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An exactly solvable model in P. R. M.: Quantization. II

by

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ABSTRACT. — The quantization of *a priori* hamiltonian predictive systems is studied. It is applied to quantize the family of N-particle relativistic models presented in ref. 1. The relativistic two-particle oscillator is studied in detail and finally our results are compared with others that already appeared in the literature.

RÉSUMÉ. — On étudie la quantification de systèmes hamiltoniens prédictifs *a priori*. Comme application, on quantifie la famille de modèles relativistes à N particules présentés dans la référence 1. On étudie en détail l'oscillateur relativiste à deux particules. Finalement on compare les résultats obtenus avec d'autres parus précédemment.

1. INTRODUCTION

In an earlier paper [*1*] we presented a family of N-particle relativistic systems in the framework of Predictive Relativistic Mechanics (PRM) and we developed the classical aspects of such systems. We also studied in detail the harmonic oscillator like interaction, and this was done for

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two reasons: on the one hand because the oscillator like N-particle relativistic systems are simple enough to be explicitly solved and, on the other hand, because some articles have recently appeared [2] [3] [4] where these models are used to give an account of phenomenological aspects of the internal structure of bosons and hadrons.

The present work is an attempt to quantize such models. The quantization of an N-particle relativistic system interacting at a distance (i. e. without an intermediary field) is a subtle problem because in dealing with a manifestly covariant formalism a set of 8N coordinates is used (coordinates and momenta are four-vectors) whereas the correct number of degrees of freedom is 6N. In a naive quantization some variables appear which have no physical meaning, for instance, the relative times. Treating these variables as independent introduces time-like oscillations, which are responsible for the undesired fact that the energy spectrum is not bounded from below and presents an infinite degeneracy. In order to avoid such a bad behavior these degrees of freedom are usually eliminated « ad hoc » [2] [3].

There are several other ways to eliminate the spurious degrees of freedom introduced in dealing with a covariant formalism. In the constraint hamiltonian formalism the phase space is a 6N submanifold of TM_4^N and the quantization procedure starts from the canonical structure defined by the Dirac bracket on the constraint manifold. The quantization procedure has also been developed in the framework of the manifestly predictive formalism of PRM [6]. This procedure, which is not manifestly covariant but is Poincaré invariant, does not introduce superfluous degrees of freedom and deals with variables which can be measured in any inertial frame.

In the present paper we follow a different approach, based on the quantization method introduced by Bel [7] and on the formalism developed by L. Schwartz [8] which, in our opinion, is especially elegant and rigorous. The « wave functions » will be elements of $\mathscr{S}'(M_4^N)$, tempered distributions on the « configuration space » spanned by the canonical coordinates q_a^μ. The Hilbert space $\mathscr{H}$ of our system will be a subspace of $\mathscr{S}'(M_4^N)$ on which the Complete Symmetry Group $\mathscr{G}$ will be unitarily represented.

The Complete Symmetry Group is the direct product of the Poincaré Group $\mathscr{P}$ and an abelian N-parameter Lie Group $\mathscr{A}_N$ associated to the motions of the N particles. Therefore, $\mathscr{P}$ is also unitarily represented on $\mathscr{H}$; hence, the quantization presented here is explicitly covariant and the transformation properties are simple. Furthermore, our « wave functions » will be eigenfunctions of the generators of $\mathscr{A}_N$ (the N hamiltonian operators), which are Casimir operators. These N conditions yield, on the one hand $N-1$ equations which eliminate the dependence on the relative times and, on the other, the wave equation of the system.

In the special case of an harmonic oscillator like interaction we obtain a mass spectrum for hadrons agreeing with results already known [18]

and we calculate explicitly the corresponding wave-functions. Finally, in section 5, we compare our results with other relativistic harmonic oscillator models that already appeared in the literature.

2. QUANTIZATION OF A PREDICTIVE RELATIVISTIC HAMILTONIAN SYSTEM

The starting point of the quantization presented here is a particular hamiltonian description of a given Poincaré invariant predictive system (PIPS). This consists of [*1*]:

i) An adapted canonical coordinate system $\{ (q_a^\mu, p_\nu^b)$; $a, b = 1 \ldots N$; $\mu, \nu = 0, \ldots, 3 \}$ i. e.: a coordinate system such that the elementary Poisson brackets are:

$$\{ q_a^\mu, p_\nu^b \} = \delta_a^b \delta_\nu^\mu, \qquad \{ q_a^\mu, q_b^\nu \} = \{ p_\mu^a, p_\nu^b \} = 0 \tag{2.1}$$

and the generating functions of the Poincaré group are

$$\left.\begin{aligned} P_\mu &= \varepsilon_a p_\mu^a \\ J_{\mu\nu} &= q_{a\mu} p_\nu^a - q_{a\nu} p_\mu^a \end{aligned}\right\} \tag{2.2}$$

ii) N scalar hamiltonian functions $\{ H_a(q, p)$, $a = 1, \ldots, N \}$ satisfying

$$\{ H_a, H_{a'} \} = 0, \qquad a \neq a' \tag{2.3}$$

The differential systems associated to these hamiltonians generate the evolution of each particle; i. e., for any $f(q, p)$:

$$\frac{\partial f}{\partial \tau_a} = \{ H_a, f \} \tag{2.4}$$

where τ_a is a scalar parameter on the world-line of particle (a). The parameter τ_a is a well known function of the propertime σ_a [*5*] [*9*].

The hamiltonian functions are integrals of motion and we have [*5*]:

$$H_a = \frac{1}{2} m_a^2 \tag{2.5}$$

iii) The physical position coordinates x_a^μ are related to the adapted canonical ones by the partial differential system:

$$\{ H_{a'}, x_a^\mu \} = 0, \qquad \forall a' \neq a \tag{2.6}$$

which has a unique solution, provided suitable Gauchy data are given.

A *quantum description* of the Poincaré invariant hamiltonian system considered will consist of:

i) a Hilbert space $(\mathscr{H}, \langle \mid \rangle)$ of tempered distributions i. e., $\mathscr{H} \subset \mathscr{S}'(M_4^N)$ and

ii) a unitary representation of the Complete Symmetry Group [*10*] $\mathscr{G}$ of our system on $\mathscr{H}$.

We need a « machinery » to assign an operator on $\mathscr{H}$ to each function $F(q, p)$ on the phase space TM_4^N. For this purpose we shall use an immediate generalization to distributions of the method proposed by Bel [*7*]. For a given $F(qp)$ we define the operator $\hat{F}$: $\mathscr{S}'(M_4^N) \to \mathscr{S}'(M_4^N)$

$$\forall \psi \in \mathscr{S}'(M_4^N) \wedge \forall \varphi \in \mathscr{S}(M_4^N) ;$$

$$\langle \hat{F}\psi, \varphi \rangle \equiv \langle \psi_q, \overline{\mathscr{F}}_q [\mathscr{F}_p(F(q' \cdot p) \cdot \varphi_{q'})] \rangle \quad (2.7)$$

where $\langle , \rangle$ is the usual dual product between a continuous linear form of $\psi \in \mathscr{S}'(M_4^N)$ and a test function $\varphi \in \mathscr{S}(M_4^N) - \mathscr{S}(M_4^N)$ in the space of the rapidly decreasing $\mathscr{C}^\infty$ functions [*8*].

$\mathscr{F}$ and $\overline{\mathscr{F}}$ are, respectively, the Fourier and the inverse Fourier transforms:

$$\mathscr{F}_q \varphi = (2\pi\hbar)^{-2N} \int \eta(p) \cdot \exp\left(\frac{i}{\hbar} q_a^\mu p_\mu^a\right) \cdot \varphi(p)$$

$$\overline{\mathscr{F}}_p \chi = (2\pi\hbar)^{-2N} \int \eta(q) \cdot \exp\left(-\frac{i}{\hbar} q_a^\mu \cdot p_\mu^a\right) \chi(q) \quad (2.8)$$

where $\eta(q) \equiv dq_1^4 \wedge \ldots \wedge dq_N^4$ is the volume element in M_4^N.

The expression on the right of eq. (2.7) must be understood as:

$$\langle \hat{F}\psi, \varphi \rangle = (2\pi\hbar)^{-4N} \int \eta(q) \cdot \eta(p) \cdot \eta(q') \cdot e^{\frac{i}{\hbar}(q'-q)p} \cdot F(q', p) \psi(q) \cdot \varphi(q')$$

We call *indicial functions* those $F(q, p)$ to which eq. (2.7) assigns a well defined continuous linear operator $\hat{F}$ on $\mathscr{S}'(M_4^N)$. In Appendix A we give the conditions under which $F(q, p)$ is an indicial function. Indeed, we show in this case that

$$F(q, p) = \exp\left(\frac{-i}{\hbar} q_a^\mu p_\mu^a\right) \cdot \hat{F} \exp\left(\frac{i}{\hbar} q_a^\mu p_\mu^a\right) \quad (2.9)$$

This allows us to consider the part of any indicial function depending on p_ν^b as tempered distribution. It is also easy to prove that any polinomial function of q_a^μ and p_ν^b is an indicial function.

We also prove in Appendix A that:

$$\widehat{q^m p^n} = q^m \left(-i\hbar \frac{\partial}{\partial q}\right)^n \quad (2.10)$$

That is to say, the « correspondence rule » (2.7) assigns to the function q_a^μ the operator « product by q_a^μ » and to p_ν^b the differential operator $-i\hbar \dfrac{\partial}{\partial q_b^\nu}$, as it is usual in quantum mechanics. Moreover, eq. (2.7) supplies a well

determined prescription to assign an operator to a product of $q's$ and $p's$: « place all the $q's$ on the left and all the $p's$ on the right, then make the above substitution ». This is a kind of « normal ordering ».

Eq. (2.7) defines a linear map θ from the space $\mathcal{I}$ of all the indicial functions onto $\mathcal{L}(\mathcal{S}'(\mathrm{M}_4^{\mathrm{N}}); \mathcal{S}'(\mathrm{M}_4^{\mathrm{N}})) \equiv \mathcal{O}_q$

$$\left.\begin{aligned} \theta : \mathcal{I} &\to \mathcal{O}_q \\ \mathrm{F}(q, p) &\to \theta(\mathrm{F}) = \hat{\mathrm{F}} \end{aligned}\right\} \tag{2.11}$$

From (2.9) it is immediate to prove that: $\theta(\mathrm{F}_1) = \theta(\mathrm{F}_2)$ iff $\mathrm{F}_1 = \mathrm{F}_2$ almost everywhere. Therefore, we shall consider θ as a linear injection of I into $\mathcal{O}_q$. This will permit us to translate the algebraic structure of $\mathcal{O}_q$ into $\mathcal{I}$.

We can define the quantum product of two indicial functions by [7]:

$$\forall \mathrm{F}, \mathrm{G} \in \mathcal{I}, \qquad \mathrm{F} \circ \mathrm{G} \equiv \theta^{-1}(\hat{\mathrm{F}} \cdot \hat{\mathrm{G}}) \tag{2.12}$$

and the quantum bracket by

$$\{ \mathrm{F}, \mathrm{G} \}_{\mathrm{QB}} \equiv \mathrm{F} \circ \mathrm{G} - \mathrm{G} \circ \mathrm{F} = \theta^{-1}([\hat{\mathrm{F}}, \hat{\mathrm{G}}]) \tag{2.13}$$

It is obvious that the main properties of the product and bracket in $\mathcal{O}_q$ are also translated into properties of the quantum product and bracket in $\mathcal{I}$. Other properties that must be mentioned are the following:

$$q_a^\mu \circ \mathrm{G} = q_a^\mu \cdot \mathrm{G} = \mathrm{G} \circ q_a^\mu + i\hbar \frac{\partial \mathrm{G}}{\partial p_\mu^a} \tag{2.14.a}$$

$$p_\mu^a \circ \mathrm{G} + i\hbar \frac{\partial \mathrm{G}}{\partial q_a^\mu} = p_\mu^a \cdot \mathrm{G} = \mathrm{G} \circ p_\mu^a \tag{2.14.b}$$

$$\lim_{\hbar \to 0} \mathrm{F} \circ \mathrm{G} = \mathrm{F} \cdot \mathrm{G} \tag{2.15}$$

$$\lim_{\hbar \to 0} \frac{1}{i\hbar} \{ \mathrm{F}, \mathrm{G} \}_{\mathrm{QB}} = \{ \mathrm{F}, \mathrm{G} \} \tag{2.16}$$

The last two identities only hold if F and G do not depend on the parameter $\hbar$. The last one is very important because it relates the quantum and Poisson brackets.

We shall say that the quantum bracket between two functions is *exact* iff:

$$\{ \mathrm{F}, \mathrm{G} \}_{\mathrm{QB}} = i\hbar \{ \mathrm{F}, \mathrm{G} \} \tag{2.17}$$

Let us now consider the Lie algebra spanned by the set of indicial functions: $\mathrm{F}_1(q, p), \ldots, \mathrm{F}_r(q, p)$ being the Poisson bracket the operation of the algebra. Let us assume that the quantum bracket between any pair of these functions is exact. Then, the Lie algebra spanned by the associated operators $\hat{\mathrm{F}}_1, \ldots, \hat{\mathrm{F}}_r$ is isomorphic to the above Lie algebra of functions.

It is our aim to represent the Complete Symmetry Group $\mathcal{G}$ on the group of unitary operators on a Hilbert space $(\mathcal{H}, \langle \, | \, \rangle)$. Therefore we

must assign a hermitian operator on $\mathscr{H}$ to each of the generating functions P_μ, $J_{\mu\nu}$, H_b. The map θ defined by (2.11) and (2.7) supplies a way of doing this if, and only if, θ preserves the commutation relation between any pair of these generating functions P_μ, $J_{\mu\nu}$, H_a. In other words: if and only if the quantum bracket of any two of them is exact. At this point it must be emphasized that this condition is not fulfilled by all possible hamiltonizations of the PIPS.

We say that a given hamiltonization is *quantizable* when the quantum bracket of any two of the generating functions P_μ, $J_{\mu\nu}$, H_a is exact.

Since P_μ and $J_{\mu\nu}$ have the simple expressions given in (2.2), it is immediate to prove that their quantum brackets with any other function are exact. Hence, taking (2.3) into account, a hamiltonization is quantizable if and only if:

$$\{ H_a, H_{a'} \}_{QB} = 0, \qquad \forall a \neq a' \tag{2.18}$$

The above relation allows us to represent the Complete Symmetry Group $\mathscr{G}$ on the space of operators on $\mathscr{S}'(M_4^N)$. A quantum description of the PIPS requires besides a Hilbert space $(\mathscr{H}, \langle\,|\,\rangle)$ such that $\mathscr{H} \subset \mathscr{S}'(M_4^N)$ and that the generators of the representation act as hermitian operators on $\mathscr{H}$. This is so because we want the representation of $\mathscr{G}$ to be unitary, but may imply some restrictions on the Hilbert product $\langle\,|\,\rangle$.

3. THE MOMENTUM REPRESENTATION

In the last section we have defined the concept of the quantum description in terms of a Hilbert space of tempered distributions on test functions of the coordinates q_a^μ. We shall call this the coordinate representation. Sometimes it will be useful the so called momentum representation (i. e.: in terms of distributions on functions of the momenta p_ν^b).

The Fourier transform $\mathscr{F}$ defines an isomorfism of $\mathscr{S}'(M_4^N)$ [*11*]:

$$\left.\begin{aligned} \mathscr{S}'(M_4^N) &\xrightarrow{\mathscr{F}} \mathscr{S}'(M_4^N) \\ \chi_p &\to \psi_q \equiv \mathscr{F}\chi \\ \forall \varphi(q) \in \mathscr{S}(M_4^N), \qquad & \langle \mathscr{F}_q \chi, \varphi(q) \rangle \equiv \langle \chi_p, \mathscr{F}_p \varphi \rangle \end{aligned}\right\} \tag{3.1}$$

where the subscripts q or p indicate the representation we are dealing with.

The transform $\mathscr{F}$ admits an inverse, $\overline{\mathscr{F}}$, and both are continuous on $\mathscr{S}'(M_4^N)$.

We can also define the operator associated to a given indicial function by:

$$\left.\begin{aligned} \mathbb{1} &\to \mathcal{O}_p \\ F(q, p) &\to \check{F} = \overline{\mathscr{F}} \circ \hat{F} \circ \mathscr{F} \end{aligned}\right\} \tag{3.2}$$

Then, taking (2.7) into account, we have:

$$\forall \rho(p) \in \mathscr{S}(\mathrm{M}_4^{\mathrm{N}}), \qquad \forall \chi_p \in \mathscr{S}'(\mathrm{M}_4^{\mathrm{N}}):$$
$$\langle \check{\mathrm{F}}\chi_p, \rho(p) \rangle = \langle \chi_p, \mathscr{F}_p(\mathrm{F}(q', p)\overline{\mathscr{F}}_q, \rho) \rangle \quad (3.3)$$

As in the coordinates representation, it can be deduced from (3.3) a kind of « correspondence rule » given by:

$$\widetilde{(q^m p^n)} = \left(i\hbar \frac{\partial}{\partial p}\right)^m \cdot p^n \quad (3.4)$$

4. APPLICATIONS TO SOME MODELS. HARMONIC OSCILLATOR

a) In ref. [*1*], which will be referred hereafter as (I) we presented a family of predictive hamiltonian models for N particle systems. The hamiltonian functions were given by (I-3.16):

$$\mathrm{H}_a = \alpha(\mathrm{P}y_a) - \beta \mathrm{P}^2 - \gamma\left[\varepsilon^{\mathrm{A}} y_{\mathrm{A}}^2 + \sum_{\mathrm{A}<\mathrm{A}'} (y_{\mathrm{A}} y_{\mathrm{A}'})\right] + \mathrm{V}(\tilde{y}_{\mathrm{B}}, \tilde{\mathrm{Z}}_{\mathrm{C}}) \quad (4.1)$$

where α, β, γ are real parameters and the variables z_{A}^{μ} and y_{ν}^{B} are respectively the relative canonical coordinates and their conjugated momenta:

$$z_{\mathrm{A}}^{\mu} \equiv q_1^{\mu} - q_{\mathrm{A}}^{\mu}, \qquad y_{\nu}^{\mathrm{A}} \equiv \frac{1}{\mathrm{N}} \mathrm{P}_{\nu} - p_{\nu}^{\mathrm{A}}, \qquad y_{\nu}^{1} \equiv - \varepsilon_{\mathrm{A}} y_{\nu}^{\mathrm{A}} \quad (4.2)$$

The indices A, B, ... run from 2 through N and the sum convention holds for all the indices if the contrary is not explicitely especified.

It is meant by $\tilde{a}^{\mu}$ the component of a^{μ} which is orthogonal to P_{μ}:

$$\tilde{a}^{\mu} = \pi_{\nu}^{\mu} a^{\nu}, \qquad \pi_{\nu}^{\mu} \equiv \eta_{\nu}^{\mu} - \frac{1}{\mathrm{P}^2} \mathrm{P}^{\mu}\mathrm{P}_{\nu} \quad (4.3)$$

For more notation details the reader is referred to (I).

Some questions immediately arise. Are these hamiltonian models quantizable ? Does this imply any restriction on the potential function $\mathrm{V}(\tilde{y}_{\mathrm{B}}, \tilde{z}_{\mathrm{C}})$? As it has been established before the necessary and sufficient condition of quantizability is:

$$\{ \mathrm{H}_a, \mathrm{H}_{a'} \}_{\mathrm{QB}} = 0, \qquad \forall a' \neq a \quad (2.18)$$

In our particular case we have:

$$\{ \mathrm{H}_a, \mathrm{H}_{a'} \}_{\mathrm{QB}} = \alpha \{ (\mathrm{P}y_a) - (\mathrm{P}y_{a'}), \mathrm{V} \}_{\mathrm{QB}} \quad (4.4)$$

The quantum bracket on the right of (4.4) is a linear combination of $\{(Py_A), V\}_{QB}$, $A = 2 \ldots, N$, which after a short calculation yields:

$$\{(Py_A), V\}_{QB} = -i\hbar \frac{\partial V}{\partial z_A^\mu} \cdot P^\mu - i\hbar y_A^\mu \cdot \frac{\partial V}{\partial X^\mu} - \hbar^2 \frac{\partial^2 V}{\partial X^\mu \partial Z_{A\mu}} \quad (4.5)$$

and, since in the case under consideration V only depends on $\tilde{z}_A^\mu$, $\tilde{y}_\nu^B$, the right hand side of (4.5) obviously vanishes. Therefore, the hamiltonian predictive model given by (4.1) is quantizable irrespectively of what potential function $V(\tilde{y}_B, \tilde{z}_C)$ is taken.

However we shall not consider here the representation of $\mathcal{G}$ on the whole $\mathcal{S}'(M_4^N)$ but in some invariant subspaces of it. These will be the proper subspaces of the operators: $\hat{P}^2$, $\hat{W}^2$ and $\hat{H}_a$, $a = 1, \ldots, N$ (respectively: the squares of the total momentum and the Pauli-Lubansky vector, and the hamiltonians). Each subspace will be labeled by $N + 2$ non negative numbers $\mathcal{H}(M, L; m_1, \ldots, m_N)$ and it will consist of all the states of the system with definite total mass M, total intrinsic angular momentum L and individual masses for the constituent particles (m_a) (*):

$$\forall \psi \in \mathcal{H}(M, L; m_1, \ldots, m_N)$$

$$\hat{P}_\mu \hat{P}^\mu \psi = -M^2 \psi \quad (4.6a)$$

$$\hat{W}_\mu \hat{W}^\mu \psi = \hbar^2 L(L+1)\psi \quad (4.6b)$$

$$\hat{H}_a \psi = \frac{1}{2} m_a^2 \psi, \qquad a = 1, \ldots, N \quad (4.6c)$$

It is clear that $\mathcal{H}(M, L; m_1, \ldots, m_N)$ so defined is not empty because the operators on the right commute mutually.

We must point out, however, that eq. (4.6) do not define

$$\mathcal{H}(M, L; m_1, \ldots, m_N)$$

because some summability conditions must be added. This will be done below.

b) From now on we shall specifically study the harmonic oscillator model presented in (I) given by the potential function:

$$V = -\frac{1}{2} k \sum_{a<a'} (\tilde{q}_a - \tilde{q}_{a'})^2 = -\frac{1}{2} k \left[(N-1)\varepsilon^A \tilde{z}_A^2 - \sum_{B \neq B'} (\tilde{z}_B \cdot \tilde{z}_B) \right] \quad (4.7)$$

In this case the simplest framework is the momentum representation.

(*) The concept of « individual mass » has no physical meaning if the system is not separable but we keep this denomination for obvious reasons.

Otherwise, since P^2 appears dividing in V, we would be forced to deal with integrodifferential equations and, hence, non-local operators. Using the « correspondence rule » (3.4) we can write equations (4.6) as:

$$(P^2 + M^2)\psi = 0 \tag{4.7,a}$$

$$\left[2\tilde{y}_\mu^A \cdot \frac{\partial}{\partial y_\mu^A} + (\tilde{y}_\mu^A \cdot \tilde{y}_\nu^B - (\tilde{y}^A \cdot \tilde{y}^B)\pi_{\mu\nu}) \cdot \frac{\partial^2}{\partial y_\nu^A \partial y_\mu^B} - L(L+1)\right]\psi = 0 \tag{4.7,b}$$

$$\left\{ \alpha(Py_a) - \beta P^2 - \gamma\left[\varepsilon^A y_A^2 + \sum_{A<A'} (y_A \cdot y_{A'})\right] + \right.$$

$$\left. + \frac{\hbar^2 k}{2}\left[(N-1)\varepsilon^A \frac{\partial^2}{\partial y_\mu^A \cdot \partial y_\nu^A} - \sum_{B \neq B'} \frac{\partial^2}{\partial y_\mu^B \cdot \partial y_\nu^{B'}}\right]\pi_{\mu\nu} - \frac{1}{2} m_a^2 \right\} \psi = 0 \tag{4.7,c}$$

where ψ is a tempered distribution on the momenta space.

After a short calculation we obtain that the N equations (4.7c) are equivalent to:

$$(Py_A - \beta_A)\psi = 0, \qquad A = 2, \ldots, N \tag{4.8a}$$

$$\left[\beta P^2 + \frac{\gamma}{2}\mathbb{\Gamma}_{AB}(y^A \cdot y^B) - \frac{\hbar^2 k N}{2}(\mathbb{\Gamma}^{-1})^{AB} \cdot \pi_{\mu\nu} \cdot \frac{1}{\partial y_\mu^A \cdot \partial y_\nu^B} + \frac{1}{2N}\varepsilon^a m_a^2\right]\psi = 0 \tag{4.8b}$$

where:

$$\beta_A = \frac{1}{2\alpha N}(N m_A^2 - \varepsilon^b m_b^2)$$

and $\mathbb{\Gamma}$ is a $(N-1) \times (N-1)$ symmetric matrix given by:

$$\mathbb{\Gamma}_{AA} = 2, \qquad \mathbb{\Gamma}_{AA'} = 1, \qquad A' \neq A = 2, \ldots, N \tag{4.9a}$$

and $\mathbb{\Gamma}^{-1}$ its inverse:

$$(\mathbb{\Gamma}^{-1})^{AA} = \frac{N-1}{N}, \qquad (\mathbb{\Gamma}^{-1})^{AA'} = -\frac{1}{N} \tag{4.9b}$$

Since $\mathbb{\Gamma}$ is symmetric, there exists an $(N-1) \times (N-1)$ matrix $\mathbb{D}$ diagonalizing it:

$$\mathbb{D}^T \cdot \mathbb{\Gamma} \cdot \mathbb{D} = \mathbb{1} \qquad (\mathbb{D}_C^A \mathbb{D}_F^B \mathbb{\Gamma}_{AB} = \delta_{CF}) \tag{4.10}$$

(The $(N-1) \times (N-1)$ matrices $\mathbb{D}$ and $\mathbb{D}^{-1}$ are given in Appendix B).

We now define the following change of variables:

$$\left.\begin{aligned} \bar{y}^A &\equiv (-P^2)^{-1/2}(Py^A) \\ \vec{u}^A &\equiv (\mathbb{D}^{-1})_B^A y_\mu^B \vec{\varepsilon}^{\,\mu}(P) \end{aligned}\right\} \tag{4.11}$$

where $\vec{\varepsilon}^{\,\mu}(P)$ are the polarization vectors associated to P_ν (see Appendix B).

Then introducing those variables into (4.7) and (4.8) and using (4.10) and Appendix B, we obtain:

$$\left[\frac{1}{2N}\varepsilon^a m_a^2+\beta P^2-\frac{\gamma}{2}\Gamma_{AB}\bar{y}^A\bar{y}^B+\frac{\gamma}{2}\varepsilon_A\vec{u}^A\cdot\vec{u}^A-\frac{\hbar^2 kN}{2}\varepsilon^A\vec{\nabla}_A\cdot\vec{\nabla}_A\right]\psi=0 \quad (4.12a)$$

$$(\bar{y}^A\cdot(-P^2)^{1/2}-\beta^A)\psi=0 \quad (4.12b)$$

$$(P^2+M^2)\psi=0 \quad (4.12c)$$

$$[(-i\hbar\vec{u}^A\times\vec{\nabla}_A)^2-\hbar^2 L(L+1)]\psi=0 \quad (4.12d)$$

where $\nabla_A^i\equiv\dfrac{\partial}{\partial u_i^A}$ and $\vec{a}\cdot\vec{b}\equiv a_i b_j\cdot\delta^{ij}$.

The general solution of eq. (4.12) can be written as:

$$\psi=\Phi\cdot\delta(P^2+M^2)\cdot\theta(P^0)\cdot\prod_{A=2}^{N}\delta(\bar{y}^A-\beta^A/M) \quad (4.13)$$

where $\Phi\in\mathscr{S}'(\mathbb{R}^{3N})$ acts on functions of $(\vec{P},\ \vec{u}^2,\ \ldots,\ \vec{u}^N)$ and it is a solution of:

$$\left(-\frac{\hbar^2 kN}{2}\varepsilon^A\vec{\nabla}_A^2+\frac{1}{2}\gamma\varepsilon_A\vec{u}^A\cdot\vec{u}^A-\zeta\right)\phi=0 \quad (4.14a)$$

$$((\vec{u}^A\times\vec{\nabla}_A)^2+L(L+1))\phi=0 \quad (4.14b)$$

with:
$$\zeta=\frac{-1}{2N}\varepsilon^a m_a^2+\beta M^2+\frac{\gamma}{8M^2\alpha^2 N}\cdot\sum_{a<a'}(m_a^2-m_{a'}^2)^2 \quad (4.15)$$

The distribution $\theta(P^0)$ in (4.13) ensures that only positive energy solutions are considered.

Since eq. (4.14*a*) is separable, we can apply the theorem (VI-B-2) of ref. [*11*] to conclude that the general solution can be written as the series:

$$\Phi=\sum_{j\in I}c_j\prod_{A=2}^{N}\prod_{i=1}^{3}f_{Ai}^{(j)}(u_i^A)$$

(summable in the sense of $\mathscr{S}'$) where the functions $f_{Ai}^{(j)}$ are solutions of the one-dimensional quantum oscillator equation.

Since ϕ is a temperated distribution, $f_{Ai}^{(j)}(u_i^A)$ must be slowly increasing functions. This implies that the range of values which can take on are:

$$\zeta_n=\hbar\sqrt{\gamma kN}\cdot\left[n+\frac{3}{2}(N-1)\right],\qquad n\in\mathbb{N} \quad (4.16)$$

and hence the mass spectrum is given by:

$$M_n^2 = \frac{1}{2\beta}\left(\frac{\varepsilon^a m_a^2}{2N} + \zeta_n\right) + \frac{1}{2\beta}\sqrt{\left(\frac{\varepsilon^a m_a^2}{2N} + \zeta_n\right)^2 - \frac{\beta\gamma}{2N\alpha^2}\sum_{a<a'}(m_a^2 - m_{a'}^2)^2} \quad (4.17)$$

where the + sign has been chosen is the square root in order to keep $M_n^2 > 0$. In the particular case $m_1 = \ldots = m_N$ we have that M_N^2 depends linearly on n. This feature of our model agrees with a well known result for hadrons [*18*].

Equation (4.14*b*) is the total angular momentum equation for the N−1 uncoupled harmonic oscillators (non-relativistic). Hence L can only take on positive integer values whose range $\mathscr{L}_{n,N}$ depends on n and N.

The explicit expression for a complete set of solutions of (4.14) is in general complicated. We give these expressions in Appendix C for the simplest cases where N = 2 or N = 3 and the lowest masses $n = 0, 1, 2$ and angular momenta $L = 0, 1, \ldots$

c) The Hilbert space $\mathscr{H}(M_n, L; m_1, \ldots, m_N)$.

As we have pointed out above, some summability conditions must be added to eq. (4.6) in order to make possible a Hilbertian structure on $\mathscr{H}(M_n, L; m_1, \ldots, m_N)$. We shall require the distribution ϕ appearing on (4.13) to be square summable with respect to the measure:

$$d^{3N}\mu_n \equiv \frac{d^3\vec{P}}{2\sqrt{M_n^2 + \vec{P}^2}} \cdot \prod_{A=2}^{N} d^3\vec{u}^A \quad (4.18)$$

We shall write: $\Phi \in \mathscr{L}^2_{\mu n}(\mathbb{R}^{3N})$, i. e.,

$$\int d^{3N}\mu_n \cdot |\Phi(\vec{P}, \vec{u}^2, \ldots, \vec{u}^N)| < +\infty$$

Notice that $d^{3N}\mu_n$ is Poincaré invariant. Indeed, the first part

$$\frac{1}{2}(M_n^2 + \vec{P}^2)^{-1/2} d^3\vec{P}$$

is Poincaré invariant and also the second $\prod_{A=2}^{N} d^3\vec{u}^A$ as can be inferred from the definition (4.11).

In fact the summability with respect to the variables $\vec{u}^2, \ldots, \vec{u}^N$ is implied by the fact that Φ is a slowly increasing solution of the harmonic oscillator equation (4.14,*a*).

Summarily, the space $\mathscr{H}(M_n, L; m_1, \ldots, m_N)$ is defined by:

$$\mathscr{H}(M_n, L; m_1, \ldots, m_N) = \left\{ \Phi(\vec{P}, \vec{u}^B) \cdot \delta(P^2 + M_n^2) \cdot \theta(P^0) \cdot \prod_{A=2}^{N} \delta(\bar{y}^A - \beta^A/M_n) / \right.$$

$$\left. \Phi \in \mathscr{L}^2_{\mu n}(\mathbb{R}^{3N}) \text{ and } \Phi \text{ is a solution of (4.14)} \right\}$$

It is obvious that the sesquilinear form:

$$\langle \psi_1 \mid \psi_2 \rangle_n = \int d^{3N}\mu_n \cdot \phi_1^*(\vec{P}, \vec{u}^B) \cdot \phi_2(\vec{P}, \vec{u}^B)$$

$$\psi_i = \Phi_i(\vec{P}, \vec{u}^B) \cdot \theta(P^0) \cdot \delta(P^2 + M_n^2) \cdot \prod_{A=2}^{N} \delta(\bar{y}^A - \beta^A/M_n), \quad i = 1, 2 \tag{4.19}$$

defines a Hilbertian structure on $\mathscr{H}(M_n, L; m_1, \ldots, m_N)$.

Also, since the measure $d^{3N}\mu_n$ does not depend on L but only on n, we can consider the space:

$$\mathscr{H}(n; m_1, \ldots, m_N) = \bigoplus_{L \in \mathscr{L}_{n,N}} \mathscr{H}(M_n, L; m_1, \ldots, m_N)$$

$$\mathscr{H}(n; m_1, \ldots, m_N) = \Big\{ \Phi(\vec{P}, \vec{u}^B) \cdot \theta(P^0) \cdot \delta(P^2 + M_n^2) \cdot \prod_{A=2}^{N} \delta(\bar{y}^A - \beta^A/M_n) /$$

$$\Phi \in \mathscr{L}^2_{\mu_n}(\mathbb{R}^{3N}) \text{ and } \Phi \text{ is a solution of (4.14,}b\text{)} \Big\}.$$

This is the space of quantum states of the cluster of N particles with total mass M_n and unspecified intrinsic angular momentum, and it admits the Hilbertian structure defined by (4.19).

Finally, we can consider the space of all the quantum states with any mass and angular momentum:

$$\mathscr{H}(m_1, \ldots, m_N) \equiv \bigoplus_{n=0}^{\infty} \mathscr{H}(n; m_1, \ldots, m_N) \tag{4.20}$$

However, the Hilbert product must be now specified:

$$\langle \psi_1 \mid \psi_2 \rangle \equiv \sum_{n=0}^{\infty} \langle \psi_1^{(n)} \mid \psi_2^{(n)} \rangle_n \tag{4.21}$$

with: $\psi_i = \sum_{n=0}^{\infty} \psi_i^{(n)}$; $\quad i = 1, 2, \quad \psi_i^{(n)} \in \mathscr{H}(n; m_1 \ldots m_N)$.

5. COMPARISON WITH OTHER MODELS AND CONCLUSIONS

There is much literature about relativistic models of particles interacting at a distance and this model is just but one among them, which has been derived in the framework of Predictive Relativistic Mechanics. We shall now compare the results obtained in the foregoing sections with

those given by some other models of two [3] [12] [13] or three [2] particles, already published.

Generally, requiring the manifest covariance of any model, implies at the classical level to deal with the 8N dimensional phase space TM_4^N. Hence, the corresponding quantum picture must be built on a Hilbert space of distributions on M_4^N – i. e.: $\psi(p_\mu^1, \ldots, p_\mu^N)$.

However, the advantage of dealing with a manifestly covariant formalism — the compactness of equations, the simplicity of the action of the Poincaré group on M_4^N, etc. — has the drawback of introducing $2N - 1$ extravariables at the classical level whose physical meaning is not clear for all of them. Indeed, whereas N of these can be easily related to the individual mass of each particle [14], the remaining $N - 1$ are related to something as vague as the so called « relative times ». (How to « measure » them ?).

At the manifestly covariant quantum level only $(N - 1)$ extravariables are necessary — the « relative times » or their conjugated momenta, depending on whether we are dealing with the coordinates or with the momenta representation, respectively — and they must be eliminated in order to avoid unpleasant features in the model. Indeed, at least since the covariant harmonic oscillator (CHO) proposed by Yukawa [15], it is well known that if the spurious degrees of freedom are not eliminated, then the energy spectrum is not bounded from below. This is due to the presence of the « ghost states » which are actually associated to the extravariables and which, in the Yukawa's CHO, correspond to « time-like » oscillations. These oscillations allow to increase negatively the energy at will.

Yukawa's CHO was defined by Klein-Gordon type equations, exhibiting a harmonic oscillator like potential, plus a supplementary condition which was imposed in order to forbid the « time-like » oscillations.

In the same direction line, mention must be made of the CHO models proposed by Kim and Noz [3] and by Feynman and coworkers [2] for $N = 2$ and $N = 2$ or 3 particles, respectively. In those models the spurious degrees of freedom are forbidden by imposing the necessary $(N - 1)$ subsidiary conditions, obtained by means of plausibility arguments.

Other methods to eliminate the extravariables are those proposed by the different constraint hamiltonian formalisms [14]. One example where it is easy to see how the Dirac formalism [16] can be applied is the CHO of Dominici and coworkers [12]. From Dirac's standpoint, the classical phase space in this two particle model is not TM_4^2 but a 13-submanifold Γ of it, defined by three constraints: two of them are second class and the remaining one is first class. The 13 degrees of freedom on Γ can be classified as follows; 6 canonical coordinates, 5 conjugated momenta and one more playing the role of « center-of-mass time ». The corresponding quantum

picture is built on a space of functions on 6 coordinates plus the « center-of -mass time », defined by a Klein Gordon type equation, where the Hamiltonian is deduced from the first class constraint.

A comparative study of the Rohrlich and Todorov formalisms [*14*] will be handled in a forthcoming paper.

In the model presented here — and in Droz-Vincent's as well [*13*] — we deal with the phase space TM_4^N, as it is usual in the manifestly covariant formalism of P. R. M., and we do not introduce any *a priori* constraints. In quantizing the model, the requirement of having unitary representations of the Complete Grup of Symmetry $\mathscr{G}$ on the Hilbert space $\mathscr{H}$ implies inmediately the N equations (4.7*c*) which are equivalent to the Klein-Gordon type equation (4.14*a*) plus (N − 1) more (4.14*b*) playing the same role as the subsidiary conditions in the above mentioned models.

So, in the case of 2 particles (N = 2), equal masses $m_1 = m_2 = m$ and parameters $\alpha = \gamma = 1/2$, $\beta = 1/8$, we obtain from (4.17) the mass spectrum:

$$M_n^2 = 4m^2 + 4\hbar\omega(n + 3/2) \tag{5.1}$$

where:
$$\omega = 2\sqrt{k} \tag{5.2}$$

The Hilbert space of states with well defined total mass M_n and total spin s is given by

$$\mathscr{H}_{n,s} \equiv \mathscr{H}(M_n, s; m_1 = m_2 = m) = \{ \Phi(\vec{P}, \vec{u}) \cdot \delta(P^2 + M_n^2) \cdot \theta(P^0) \cdot \delta(\bar{y}) / \Phi \in \mathscr{L}^2_{\mu_n}(\mathbb{R}^6) \} \tag{5.3}$$

with the Hilbert product:

$$\langle \psi_1 \mid \psi_2 \rangle_n = \int d^6\mu_n \Phi_1^*(\vec{P}, \vec{u}) \cdot \Phi_2(\vec{P}, \vec{u}) ; \qquad \psi_1, \psi_2 \in \mathscr{H}_{n,s} \tag{5.3'}$$

where $\Phi(\vec{P}, \vec{u})$ is a solution of:

$$\left[- \hbar^2 k \vec{\nabla}_u^2 + \frac{1}{4} \vec{u}^2 - \frac{\hbar\omega}{2}\left(n + \frac{3}{2}\right)\right]\Phi = 0 \tag{5.4a}$$

$$[(\vec{u} \times \vec{\nabla}_u)^2 + s(s + 1)]\Phi = 0 \tag{5.4b}$$

and $d^6\mu_n$ is the measure:

$$d^6\mu_n = \frac{1}{2}(\vec{P}^2 + M_n^2)^{-1/2} d^3\vec{P} \cdot d^3\vec{u} \tag{5.5}$$

The Hilbert space of states with definite mass M_n is given by:

$$\mathscr{H}_n = \bigoplus_s \mathscr{H}_{n,s}$$

whose elements are of the form:

$$\psi = \Phi(\vec{P}, \vec{u}) \cdot \delta(P^2 + M_n^2) \cdot \theta(P^0) \delta(\bar{y})$$

$\Phi(\vec{P}, \vec{u})$ being a solution of (5.4*a*).

Since the measure of $\mathscr{H}_{n,s}$ does not depen on s, the same $d^6\mu_n$ defines the Hilbert product on $\mathscr{H}_n$.

Notice that the variable $\bar{y}$ is completely spurious and does not play any role in this quantum picture. It is like in Dirac's constraint formalism as we have commented above.

We have to point out also that in this particular case (N = 2) our model recovers the main features of Droz-Vincent's one [*13*]. It must be emphasized, however that the quantum model of Droz-Vincent is not completely defined. Indeed, although he obtains a mass spectrum and gives the space of wave functions, he does not take care of defining a Hilbert structure on it. This leads Droz-Vincent [*17*] and also other authors [*12*] not to be aware of the fact that this predictive model reproduces the main features of the CHO of Kim and Noz. Indeed, although the space of wave functions here obtained $\mathscr{H}_n$ and the one given by Kim and Noz [*3*] $\mathscr{H}_n^{\mathrm{KN}}$ are different, so are the Hilbert products, but in such a way that there is an isometric mapping which relates $\mathscr{H}_n$ with $\mathscr{H}_n^{\mathrm{KN}}$, for each $n \in \mathbb{N}$

$$\begin{array}{ccc} & \mathscr{H}_n & \to \ \mathscr{H}_n^{\mathrm{KN}} \\ \Phi(\vec{\mathrm{P}}, \vec{u})\cdot\delta(\mathrm{P}^2+\mathrm{M}_n^2)\theta(\mathrm{P}^0)\cdot\delta(\bar{y}) & \to & \Phi(\vec{\mathrm{P}}, \vec{u})\cdot\delta(\mathrm{P}^2+\mathrm{M}_n^2)\cdot\theta(\mathrm{P}^0)\cdot\exp\left(-\bar{y}^2/2\hbar k\right) \end{array}$$

Hence, both models are quite equivalent from the quantum point of view.

Finally we want to emphasize the main points in which the quantization method presented here is based. These ideas have been developed by L. Schwartz [*8*] in the case of free particles:

i) We are interested in the irreducible unitary representations of the symmetry group of our system (in our case the complete symmetry group: $\mathscr{G} = \mathscr{P} \times \mathscr{A}_{\mathrm{N}}$).

ii) On a Hilbert space of tempered distributions $\mathscr{H} \subset \mathscr{S}'(\mathrm{M}_4^{\mathrm{N}})$.

The requirement (*i*) implies that the 10 + N generators of the infinitesimal transformations of $\mathscr{G}$ are hermitian operators on $\mathscr{H}$ and therefore they are associated to 10 + N observables. Furthermore, we want to stress that (*ii*) not only implies choosing a subspace $\mathscr{H} \subset \mathscr{S}'(\mathrm{M}_4^{\mathrm{N}})$, but also a Hilbertian structure on it, which will allow to calculate expectation values of observables. Consequently, to compare two quantum models requires to consider together the wave function space and the Hilbert product.

Some problems are still open: the definition of the operators $\hat{x}_a^\mu$ (individual position of each particle), a probabilistic interpretation of the model, etc. After that we shall be able to apply these results to compute form factors, anomalous magnetic momenta, etc. of hadrons in the framework of a quark model.

APPENDIX A

1) Conditions on the function $F(q, p)$ in order that $\hat{F}$ be a well defined continuous linear operator on $\mathscr{S}(M_4^N)$, i. e.: to be an indicial function. It is easy to see from the definition (2.7) that:

i) $F(q,p)\cdot\varphi(q)\in\mathscr{S}_q(M_4^N)$, $\forall p\in M_4^N$, $\forall\varphi\in\mathscr{S}(M_4^N)\Leftrightarrow F(q,p)$ is a slowly increasing function of the variables q.

ii) $\mathscr{F}_p(F(q,p)\cdot\varphi(q))\in\mathscr{S}_p(M_4^N)$, $\forall\varphi\in\mathscr{S}(M_4^N)\Leftrightarrow F(q,p)$ is a slowly increasing function on the variables p.

Both conditions are fulfilled when $F(q, p)$ is a polynomial.

2) Proof of equation (2.9):

Let $\hat{F}$ be the continuous linear operator on $\mathscr{S}'_q(M_4^N)$ associated to the indicial function $F(q, p)$ and consider the slowly increasing functions $\exp\left(\pm\frac{i}{\hbar}q_a^\mu\cdot p_\mu^a\right)$, which are also tempered distributions $\mathscr{S}'_q(M_4^N)$. We then have:

$$\exp\left(-\frac{i}{\hbar}q_a^\mu p_\mu^a\right)\cdot\hat{F}\exp\left(\frac{i}{\hbar}q_a^\mu p_\mu^a\right)\in\mathscr{S}'_q(M_4^N)$$

Using the definition (2.7) and some properties of tempered distributions we can write:

$$\begin{aligned}\langle e^{-\frac{i}{\hbar}q_a^\mu p_\mu^a}\hat{F}e^{\frac{i}{\hbar}q_a^\mu\cdot p_\mu^a},\varphi(q)\rangle &= \langle e^{\frac{i}{\hbar}q_a^\mu p_\mu^a},\overline{\mathscr{F}}_q[\mathscr{F}_{p'}[F(q',p')e^{-\frac{i}{\hbar}q_a^\mu p_\mu^a}\varphi(q')]]\rangle\\ &= (2\pi\hbar)^{2N}\langle\delta^{4N}(p-p'),\mathscr{F}_{p'}[F(q',p')e^{-\frac{i}{\hbar}q_a^\mu p_\mu^a}\varphi(q')]\rangle = \langle F(q,p),\varphi(q)\rangle\end{aligned}$$

3) Proof of equation (2.10):

$$\begin{aligned}\langle\widehat{q^m p^n}\psi_q,\varphi_q\rangle &= \langle\psi_q,\overline{\mathscr{F}}_q\{\mathscr{F}_p(q'^m p^n\varphi(q'))\}\rangle =\\ &= \langle\psi_q,\overline{\mathscr{F}}_q\{p^n(-i\hbar\partial_p)^m\mathscr{F}_p\varphi\}\rangle = \langle\psi_q,(i\hbar\partial_q)^n\overline{\mathscr{F}}_q\{(-i\hbar\partial_p)^m\mathscr{F}_p\varphi\}\rangle =\\ &= \langle(-i\hbar\partial_q)^n\psi_q,\ q^m\overline{\mathscr{F}}_q(\mathscr{F}_p\varphi)\rangle = \langle q^m(-i\hbar\partial_q)^n\psi_q,\varphi_q\rangle\end{aligned}$$

APPENDIX B

1) Diagonalization of Γ.

The characteristic and minimum polinomials of the matrix Γ are, respectively:

$$\det(\Gamma - \lambda\mathbb{1}) = (N - \lambda)(1 - \lambda)^{N-2}$$
$$(\Gamma - N\cdot\mathbb{1})\ (\Gamma - \mathbb{1}) = 0$$

Hence, Γ can be diagonalized and we can find a basis of $N - 1$ eigenvectors: one for the eigenvalue N and the remaining $(N - 2)$ corresponding to the eigenvalue 1.

Due to this degeneracy the matrix equation $\mathbb{D}^T\cdot\Gamma\cdot\mathbb{D} = \mathbb{1}$ admits a great many of solutions $\mathbb{D}$ and a particular one is given by:

$$\mathbb{D}_{A,2} = N^{-1/2}\cdot(N-1)^{-1/2}\quad,\qquad A = 2,\ldots,N$$

$$\mathbb{D}_{A,B} = \begin{cases} (B-1)^{-1/2}\cdot(B-2)^{-1/2}, & A < B \\ -(B-1)^{-1/2}(B-2)^{1/2}, & A = B \\ 0\quad, & A > B,\quad B = 3,\ldots,N \end{cases}$$

2) Polarization vectors:

In the center-of-mass frame $\overset{\circ}{P}{}^{\nu} = (M, 0, 0, 0)$ with $M = (-P^2)^{1/2}$ we take three independent space-like fourvectors $\varepsilon^{i\mu}(\overset{\circ}{P})$ which, together with $\overset{\circ}{P}$ form an orthogonal basis of M_4:

$$\varepsilon^{i\mu}(\overset{\circ}{P}) = (0, \delta^{ij})$$

In any other reference frame the polarization vectors are defined by the transforms of these vectors under the boost $L^{\mu}_{\nu}(P)$ which brings $(M, 0, 0, 0)$ into $(P^0, \vec{P})$:

$$\varepsilon^{i\mu}(P) = L^{\mu}_{\nu}(P)\cdot\varepsilon^{i\nu}(\overset{\circ}{P})$$

Also, $\varepsilon^{i\mu}(P)$, $i = 1, 2, 3$ and P^{μ} form an orthogonal basis of M_4. Some interesting properties of the polarization vectors are the following:

$$\varepsilon^{i\mu}(P)P_{\mu} = 0\ ;\qquad \varepsilon^{i\mu}(P)\cdot\varepsilon^{j}_{\mu}(P) = \delta^{ij}$$

$$\varepsilon^{i\mu}(P)\varepsilon^{\nu}_{i}(P) = \pi^{\mu\nu} = \eta^{\mu\nu} - \frac{P^{\mu}P^{\nu}}{P^2}$$

$$\frac{1}{M}\,\varepsilon_{\mu\nu\rho\lambda}\cdot\varepsilon^{i\nu}(P)\cdot\varepsilon^{j\rho}(P)\cdot\varepsilon^{k\lambda}(P) = -\,\varepsilon^{ijk}$$

$$\varepsilon^{i\mu}(\Lambda P) = D^{i}_{j}(\Lambda, P)\Lambda^{\mu}_{\nu}\varepsilon^{j\nu}(P)$$

where $\Lambda^{\mu}_{\nu}\varepsilon\mathscr{L}^{\uparrow}_{+}$ and D^{i}_{j} is the Wigner rotation associated to (Λ, P).

APPENDIX C

The equations (4.14) have the same form as the wave equation of (N − 1) uncoupled quantum harmonic oscillators. Using the separability of this equation we can write:

$$\Phi(\vec{u}^2, \ldots, \vec{u}^N) = \prod_{A=2} \Phi^A(u_A, \theta_A, \varphi_A) \tag{C.1}$$

where $(u_A, \theta_A, \varphi_A)$ are the spherical coordinates of $\vec{u}^A$.

Then, substituting this in (4.14), it yields:

$$\left(- \zeta_{n_A} - \frac{1}{2} \hbar^2 k N \vec{\nabla}_A^2 + \frac{1}{2} \gamma (\vec{u}^A)^2 \right) \Phi^A(u_A, \theta_A, \varphi_A) = 0$$

with: $$\varepsilon^A \zeta_{n_A} = \zeta_n, \qquad \varepsilon^A n_A = n$$

The solutions of this last equation are the classical solutions of the isotropic three-dimensional harmonic oscillator:

$$\Phi_{n_A L_A \mu_A}(u_A, \theta_A, \varphi_A) = C_{n_A L_A} \cdot R_{n_A L A}(u_A) \cdot Y_{L_A}^{\mu_A}(\theta_A, \varphi_A) \tag{C.2}$$

$$\rho^2 = \sqrt{\frac{\gamma}{N\hbar^2 k}}, \qquad C_{nL} = \sqrt{\frac{2\rho^{2L+3} \Gamma[(n + L + 3)/2]}{\left(\frac{n - L}{2}\right)! \, \Gamma^2(L + 3/2)}}$$

$$R_{nL}(r) = r^L \cdot M\left(- \frac{n - L}{2}, L + \frac{3}{2}, \rho^2 r^2 \right) \cdot \exp\left[- \frac{\rho^2 r^2}{2} \right] \tag{C.3}$$

where $M(-n, m, x)$ are the Kummer functions of integer index [20].

$$n_A = 0, 1, 2, \ldots$$
$$L_A = n_A, n_A - 2, n_A - 4, \ldots, n_A - 2[n_A/2]$$
$$\mu_A = - L_A, - L_A + 1, \ldots, L_A - 1, L_A$$

and $[\alpha]$ means the integral part of α.

The total angular momentum operator is:

$$\hat{L}^i = - i\hbar \varepsilon^A (\vec{u}_A \wedge \vec{\nabla}_A)^i = \varepsilon^i_\mu(P) \cdot \hat{W}^\mu$$

We want our functions $\Phi(\vec{u}^2, \ldots, \vec{u}^N)$ to be eigenfunctions of $\vec{L}^2 = \hat{W}^2$ and $\hat{L}_3$. Since $\Phi_{n_A L_A \mu_A}(u_A, \theta_A, \varphi_A)$ are eigenfunctions of $(\vec{u}_A \wedge \vec{\nabla}_A)^2$ and $(\vec{u}_A \wedge \vec{\nabla}_A)_3$ $(A = 2, \ldots, N)$, we must compose the N − 1 angular momenta L_A to obtain the total angular momentum L and the corresponding third component μ.

a) *Wave functions for* N = 2.

In this case only one function of type (C.2) is involved: $\Phi_{nL\mu}(\vec{u})$ and no more quantum numbers must be considered.

The quantum number L runs through the range of values:

$$\mathscr{L}_{n,2} = \left\{ n, n - 2, n - 4, \ldots, n - 2\left[\frac{n}{2}\right] \right\}; \qquad \left[\frac{n}{2}\right] + 1 \quad \text{values of L for each } n.$$

The radial part of the wave functions corresponding to the few first quantum numbers are:

$$R_{00}(r) = \exp\left(-\frac{\rho^2 r^2}{2}\right), \qquad R_{11}(r) = rR_{00}(r)$$

$$R_{22}(r) = r^2 R_{00}(r) \quad , \qquad R_{20}(r) = \left(1 - \frac{2}{5}\rho^2 r^2\right) \cdot R_{00}(r)$$

b) *Wave functions for* N = 3.

In this case two functions of type (C.2) are necessary and, since we are looking for eigenfunctions of $\vec{L}^2$ and $\hat{L}_3$, we have to couple them using the Clebsch-Gordan coefficients:

$$\Phi_{n,L,\mu(n_2,L_2,L_3)}(\vec{u}^2, \vec{u}^3) = \sum_{\mu_2,\mu_3} C(L_2, L_3, L; \mu_2, \mu_3, \mu) \cdot \Phi_{n_2 L_2 \mu_2}(\vec{u}^2) \cdot \Phi_{n-n_2 L_3 \mu_3}(\vec{u}^3)$$

It is easily seen that:

$$\mathscr{L}_{n,3} = \left\{ n, n-1, n-2, \ldots, n-2\left[\frac{n}{2}\right] \right\}; \qquad 2\left[\frac{n}{2}\right] + 1 \quad \text{values of L for each } n.$$

But there also appear the internal quantum numbers (n_2, L_2, L_3) which account for how the two oscillators are coupled. Therefore, there is a certain degeneracy which for the few first values of (n, L) is given in the following table:

n	L	n_2	L_2	L_3	Degeneracy
0	0	0	0	0	1
1	1	0	0	1	2
		1	1	0	
2	2	2	2	0	3
		1	1	1	
		0	0	2	
2	1	1	1	1	1
2	0	2	0	0	3
		1	1	1	
		0	0	0	

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