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A rigorous approach to relativistic corrections of bound state energies for spin-1/2 particles

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ABSTRACT. — Under fairly general conditions on the interactions we prove holomorphy of the Dirac resolvent around its nonrelativistic limit. As a consequence, perturbation theory in terms of resolvents (instead of Hamiltonians) yields holomorphy of Dirac eigenvalues and eigenfunctions with respect to c^{-1} and a new method of calculating relativistic corrections to bound state energies. Due to a formulation in an abstract setting our method is applicable in many different concrete situations. In particular our approach covers the case of the relativistic hydrogen atom in external electromagnetic fields.

RÉSUMÉ. — Sous des hypothèses assez générales sur les interactions, on démontre l'holomorphie de la résolvante de l'opérateur de Dirac au voisinage de sa limite non relativiste. En conséquence, la théorie des perturbations en termes des résolvantes (au lieu des hamiltoniens), fournit l'holomorphie par rapport à c^{-1} des valeurs propres et des fonctions propres de l'opérateur de Dirac, et une nouvelle méthode de calcul des corrections relativistes aux énergies de liaison. Grâce à sa formulation dans un cadre

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abstrait, notre méthode s'applique à un grand nombre de situations concrètes différentes. En particulier elle couvre le cas de l'atome d'hydrogène relativiste dans des champs électromagnétiques externes.

1. INTRODUCTION

The aim of this paper is to discuss the family of abstract Dirac operators H(c), $c \in \mathbb{R}\setminus\{0\}$, (cf. eq. (2.10)) in a neighbourhood of $c^{-1} = 0$. In applications c is the velocity of light and H(c) describes the behaviour of spin-1/2 particles in external electromagnetic fields. There exist formal schemes trying to expand the concrete Dirac operator $H^D(c)$ (eq. (3.15)) into powers of c^{-1} and showing that the nonrelativistic limit $c^{-1} = 0$ is formally given by the Pauli operator H_∞ (cf. eq. (3.16)) [*4*] [*13*]. In spite of the fact that $H^D(c)$ as a function of c^{-1} is not holomorphic around $c^{-1} = 0$ one has obtained relativistic perturbations of H_∞ in form of a power series in c^{-2}. These corrections are, however, very singular. Adding the c^{-2}-perturbation already destroys all spectral properties of H_∞, lower semiboundedness turns into upper semiboundedness, and the discrete eigenvalues dissolve into a continuous spectrum in the interval $(-\infty, mc^2/2]$. Higher order corrections make the situation even worse. Nevertheless, the results on the relativistic corrections of the eigenvalues calculated by means of formally applied perturbation theory can be interpreted in terms of spectral concentration [*5*] [*6*] and yield reasonable numerical results [*13*]. This approach is not quite satisfactory for the following reasons:

i) It seems to be difficult if not impossible to justify the formal manipulations of [*4*], and *ii*) it is well-known that the Dirac operator e. g. in the hydrogen case has eigenvalues which are holomorphic functions with respect to c^{-2}. Therefore, since bound states are turned into resonances, the addition of relativistic « perturbations » to H_∞ qualitatively does not reproduce the actual properties of the Dirac operator $H^D(c)$.

In the following we present an alternative method of investigating the dependence of eigenvalues and eigenfunctions of H(c) on the parameter c^{-1}. Our approach is based on an extension of the results of Veselic [*24*] and Hunziker [*11*] concerning the holomorphy of the Dirac resolvent $(H(c) - mc^2 - z)^{-1}$ with respect to c^{-1} under certain conditions on the external fields. In section 2 we prove holomorphy of the Dirac resolvent and of related quantities for a wide class of interactions including e. g. Coulomb-like potentials. It turns out that the Dirac resolvent converges in norm to the Pauli resolvent times a projector (cf. eq. (2.17)). After deriving the explicit expansion of the Dirac resolvent around its non-relativistic limit and reformulating perturbation theory in terms of resolvents we are able to obtain relativistic correction formulas (2.30), (2.32) for the eigenvalues of the Pauli operator. We conclude section 2 by sketching the derivation of estimates on the convergence radius of the expansion of the resolvent into powers of c^{-1}.

Since all calculations of section 2 are performed in the abstract representation introduced in refs. [*11*] [*2*] our results are applicable for a great variety of situations including Dirac operators over Riemannian manifolds [*2*]. In section 3 we discuss applications to concrete realizations of the abstract Dirac operator H(c) and investigate the relations to the conventional approach of refs. [*4*] [*13*]. A short outline of these results already appeared in ref. [*7*].

We finally mention some related work. Strong convergence of the unitary groups associated with the Klein-Gordon and Dirac theory has been derived by Cirincione and Chernoff [*2*]. Schöne [*18*] investigated solutions of the Klein-Gordon and Dirac equations in the nonrelativistic limit by means of different methods. An investigation of scattering theory as $c^{-1} \to 0$ can be found in refs. [*23*] [*26*]. Quite recently Veselic [*24*] gave a detailed description of the Klein-Gordon case. The nonrelativistic limit of proper-time quantum-mechanics has been treated by Horwitz and Totbart [*10*], Steinmann [*20*] discussed a c^{-1}-expansion of bound state energies in quantum-electrodynamics which partly motivated our work.

2. THE ABSTRACT APPROACH

Following Hunziker [*11*] and Cirincione and Chernoff [*2*] we introduce an abstract setting for the Dirac operator.

Let α and β be self-adjoint operators in some (complex) Hilbert space $\mathscr{H}$ and assume

$$\beta^2 = 1 \tag{2.1}$$

(i. e. $\beta \in \mathscr{B}(\mathscr{H})$) and the commutation relation

$$\alpha\beta + \beta\alpha \subseteq 0 . \tag{2.2}$$

Due to eq. (2.1) $P_\pm$ defined by

$$P_\pm = (1 \pm \beta)/2 \tag{2.3}$$

obey

$$P_\pm^2 = P_\pm , \qquad P_+P_- = P_-P_+ = 0 \tag{2.4}$$

and we introduce

$$\mathscr{H}_\pm = P_\pm \mathscr{H} , \qquad \mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_- . \tag{2.5}$$

As a consequence of self-adjointness and (2.1)-(2.4) α and β can be written with respect to the decomposition (2.5)

$$\alpha = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6}$$

where A is a densely defined closed operator from $\mathcal{H}_+$ into $\mathcal{H}_-$. Next we define

$$H^0(c) = c\alpha + mc^2\beta, \qquad \mathcal{D}(H^0(c)) = \mathcal{D}(\alpha), \qquad c \in \mathbb{R}\backslash\{0\}, \qquad m > 0 \quad (2.7)$$

and

$$V = \begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix} \quad (2.8)$$

where $V_\pm$ denote self-adjoint operators in $\mathcal{H}_\pm$ (note that obviously V commutes with β). If in addition we assume relative boundedness of V_+ (resp. V_-) with respect to A (resp. A^*) i. e.

$$\mathcal{D}(V_+) \supseteq \mathcal{D}(A), \qquad \mathcal{D}(V_-) \supseteq \mathcal{D}(A^*) \quad (2.9)$$

then the abstract Dirac operator $H(c)$ defined as

$$H(c) = H^0(c) + V, \qquad \mathcal{D}(H(c)) = \mathcal{D}(\alpha) \quad (2.10)$$

is self-adjoint for $|c|$ large enough.

Finally we also introduce

$$H_+^0 = (2m)^{-1}A^*A, \qquad H_-^0 = (2m)^{-1}AA^* \quad (2.11)$$

and the pair of abstract Pauli operators

$$H_+ = H_+^0 + V_+, \qquad \mathcal{D}(H_+) = \mathcal{D}(A^*A), \quad (2.12)$$

$$H_- = H_-^0 + V_-, \qquad \mathcal{D}(H_-) = \mathcal{D}(AA^*). \quad (2.13)$$

Due to assumption (2.9) V_+ (resp. V_-) is infinitesimally bounded with respect to H_+^0 (resp. H_-^0) and hence $H_\pm$ are self-adjoint. Clearly $H^0(c)$ has the spectral gap $(-mc^2, +mc^2)$ and by (2.9) one can prove that for $|c|$ large enough also $H(c)$ has a spectral gap containing zero (cf. [2]).

Now we state some commutation formulas proved by Deift [3] which are needed several times later:

$$\begin{aligned} (A^*A - z)^{-1}A^* &\subseteq A^*(AA^* - z)^{-1}, \\ (AA^* - z)^{-1}A &\subseteq A(A^*A - z)^{-1}, \\ A(A^*A - z)^{-1}A^* &\subseteq 1 + z(AA^* - z)^{-1}, \\ A^*(AA^* - z)^{-1}A &\subseteq 1 + z(A^*A - z)^{-1}, \end{aligned}$$

for all $z \in \rho(A^*A)\backslash\{0\} = \rho(AA^*)\backslash\{0\}$. (2.14)

The bounded operators on the right hand side of (2.14) are just the closures of the densely defined operators on the left hand side. In the following we freely use such densely defined operators instead of their bounded closures if no confusions arise.

Now we are prepared to state

THEOREM 2.1. — Let $H(c)$ be defined as in eq. (2.10) and assume

$z \in \mathbb{C}\backslash\mathbb{R}$. Then $(\mathrm{H}(c) - mc^2 - z)^{-1}$ is holomorphic with respect to c^{-1} around $c^{-1} = 0$, and

$$(\mathrm{H}(c) - mc^2 - z)^{-1} = \\ = \left\{ 1 + \begin{pmatrix} 0 & (2mc)^{-1}(\mathrm{H}_+ - z)^{-1}\mathrm{A}^*(\mathrm{V}_- - z) \\ (2mc)^{-1}\mathrm{A}(\mathrm{H}_+^0 - z)^{-1}\mathrm{V}_+ & (2mc^2)^{-1}z(\mathrm{H}_-^0 - z)^{-1}(\mathrm{V}_- - z) \end{pmatrix} \right\}^{-1} \cdot \\ \cdot \begin{pmatrix} (\mathrm{H}_+ - z)^{-1} & (2mc)^{-1}(\mathrm{H}_+ - z)^{-1}\mathrm{A}^* \\ (2mc)^{-1}\mathrm{A}(\mathrm{H}_+^0 - z)^{-1} & (2mc^2)^{-1}z(\mathrm{H}_-^0 - z)^{-1} \end{pmatrix} \tag{2.15}$$

Proof. — Let $z \in \mathbb{C}\backslash\mathbb{R}$ then

$$(\mathrm{H}(c) - mc^2 - z)^{-1} = \left(\begin{pmatrix} -z & c\mathrm{A}^* \\ c\mathrm{A} & -2mc^2 \end{pmatrix} + \begin{pmatrix} \mathrm{V}_+ & 0 \\ 0 & \mathrm{V}_- - z \end{pmatrix} \right)^{-1}.$$

From (2.14) one infers

$$\begin{pmatrix} -z & c\mathrm{A}^* \\ c\mathrm{A} & -2mc^2 \end{pmatrix}^{-1} = \begin{pmatrix} 2m(\mathrm{A}^*\mathrm{A} - 2mz)^{-1} & c^{-1}\mathrm{A}^*(\mathrm{A}\mathrm{A}^* - 2mz)^{-1} \\ c^{-1}\mathrm{A}(\mathrm{A}^*\mathrm{A} - 2mz)^{-1} & c^{-2}z(\mathrm{A}\mathrm{A}^* - 2mz)^{-1} \end{pmatrix} \in \mathscr{B}(\mathscr{H})$$

and thus

$$(\mathrm{H}(c) - mc^2 - z)^{-1} = \\ = \left\{ 1 + \begin{pmatrix} 2m(\mathrm{A}^*\mathrm{A} - 2mz)^{-1}\mathrm{V}_+ & c^{-1}\mathrm{A}^*(\mathrm{A}\mathrm{A}^* - 2mz)^{-1}\mathrm{V}_- - z) \\ c^{-1}\mathrm{A}(\mathrm{A}^*\mathrm{A} - 2mz)^{-1}\mathrm{V}_+ & c^{-2}z(\mathrm{A}\mathrm{A}^* - 2mz)^{-1}(\mathrm{V}_- - z) \end{pmatrix} \right\}^{-1} \\ \cdot \begin{pmatrix} 2m(\mathrm{A}^*\mathrm{A} - 2mz)^{-1} & c^{-1}\mathrm{A}^*(\mathrm{A}\mathrm{A}^* - 2mz)^{-1} \\ c^{-1}\mathrm{A}(\mathrm{A}^*\mathrm{A} - 2mz)^{-1} & c^{-2}z(\mathrm{A}\mathrm{A}^* - 2mz)^{-1} \end{pmatrix}. \tag{2.16}$$

Since for $|\,\mathrm{Im}\, z\,|$ large enough

$$\begin{pmatrix} 1 + 2m(\mathrm{A}^*\mathrm{A} - 2mz)^{-1}\mathrm{V}_+ & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} [1 + 2m(\mathrm{A}^*\mathrm{A} - 2mz)^{-1}\mathrm{V}_+]^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{B}(\mathscr{H})$$

(2.16) implies eq. (2.15) for $|\,\mathrm{Im}\, z\,|$ large enough. Note that all matrix elements in (2.15) have bounded closures, e. g.

$$(\mathrm{H}_+ - z)^{-1}\mathrm{A}^*\mathrm{V}_- \subseteq \{ \mathrm{V}_- [\mathrm{A}(\mathrm{H}_+ - \bar{z})^{-1}] \}^*$$

and

$$\|\mathrm{V}_- [\mathrm{A}(\mathrm{H}_+ - \bar{z})^{-1}]\| \leq \|\mathrm{V}_-((\mathrm{A}\mathrm{A}^*)^{1/2} + 1)^{-1}\| \, \|((\mathrm{A}\mathrm{A}^*)^{1/2} + 1)\mathrm{A}(\mathrm{H}_+ - \bar{z})^{-1}\| < \infty$$

since $\mathscr{D}(\mathrm{H}_+) = \mathscr{D}(\mathrm{A}^*\mathrm{A})$ and $\mathscr{D}((\mathrm{A}\mathrm{A}^*)^{1/2}) = \mathscr{D}(\mathrm{A}^*)$. Finally holomorphic continuation of both sides in (2.15) with respect to $z \in \mathbb{C}\backslash\mathbb{R}$ removes the restriction on $|\,\mathrm{Im}\, z\,|$. Consequently (2.15) proves holomorphy of $(\mathrm{H}(c) - mc^2 - z)^{-1}$ in c^{-1} around $c^{-1} = 0$ (in a z-dependent neighbourhood of $c^{-1} = 0$) for $z \in \mathbb{C}\backslash\mathbb{R}$.

Since the nonrelativistic limit $c \to \infty$ is of particular interest we state

COROLLARY 2.1. — As $c \to \infty$ the Dirac operator (rest energy subtracted) $H(c) - mc^2$ converges in norm resolvent sense to the Pauli operator H_+ times the projector onto $\mathcal{H}_+$

$$n\text{-}\lim_{|c|\to\infty} (H(c)-mc^2-z)^{-1}=(H_+-z)^{-1}\otimes\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad z\in\mathbb{C}\backslash\mathbb{R}. \quad (2.17)$$

In particular $\lambda_0\in\sigma(H_+)$ (resp. $\sigma_{ess}(H_+)$) if and only if there exists a sequence $\lambda(c)\in\sigma(H(c)-mc^2)$ (resp. $\sigma_{ess}(H(c)-mc^2)$) which converges to λ_0 as $c \to \infty$.

Proof. — Eq. (2.17) results from (2.15). Taking $-z_0>0$ large enough then $z_0\in\rho(H_+)\cap\rho(H(c)-mc^2)$ for $|c|$ large enough and convergence of the spectrum (as well as the essential spectrum) follows from norm convergence of self-adjoint operators ([*16*, p. 289], [*25*, p. 272]). □

Expanding the right hand side of (2.15) yields

$$(H(c)-mc^2-z)^{-1}= \\ =\begin{pmatrix}(H_+-z)^{-1} & 0\\ 0 & 0\end{pmatrix}+c^{-1}\begin{pmatrix}0 & (2m)^{-1}(H_+-z)^{-1}A^*\\ (2m)^{-1}A(H_+-z)^{-1} & 0\end{pmatrix}+ \\ +O(c^{-2}). \quad (2.18)$$

Theorem 2.1 and Corollary 2.1 describe the Dirac resolvent and its non-relativistic limit. To treat relativistic bound state energies we need some additional results. Following Hunziker [*11*] we first give

LEMMA 2.1. — Let

$$B(c)=\begin{pmatrix}1 & 0\\ 0 & c\end{pmatrix}$$

then under the conditions of Theorem 2.1, $B(c)(H(c)-mc^2-z)^{-1}B(c)^{-1}$ is holomorphic with respect to c^{-2} around $c^{-2}=0$ for fixed $z\in\mathbb{C}\backslash\mathbb{R}$

$$B(c)(H(c)-mc^2-z)^{-1}B(c)^{-1}= \\ =\left\{1+\begin{pmatrix}0 & (2mc^2)^{-1}(H_+-z)^{-1}A^*(V_--z)\\ 0 & (2mc^2)^{-1}[(2m)^{-1}A(H_+-z)^{-1}A^*-1](V_--z)\end{pmatrix}\right\}^{-1}\cdot \\ \cdot\begin{pmatrix}(H_+-z)^{-1} & (2mc^2)^{-1}(H_+-z)^{-1}A^*\\ (2m)^{-1}A(H_+-z)^{-1} & (2mc^2)^{-1}[(2m)^{-1}A(H_+-z)^{-1}A^*-1]\end{pmatrix}. \quad (2.19)$$

Proof. — Eq. (2.15) implies

$$B(c)(H(c)-mc^2-z)^{-1}B(c)^{-1}= \\ =\left\{1+\begin{pmatrix}0 & (2mc^2)^{-1}(H_+-z)^{-1}A^*(V_--z)\\ (2m)^{-1}A(H_+^0-z)^{-1}V_+ & (2mc^2)^{-1}z(H_-^0-z)^{-1}(V_--z)\end{pmatrix}\right\}^{-1}\cdot \\ \cdot\begin{pmatrix}(H_+-z)^{-1} & (2mc^2)^{-1}(H_+-z)^{-1}A^*\\ (2m)^{-1}A(H_+^0-z)^{-1} & (2mc^2)^{-1}z(H_-^0-z)^{-1}\end{pmatrix}.$$

From

$$\begin{pmatrix} 1 & 0 \\ (2m)^{-1}A(H_+^0 - z)^{-1}V_+ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -(2m)^{-1}A(H_+^0 - z)^{-1}V_+ & 1 \end{pmatrix}$$

one infers

$$B(c)(H(c) - mc^2 - z)^{-1}B(c)^{-1} =$$

$$= \left\{ 1 + \begin{pmatrix} 0 & \mathscr{K} \\ 0 & \mathscr{L} \end{pmatrix} \right\}^{-1} \cdot \begin{pmatrix} (H_+ - z)^{-1} & (2mc^2)^{-1}(H_+ - z)^{-1}A^* \\ \mathscr{M} & \mathscr{N} \end{pmatrix}$$

$$\mathscr{K} = (2mc^2)^{-1}(H_+ - z)^{-1}A^*(V_- - z)$$

$$\mathscr{L} = -(2mc)^{-2} A(H_+^0 - z)^{-1}V_+(H_+ - z)^{-1}A^*(V_- - z) + (2mc^2)^{-1}z(H_-^0 - z)^{-1}(V_- - z)$$

$$\mathscr{M} = -(2m)^{-1}A(H_+^0 - z)^{-1}V_+(H_+ - z)^{-1} + (2m)^{-1}A(H_+^0 - z)^{-1}$$

$$\mathscr{N} = -(2mc)^{-2}A(H_+^0 - z)^{-1}V_+(H_+ - z)^{-1}A^* + (2mc^2)^{-1}z(H_-^0 - z)^{-1}$$

This proves (2.19) if we use

$$-(H_+^0 - z)^{-1}V_+(H_+ - z)^{-1} = (H_+ - z)^{-1} - (H_+^0 - z)^{-1}$$

and eqs. (2.14). □

First order expansion of (2.19) yields

$$B(c)(H(c) - mc^2 - z)^{-1}B(c)^{-1} = \begin{pmatrix} (H_+ - z)^{-1} & 0 \\ (2m)^{-1}A(H_+ - z)^{-1} & 0 \end{pmatrix}$$

$$+ c^{-2}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} + 0(c^{-4}) \equiv R_0(z) + c^{-2}R_1(z) + 0(c^{-4}). \quad (2.20)$$

$$R_{11} = (2m)^{-2}(H_+ - z)^{-1}A^*(z - V_-)A(H_+ - z)^{-1}$$

$$R_{12} = (2m)^{-1}(H_+ - z)^{-1}A^*$$

$$R_{21} = (2m)^{-2}[(2m)^{-1}A(H_+ - z)^{-1}A^* - 1](z - V_-)A(H_+ - z)^{-1}$$

$$R_{22} = (2m)^{-1}[(2m)^{-1}A(H_+ - z)^{-1}A^* - 1]$$

Our approach to calculate relativistic eigenvalue corrections is based on the idea to use perturbation theory of the Dirac resolvent (instead of the Dirac operator) since it is holomorphic in c^{-1} whereas the Dirac Hamiltonian is not. While this procedure is possible in principle we prefer to use $B(c)(H(c) - mc^2 - z)^{-1}B(c)^{-1}$ instead of $(H(c) - mc^2 - z)^{-1}$ since it directly leads to relativistic bound state corrections in terms of c^{-2} instead of c^{-1}. Thus we need the spectral properties of the « unperturbed » operator $R_0(z)$ of (2.20).

LEMMA 2.2. — Assume the conditions of Theorem 2.1. Then

$$\sigma((H_+ - z)^{-1})\backslash\{0\} = \sigma(R_0(z))\backslash\{0\}, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \quad (2.21)$$

In addition $0 \in \sigma_P(R_0(z))$ with geometric multiplicity equal to dim $\mathcal{H}_-$, and

$$R_0(z)\begin{pmatrix} 0 \\ g_- \end{pmatrix} = 0 \qquad \text{for all } g_- \in \mathcal{H}_-. \tag{2.22}$$

Moreover,

$$(H_+ - z)^{-1} f_0 = \lambda_0 f_0, \qquad f_0 \in \mathcal{H}_+ \tag{2.23}$$

implies

$$R_0(z)\begin{pmatrix} f_0 \\ (2m)^{-1} A f_0 \end{pmatrix} = \lambda_0 \begin{pmatrix} f_0 \\ (2m)^{-1} A f_0 \end{pmatrix}. \tag{2.24}$$

Conversely,

$$R_0(z)\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \lambda \begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \qquad \lambda \neq 0, \qquad f_\pm \in \mathcal{H}_\pm \tag{2.25}$$

implies

$$(H_+ - z)^{-1} f_+ = \lambda f_+ \qquad \text{and} \qquad f_- = (2m)^{-1} A f_+. \tag{2.26}$$

In particular all discrete eigenvalues λ_0 of $(H_+ - z)^{-1}$ are semisimple eigenvalues of $R_0(z)$ i. e. algebraic and geometric multiplicity of λ_0 as an eigenvalue of $R_0(z)$ coincide.

Proof. — Let $\zeta \neq 0$ then

$$(R_0(z) - \zeta)^{-1} = \begin{pmatrix} [(H_+ - z)^{-1} - \zeta]^{-1} & 0 \\ \zeta^{-1}(2m)^{-1} A (H_+ - z)^{-1} [(H_+ - z)^{-1} - \zeta]^{-1} & -\zeta^{-1} \end{pmatrix} \tag{2.27}$$

proves (2.21). Eqs. (2.22)-(2.26) follow by direct computation. That discrete eigenvalues λ_0 of $(H_+ - z)^{-1}$ are semisimple eigenvalues of $R_0(z)$ follows from (2.27) since due to normality of $(H_+ - z)^{-1}$, $[(H_+ - z)^{-1} - \zeta]^{-1}$ has precisely a first order pole at $\zeta = \lambda_c$ (cf. [*12*], ch. I.5.4]).

REMARK 2.1. — In concrete applications (cf. section 3) dim $\mathcal{H}_- = \infty$ and thus 0 is an infinitely degenerate eigenvalue of $R_0(z)$. Of course $0 \in \sigma((H_+ - z)^{-1})$ (and then $0 \in \sigma_{ess}((H_+ - z)^{-1})$) if and only if H_+ is unbounded [*17*, p. 109].

After these preliminaries we obtain the following characterization of relativistic bound state energies:

THEOREM 2.2. — Assume the conditions of Theorem 2.1.

a) Let $E_0 \in \sigma_d(H_+)$ be an isolated nondegenerate eigenvalue of H_+ and assume $H_+ f_0 = E_0 f_0$, $f_0 \in \mathcal{D}(H_+)$, $\| f_0 \| = 1$. Then, for c^{-2} small enough there is a unique (simple) eigenvalue $E(c^{-2}) \in \sigma_d(H(c) - mc^2)$ with $E(0) = E_0$ which is holomorphic with respect to c^{-2} at $c^{-2} = 0$. Moreover, there are eigenvectors $f_\pm(c^{-2}) \in \mathcal{H}_\pm$ holomorphic at $c^{-2} = 0$ such that

$$(H(c) - mc^2)\begin{pmatrix} f_+(c^{-2}) \\ c^{-1} f_-(c^{-2}) \end{pmatrix} = E(c^{-2}) \begin{pmatrix} f_+(c^{-2}) \\ c^{-1} f_-(c^{-2}) \end{pmatrix} \tag{2.28}$$

and

$$f_+(0) = f_0 .$$

Writing

$$E(c^{-2}) = E_0 + c^{-2}E_1 + c^{-4}E_2 + 0(c^{-6}) \tag{2.29}$$

one obtains

$$E_1 = (2m)^{-2}(Af_0, (V_- - E_0)Af_0) . \tag{2.30}$$

b) If $E_0 \in \sigma_d(H_+)$ is a discrete eigenvalue of H_+ with multiplicity $s_0 \geq 2$ then, for c^{-2} small enough, $H(c) - mc^2$ has precisely s_0 eigenvalues (counting multiplicity) near $c^{-2}=0$. All eigenvalues $E_j(c^{-1})$, $1 \leq j \leq s_0$, are holomorphic in c^{-1} and

$$E_j(c^{-1}) = E_0 + \sum_{l=2}^{\infty} (c^{-1})^l E_{j,l}, \qquad 1 \leq j \leq s_0 . \tag{2.31}$$

Moreover, if $E_j(c^{-1})$ is an eigenvalue then also $E_j(-c^{-1})$ is an eigenvalue of $H(c)-mc^2$.

Proof. — Uniqueness and holomorphy of $E(c^{-2})$, $f_\pm(c^{-2})$ etc. follows from Lemmas 2.1 and 2.2 and nondegenerate perturbation theory [*1*] [*12*] [*17*]. Insertion of (2.29) into the left hand side of the following first order perturbation formula

$$(E(c^{-2})-z)^{-1}=(E_0-z)^{-1}+c^{-2}\left(\begin{pmatrix} f_0 \\ 0 \end{pmatrix}, \quad R_1(z)\begin{pmatrix} f_0 \\ (2m)^{-1}Af_0 \end{pmatrix}\right)+0(c^{-4}),$$

where $\begin{pmatrix} f_0 \\ 0 \end{pmatrix}$ represents the eigenvector of $R_0(z)^*$ to the simple eigenvalue $(E_0 - z)^{-1}$, proves (2.30). That all $E_j(c^{-1})$ of part *b)* are holomorphic near $c^{-1} = 0$ follows from Theorem 2.1 and normality of $(H(c)-mc^2-z)^{-1}$ [*12*, p. 71]. Since by Lemma 2.2 $(E_0-z)^{-1}$ is a semisimple eigenvalue of $R_0(z)$ one gets $E_j(c^{-1})-E_0=0(c^{-2})$ by the reduction process described in [*1*] [*12*]. The last statement finally follows from

$$B(c)(H(c)-mc^2-z)^{-1}B(c)^{-1}=B(-c)(H(-c)-mc^2-z)^{-1}B(-c)^{-1} . \qquad \square$$

REMARK 2.2. — For simplicity we only derived E_1 in Theorem 2.2 *a)*. A straightforward computation taking into account all spectral properties of $R_0(z)$ (in particular the point $0 \in \sigma_P(R_0(z))$ e. g. yields for the second coefficient E_2 in (2.29)

$$E_2= -(2m)^{-4}(Af_0, (V_- -E_0)AR_+^{\perp}(E_0)A^*(V_- -E_0)Af_0)+ \\ +(2m)^{-3}(Af_0, (V_- -E_0)^2Af_0)+(2m)^{-1}E_1(f_0, (V_+ -E_0)f_0) \tag{2.32}$$

where $R_+^{\perp}(E_0)$ denotes the reduced resolvent

$$R_+^{\perp}(E_0) = n\text{-}\lim_{\varepsilon\to 0} (H_+ -E_0+i\varepsilon)^{-1}(1-P_0), \qquad P_0=(f_0,.)f_0 .$$

We note that under stronger assumptions on the interaction holomorphy of the Dirac resolvent and relativistic eigenvalues with respect to c^{-1} using pseudo-resolvent techniques has been proved by Veselic [22]. Holomorphy of the Dirac resolvent, eq. (2.18), $R_0(z)$ in (2.20), and (2.28) of Theorem 2.2 have been derived by Hunziker [11] using somewhat different methods and relative compactness assumptions. Within this abstract approach eqs. (2.15), (2.19), and particularly (2.30) are new. Strong continuity of the Dirac resolvent (cf. (2.17)) and the corresponding unitary group in the nonrelativistic limit under assumptions (2.8) and (2.9) is a result of Cirincione and Chernoff [2].

REMARK 2.3. — Of course all results of this section have direct analogues if one adds the rest energy mc^2 to H(c). E. g. eq. (2.17) has to be replaced by

$$n\text{-}\lim_{|c|\to\infty} (\mathrm{H}(c)+mc^2-z)^{-1} = -(\mathrm{H}_-^0 - \mathrm{V}_- + z)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}. \qquad (2.33)$$

Finally we sketch how to obtain norm estimates of various quantities which in particular imply an estimate of the convergence radius of the expansion (2.19). For simplicity assume $\mathrm{A} = \mathrm{A}^*$ and $\mathrm{V}_+ = \mathrm{V}_- = \mathrm{V}$. Then (2.19) may be written as

$$\mathrm{B}(c)(\mathrm{H}(c) - mc^2 - z)^{-1}\mathrm{B}(c)^{-1} = \begin{pmatrix} \rho(z) & 0 \\ (2m)^{-1}\beta(z) & 0 \end{pmatrix} + \\ + \sum_{n=0}^{\infty} (2mc^2)^{-(n+1)} \begin{pmatrix} \gamma(\bar z)^* [\alpha(z)]^n (2m)^{-1}\beta(z) & \beta(\bar z)^* [\alpha(z)^*]^n \\ [\alpha(z)]^{n+1}(2m)^{-1}\beta(z) & [\alpha(z)]^n \delta(z) \end{pmatrix} \qquad (2.34)$$

where

$$\begin{aligned} \rho(z) &= ((2m)^{-1}\mathrm{A}^2 + \mathrm{V} - z)^{-1}, \qquad \rho_0(z) = ((2m)^{-1}\mathrm{A}^2 - z)^{-1}, \\ \alpha(z) &= [(2m)^{-1}\mathrm{A}\rho(z)\mathrm{A} - 1](z - \mathrm{V}), \\ \beta(z) &= \mathrm{A}\rho(z), \\ \gamma(z) &= (z - \mathrm{V})\mathrm{A}\rho(z), \\ \delta(z) &= (2m)^{-1}\mathrm{A}\rho(z)\mathrm{A} - 1. \end{aligned} \qquad (2.35)$$

Assuming $\mathrm{Re}\, z = 0$ we immediately get from (cf. assumption (2.9))

$$\| \mathrm{V}f \|^2 \le a_0 \| \mathrm{A}f \|^2 + b_0 \| f \|^2 \qquad (2.36)$$

and

$$\| \rho_0(z) \|^2 \le | \mathrm{Im}\, z |^{-2}, \quad \| \mathrm{A}\rho_0(z) \|^2 \le m | \mathrm{Im}\, z |^{-1}, \quad \| \mathrm{A}^2\rho_0(z) \|^2 \le 4m^2$$

that

$$\begin{aligned} \| \mathrm{V}\rho_0(z) \|^2 &\le ma_0 | \mathrm{Im}\, z |^{-1} + b_0 | \mathrm{Im}\, z |^{-2}, \\ \| \mathrm{VA}\rho_0(z) \|^2 &\le 4m^2 a_0 + mb_0 | \mathrm{Im}\, z |^{-1} \end{aligned}$$

and consequently

$$\| V\rho(z) \|^2 \leq a_0 \| \rho(\overline{z}) A^2 \rho(z) \| + b_0 | Im\, z |^{-2}$$
$$\leq 2ma_0 | Im\, z |^{-1} [2 + \| V\rho(z) \|] + b_0 | Im\, z |^{-2}$$

implying

$$\| V\rho(z) \| \leq ma_0 | Im\, z |^{-1} + (m^2 a_0^2 | Im\, z |^{-2} + 4ma_0 | Im\, z |^{-1} + b_0 | Im\, z |^{-2})^{1/2}. \tag{2.37}$$

From

$$\| ((2m)^{-1} A^2 - z)\rho(z) \| \leq 1 + \| V\rho(z) \|$$

we infer

$$\| VA\rho(z) \| \leq (4m^2 a_0 + mb_0 | Im\, z |^{-1})^{1/2} (1 + \| V\rho(z) \|).$$

Summing up we obtain

$$\| \alpha(z) \| \leq | Im\, z | [1 + (ma_0 | Im\, z |^{-1} + b_0 | Im\, z |^{-2})^{1/2}] + (2m)^{-1} (4m^2 a_0 + mb_0 | Im\, z |^{-1})^{1/2} \| \gamma(z) \|, \tag{2.38}$$

$$\| \beta(z) \| \leq (2m | Im\, z |^{-1})^{1/2} (2 + \| V\rho(z) \|)^{1/2}, \tag{2.39}$$

$$\| \gamma(z) \| \leq (2m | Im\, z |)^{1/2} (2 + \| V\rho(z) \|)^{1/2} + (4m^2 a_0 + mb_0 | Im\, z |^{-1})^{1/2} (1 + \| V\rho(z) \|), \tag{2.40}$$

$$\| \delta(z) \| \leq 1 + (2m | Im\, z |)^{-1/2} (4m^2 a_0 + mb_0 | Im\, z |^{-1})^{1/2} (2 + \| V\rho(z) \|)^{1/2} \tag{2.41}$$

Thus the expansion (2.34) (resp. (2.19)) converges in norm if $\| \alpha(z) \| < 2mc^2$.

3. CONCRETE REALIZATIONS

As shown in [2] the abstract approach of section 2 is general enough to cover Dirac operators over Riemannian manifolds. Here we restrict ourselves to the Dirac operator in flat space and first discuss the case of vanishing external vector potentials and spherically symmetric electrostatic potentials $V(r)$.

We define

$$\mathscr{H} = L^2((0, \infty)) \otimes \mathbb{C}^2, \qquad \mathscr{H}_{\pm} = L^2((0, \infty)) \tag{3.1}$$

and

$$\dot{A}_{j,\kappa} = \frac{d}{dr} + \frac{\kappa}{r}, \quad \mathscr{D}(\dot{A}_{j,\kappa}) = C_0^{\infty}((0, \infty)), \quad \kappa = \pm (j + 1/2), \quad 2j = 1, 3, 5, \ldots \tag{3.2}$$

Let $A_{j,\kappa}$ denote the closure of $\dot{A}_{j,\kappa}$ and

$$h_{j,\kappa}^{0,D}(c) = \begin{pmatrix} mc^2 & cA_{j,\kappa}^* \\ cA_{j,\kappa} & -mc^2 \end{pmatrix}. \tag{3.3}$$

Next assume

$$V_{\pm} = V = \gamma r^{-1} + V_1 + V_2, \qquad \gamma \in \mathbb{R} \tag{3.4}$$

where V_1 is locally uniformly square integrable, i. e. for all $A \geq 0$

$$\int_A^{A+1} dr \, | V_1(r) |^2 \leq M < \infty, \qquad \text{M independent of A} \tag{3.5}$$

and

$$| V_2(r) | \leq ar^{-1} \qquad \text{for some } a > 0 . \tag{3.6}$$

Then the radial Dirac operator $h^D_{j,\kappa}(c)$ in $L^2((0, \infty)) \otimes \mathbb{C}^2$, defined by

$$h^D_{j,\kappa}(c) = \begin{pmatrix} mc^2 + V & cA^*_{j,\kappa} \\ cA_{j,\kappa} & -mc^2 + V \end{pmatrix}, \qquad \mathscr{D}(h^D_{j,\kappa}(c)) = \mathscr{D}(h^{0,D}_{j,\kappa}(c)) \tag{3.7}$$

is self-adjoint for $|c|$ large enough (cf. section 2), and the radial Pauli operators $h_{\pm,j,\kappa}$ in $L^2((0, \infty))$ are given as the Friedrichs extension of

$$-\frac{1}{2m}\frac{d^2}{dr^2} + \frac{\kappa(\kappa \pm 1)}{2mr^2} + V \Big|_{C_0^\infty((0,\infty))} . \tag{3.8}$$

$\left(\text{Note that } \mathscr{D}(V) \supseteq \mathscr{D}\left(\frac{d}{dr}\Big|_{C_0^\infty((0,\infty))}\right) = \mathscr{D}(A_{j,\kappa}) \text{ by [12, p. 346] and [8,}\right.$ Lemma 1].) The full Dirac operator $H^D(c)$ in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and the full Pauli operator H_∞ in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ are then defined as direct sums over $h^D_{j,\kappa}(c)$ and $h_{\pm,j,\kappa}$. Since under conditions (3.4)-(3.6) discrete eigenvalues of $h_{\pm,j,\kappa}$ are simple ([*14*, p. 133]), Theorem 2.2*a*) applies and we obtain.

COROLLARY 3.1. — Let $e^{(0)}_{j,\kappa}$ be a discrete eigenvalue of $h_{+,j,\kappa}$,

$$h_{+,j,\kappa} f^{(0)}_{j,\kappa} = e^{(0)}_{j,\kappa} f^{(0)}_{j,\kappa}, \qquad f^{(0)}_{j,\kappa} \in \mathscr{D}(h_{+,j,\kappa}),$$

$\| f^{(0)}_{j,\kappa} \| = 1$ then, for c^{-2} small enough, there is precisely one (simple) eigenvalue $e_{j,\kappa}(c^{-2})$ of $h^D_{j,\kappa}(c) - mc^2$ near $c^{-2} = 0$ which is holomorphic in c^{-2} with

$$e_{j,\kappa}(c^{-2}) = e^{(0)}_{j,\kappa} + c^{-2} e^{(1)}_{j,\kappa} + 0(c^{-4}), \qquad e^{(1)}_{j,\kappa} = (2m)^{-2}(A_{j,\kappa} f^{(0)}_{j,\kappa}, (V - e^{(0)}_{j,\kappa}) A_{j,\kappa} f^{(0)}_{j,\kappa}) . \tag{3.9}$$

Using different methods, employing in particular holomorphy of eigenfunctions of $h^D_{j,\kappa}(c) - mc^2$ with respect to c^{-2}, Titchmarsh [*21*] derived the result (3.9).

Next we discuss non-spherically symmetric interactions, Let

$$\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \qquad \mathscr{H}_\pm = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \tag{3.10}$$

and assume [*15*]

$$A_k \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3), \quad p > 3, \quad 1 \leq k \leq 3 \quad V_\pm = V = V_1 + V_2 \tag{3.11}$$

where

$$V_1 \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) . \qquad p > 3 \tag{3.12}$$

and V_2 is a finite sum of Coulomb potentials

$$V_2(\underline{x}) = \sum_{j=1}^{N} \gamma_j | \underline{x} - \underline{x}_j |^{-1}, \qquad \gamma_j \in \mathbb{R}. \tag{3.13}$$

With

$$A = A^* = (-i\nabla - \underline{A}) \otimes \underline{\sigma}, \qquad \mathcal{D}(A) = \mathcal{D}(|\nabla| \otimes 1) \tag{3.14}$$

where σ_k, $1 \leq k \leq 3$, denote the usual Pauli matrices [*13*] we define the Dirac operator

$$H^D(c) = c(-i\nabla - \underline{A}) \otimes \underline{\sigma} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + mc^2 1 \otimes 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + V \otimes 1 \otimes 1,$$
$$\mathcal{D}(H^D(c)) = \mathcal{D}(|\nabla| \otimes 1 \otimes 1). \tag{3.15}$$

$H^D(c)$ is self-adjoint for $|c|$ large enough (cf. section 2). The Pauli Hamiltonian $H_{\pm} = H_{\infty}$ then reads

$$H_{\infty} = (2m)^{-1}[(-i\nabla - \underline{A}) \otimes \underline{\sigma}]^2 + V \otimes 1, \qquad \mathcal{D}(H_{\infty}) = \mathcal{D}(\Delta \otimes 1). \tag{3.16}$$

Thus Theorem 2.2 implies

COROLLARY 3.2. — *a*) Assume E_0 to be a discrete, simple eigenvalue of H_{∞}, $H_{\infty} f_0 = E_0 f_0$, $f_0 \in \mathcal{D}(H_{\infty})$, $\| f_0 \| = 1$. Then, for c^{-2} small enough, there is a unique (simple) eigenvalue $E(c^{-2})$ of $H^D(c) - mc^2$ near $c^{-2} = 0$ which is holomorphic in c^{-2} with

$$E(c^{-2}) = E_0 + c^{-2} E_1 + 0(c^{-4}),$$
$$E_1 = (2m)^{-2}([(-i\nabla - \underline{A}) \otimes \underline{\sigma}] f_0, \ (V \otimes 1 - E_0)[(-i\nabla - \underline{A}) \otimes \underline{\sigma}] f_0). \tag{3.17}$$

b) If $E_0 \in \sigma_d(H_{\infty})$ is a discrete eigenvalue of H_{∞} with multiplicity $s_0 \geq 2$ then, for c^{-2} small enough, $H(c) - mc^2$ has precisely s_0 eigenvalues (counting multiplicity) near $c^{-2} = 0$. All eigenvalues $E_j(c^{-1})$, $1 \leq j \leq s_0$, are holomorphic in c^{-1} with

$$E_j(c^{-1}) = E_0 + \sum_{l=2}^{\infty} (c^{-1})^l E_{j,l}, \qquad 1 \leq j \leq s_0. \tag{3.18}$$

If $E_j(c^{-1})$ is an eigenvalue then also $E_j(-c^{-1})$ is an eigenvalue of $H(c) - mc^2$.

In the special case of vanishing external vector potentials $\underline{A} = 0$ eq. (3.17) has been derived by Sewell [*19*] using formal expansions of wave functions in powers of c^{-2}.

At this point it is interesting to remark that in the physical literature relativistic corrections to bound state energies are usually obtained by completely different means. Following the procedure of Foldy and Wouthuysen [4] one adds a first order relativistic correction to the Pauli operator H_∞ to get

$$H^{(1)}(c) = H_\infty - c^{-2}\{(2m)^{-3}[(-i\nabla - \underline{A}) \otimes \underline{\sigma}]^4 - \\ - (2m)^{-2}[(\nabla V) \wedge (-i\nabla - \underline{A})] \otimes \underline{\sigma} - (8m^2)^{-1}(\Delta V) \otimes 1\}, \quad (3.19)$$
$$\mathscr{D}(H^{(1)}(c)) = \mathscr{D}((-\Delta)^2 \otimes 1)$$

where for simplicity we assume

$$V, A_k \in \mathscr{S}(\mathbb{R}^3), \qquad 1 \leq k \leq 3. \quad (3.20)$$

Relativistic corrections to bound state energies of the Pauli operator H_∞ are then obtained formally from (3.19) by means of perturbation theory [4] [13].

Under conditions (3.20) H(c) is self-adjoint and (cf. [5] [6])

$$\sigma_{ess}(H_\infty) = [0, \infty) \quad (3.21)$$

but

$$\sigma_{ess}(H^{(1)}(c)) = (-\infty, mc^2/2]. \quad (3.22)$$

(Of course these conditions can be relaxed regarding smoothness and their behaviour at infinity but Coulomb-type interactions $|\underline{x}|^{-1}$ near the origin are certainly excluded due to the term (ΔV) in (3.19).) Thus $H^{(1)}(c)$ (in contrast to H_∞) certainly has no negative discrete eigenvalues. Nevertheless one can prove first order spectral concentration of $H^{(1)}(c)$ at the negative eigenvalues of H_∞ in the nonrelativistic limit. More explicitly we have ([5] [6]).

PROPOSITION 3.1. — Under conditions (3.20) in addition to that of Corollary (3.2 *a*) there is a first order pseudo-eigenvalue $\hat{E}(c^{-2})$ of $H^{(1)}(c)$

$$\hat{E}(c^{-2}) = E_0 + c^{-2}\hat{E}_1, \quad \hat{E}_1 = -(2m)^{-3} \| [(-i\nabla - \underline{A}) \otimes \underline{\sigma}]^2 f_0 \|^2 + \\ + (2m)^{-2}\left(f_0, \left\{ [(\nabla V) \wedge (-i\nabla - \underline{A})] \otimes \underline{\sigma} + \frac{1}{2}(\Delta V) \otimes 1 \right\} f_0\right). \quad (3.23)$$

In particular $\hat{E}_1$ coincides with E_1 of Corollary 3.2 *a*).

Proof. — Eq. (3.23) has been proved in [6]. Using $E_0 f_0 = H_\infty f_0$ in E_1 of (3.17) and calculating various commutators shows $E_1 = \hat{E}_1$. □

Thus in the presence of smooth electromagnetic potentials $\underline{A}$, V (excluding the important case of hydrogen-type systems) both approaches yield the same relativistic bound state energy corrections to first order in c^{-2}. But obviously there are important differences from a conceptual point of view: The method described in Corollary 3.2 starts from a discrete eigenvalue E_0 of the nonrelativistic Pauli operator and proves that eigenvalues $E(c^{-2})$ of the Dirac Hamiltonian (rest energy subtracted) are holomorphic around their nonrelativistic limit $c \to \infty$. The Foldy-Wouthuysen method also starts from a discrete eigenvalue E_0 of the Pauli operator and then shows the existence of first order pseudo-eigenvalues of some appropriately relativistically corrected Hamiltonian. These pseudo-eigenvalues are not discrete eigenvalues but intuitively they correspond to resonances whose width become arbitrarily small as $c \to \infty$.

We finally note that our concept of expanding the Dirac resolvent around its nonrelativistic limit not only includes a description of Coulomb-type interactions but trivially generalizes to nonlocal interactions V (cf. the example in [7]) and also allows a discussion of anomalous electric (resp. magnetic) moments δ (resp. μ) of a charged particle in external vector potentials $\underline{A}$, V. In this case the Dirac Hamiltonian reads

$$H^{D}_{\delta,\mu}(c) = \begin{pmatrix} V_+(c^{-1}) + mc^2 & cA(c^{-1})^* \\ cA(c^{-1}) & V_-(c^{-1}) - mc^2 \end{pmatrix} \tag{3.24}$$

where

$$V_\pm(c^{-1}) = V \otimes 1 \mp (2m)^{-1}[\mu(\nabla \wedge \underline{A}) + c^{-1}\delta(\nabla V)] \otimes \underline{\sigma},$$
$$A(c^{-1}) = \{(-i\nabla - \underline{A}) - (2mc)^{-1} i[-c^{-1}\mu(\nabla V) + \delta(\nabla \wedge \underline{A})]\} \otimes \underline{\sigma} \tag{3.25}$$

and V, A_k, $(\nabla \wedge \underline{A})_k$, $(\nabla V)_k$, $1 \leq k \leq 3$ are assumed to be infinitesimally bounded with respect to $|\nabla|$ (e. g. each operator acts by a function in $L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $p > 3$ [15]). Since a detailed treatment of anomalous moment interactions including hydrogen-type systems will appear elsewhere [9] we only note that in the nonrelativistic limit $c \to \infty$

$$n\text{-}\lim_{c\to\infty} (H^{D}_{\delta,\mu}(c) - mc^2 - z)^{-1} = (H_\infty - (2m)^{-1}\mu(\nabla \wedge \underline{A}) \otimes \underline{\sigma} - z)^{-1}. \tag{3.26}$$

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