$, and a perturbative φ_4^4 theory: a *renormalization scheme* on $[0, \varepsilon]$ (or $[-\varepsilon, 0]$) is a sequence of functions of a variable g which are holomorphic in a domain containing $[0, \varepsilon]$ (or $[-\varepsilon, 0]$), $G_N(g)$, $M_N(g)$, $A_N(g)$ and have power series at $g = 0$ given by

$$G_N(g) = g + \sum_{k=2}^{\infty} g^k \gamma_k^*(N), \qquad M_N(g) = \sum_{k=2}^{\infty} g^k m_k^*(N),$$

$$A_N(g) = \sum_{k=2}^{\infty} g^k \alpha_k^*(N) \qquad (2.25)$$

such that $\gamma_k^*(N) \equiv \gamma_k^\alpha(N)$, $m_k^*(N) \equiv m_k^\alpha(N)$, $\alpha_k^*(N) \equiv \alpha_k^\alpha(N)$ for $N \geq N_k$ sufficiently large (where γ^α, m^α, α^α are the coefficients of the formal power series for the bare constants which lead to the chosen perturbative φ_4^4 theory, see Definition 2).

If for all $g \in [0, \varepsilon]$ (or $[-\varepsilon, 0]$) and, given g, for all N large enough it is true that

$$G_N(g) > 0 \tag{2.26}$$

we say that the renormalization scheme is *admissible*.

The above definition can certainly be generalized (e. g. replacing « holomorphic » by C^∞): however it already contains so much structure and interesting open problems that we do not generalize it in this paper.

A typical renormalization scheme is what we call a « truncation scheme »

$$G_N(g) = g + \sum_{k=2}^{\tau_G(N)} g^k \gamma_k^\alpha(N) \tag{2.27}$$

$$M_N(g) = \sum_{k=2}^{\tau_M(N)} g^k m_k^\alpha(N) \tag{2.27}$$

$$A_N(g) = \sum_{k=2}^{\tau_A(N)} g^k \alpha_k^\alpha(N) \tag{2.27}$$

and $\lim_{N\to\infty} \tau_j(N) = \infty$. We shall denote such schemes as (τ_G, τ_M, τ_A).

However the above definition allows quite a few subtleties and trivialities. To give an idea of what we have in mind just pretend for a moment that.

$$\gamma_k(N) = N^{k-1}, \qquad \alpha_k(N) = -N^{k-1}, \qquad m_k(N) = N^{k-1} \tag{2.28}$$

for $\alpha = PV$ or l. Then a non truncation renormalization scheme could be, on $[-\varepsilon, 0]$:

$$G_N(g) = \frac{g}{1-Ng}, \qquad A_N(g) = \frac{-gNg}{1-Ng}, \qquad M_N(g) = \frac{gNg}{1-Ng} \tag{2.29}$$

This scheme would not be admissible: however the corresponding truncation schemes would be admissible on $[0, \varepsilon]$ and, if $\tau_G(N)$ is even, also on $[-\varepsilon, 0]$.

If we change (2.29), replacing G_N by G_N^* defined by

$$G_N^*(g) = \frac{g}{1-Ng} + (2N)!g^{2N} \tag{2.30}$$

we see that the renormalization scheme obtained in this way is admissible on $[-\varepsilon, 0]$.

As we shall see a case more realistic than the one in (2.28) is, for each $k = 1, 2, \ldots,$ and when $N \to \infty$:

$$\gamma_k^\alpha(N) \sim (AN)^{k-1}, \quad m_k^\alpha(N) \sim C(AN)^{k-1}, \quad -\alpha_k^\alpha(N) \sim B(AN)^{k-1} \quad (2.31)$$

Then if τ_G grows slowly enough with N (to $+\infty$) and τ_M, $\tau_A \to +\infty$ as $N \to \infty$ (at any rate) the truncation scheme (τ_G, τ_M, τ_A) is admissible on $[0, 1]$, say, and on $[-1, 0]$ also provided τ_G is even.

The possibility of manipulations of a renormalization scheme, like in (2.30), tells us that most renormalization schemes are uninteresting: this remark leads to the following definition:

DEFINITION 4. — An admissible renormalization scheme is called « perturbatively singular » if the $\lim_{N\to\infty} P_{N,int}^\alpha \equiv P_\infty$ exists for all $g \in [0, \varepsilon]$ in the sense that the Schwinger functions of the l. h. s. converge to those of the r. h. s. but their derivatives with respect to g at $g = 0$ do not converge to those given by the renormalized perturbation theory. An admissible renormalization scheme will be called « perturbatively » or « weakly » trivial if P_∞ is a gaussian measure or if its Schwinger functions vanish at non coinciding points. In both cases indeed P_∞ will not be an interesting model for realistic interacting and propagating fields.

The triviality conjecture then implies that all the renormalization schemes are weakly trivial. The existence of a non singular renormalization scheme for the B. P. H. Z. choice of the bare constants would imply the falsity of this triviality conjecture.

Avoiding the formulation of new conjectures but trying to see what really happens did not lead us very far. We have been able to prove:

PROPOSITION 2. — Consider the B. P. H. Z. choice of the formal bare constants and $\alpha = PV$ or $\alpha = l$;

i) There are three positive constants A, B, C, such that for any fixed $k \geq 2$, one has:

$$\lim_{N\to\infty} \frac{\gamma_k^\alpha(N)}{(AN)^{k-1}} = 1 \quad (2.32)$$

$$\lim_{N\to\infty} \frac{-\alpha_k^\alpha(N)}{B(AN)^{k-1}} = 1 \quad (2.33)$$

$$\lim_{N\to\infty} \frac{m_k^\alpha(N)}{C(AN)^{k-1}} = 1 \quad (2.34)$$

ii) There are admissible renormalization schemes of the truncation type for $\alpha = PV$ or $\alpha = l$, and for $[0, \varepsilon]$ or $[-\varepsilon, 0]$.

iii) Among the admissible truncation renormalization schemes, many are trivial (and can be explicitly described).

Statement *ii*) follows immediately from *i*), which is proved in the Appendix. Statement *iii*) is proved in some detail in Sect. 3 and Sect. 4. The results on the case $\alpha = l$ were communicated to us by Alan Sokal (see Sect. 4).

In all the instances in which we can prove admissibility of a truncation scheme we find that:

$$\lim_{N\to\infty} G_N(g) = +\infty, \qquad \lim_{N\to\infty} A_N(g) = -\infty \tag{2.35}$$

The fact that $G_N > 0$ implies $1 + A_N < 0$ in these cases might be a more general property of many classes of renormalization schemes: if the triviality conjectures for the *ferromagnetic* φ_4^4 models emerging from [*5*] [*6*] are supposed to hold, it would be precisely a necessary condition for a non-trivial φ_4^4 theory, generated by a renormalization scheme in the way we propose, to satisfy $G_N > 0$ and $1 + A_N < 0$. On the other hand G_N and $-A_N$ need not tend to $+\infty$ for any constructive reason; they may rather behave as the trivial resummed « examples » (2.29) suggest. These two questions are quite interesting and in our opinion deserve further careful investigation (see Sect. 5).

The behavior (2.33), whether or not $G_N > 0$ really implies, under rather general conditions, $1 + A_N < 0$, suggest that the antiferromagnetic cases ($A_N < -1$ or $\beta < 0$, $\alpha = PV$ or $\alpha = l$), should be considered at least on equal footing as the corresponding ferromagnetic ones (which tend to provide trivial theories, as shown in [*5*] [*6*]). It is not inconceivable that consistency with renormalization theory (if $d \geq 4$) requires antiferromagnetic couplings (i. e. huge field strength renormalization). The super-renormalizable cases $d < 0$ are not a reliable guide for this situation since they do not need any field strength renormalization ([1]).

In Sect. 5 we try to make less absurd the throught that this might be a possible mechanism for the construction of a non trivial φ_4^4 theory.

In Sect. 6 we pursue the analysis of the notion of renormalization scheme by deriving another simple result.

PROPOSITION 3. — There exist non-trivial (singular), non ferromagnetic renormalization schemes for φ_4^4 theories of type *el.*

The formal bare constants of the renormalization schemes found in the proof do not correspond to a truncation of the B. P. H. Z. ones, and the φ_4^4 theory constructed by Prop. 3 does not admit a quantum field interpretation. We regard the above Proposition therefore only as a sign of the interest of the antiferromagnetic couplings and as a strong argument showing

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([1]) It might be relevant to notice that the very word « field strength renormalization » is misleading in the case of antiferromagnetic couplings. One uses this word because a gradient term like $A(\partial\varphi)^2$ can be changed into the usual gradient term $(\partial\varphi)^2$ by a rescaling of the field $\varphi \to \sqrt{A}\varphi$. But this rescaling is ill-defined precisely in the antiferromagnetic case $A < 0$!

that it will not be easy to prove the triviality conjecture on φ_4^4 without reference to perturbation theory or without the use of some kind of general axiomatic arguments, independent of Lagrangian field theory, e. g. proving inconsistency of 4-dimensional interacting scalar field theory (see Sect. 8 for a more complete discussion). In fact. Prop. 3 shows the falsity of the (ordinary) triviality conjecture if one does not restrict it to the theories which admit a quantum field interpretation (at least for theories of type *el*, which are formally equivalent to the theories of type *l*, for which, however, we cannot say much). In Sect. 7 we outline a project of constructive analysis of φ_4^4 which, although probably not new to many, does not seem to have been written down in a precise form.

Finally we stress that the bounds (2.32)-(2.34), which are our main technical result, are well known in the physics literature as they say that the leading divergence of the counterterms of order k is $(AN)^{k-1}$; recalling that N is the logarithm of the cutoff, this is nothing but the « leading-log » behavior considered already in [1] and widely discussed thereafter. In its simplest form, the « Landau argument » is just a summation of the power series (2.22) in which $\gamma_k^\alpha(N)$, according to (2.32), has been replaced by $(AN)^{k-1}$:

$$G_N = g + \sum_{k=2}^{\infty} g^k (AN)^{k-1} = g(1-gAN)^{-1} \Rightarrow g = G_N(1 + G_N AN)^{-1} \quad (2.36)$$

Indeed $(AN)^{k-1}$ is the « leading-log » term found in the divergence of the sum of the bare amplitudes at order k. From (2.36) Landau concluded that $\lim_{N\to\infty} g = 0$, no matter how the bare coupling constant $G_N > 0$ varies with N; the vacuum polarization « screens » the interaction, and the resulting theory is a free field. However the mass and field strength renormalization have not been as extensively studied as the coupling constant renormalization and this may explain why the negative sign in (2.34) has not been previously emphasized. In our opinion one cannot give credit to the Landau (or one-loop renormalization group) argument without worrying about this negative sign which arises from the same approximation on which these arguments are based, and is therefore independent of the particular choice of the renormalization, exactly to the same extent as they are.

The new derivation we give in the Appendix of this well-known « leading-log » behavior has the following interesting features which justify our including it:

i) it is completely rigorous and original: we analyze « leading-log » ultraviolet divergences graph by graph, by performing Hepp's sector splitting and the classification of forests according to the new methods introduced in [11]. We extract the exact coefficient for any of the relevant contributions, then sum up all the contributions at a given order, weighted by the exact symmetry factors of the corresponding graphs and show that the result of this tedious computation is a nice geometric series. This « brute force » method is technically more complicated but conceptually simpler than the use of a renormalization group argument. We do not know of such a completely rigorous analysis in the literature (if however it exists, we apologize to its author); in particular the early graphical derivations of leading log behavior by Landau et al. [*1*] [*2*] probably cannot be considered 100 % rigorous simply because there was no rigorous theory of perturbative renormalization at that time.

ii) It is potentially more powerful and general than the derivations based on a renormalization group analysis which retains only the first term in the β function. Indeed it may yield reasonably accurate information on the way the limit is approached in (2.32)-(2.34). This depends obviously on finding rigorous bounds on all non-leading-log contributions to $\gamma_k^\alpha(N)$, $\alpha_k^\alpha(N)$, $m_k^\alpha(N)$. Although we did not try to find actually these bounds it is quite clear that one can obtain them by using the full combinatoric machinery developed in [*11*], which is able to provide reasonable bounds on contributions associated to any graph, sector, and equivalence class of renormalization forests. Up to now, it is the only method we know of in order to get rigorous information on the precise possible rates of truncation $\tau(N)$ for the admissible schemes of truncation type introduced in Prop. 2, *ii*) (see the end of the Appendix for more details).

3. EXAMPLES OF TRIVIALITY IN THE FRAMEWORK OF THE PV REGULARIZATION

From Prop. 2, *i*) one deduce that by choosing $(\tau_G, \tau_M, \tau_A) \to +\infty$ very slowly we can obtain in (2.27) $G_N(g) \to \infty$, $M_N(g) \to +\infty$, $-A_N(g) \to +\infty$ as:

$$g(gAN)^{\tau_G - 1}, \qquad gB(gAN)^{\tau_M - 1}, \qquad gC(gAN)^{\tau_A - 1} \tag{3.1}$$

and the ratios between any two of these three quantities can obviously be made to tend to any of the three choices 0, positive constant, $+\infty$, for all $g > 0$, by simply adjusting the integer-valued functions (τ_G, τ_M, τ_A).

We now look at (2.11) and note that the covariance C^1 of $\varphi_{\mathbf{x}}$ has the form

$$C^1(\mathbf{x} - \mathbf{y}) = \frac{1}{\rho_N} \sum_{\mathbf{n} \in \mathbf{Z}^4} C^{*1}[\sqrt{\xi_N}(\mathbf{x} - \mathbf{y} - 2\pi\mathbf{n})] \tag{3.2}$$

where

$$C^{*1}(\mathbf{x} - \mathbf{y}) = \int \frac{d\mathbf{p}}{(2\pi)^4} \frac{\exp i\mathbf{p}.(\mathbf{x} - \mathbf{y})}{\omega^2 + (\mathbf{p}^2 - 1)} \tag{3.3}$$

Therefore it is natural to define

$$\varphi_{\mathbf{x}} = \rho_N^{-1/2}\psi_{\sqrt{\xi_N}\mathbf{x}} \tag{3.4}$$

with $\psi_{\mathbf{x}}$ being a field on $\sqrt{\xi_N}\mathbf{T}^4$.

In terms of the field ψ, whose covariance is not too different from (3.3) and differs from it by a regular correction of order $\exp - \sqrt{2\xi_N}$ (coming from the sum in (3.2)) the measure (2.13) is rewritten:

$$P^{PV}_{N,int}(d\varphi) = Z^{-1}\left[\exp - \int_{\sqrt{\xi_N}\mathbf{T}^4} d\mathbf{y}(G : \psi_{\mathbf{y}}^4 : + M : \psi_{\mathbf{y}}^2 :)\right] P^1(d\psi) \tag{3.5}$$

with : : denoting the Wick ordering with respect to P^1 and:

$$G = G_N/(\xi_N\rho_N)^2 \tag{3.6}$$

$$M = \frac{1}{\rho_N\xi_N^2}\left[-6G_N(C_N^{PV} - C_N^1) + (1/2)a^{-2}M_N + \frac{(1+\rho_N)}{2} - \frac{(1+2\rho_N + A_N)^2}{8\rho_N}(1+\omega^2)\right] \tag{3.6}$$

and a simple calculation shows, from (2.14)

$$C_N^{PV} - C_N^1 = c_\omega \rho_N^{-1}\log \xi_N\rho_N + O(\rho_N^{-1}) \tag{3.7}$$

and

$$G = (4G_N/A_N^2)[1 + \delta_N^1] \tag{3.8}$$

$$M = (4G_N/A_N^2)(-6c_\omega \log|A_N| + c'_\omega)(1+\delta_N^2) + c''_\omega M_N A_N^{-2}(1+\delta_N^3) + c'''_\omega + \delta_N^4 \tag{3.8}$$

where $c_\omega, c'_\omega, c''_\omega, c'''_\omega$ are suitable constants and $\lim_{N\to\infty} \delta_N^j = 0$; actually c_ω is ω-independent.

We now arrange the choice of G_N, A_N, M_N so that $G \to 0$, $M \to 0$. This will imply that the field ψ has a limit distribution as $N \to \infty$ which is gaussian, hence the field ψ will be trivial.

This can be inferred from the fact that we can, in principle, perform the whole constructive theory of the (superrenormalizable « $P(\varphi)_2$-like »): $\psi_{\mathbf{x}}^4$: interaction and conclude that if $G \to 0$, $M \to 0$ its mass gap is uniform. Taking into account the speed at which ρ_N approaches zero and ξ_N infinity and the relation (3.4), this immediately implies that the field φ has all its Schwinger functions going to zero as $N \to \infty$ (i. e. $P^{PV}_{N,int}$ converges in this case to $\delta_0(d\varphi)$, the Dirac measure on the null field).

It seems likely that all the other choices of the truncations which lead to $G_N \to +\infty$, $M_N \to +\infty$, $A_N \to -\infty$ yield a trivial theory. We do not, however, analyze this point in further detail (the discussion would involve developing in detail the theory of the ψ-field with: $\psi_{\mathbf{x}}^4$: and :$\psi_{\mathbf{x}}^2$: interaction with singular coupling constants; a task that might be hard in full generality). As we shall see in the next section the analogous discussion in the lattice regularization is easier (elementary in fact) and can be carried out completely.

As a final remark on this section we note that it is possible to modify M_N so that the constant M described in (3.8) goes to zero as $N \to \infty$ while G_N/A_N^2 converges to a positive limit: $G \to B^{-2}g^{-1}$. To achieve this, however, one has to abandon the frame of the truncation schemes (in fact $c_\omega \neq 0$ implies that M_N cannot be a truncation of a series like (2.22) if G_N, A_N are of truncation type as above and if one whishes $\lim_{N\to\infty} M_N = 0$).

Then using the fact that the field ψ has a: ψ^4: theory which is super-renormalizable it is quite clear that ψ will have a limit distribution as $N \to \infty$ which is non gaussian and not analytic in g (though analytic in g^{-1} for g large). Since the ψ-field is obtained by suitable scalings from the φ-field we can interpret this as saying that in the frame of the Pauli-Villars regularization it is possible to construct an interaction of fourth order in the field and introduce suitable counterterms so that the field admits a non trivial ultraviolet limit. It is however clear that this ultraviolet limit does not admit a quantum field interpretation (since it is a perturbation of a field with covariance given by a high order polynomial in the Laplace operator and therefore does not verify the Källen-Lehmann representation, for instance).

However this remark tells us that the triviality conjecture in the form in which we first stated it cannot hold for the PV-theories. Nothing however can be said from the above analysis about the « physical » triviality conjecture (see also the entire Sect. 6 for related discussions).

We finally note that the PV-regularization is intrinsically « antiferromagnetic » as it appears from its discretized versions on the lattice and this seems essential in producing the construction discussed above.

4. EXAMPLES OF TRIVIALITY IN THE LATTICE REGULARIZATION

We shall rewrite the regularized measure (2.5) in the form (2.7).

Suppose that $G_N(g)$, $M_N(g)$, $A_N(g)$ are given and constitute an admissible renormalization scheme with $G_N \to +\infty$, $M_N \to +\infty$, $A_N \to -\infty$.

Then we perform the following change of coordinates:

$$\varphi_{\mathbf{x}} = (-1)^{|\mathbf{x}|}\psi_{\mathbf{x}}, \qquad |\mathbf{x}| = \sum_{i=1}^{4} |\mathbf{x}_i| \tag{4.1}$$

This well known change of coordinates produces a field ψ whose distribution is the same as (2.7) but with β replaced by $-\beta$. So, since $\beta = a^2(1 + A_N) < 0$, the field ψ is ferromagnetic and we can apply to it the inequalities (7) and (12) of reference [*6*]. Transferring such inequalities back into relations involving the field φ we find that the infrared inequality (7) of Ref. [6] becomes, if one keeps the notations of [6]:

$$\langle \mid \varphi(k) \mid^2 \rangle \leq \frac{a^2}{\zeta(a) 2 \sum_{i=1}^{d} (1 + \cos k_i . a)} \quad \text{for any } k \neq 0 \qquad (4.2)$$

where $\zeta(a) = \mid \beta a^{-2} \mid \equiv - (1 + A_N)$ (the antiferromagnetic coupling comes in via the crucial $+$ sign in front of the cosine in (4.2)) (²).

Clearly $a^2 \zeta(a)^{-1}$ goes to zero when $N \to \infty$ and (4.2) tells us that φ goes to zero in the sense of distributions at least on the test functions f whose Fourier transform excludes a vicinity of the origin. Since this property is false in all the known perturbative φ_4^4 (e. g. in the B. P. H. Z. theory) we conclude that the above remark can be used to show the weak triviality of all the renormalization schemes with $A_N \to -\infty$ or even

$$- (1 + A_N) \; a^{-2} = \zeta(a) a^{-2} \to +\infty .$$

In particular, all the truncation schemes associated with the B. P. H. Z. theory which were discussed in the preceding sections (hence those with the slowly growing secondary cutoffs τ_G, τ_M, τ_A) are shown in this way to be weakly trivial.

5. SOME PROPERTIES THAT A NON TRIVIAL RENORMALIZATION SCHEME SHOULD HAVE

We pursue the analysis in the frame of the lattice regularization.

The discussion of Sect. 4 and of Ref. [6] shows that the only possibility for having a non trivial φ_4^4 theory of l-type (i. e. as a limit of a lattice regularized theory with φ^4 interaction and nearest neighbor coupling) is that $\beta a^{-2} = \zeta(a) > 0$ does not tend to zero as $N \to \infty$ in the ferromagnetic case (see [6], page 286), or that $\zeta(a) a^{-2}$ does not tend to $+ \infty$ in the antiferromagnetic case.

Since the analysis of [5] [6] gives hope to exclude all the ferromagnetic cases, one is led to consider the antiferromagnetic cases (with $\beta < 0$) in which

$$\lim_{N \to \infty} \zeta(a) . a^{-2} = \zeta \geq 0 \qquad (5.1)$$

(²) It is interesting to notice that there is no analogue of transformation (4.1) if the lattice is not a regular cubic lattice. In more complicated cases, which we never consider in this paper, the full complexity of frustation effects due to antiferromagnetism arises and it becomes very difficult to relate these theories to any of the triviality results of [5] [6]. In particular it is not known whether (5.1) has to be satisfied to keep open the possibility of a non trivial continuum limit. This is just another open problem among the many listed in this paper which might attract the courageous reader...

as the most interesting for the possibility of a non trivial ultraviolet limit.

It is not known if admissible renormalization schemes exist in which (5.1) holds. Assuming so, one can ask whether they would fall under the wrath of the triviality theorems as time goes on. This seems unlikely. We are going to give some qualitative argument to support this assertion.

The triviality results for $\beta > 0$ proceed from the basic assumptions:

1) The spontaneous magnetization of the ∞-volume lattice vanishes.

2) The value of $\langle \varphi_0 \varphi_{\mathbf{x}} \rangle$ is, for each $\mathbf{x} \in$ e. g. $\{ \mathbf{x} \mid 0 < |\mathbf{x}| < 1 \}$, uniformly bounded as $a \to 0$ (or $N \to \infty$).

The assumption 2) is reasonable if $\beta < 0$ because in this case one expects from various correlations inequalities $\langle \varphi_0 \varphi_{\mathbf{x}} \rangle$ to be a positive, monotonically decreasing (3) function of $\mathbf{x}$ and its convergence towards the continuum two-point function to be a smooth, pointwise convergence (see e. g. [*12*] and references therein). However if $\beta < 0$ the field tends to oscillate wildly and probably it has to be thought close to a white noise. If we therefore supposed 2) also in this antiferromagnetic case then triviality would easily follow from the basic inequality (12) of [*6*] (i. e. this inequality, which remains true in the antiferromagnetic case as well because of the exact transformation (4.1), combined with 2) and (5.1), forces $u^{(4)}$ to go to zero, and the resulting theory would be gaussian).

However the limit of the Schwinger functions might well be approached only in the sense of the distributions and through unbounded oscillations (violating assumption 2)), as suggested in [*7*], and nothing in [*5*] [*6*] seems against such a possibility.

Although this mechanism of convergence may appear at first sight unusual and strange, we have been unable so far to find any argument against it: in some sense the strongest argument for our belief that it should deserve attention (particularly if all the ferromagnetic theories turn out to be trivial) is that it looks more compatible with perturbation theory, as the results of [*5*] [*6*] seem also to suggest.

6. A TRIVIAL EXAMPLE OF HOW ANTIFERROMAGNETISM CAN LEAD TO NON TRIVIALITY

In this section we show that among the φ_4^4 theories of *el* type (i. e. with lattice regularization and non nearest neighbor coupling) there are some

(3) The rigorous statements on monotonicity of the two point function follow from the Schrader and the Messager Miracle-Sole inequalities; for an exact formulation, see [*12*] and references therein.

non trivial renormalization schemes. Clearly this is not a very exciting result because the renormalization schemes that we exhibit are singular in the sense of Def. 4. Their existence, however, shows that it will be quite hard to prove the physical triviality of the φ_4^4 theories of *el* type (although to show that, conversely, they are not always trivial is also probably very hard !).

The model is very simple and inspired directly by the lattice discretization of the Pauli-Villars free field. We choose A_N and M_N so that the measure (2.16) can be written:

$$Z^{-1}\left(\exp - a^4 \sum_{\mathbf{x}} G_N(g) : \varphi_{\mathbf{x}}^4 :_0 \right) P_N^0(d\varphi) \tag{6.1}$$

where P_N^0 is a gaussian measure with covariance $C^0(\mathbf{x}, \mathbf{y})$ given by (2.1) with $\mathbf{C}$ replaced by:

$$\mathbf{C}(\mathbf{x}, \mathbf{y}) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{\exp i.\mathbf{p}(\mathbf{x}-\mathbf{y})}{1 + \sum_{\mathbf{n}} A_N(\mathbf{n}) \frac{(1-\cos a\mathbf{p}.\mathbf{n})}{a^2} + 2\sum_{i=1}^{4} \frac{(1-\cos \mathbf{p}_i.a)}{a^2}} \tag{6.2}$$

and the denominator in the r. h. s. of (6.2) converges as $a \to 0$ to $1+\mathbf{p}^2(1+\mathbf{p}^2)$.

For instance we choose:

$$A_N(\pm\mathbf{e}_i) = 16a^{-2}, \quad A_N(\pm 2\mathbf{e}_i) = -a^{-2}, \quad A_N(\pm\mathbf{e}_i \pm \mathbf{e}_j) = -2a^{-2} \tag{6.3}$$

and $A_N(\mathbf{n}) \equiv 0$ for $|\mathbf{n}| > 2$. Note that if we look at the formally corresponding theory of type l (according to the rule (2.23)) we find that the corresponding A_N is 0, hence there is no field strength renormalization and the continuum limit is therefore ill-defined. This shows that one should be careful in using the notion of corresponding theories; it does not distinguish between theories which have the same quadratic momentum behavior in their covariance.

To continue we fix any formal renormalization theory, i. e. three formal power series for G_N, M_N, A_N producing a perturbative φ_4^4; for instance the B. P. H. Z. series. Then we set (quite arbitrarily):

$$G_N = g + \frac{\sum_{j=2}^{\tau_G(N)} g^j \gamma_j(N)}{1 + g^N N!} \tag{6.4}$$

$$a^{-2}M_N(g) = \frac{\sum_{j=2}^{\tau_M(N)} g^j m_j(N)}{1 + g^N N!} \tag{6.4}$$

$$A_N(\pm \mathbf{e}_i) = 16a^{-2}\frac{g^N N!}{1+g^N N!} + \frac{\sum_{j=2}^{\tau_A(N)} g^j \alpha_j(N)}{1+g^N N!} \tag{6.4}$$

$$A_N(\pm 2e_i) = -a^{-2}\frac{g^N N!}{1+g^N N!}, \quad A_N(\pm e_i \pm e_j) = -2a^{-2}\frac{g^N N!}{1+g^N N!} \tag{6.4}$$

It is then easy to check that the measure (6.1) converges to the $g:\varphi_{\mathbf{x}}^4:_{0*}$ theory with free field described by the gaussian measure with covariance C^{0*} given by (2.1) in which **C** has been replaced by:

$$\mathbf{C}^{0*}(\mathbf{x}, \mathbf{y}) = \int \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{\exp i\,\mathbf{p}.(\mathbf{x}-\mathbf{y})}{1+\mathbf{p}^2(1+\mathbf{p}^2)} \tag{6.5}$$

provided $\tau_G, \tau_M, \tau_A \to +\infty$ slowly enough.

This theory is superrenormalizable (in the same sense as: $P(\varphi)_2$:) and its Schwinger functions should admit an asymptotic series which, however, will turn out different from the perturbative series associated with the formal power series used for G_N, M_N, A_N. This means that the renormalization scheme that we built in (6.4) are singular in the sense of Def. 4. Clearly this is not too surprising because the scheme we used has been « manipulated » in such a way that it has nothing to do any more with the original coefficients $\gamma_j(N)$, $m_j(N)$, $\alpha_j(N)$. In particular it is not a scheme of truncation type. Unfortunately we do not know any examples of a non trivial renormalization scheme of truncation type, which would be more likely to be non singular, and therefore to verify the axioms.

7. SOME PROPOSALS FOR FURTHER INVESTIGATION

If one believes that the renormalization counterterms can have the effect of changing the field from ferromagnetic into antiferromagnetic or mixed one should try to exploit this fact appropriately in the construction of a (non trivial) φ_4^4.

The antiferromagnetic couplings will tend to force the field to oscillate very strongly on the scale of the cutoff: therefore we may expect that what shall converge to a limit will not be the field itself but, rather, some average of it (convergence in the sense of distributions).

Let $b = a2^B$, B integer, be a new length scale and define for $\mathbf{x} \in b\mathbf{Z}^4$:

$$\psi_{\mathbf{x}} = 2^{-4B}\sum_{\xi\in\beta(\mathbf{x})} \varphi_\xi \equiv (b/a)^{-4}\sum_{\xi\in\beta(\mathbf{x})} \varphi_\xi \equiv \varphi_{\mathbf{x}}^* \tag{7.1}$$

where $\beta(\mathbf{x}) = \{\, \xi \mid x_s \leq \xi_s < x_s + b, \xi \in a\mathbf{Z}^4, s = 1, 2, 3, 4 \,\}$.

The fields $\psi_{\mathbf{x}}$ and $\varphi_{\mathbf{x}}$ are very close in distribution with respect to the measure $P_N^l(d\varphi)$ if $N \to \infty$, $b \to 0$ (no matter which value is taken by the ratio b/a); however this needs not be the case any more when the fields' distribution is altered by the interaction. If the interaction turns the field φ into an antiferromagnetic field we may imagine that ψ and φ have very different interacting distributions and ψ will be smaller and smoother than φ. The difference between the interacting φ and ψ and the difference in scale between b and a can be used, perhaps, to take into account simultaneously the field strength and the coupling constant renormalizations.

We imagine extending the notion of φ_4^4 theory further by considering the measure:

$$P_N(d\varphi) = Z^{-1} \exp\left\{ - b^4 \sum_{\mathbf{x}\in b\mathbf{Z}^4\cap\mathbf{T}^4} (G_N : \psi_{\mathbf{x}}^4 : + (1/2)b^{-2}M_N : \psi_{\mathbf{x}}^2 :) \right.$$
$$\left. -(1/2)a^4 \sum_{\xi\in a\mathbf{Z}^4\cap\mathbf{T}^4} \left[a^{-2}M'_N : \varphi_\xi^2 : + \sum_{\mathbf{n}} \frac{A_N(\mathbf{n})}{2} : \left(\frac{\varphi_{\xi+\mathbf{n}a}-\varphi_\xi}{a}\right)^2 : \right] \right\} P_N^l(d\varphi). \quad (7.2)$$

with $A_N(\mathbf{n}) \equiv 0$ for $a\,|\,\mathbf{n}\,| > b$ (see (2.16)).

We call $P_N^*(d\psi)$ the ψ distribution in the measure (7.2) and we shall extend the notion of φ_4^4 theory as meaning also any limit P^*, as $N \to \infty$, of a sequence of measures of the form P_N^*.

First we observe that if $G_N = M_N = M'_N = A_N \equiv 0$ then the Schwinger functions of the field φ with respect to the measure $P_N^l(d\varphi)$ and those of the field ψ with respect to the measure $P_N^{*l}(d\psi)$ coincide in the limit $N \to \infty$ provided $\lim_{N\to\infty} b = 0$.

Secondly one notes that there are three formal power series

$$G_N(g) = g + \sum_{k=2}^{\infty} g^k \gamma_k^*(N) \quad (7.3)$$

$$M_N(g) = \sum_{k=2}^{\infty} g^k m_k^*(N), \qquad M'_N(g) = \sum_{k=2}^{\infty} g^k m_k'^*(N) \quad (7.3)$$

$$A_N(g, \mathbf{n}) = \sum_{k=2}^{\infty} g^k \alpha_k^*(N, \mathbf{n}) \quad (7.3)$$

such that if $b \to 0$ as $N \to \infty$, the formal power series in g for the Schwinger functions for the ψ-fields distributed as in (7.2) converge, order by order, as $N \to \infty$; furthermore the coefficients in (7.3) can be suitably chosen to obtain that the ψ-fields Schwinger functions converge as formal power series to the φ_4^4-theory Schwinger functions prescribed by the B. P. H. Z. choice of the counterterms. To obtain this it suffices to choose $\gamma_k^*(N)$,

$m_k^*(N) + m_k'^*(N)$, $\alpha_k^*(N, \mathbf{n}) = \delta_{|\mathbf{n}|,1} . \alpha_k^*(N)$ as in the B. P. H. Z. theory for φ_4^4 of type l.

This result is quite simple to obtain and it follows from the above mentioned fact that with respect to the free field measure $P^l(d\varphi)$ the fields φ and ψ are essentially equally distributed. But in perturbation theory what counts is precisely the free distribution of the field under examination and therefore it is not surprising that one cannot distinguish between φ and ψ from a perturbative point of view.

So we think that it would be reasonable to call the limits as $N \to \infty$ of the measures $P_N^*(d\psi)$ by the name of φ_4^4 theories; to distinguish them from the ones of type l, el, PV already considered we shall call them of type $\alpha = sl$ (« special lattice »). The notion of renormalization scheme, perturbative triviality or singularity are extended to this new case in the obvious way.

Imagine to have fixed a sequence $b \to 0$, $b = a^{2B}$, and a renormalization scheme: to study the actual convergence as $N \to \infty$ of $P_N^*(d\psi)$ we can think of considering $P_N^*(d\psi)$ as given by:

$$P_N^*(d\psi) = Z^{*-1} \left\{ \exp - gb^4 \sum_{\xi \in b\mathbf{Z}^4 \cap \mathbf{T}^4} :\psi_\xi^4: - M(g) b^4 \sum_{\xi \in b\mathbf{Z}^4 \cap \mathbf{T}^4} :\psi_\xi^2: \right\} P_{N,g}^{*l}(d\psi) \tag{7.4}$$

where the measure $P_{N,g}^{*l}(d\psi)$ is obtained by associating the quadratic counterterms in φ with the measure $P_N^l(d\varphi)$ in (7.2) and then integrating over φ at fixed ψ. The properties of the ψ-field in the distribution $P_{N,g}^{*l}(d\psi)$ may be radically different from those of the same field in the distribution $P_N^l(d\varphi)$ as already noted. In particular $|\psi_\xi|$ can be much less than $|\varphi_\xi|$; and this could be thought as a new way of taking the coupling constant renormalization into account (if $|\psi|/|\varphi| \sim \delta$, then the bare coupling constant can be thought as being $g\delta^4$ where δ depends on the quadratic counterterms, hence on g and N). We may also hope, particularly if we believe that the field φ is strongly oscillating, that the field ψ is more regular, i. e. that it has a covariance of the form (2.1) with $\mathbf{C}$ replaced by a lattice approximation to:

$$\int \frac{d^4\mathbf{p}}{(2\pi)^4} \frac{\exp i\,\mathbf{p}.(\mathbf{x}-\mathbf{y})}{1 + \mathbf{p}^2(1 + f_g(\mathbf{p}))} \tag{7.5}$$

for large N, where $f_g(\mathbf{p})$ is some unknown function such that $f_g(\mathbf{p}) \to +\infty$ as $|\mathbf{p}| \to \infty$ not too slowly, e. g. in such a way that:

$$\liminf_{\mathbf{p} \to \infty} f_g(\mathbf{p})/\log \mathbf{p}^2 > 0 \tag{7.6}$$

If this were true we could probably construct the corresponding ψ_4^4 field approximated by (7.4) because that theory is « logarithmically » asymptotically free.

The construction would however be, in the end, dependent on the arbitrary function $f_g(\mathbf{p})$: one should then determine f_g so that the theory admits a quantum field interpretation. This last step seems obscure to us: we hope to understand more about it by studying the problem of constructing the ψ_4^4 field (7.4) with $P_{N,g}^{*l}$ given by (7.5) (7.6). A clear understanding of the latter problem seems important to us as, in any event, it would show that one really has in his hands the techniques to deal constructively with a renormalizable (not superrenormalizable) « logarithmically » asymptotically free theory.

If one takes the above discussion literally one should first investigate whether ψ-fields with covariance given by (7.5) (7.6) can be realized from φ-fields of the form (7.2) with M'_N, $A_N \equiv 0$ and suitable b. If this is not true one could, nevertheless, pursue the program of determining f_g so that the field (7.4) has a limit as as $N \to \infty$ which verifies enough properties to admit a quantum field interpretation.

Another problem that should be analyzed is whether there is a formal power series for $f_g(\mathbf{p}^2)$ with coefficients which converge as $N \to \infty$ and which leads to Schwinger functions for ψ expressed as formal power series in g termwise convergent to a limit coinciding with the B. P. H. Z. renormalized formal power series for perturbative φ_4^4.

8. IS DESTRUCTIVE FIELD THEORY POSSIBLE ?

In this section with a slightly provocative title we would like to include, mainly for completeness, a general discussion on the precise mathematical content one can give to the notion of triviality of φ_4^4. We decided to include it, not because it presents any new idea, but because we have not found it in existing papers. In the previous sections we discussed what are the open problems concerning the « ordinary » φ_4^4-theories as defined in Sect. 2, Def. 1. Here we want to take a different, still more general, point of view. What could be called, in the broadest possible sense, a physical φ_4^4 euclidean field theory? It has to be a 4-dimensional purely scalar theory, and to verify some axioms like Osterwalder-Schrader's or Nelson's axioms [*13*] which ensure the existence of a corresponding Wightman theory in Minkowski space. Clearly this is not enough because one wants a φ_4^4 theory to correspond at least in some sense to a φ_4^4 lagrangian. Considering the historical and experimental importance of perturbation theory in field theories like QED_4 one would like a non trivial φ_4^4 to be at least asymptotic to the renormalized perturbation series computed according to a given prescription (B. P. H. Z., analytic renormalization, dimensional renormalization, or any other). Therefore we propose the following definition:

DEFINITION 5. — An (extended) $g\varphi_4^4$ euclidean field theory is a one-parameter family of theories, with $g \in [0, \varepsilon]$ or $g \in [-\varepsilon, 0]$, satisfying euclidean axioms, and such that their Schwinger functions are C^∞ at $g=0$ with the renormalized perturbation series (2.20) as Taylor series at $g = 0$.

Such theories are obviously non gaussian for small g; therefore with respect to this definition the sentence « φ_4^4 is trivial » means:

Strong triviality conjecture (or S. T. C.)

There does not exist any $g\varphi_4^4$ theory in the sense of Def. 5 ([4]).

Note that although the S. T. C. seems stronger than the (ordinary) triviality conjecture (T. C.) formulated in Sec. 2 after Def. 1, we have neither S. T. C. $\Rightarrow$ T. C. nor T. C. $\Rightarrow$ S. T. C.

We are not convinced that very strong arguments support the S. T. C. at present. Even after the works [*1*]-[*6*] and [*12*], we do not see how one could try to attack the proof of the S. T. C. with any idea or method existing today. Indeed there is a basic difference between constructive and destructive field theory (in the sense of [*12*]) which is to the advantage of the first one: namely in constructive field theory one has only to exhibit a particular construction with a particular cutoff, a particular bare action etc., which works for a given model. In contrast it is the burden of « destructive field theory » that in order to prove a negative statement like the S. T. C. it has to study in principle all possible ways to go to the continuum, including all possible regularizations, all possible bare actions etc., or to find an argument which works directly in the continuum. Since the first possibility sounds like an infinite and therefore impossible program let us discard it and examine the second. If we are to work directly in the continuum we cannot rely on the special form of the φ^4 lagrangian, since it is ill defined without the help of a cutoff. Therefore we do not know how to use the condition on asymptoticity in Def. 5 and the only possibility we see to prove in this way the S. T. C. is to prove something still stronger, namely the following:

Super-strong Triviality Conjecture (or S. S. T. C...).

There does not exist any 4-dimensional self-interacting purely scalar field theory.

If one uses the elegant version of euclidean axioms due to Nelson [*13*] the S. S. T. C. would imply that there is no non-gaussian covariant. Markov

([4]) Since there is always an infinite number of families of functions asymptotic to any given family of power series like (2.20), the S. T. C. simply states that in the case of the power series (2.20) none of these families satisfy, say. Osterwalder-Schrader's axioms.

process of Nelson's type on $S'(\mathbf{R}^4)$, a statement which is very general and apparently so far from any argument based on power counting, renormalizability of φ_4^4, and the fact that φ_4^4 is not asymptotically free, that we do not see at the moment any compelling reason to believe it at all. It is however an important question for mathematicians as well as physicists, and therefore one should expect that research will probably continue on this issue until a definitive proof or disproof of the S. S. T. C. has been found, even if the interest of most particle physicists has shifted in the last decade towards non abelian four dimensional gauge theory. It is clearly a very difficult problem (the existence of non-gaussian Markov processes satisfying Nelson's axioms on S' in dimensions lower than 4 has been proved only for some simple models like the sine-gordon model, and the global Markov property has not yet been verified for φ_2^4 and φ_3^4, although these theories have been shown to exist and to satisfy Osterwalder-Schrader's axioms). However serious attempts have been already made in the framework of the structural program (see [*14*] for a recent review) to obtain general theorems on quantum field theory which are independent of any particular Lagrangian formulation; they may yield in the future to such results as the S. S. T. C. (or its disproof).

Finally we would like to mention that there are other possible non-standard approaches to the construction of a non trivial φ_4^4 beside the ones based on the idea of « antiferromagnetism » that we have advocated in this paper. Even if one agrees to restrict the search for φ_4^4 theories to continuum limits of lattice theories with, say, nearest neighbor interaction, there is no overwhelming reason to use only bare actions of the φ^4 type as in (2.5). As pointed out in [*15*], lattice lagrangians are not always related in an obvious manner to their continuum limit counterparts. In the case of φ_4^4 one may believe that the triviality one seems to encounter in the standard approach of [*5*] [*6*] is related to the Landau argument ([5]). In its modern version [*17*] [*18*] this argument says that there should be singularities, called « renormalons », on the positive real axis for the analytic continuation of the Borel transform of the perturbative φ_4^4-series (this Borel transform is a well defined object at least near the origin [*11*]). If this is true, it destroys Borel summability in the ordinary sense, and one can expect a non trivial φ_4^4 to be only « weakly » asymptotic to its perturbative series. But this suggests also that if there is a way to remove these ultraviolet « renormalons » which preserves the basic axioms like O. S.

([5]) A proof of this, i. e. a rigorous connection between the Landau or the renormalization group arguments on triviality and the methods of Ref. [*5*] [*6*] would of course be extremely interesting. It might require the proof of new correlation inequalities based on the special form of the field equations [*16*], such as the ones which were conjectured in [*12*].

positivity, this might very well be the clue to the construction of a non trivial φ_4^4, and perhaps a big step forwards for the construction of non renormalizable field theories like φ_5^4 as well. In fact Parisi has proposed [*18*] to remove the « renormalons » precisely via the introduction of local operators of higher degree (like φ^6) in the bare lagrangian, with coefficients which are exponentially small in g as $g \to 0$. Therefore they should not show up in perturbation theory preserving the possibility for the final continuum theory to be asymptotic to the usual φ_4^4 perturbative series. Their role might however be crucial in permitting the cancellation of the ultraviolet « renormalons ». We simply remark that the triviality arguments in [*5*] [*6*] are limited to theories with a purely φ^4 bare action and therefore do not cover these kind of actions. In our opinion they are certainly as good candidates as the ones we proposed in the last sections for the construction of a non trivial φ_4^4. Progress in this direction might require a rigorous proof of existence and a better understanding of the structure of the « renormalon » singularities; we notice that this program has recently been carried out in the simpler context of « infrared renormalons » for the non-linear σ-model in 2 dimensions [*19*].

APPENDIX

Throughout this appendix, we assume the reader's familiarity with perturbative renormalization techniques in α-parametric space ([11] [20] [21]). Most of the time the results and notations of Ref. [11] will be used without any further explanation. We fix the regularization to be Pauli-Villars and we will come back briefly to the lattice regularization at the end of the appendix. A graph will always mean an unlabeled graph. A graph with labeled vertices will always be called a diagram.

Our aim is to prove Proposition 2 in Sect. 2. We should therefore study the dependence on the cutoff N of the counterterms used to build the power series (2.22). These counterterms are themselves built from contributions associated to proper (one-particle irreducible) superficially divergent graphs. More precisely one may write, if $k(G)$ is the number of internal vertices:

$$g^k\gamma_k(N) = (-g)^k \sum_{G/k(G)=k} c_G(N) \tag{A.1}$$

$$g^k\alpha_k(N) = (-g)^k \sum_{G/k(G)=k} d_G^1(N) \tag{A.1}$$

$$g^k m_k(N) = (-g)^k 2^{-2N} \sum_{G/k(G)=k} d_G^0(N) \tag{A.1}$$

where the sum is performed over proper « quadrupeds » in the first line and over proper « bipeds » in the second and third lines.

In the B. P. H. Z. subtraction scheme at 0 external momenta, these counterterms are defined by integrals in the α-parametric space. If we define $Z_G = exp\,[-V_G(p,\alpha)/U_G^2(\alpha)]$, the usual α-parametric integrand associated to G, and $d\mu^N(\alpha) = \prod_i [exp\,(-\alpha_i) - exp\,(-2^{2N}\alpha_i)]d\alpha_i$, the cutoff measure in α-space, we have:

$$c_G(N) = -\int_0^\infty d\mu^N(\alpha)(-\tau_G) \sum_{\mathbf{F}/G\notin\mathbf{F}} \prod_{F\in\mathbf{F}} (-\tau_F)Z_G \tag{A.2}$$

$$d_G^1(N) = -\int_0^\infty d\mu^N(\alpha)(-\tau_G^1) \sum_{\mathbf{F}/G\notin\mathbf{F}} \prod_{F\in\mathbf{F}} (-\tau_F)Z_G \tag{A.3}$$

$$d_G^0(N) = -\int_0^\infty d\mu^N(\alpha)(-\tau_G^0) \sum_{\mathbf{F}/G\notin\mathbf{F}} \prod_{F\in\mathbf{F}} (-\tau_F)Z_G \tag{A.4}$$

The sums in (A.2)-(A.4) are performed on the proper divergent forests $\mathbf{F}$ which do not contain G. Note that if G is a biped, it is quadratically divergent and we may write $\tau_G = \tau_G^0 + \tau_G^1$, where τ_G^0 is the Taylor operator which retains only the first term in the appropriate Taylor series defined in [11], and τ_G^1 retains only the second one. These operators are the ones which appear in formulae (A.3) and (A.4) and they are defined in [11]. The minus sign in front of (A.2)-(A.4) compensates the minus sign in front of G_N in (2.5). As in Ref. [11] we may get rid of non-essential subtractions associated to « open quadrupeds ». We use the following formulae:

$$\tau_F(1 - \tau_{F^*})Z_G \equiv 0 \tag{A.5}$$

$$\tau_{F^*}^1\tau_F Z_G \equiv 0 \tag{A.6}$$

where F is an open quadruped and F^* is its closure. (A.5) is proved in [11] and (A.6) is easy to prove and left as an exercise to the reader. Therefore in (A.2) and (A.3) the summations can be performed only on the closed divergent forests. The mass renormalization (A.4) will be studied separately. Apparently there are contributions from non closed graphs, but a careful study reveals that they disappear also since we used Wick ordered polynomials in the fields in the definition (2.5) of the interacting measure of our theory.

As in [11] we perform the Hepp's sectors splitting to obtain:

$$c_G(N) = \sum_{\sigma,\mathbf{F}} c_G^{\sigma,\mathbf{F}}(N) \tag{A.7}$$

$$d_G^1(N) = \sum_{\sigma,\mathbf{F}} d_G^{1,\sigma,\mathbf{F}}(N) \tag{A.8}$$

where:

$$c_G^{\sigma,\mathbf{F}}(N) = \int_{h_\sigma} d\mu^N(\beta) \prod_i \beta_i^{i-1}(+\tau_G) \prod_{F\in\mathbf{F}} (-\tau_F) \prod_{F\in\mathbf{H}(\mathbf{F}),\, F\neq G} (1-\tau_F) Z_G(\beta) \tag{A.9}$$

$$d_G^{1,\sigma,\mathbf{F}}(N) = \int_{h_\sigma} d\mu^N(\beta) \prod_i \beta_i^{i-1}(+\tau_G^1) \prod_{F\in\mathbf{F}} (-\tau_F) \prod_{F\in\mathbf{H}(\mathbf{F}),\, F\neq G} (1-\tau_F) Z_G(\beta) \tag{A.10}$$

with $h_\sigma = \{ \beta \mid 0 \le \beta_i \le 1 \text{ for } i < l(G),\ 0 \le \beta_l < \infty \}$, and:

$$d\mu^N(\beta) = \prod_i d\beta_i \left[exp\left(- \prod_{j=i}^{l} \beta_j \right) - exp\left(- 2^{2N} \prod_{j=i}^{l} \beta_j \right) \right] \tag{A.11}$$

In (A.7) and (A.8) the sum is performed over sectors σ and closed divergent forests which are skeleton forests for σ.

The divergence of (A.9) or (A.10) when $N \to \infty$ comes obviously from the $-\tau_G$ which replaces the operator $(1-\tau_G)$ in the corresponding formulae for renormalized amplitudes ([11]). Since the integrand Z_G has the FINE property [22], we may write (A.9) and (A.10) as:

$$c_G^{\sigma,\mathbf{F}}(N) = \int_{h_\sigma} d\mu^N(\beta) \prod_i \beta_i^{\omega_i(G,\sigma,\mathbf{F})} Z_G^{\sigma,\mathbf{F}}(\beta) \tag{A.12}$$

$$d_G^{1,\sigma,\mathbf{F}}(N) = \int_{h_\sigma} d\mu^N(\beta) \prod_i \beta_i^{\omega_i^1(G,\sigma,\mathbf{F})} Z_{G,1}^{\sigma,\mathbf{F}}(\beta) \tag{A.13}$$

with $\omega_i(G, \sigma, \mathbf{F})$ and $\omega_i^1(G, \sigma, \mathbf{F})$ integers, and $Z_G^{\sigma,\mathbf{F}}$ and $Z_{G,1}^{\sigma,\mathbf{F}}$ regular functions of the β's with a finite non zero limit at $\beta_i \equiv 0$. The divergence of these integrals when $N \to \infty$ is therefore related to the existence of negative ω_i's.

LEMMA A.1. — For any i, G, σ, $\mathbf{F}$, $\omega_i(G, \sigma, \mathbf{F}) \ge -1$ and $\omega_i^1(G, \sigma, \mathbf{F}) \ge -1$.

Proof. — We go back to the proof of Lemma III.8 in [11]. The only difference here is that we obtain $\omega_i \ge i + \dot\eta_i$ (resp. $\omega_i^1 \ge i + \dot\eta_i^1$) where $i + \dot\eta_i$ (resp. $i + \dot\eta_i^1$) is defined by the right hand side of equation (III.60) in [11], but with G missing in the third sum, over elements of Q′ (resp. with the factor 2 associated to G in the second sum, over elements of B′, replaced by a factor 1). Since it was proved in [11] that the right hand side of (III.60) was a positive integer, one gets Lemma A.1.

Let us call $k(G, \sigma, \mathbf{F})$, (resp. $k^1(G, \sigma, \mathbf{F})$) the number of factors $\omega_i(G, \sigma, \mathbf{F})$ (resp. $\omega_i^1(G, \sigma, \mathbf{F})$) which take the value -1 in (A.12) (resp. (A.13)). Then:

LEMMA A.2. — When $N \to \infty$, $c_G(\sigma, \mathbf{F})$ and $d_G^1(\sigma, \mathbf{F})$ behave asymptotically at most like $N^{k(G,\sigma,\mathbf{F})}$ and $N^{k^1(G,\sigma,\mathbf{F})}$ respectively.

Proof.— Since $Z_G^{\sigma,\mathbf{F}}$ is regular and independent of N, a simple inductive argument proves the statement. One could also use a more sophisticated Mellin transform argument [*22*].

To find the graphs with most divergent behavior as $N \to \infty$ we have to introduce the notion of a « parquet graph », related to the « parquet approximation » of Landau *et al.* [*1*], which is nothing but the sum of the amplitudes associated to such graphs.

DEFINITION A.3. — A parquet quadruped (resp. biped) G with k vertices is a quadruped (resp. biped) such that there exists a closed divergent forest $\mathbf{F} = \{ F_1 = G, F_2, \ldots, F_{k-1} \}$ with $k-1$ elements. Any such forest is called a complete forest of G.

It is not hard to see that by « minimality » of the graphs G_0 and G_1 in Fig. 1 among divergent graphs, for $2 \leq i \leq k-1$, $F_i/\mathbf{F}$ is isomorphic to G_0, and if G is a quadruped, (resp.

FIG. 1.

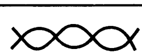
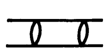
Parquet graph	Symmetry factor S	Leading log Coeff L	Product SL
	3/2	1	3/2
	3/4	1	3/4
	3	1/2	3/2
total, third order	15/4		9/4 = (3/2)^2
	3/8	1	3/8
	3/4	1/3	1/4
	3/2	1/2	3/4
	3/2	1/3	1/2
	3/2	1/3	1/2
	6	1/6	1
total, fourth order	93/8		27/8 = (3/2)^3

FIG. 2.

a biped) $F_1/\mathbf{F}$ is isomorphic to G_0 (resp. to G_1). Therefore one can generate all the parquet quadrupeds at order k by replacing any arbitrary vertex in any parquet quadruped of order $k-1$ by the full graph G_0, in all possible ways. Up to order 4, the graphs one obtains are pictured in Fig. 2, with the corresponding combinatoric factors. The parquet bipeds at order k are obtained by pasting together two external lines of the parquet quadrupeds of order k.

At a given order k the total number of graphs is roughly $(const.)^k k\,!$. But the parquet graphs are only a few compared to all others. Indeed:

LEMMA A.4. — The total number of parquet graphs at order k (counted with their right combinatoric factors) is bounded by $(const.)^k$.

Proof. — This is just a trivial consequence of the more general Theorem II in [*11*].

From now on, we shall often restrict ourselves to the case of quadrupeds when the corresponding statements for the bipeds are obtained by the trivial substitutions $c_G \to d_G^1$, $\omega_i(G, \sigma, \mathbf{F}) \to \omega_i^1(G, \sigma, \mathbf{F})$, parquet quadruped $\to$ parquet biped, etc...

LEMMA A.5. — For any G, $k(G, \sigma, \mathbf{F}) \leq k(G) - 1$; moreover, if $k(G, \sigma, \mathbf{F}) = k(G)-1$, G is a parquet quadruped and $\mathbf{F} \cup \{ G \}$ is a complete forest of G.

Proof. — Using inequality (III.59) in [*11*] one verifies that $\omega_i(G, \sigma, \mathbf{F})$ cannot be -1 unless $G_i^\sigma/\mathbf{F} = G/\mathbf{F}$, and unless for any $F \in \mathbf{F}$, $G_i^\sigma \cap F/\mathbf{F} = \phi$ or $G_i^\sigma \cap F/\mathbf{F} = F/\mathbf{F}$; therefore if $\omega_i(G, \sigma, \mathbf{F}) = -1$, G_i^σ is the union of $G/\mathbf{F}$ and a certain number of $F/\mathbf{F}$, $F \in \mathbf{F}$. Since the family G_i^σ is completely ordered by inclusion, this proves that $k(G, \sigma, \mathbf{F}) \leq 1 + |\, \mathbf{F} \,|$, where $|\, \mathbf{F} \,|$ is the number of elements of $\mathbf{F}$. By Definition A.3 this shows the Lemma.

DEFINITION A.6. — Given a parquet graph G with k vertices and $\mathbf{F}$ a complete forest of G, we say that a sector σ is compatible with $\mathbf{F}$ if and only if:

a) There exists a numbering $\bar{\sigma}$: $[1, k-1] \to \mathbf{F}$ of the forest $\mathbf{F}$, such that if we note F_i the graph $\bar{\sigma}(i)$:

$$G_{2p}^\sigma = \bigcup_{1 \leq i \leq p} F_i/\mathbf{F} \tag{A.14}$$

b) $\mathbf{F} - \{ G \}$ is a skeleton forest for σ.

Remark. — *i*) If G is a biped, equation (A.14) has to be replaced by $G_{2p+1}^\sigma = \bigcup_{1 \leq i \leq p} F_i/\mathbf{F}$.

ii) Condition *b*) implies that $F_1 = G$, and that $F_{k-1}/\mathbf{F} = F_{k-1}$, isomorphic to G_0. This is trivial if one remembers carefully the definition of skeleton forests in [*11*], Sect. III.1.

LEMMA A.7. — For a parquet graph G and a complete forest $\mathbf{F}$, we have:

$$\mathbf{F} - \{ G \} \text{ skeleton for } \sigma \text{ and } k(G, \sigma, \mathbf{F}) = k(G) - 1 \Leftrightarrow \sigma \text{ compatible with } \mathbf{F}.$$

Proof. — Suppose $k(G, \sigma, \mathbf{F} - \{ G \}) = k(G) - 1$, and G is a quadruped. We repeat the argument in the proof of Lemma A.5. For any i such that $\omega_i(G, \sigma, \mathbf{F}) = -1$, G_i^σ has to be the union $\bigcup_{F \in S(i)} F/\mathbf{F}$ over a certain subset $S(i)$ of $\mathbf{F}$ of reduced graphs of $\mathbf{F}$. Since any $F/\mathbf{F}$ is isomorphic to G_0 which has two lines, i has to be even. Since there are exactly $k-1$ even indices between 1 and $l = 2k-2$, $\omega_{2p}(G, \sigma, \mathbf{F})$ has to be -1 for any $1 \leq p \leq k-1$. Defining $\bar{\sigma}(p) = S(2p) - S(2p-2)$ we see that (A.14) is satisfied.

Conversely, let σ be compatible with $\mathbf{F}$. Let us number the lines of G so that

$$F_p/\mathbf{F} = \{ 2p-1, 2p \}.$$

Since $\tau_G Z_G = U_G^{-2}$ and $\tau_S[U_G^{-2}] = U_S^{-2}U_{G/S}^{-2}$, we have in α parametric space:

$$\prod_{p=1}^{k-1}(-\tau_{F_p})Z_G = (-1)^{k-1}\prod_{p=1}^{k-1}(\alpha_{2p-1}+\alpha_{2p})^{-2} \tag{A.15}$$

Therefore formula (A.12) simplifies to:

$$c_G^{\sigma,\mathbf{F}} = (-1)^k\int_{h_\sigma} d\mu^N(\beta)\prod_{p=1}^{k-1}\beta_{2p}^{-1}(1+\beta_{2p-1})^2 \equiv (-1)^k c_k(N) \tag{A.16}$$

which proves that $k(G, \sigma, \mathbf{F} - \{G\}) = k - 1$.

The generalization to the biped case is slightly different. We have to number the lines of G so that $F_1/\mathbf{F} = G/\mathbf{F} = \{1, 2, 3\}$, and $F_p/\mathbf{F} = \{2p, 2p+1\}$ for $p \geq 2$.

The analogous of (A.14) is:

$$(-\tau_{F_1}^1)\prod_{p=2}^{k-1}(-\tau_{F_p})Z_G = (-1)^k\frac{\alpha_1\alpha_2\alpha_3}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^3}\prod_{p=2}^{k-1}(\alpha_{2p}+\alpha_{2p+1})^{-2} \tag{A.17}$$

There is one extra factor (-1) in (A.17) due to the derivation of $\exp\left(\frac{-V_G(p,\alpha)}{U_G(\alpha)}\right)$ in the action of $\tau_{F_1}^1 = \tau_G^1$ on Z_G. The analogous of (A.16) is therefore:

$$d_G^{1,\sigma,\mathbf{F}} = (-1)^{k+1}\int_{h_\sigma} d\mu^N(\beta)\frac{\beta_1}{\beta_3(1+\beta_1+\beta_1\beta_2)^3}\prod_{p=2}^{k-1}\beta_{2p+1}^{-1}(1+\beta_{2p})^{-2} \equiv (-1)^{k+1}d_k^1(N) \tag{A.18}$$

An important consequence of this proof of Lemma A.7 is that the contributions $c_G^{\sigma,\mathbf{F}}$ and $d_G^{1,\sigma,\mathbf{F}}$ depend only on $k(G)$, not on the particular structure of G, σ, $\mathbf{F}$. Moreover from formulae (A.16) and (A.18) one can easily deduce that c_k and d_k^1 behave really like N^{k-1} at large N; more precisely:

$$\lim_{N\to\infty}\frac{c_k(N)}{N^{k-1}} = c_k > 0\,; \qquad \lim_{N\to\infty}\frac{d_k^1(N)}{N^{k-1}} = d_k^1 > 0 \tag{A.19}$$

In the next Lemma, we evaluate these constants c_k and d_k^1; the computation of the leading-log behavior of $\gamma_k(N)$ and $\alpha_k(N)$ will therefore be reduced to a pure combinatoric problem. First let us remark that by an elementary analysis:

$$\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2\frac{[\exp-(\alpha_1+\alpha_2)-\exp-2^{2N}(\alpha_1+\alpha_2)]}{(\alpha_1+\alpha_2)^2} = aN + b(N) \tag{A.20}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\frac{[\exp-(\alpha_1+\alpha_2+\alpha_3)-\exp-2^{2N}(\alpha_1+\alpha_2+\alpha_3)]\alpha_1\alpha_2\alpha_3}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^3} = a'N + b'(N) \tag{A.21}$$

where $b(N)$ and $b'(N)$ are bounded as $N \to \infty$. Then we claim:

LEMMA A.8. —

$$c_k = (a/2)^{k-1}\frac{1}{(k-1)\,!} \tag{A.22}$$

$$d_k^1 = a'(a/2)^{k-2}\frac{1}{(k-1)\,!} \tag{A.23}$$

Proof. — Let us prove only (A.22) since the proof of (A.23) is similar. By (A.15), we have for G parquet quadruped and $\mathbf{F}$ complete forest:

$$\int_0^\infty d\mu^N(\alpha) \prod_{p=1}^{k-1} (-\tau_{F_p}) Z_G = [aN + b(N)]^{k-1} \tag{A.24}$$

But if we develop the left hand side of (A.24) in Hepp's sectors, we can analyze its leading-log behavior as $N \to \infty$ in the same way as we did for an ordinary graph, and conclude that it is a sum over all sectors satisfying condition *a)* in Definition A.6 of terms attached to these sectors; furthermore any of these terms is equal to c_k. Since there are $2^{k-1}(k-1)!$ sectors verifying (A.14) and since the leading-log behavior in the right hand side of (A.24) is obviously $(aN)^{k-1}$, equation (A.22) must hold.

Now the proof of formulae (2.32) and (2.33) is reduced to a combinatoric problem which is solved by one further Lemma. It is easy to verify that putting together formula (A.1) and Lemmas A.1 to A.9 leads indeed to (2.32) and (2.33).

LEMMA A.9. — There exists three positive constants b, c, d, such that:

$$\# \{ (G, \sigma, \mathbf{F}) \}_{4,k} = cb^{k-1}(k-1)! \tag{A.25}$$

$$\# \{ (G, \sigma, \mathbf{F}) \}_{2,k} = db^{k-1}(k-1)! \tag{A.26}$$

where the left hand side of (A.25) is the number of triplets made of a parquet quadruped with k vertices, counted with its right multiplicity factor, a complete forest $\mathbf{F}$ of G and a sector σ compatible with $\mathbf{F}$ (in (A.26) G is a biped instead of a quadruped).

Proof. — We prove only (A.25), by induction on k, since the proof of (A.26) is almost identical. We return to diagrams, which are directly associated to Wick contractions and therefore easier to count without mistakes; since there are $k!$ diagrams associated to a graph G, we have to show, for $k \geqslant 3$:

$$\# \{ (\Gamma, \mathbf{F}, \sigma) \}_{4,k} = k(k-1)b [\# \{ (\Gamma, \mathbf{F}, \sigma) \}_{4,k-1}] \tag{A.27}$$

But by Remark *ii)* at the end of Definition (A.6) there is a natural mapping $\mathbf{f}$ from $\{ (\Gamma, \mathbf{F}, \sigma) \}_{4,k}$ into $\{ (\Gamma, \mathbf{F}, \sigma) \}_{4,k-1}$: $\mathbf{f}$ indeed associates to $(\Gamma, \mathbf{F}, \sigma)$ the triplet $(\Gamma', \mathbf{F}', \sigma')$ with $\Gamma' = \Gamma/F_{k-1}$, $\mathbf{F}' = \{ F_p/F_{k-1} \}$, $\sigma' = \sigma$ restricted to Γ'. It is not hard to verify that if $\mathbf{F}$ is skeleton for σ, with this definition of $\mathbf{f}$, $\mathbf{F}'$ is skeleton for σ'. To specify completely the set of Wick contractions which determine Γ', we have also to fix the numbering of the vertices of Γ' and of every half-line attached to them. Suppose the two vertices of F_{k-1} in Γ have indices i and $j > i$; the half-lines attached to i (resp. to j) are numbered as $i1, i2, i3, i4$ (resp. as $j1, j2, j3, j4$). Since F_{k-1} is isomorphic to G_0, we have two half-lines, say $i2$ and $i4$, attached to respectively $j3$ and $j2$ for instance. Then we number the vertex in Γ' corresponding to the reduction of F_{k-1} as i; and we number its half-lines in the following way: the lines $i1$ and $i3$ need not to be changed and keep their initial numbers; the other ones, which are $i2$ and $i4$ in the example chosen, contract to this half-line which in Γ contracted to $j4$ and to $j1$; this choice corresponds in fact to associate disjoint ordered pairs of $\{ j1, j2, j3, j4, \}$ in the unique way which respects relative ordering. Finally the vertices of Γ' constructed in this way have indices running from 1 to k with one missing value; we renumber them from 1 to $k-1$, preserving the order. This ends the complete specification of Γ'. It is easy to verify that f is onto; moreover for any $(\Gamma', \mathbf{F}', \sigma')$, we will show that $\{ \mathbf{f}^{-1}(\Gamma', \mathbf{F}', \sigma') \}$ has exactly $72 . k(k-1)$ elements. Indeed to construct Γ from Γ', we can choose any pair of indices i and j between 1 and k (there are $k(k-1)/2$ possible choices). The lowest one, say i, tells us where to insert $G_0 = F_{k-1}$ in Γ'; the second one, what number we should assign to the second vertex of F_{k-1} in Γ. It is not difficult now to verify that there are exactly 72 ways to number the half-lines of i and j in Γ so that the reverse process described above gives back the initial numbering of the half-lines of i in Γ', therefore so that $\mathbf{f}(\Gamma) = \Gamma'$. Please note that F is uniquely determined by F′, and that there are exactly two possible σ's corresponding to σ'. This ends the proof.

Taking together Lemma A.8 and A.9 we conclude that the constant A which tells the leading-log behavior in (2.32) and (2.33) is exactly $[3a/2].4\,!$, where a is defined by (A.20). We can get rid of the 4 ! by using the reduced coupling constant $g' = 4\,!\,g$. Then the relevant combinatoric factor which exponentiates is just 3/2. This is shown in detail up to fourth order in Fig. 2 where the different parquet quadrupeds with their multiplicities are pictured. It is striking to see that the number of parquet graphs itself does not seem to obey any simple recursive relation, in contrast with the number of triplets (Γ, F, σ), which is the right combinatoric quantity for the leading-log behavior.

The reader already familiar with this behavior may find our derivation of it quite lengthy and cumbersome, but we did not know of any such proof in the existing literature, with a detailed description of the contributions of each particular graph and a precise solution of the couting problem. However we are not going to reproduce this painful computation for the coefficient $m_k(N)$ in equation (2.34); it is easy to convince oneself that this computation involves parquet bipeds and is almost similar to the one for $\alpha_k(N)$, with the two remarkable changes that a divergent factor 2^{2N} is present in $d_G^0(N)$ (but not in $m_k(N)$, according to (A.1)) and that the sign is different, being now positive as for γ_k.

Still another problem would be to verify the results of this Appendix when the lattice regularization is used instead of the Pauli-Villars. But since the computation of γ_k, α_k, m_k involve only sums over finitely many graphs with a bounded number of propagators, and since the lattice and the Pauli-Villars propagator converge uniformly to the continuum propagator as the cutoff is removed, the answer has to be the same in both cases.

A more interesting and final remark is that it should be possible to prove reasonable upper bounds on all non-leading-log contributions to the counterterms by using the full machinery of Ref. [*11*]. We conjecture that for instance the terms in γ_k which are proportional to $N^p, p < k - 1$, are absolutely bounded by $(k - p)\,!\,(const.)^k . N^p$, where the constant might however be quite large (bigger than A) for technical reasons in the evaluations. If true, this conjecture would prove admissibility of the truncations schemes which verify:

$$\tau_G(N) \leq const. \log (N) \tag{A.28}$$

a rather strong condition however, if one remembers that N is already the log of the cutoff. If the conjecture of Parisi [*18*] that the « renormalon » singularity dominates the « instanton » singularity in the Borel plane of φ_4^4 is correct, a weaker condition should be sufficient in order to have admissibility, namely:

$$\tau_G(N) \leq const.\ N \tag{A.29}$$

However to prove the Parisi conjecture requires a finer analysis than the one available in [*11*] and is in our opinion a very difficult program, although probably not out of reach.

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Note added

After completion of this paper we learned from J. Fröhlich that one can use the lattice Källen-Lehman representation to show that the continuum limit of the two-point function for a lattice model with *nearest-neighbor* antiferromagnetic interaction of the type considered in Sect. 5 is generally a Dirac function, the resulting theory is therefore uninteresting (white noise). This argument does not extend to the other cases considered in this paper, for which the mechanism of convergence through unbounded oscillations considered in Sect. 5 might be relevant. We thank J. Fröhlich for communicating to us this result.

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