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Bound states and propagating states for time-dependent Hamiltonians

by

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ABSTRACT. — The notions of the continuous and point-spectral subspaces are carried over to quantum mechanical systems governed by time-dependent Hamiltonians. The relation to the geometric characterization of « bound » states and « propagating » states is discussed generalizing a theorem of Ruelle. We study the problem of absence or existence of bound states for various models. Most results concern the time-periodic case.

RÉSUMÉ. — Les notions de sous-espaces spectraux continu et ponctuel sont étendues aux systèmes quantiques gouvernés par des Hamiltoniens dépendant du temps. On discute leur relation avec la caractérisation géométrique d'états liés et d'états qui se propagent, généralisant ainsi un théorème de Ruelle. On étudie le problème de l'absence ou de l'existence d'états liés pour divers modèles. La plupart des résultats concerne le cas périodique en temps.

I. INTRODUCTION AND PRELIMINARIES

We study the characterization of bound states and propagating states for quantum mechanical systems with time-dependent Hamiltonians $H(t)$. The time evolution of the states is determined by the Schrödinger equation

$$(1.1) \quad i \frac{d\Psi(t)}{dt} = H(t)\Psi(t)$$

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where $H(t)$ is a family of self-adjoint operators on the underlying Hilbert space \mathcal{H} and $\Psi(t) \in \mathcal{H}$ for all t . Under suitable conditions on $H(t)$ there exists a solution of the initial value problem $\Psi(0) = \Psi$:

$$(1.2) \quad \Psi(t) = U(t, 0)\Psi.$$

The *propagators*, or *time evolution operators* $U(t, s)$ form a jointly strongly continuous family of unitary operators satisfying

$$(1.3) \quad \begin{aligned} U(t, r)U(r, s) &= U(t, s) \quad \text{for all } r, s, t, \\ U(t, t) &= \mathbb{1} \quad \text{for all } t. \end{aligned}$$

Usually $H(t)$ is independent of the time t , however time-dependent Hamiltonians arise naturally as approximations in complex systems governed by a time-independent Hamiltonian. A small subsystem is singled out, whose action on the larger part of the system can be neglected. If the motion of the larger part is known, then its action on the small subsystem can be approximately described by a force which depends explicitly on the time. Examples for this are: (i) interactions which are turned on or off at finite times or asymptotically; (ii) external electromagnetic fields acting on atoms, molecules, etc.; they are often periodic in time; (iii) random perturbations, caused e. g. by thermal fluctuations; (iv) the « charge transfer » — or « impact parameter model »: this is an approximation of the three-body problem. Two heavy particles (atoms, ions) move along prescribed trajectories and one studies the motion of the third light particle (electron) in their field. For the last example the problem has recently been studied by Hagedorn [5] and we shall not treat it here although our concepts can easily be generalized to cover it as well.

If $H(t) = H$ is independent of t the time evolution operators form a one-parameter unitary group which is obtained by the functional calculus

$$(1.4) \quad U(t, s) = \exp \{ -iH(t - s) \}.$$

In this case the spectral theorem allows to distinguish between states with qualitatively different time evolution. To the self-adjoint H here corresponds a direct sum decomposition

$$(1.5) \quad \mathcal{H} = \mathcal{H}^{pp}(H) \oplus \mathcal{H}^{cont}(H)$$

where the point spectral subspace $\mathcal{H}^{pp}(H)$ is spanned by the eigenvectors of H , and $\mathcal{H}^{cont}(H)$ is the continuous spectral subspace of H . (Our terminology mainly follows [10]). The spectral properties of a state with respect to H are identical to those with respect to the time evolution operator $\exp(-iHt)$ for any $t \neq 0$. Moreover the trajectory $\{ \Psi(t) | t \in \mathbb{R} \}$ is a precompact set in \mathcal{H} (approximately finite dimensional), if $\Psi \in \mathcal{H}^{pp}(H)$,

whereas the trajectory will leave any compact subset of \mathcal{H} in the time average, if $\Psi \in \mathcal{H}^{\text{cont}}(\mathbf{H})$:

$$(1.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \| Ce^{-i\mathbf{H}t} \Psi \| = 0$$

for any compact operator C . This is true for any self-adjoint operator \mathbf{H} .

If we consider the Hilbert space

$$(1.7) \quad \mathcal{H} = L^2(\mathbb{R}^v, d^v x)$$

and if e. g. \mathbf{H} is a Schrödinger operator of the form

$$(1.8) \quad \mathbf{H} = \mathbf{H}_0 + V(x), \quad \mathbf{H}_0 = \frac{p^2}{2m} = -\frac{1}{2m} \Delta$$

then the spectral characterization can be given equivalently in terms of the localization of states in « x -space » as was shown by Ruelle [11], for extensions see [1], Appendix to XI.17 in [10], Section IV in [2]. The theorem was called RAGE-theorem in [10].

Denote by $F(|x| < R)$ the multiplication operator in x -space with the characteristic function of the ball of radius R (for other regions analogously). The RAGE-theorem then states for an extremely wide class of potentials

$$(1.9) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \| F(|x| < R) e^{-i\mathbf{H}t} \Psi \| = 0$$

for any $R < \infty$ and $\Psi \in \mathcal{H}^{\text{cont}}(\mathbf{H})$. On the other hand for any $\Psi \in \mathcal{H}^{\text{pp}}(\mathbf{H})$, $\varepsilon > 0$, there is an $R(\varepsilon)$ such that

$$(1.10) \quad \sup_{t \in \mathbb{R}} \| F(|x| > R(\varepsilon)) e^{-i\mathbf{H}t} \Psi \| < \varepsilon.$$

The properties (1.9) and (1.10) can be called the *geometric characterization of propagating states and bound states*, respectively. This justifies to call a state Ψ a bound state (staying essentially in a bounded region of space uniformly in time) if and only if $\Psi \in \mathcal{H}^{\text{pp}}(\mathbf{H})$. The main ingredients of the proof are *energy conservation* of the time evolution and *local compactness*:

$$(1.11) \quad F(|x| < R)(\mathbf{H} + i)^{-1} \quad \text{is compact} \quad \forall R < \infty.$$

(See e. g. Section III in [2] and references given there for a discussion of this notion.)

In the present paper we study which of these notions can be generalized to the time dependent case. The spectral decomposition

$$\mathcal{H} = \mathcal{H}^{\text{pp}}(\mathbf{H}(t)) \oplus \mathcal{H}^{\text{cont}}(\mathbf{H}(t))$$

in general does not have a significant influence on the time evolution, similarly with the decomposition w. r. t. $U(t, s)$. However, it is still meaningful

to study initial values giving rise to solutions with precompact trajectories. We define the set of bound states as

$$(1.12) \quad \mathcal{H}_+^p = \{ \Psi \in \mathcal{H} \mid \{ \Psi(t) \mid t > 0 \} \text{ is precompact in } \mathcal{H} \},$$

and analogously for negative times. In general the states which leave in the time average any compact subset of the Hilbert space (1.6) form a proper closed subspace of the orthogonal complement of \mathcal{H}_+^p . If, however, the time dependence is periodic,

$$(1.13) \quad H(t + T) = H(t) \quad \text{for} \quad t \in \mathbb{R}, \quad \text{some } T > 0,$$

then \mathcal{H}_+^p coincides with the point spectral subspace of the time evolution operator for one period,

$$(1.14) \quad \mathcal{H}_+^p = \mathcal{H}^{pp}(U(T, 0)) = \mathcal{H}_-^p.$$

Moreover

$$(1.15) \quad (\mathcal{H}_+^p)^\perp = \{ \Psi \mid \Psi \text{ satisfies (1.6)} \}.$$

Therefore \mathcal{H}_+^p is the proper generalization of the point spectral characterization of states. This is discussed in Section II.

It is easy to see that any state in \mathcal{H}_+^p is localized uniformly for positive times (1.10). However, the geometric characterization (1.9) of its orthogonal complement, even if (1.6) holds, need not be true in general. In the time independent case this can happen only for very special potentials which oscillate near a strong singularity, cf. Pearson's model of local adsorption [9]. Local compactness (1.11) excludes this pathology. As our example in Section IV shows even relatively compact time dependent perturbations can cause an infinite increase of kinetic energy of a system. Thus a system which leaves any compact subset of the Hilbert space may still be localized in x -space. The simple argument that (1.6) implies (1.9) fails in general. For our results in Section III we have to assume uniform boundedness of some kind of energy.

There is, however, a different mechanism which may still imply the geometric decay (1.9). A particle with high kinetic energy should travel fast and should leave a bounded region quickly, provided the potential can neither trap the particle nor reflect it back within too short a time. In Section VI under strong assumptions on the potential we can implement this picture and prove that the geometric characterization (1.9) holds for any state orthogonal to \mathcal{H}^p .

A direct proof of the equivalence of the geometric and spectral characterizations is given for special examples of periodic time evolutions in Sections IV, V, and VII. There we also study the question whether bound states or propagating states exist at all and how this changes under perturbations. If e. g. a harmonic oscillator is perturbed by a resonantly alternating Stark field the resulting system does not have any bound state. For non resonant

perturbations all states are bound states. On the other side, if the time periodic perturbation has compact support in space, then we can show stability of the bound states only if the frequencies are relatively rational. This indicates that for a complete understanding of these questions one needs more powerful methods. An interesting open question is whether there exist bound states for atoms in exterior electromagnetic radiation of low frequency.

In the following we will mostly use the time evolution operators $U(t, s)$. The following result (Theorem X. 70 in [10]) contains convenient sufficient conditions on $H(t)$ for the existence of $U(t, s)$. For weaker conditions see e. g. the references in [10].

1.1. THEOREM. — Let $H(t)$ be a self-adjoint operator-valued function of $t \in \mathbb{R}$ such that (α) the domain \mathcal{D} of $H(t)$ is independent of t
 (β) the function

$$t, s \rightarrow (t - s)^{-1} \{ (i + H(t))(i + H(s))^{-1} - \mathbb{1} \}$$

extends to a jointly strongly continuous bounded operator-valued function on \mathbb{R}^2 .

Then there exists a unique propagator U satisfying (1.3) such that $U(t, s)\mathcal{D} \subset \mathcal{D}$ and

$$(1.16) \quad i \frac{d}{dt} U(t, s)\Psi = H(t)U(t, s)\Psi, \quad \Psi \in \mathcal{D}.$$

Moreover one easily sees that if $H(t)$ is T periodic

$$(1.17) \quad H(t + T) = H(t) \quad \text{for all } t$$

then the propagator has the properties

$$(1.18) \quad U(t + T, s + T) = U(t, s) \quad \text{for all } t, s;$$

$$(1.19) \quad U(t + nT, s) = U(t, s)[U(s + T, s)]^n.$$

Thus it is sufficient to know $U(t, s)$ for one period $t \in [s, s + T]$, any s . In particular $U(s + T, s)$ is called the *monodromy* operator, or *Floquet* operator. In particular the propagator can be written in the (highly non-unique) Floquet form

$$(1.20) \quad U(t, 0) = P(t) \exp(-iGt)$$

with a self-adjoint operator G and a strongly continuous T -periodic function $P(t)$ which satisfies

$$(1.21) \quad P(0) = P(nT) = 1 \quad \text{for all } n \in \mathbb{Z}.$$

Finally let us note that the RAGE-problem for time dependent periodic potentials was treated independently in preprints by Howland [7] and

Veselic [14] (the results of the latter are partially contained in the present paper). However, both of them contained an error of the same kind, namely the tacit assumption of the time boundedness of the energy. The second author is indebted to B. Najman, Zagreb for pointing out the mentioned error in [14]. Helpful discussions with I. Herbst, B. Simon and K. Yajima are gratefully acknowledged. We would like to thank Lenart Carleson for his kind hospitality at the Institut Mittag-Leffler.

Throughout this paper we shall freely use the terminology and notations from Reed and Simon [10].

II. STATES WITH PRECOMPACT TRAJECTORIES

Here we shall extend the *spectral* definition of the point spectral subspace \mathcal{H}^{pp} to the case of a general unitary time evolution as defined by (1.3). The geometrical characterization will be discussed in the next section.

2.1. DEFINITION. — We define \mathcal{H}_\pm^p as the set of all states Ψ for which

$$(2.1) \quad \{ U(t, 0)\Psi, t \geq 0 \}$$

is precompact in \mathcal{H} .

2.2. REMARKS. — (i) By the strong continuity of $U(t, s)$ the above definition depends only on the behavior of $U(t, 0)\Psi$ for sufficiently large $t(-t)$. (ii) Both \mathcal{H}_\pm^p are closed linear subspaces of \mathcal{H} . The linearity is evident whereas the closedness is a consequence of the uniform boundedness of $U(t, 0)$ and the fact that the closure of a precompact set is compact. (iii) By requiring the precompactness of

$$\{ U(t, s)\Psi, t \geq 0 \}$$

a whole family $\mathcal{H}_\pm^p(s)$ of subspaces of bound states is defined. They are obviously connected by

$$(2.2) \quad \mathcal{H}_\pm^p(s) = U(s, t)\mathcal{H}_\pm^p(t)$$

and are therefore isomorphic. There is no loss in generality to restrict oneself to the case $s = 0$, i. e. to consider

$$\mathcal{H}_\pm^p(0) = \mathcal{H}_\pm^p.$$

The same restriction will be made in all similar cases without special mentioning.

2.3. THEOREM. — Let $U(t, s)$ be T -periodic as in (1.18). Then

$$\mathcal{H}_\pm^p = \mathcal{H}_\pm^p = \mathcal{H}^{pp}(U(T, 0)).$$

Proof. — To prove the theorem we need the following uniform estimate, which is a generalization of Lemma 4.2 in [2] to the time periodic case.

2.4. LEMMA. — Let $U(t, s)$ be as in Th. 2.3 and denote by P^{cont} the orthogonal projection onto $\mathcal{H}^{cont}(U(T, 0))$. Then for any compact operator C .

$$(2.3) \quad \lim_{|\tau| \rightarrow \infty} \left\| \frac{1}{\tau} \int_0^\tau U^*(t, 0)CU(t, 0)P^{cont} dt \right\| = 0.$$

Proof of Lemma 2.4. — We write

$$\tau = \sigma + nT, \quad \sigma \in [0, T).$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau U^*(t, 0)CU(t, 0)P^{cont} dt \right\| \leq \frac{T}{\tau} \|C\| + \left\| \frac{1}{nT} \int_0^{nT} U^*(t, 0)CU(t, 0)P^{cont} dt \right\|,$$

where the first summand tends to zero with $|\tau| \rightarrow \infty$. For the second summand we have

$$(2.4) \quad \begin{aligned} & \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{T} \int_{jT}^{(j+1)T} U^*(t, 0)CU(t, 0)P^{cont} dt = \\ & = \frac{1}{n} \sum_{j=0}^{n-1} U^*(T, 0)^j \left\{ \frac{1}{T} \int_0^T U^*(t, 0)CU(t, 0) dt \right\} U(T, 0)^j P^{cont} = \\ & = \frac{1}{n} \sum_{j=0}^{n-1} U^*(T, 0)^j C' U(T, 0)^j P^{cont}, \end{aligned}$$

where

$$C' = \frac{1}{T} \int_0^T U^*(t, 0)CU(t, 0) dt$$

is again compact as an integral of a norm-continuous compact operator-valued function. Since C' is approximated by finite rank operators, it is sufficient to estimate (2.4) for a rank 1 operator $C'' = (\Phi, \cdot)\Psi, \|\Phi\| = 1$. Since P^{cont} commutes with $U(T, 0)$ it is sufficient to consider $\Phi \in \mathcal{H}^{cont}(U(T, 0))$.

We have

$$(2.5) \quad \begin{aligned} \left\| \frac{1}{n} \sum_{j=0}^{n-1} U^*(T, 0)^j C'' U(T, 0)^j P^{cont} \right\|^2 & \leq \frac{1}{n^2} \sum_{j,k=0}^{n-1} |(\Phi, U(T, 0)^{j-k}\Phi)| \\ & \leq \left\{ \frac{1}{n^2} \sum_{k,j=0}^{n-1} |(\Phi, U(T, 0)^{j-k}\Phi)|^2 \right\}^{1/2}, \end{aligned}$$

where the last step uses the Schwarz inequality. With the spectral representation

$$(2.6) \quad U(\mathbf{T}, 0) = \int_0^{2\pi} e^{i\lambda} d\mathbf{E}(\lambda)$$

the curly bracket in (2.5) becomes

$$(2.7) \quad \int_0^{2\pi} \int_0^{2\pi} f_n(\lambda - \mu) d(\Phi, \mathbf{E}(\lambda)\Phi) d(\Phi, \mathbf{E}(\mu)\Phi)$$

where

$$f_n(\lambda - \mu) = \left| \frac{1}{n} \frac{1 - e^{in(\lambda - \mu)}}{1 - e^{i(\lambda - \mu)}} \right|^2$$

is uniformly absolutely bounded by one. Moreover for any $\delta > 0$ we have

$$|f_n(\lambda - \mu)| \leq \frac{\text{const}(\delta)}{n^2},$$

if $|\lambda - \mu - 2\pi k| > \delta$, $k = -1, 0, 1$.

The region of integration in (2.7) can accordingly be split and the integral itself is bounded by

$$(2.8) \quad \sum_{k=-1}^1 \iint_{|\lambda - \mu - k2\pi| < \delta} d(\Phi, \mathbf{E}(\mu)\Phi) d(\Phi, \mathbf{E}(\lambda)\Phi) + \frac{\text{const}(\delta)}{n^2}$$

where we took into account that

$$\int_0^{2\pi} \int_0^{2\pi} d(\Phi, \mathbf{E}(\mu)\Phi) d(\Phi, \mathbf{E}(\lambda)\Phi) = 1.$$

By making the natural extension

$$\mathbf{E}(\lambda) = 0 \quad \text{for } \lambda < 0, \quad \mathbf{E}(\lambda) = \mathbf{I} \quad \text{for } \lambda > 2\pi$$

the first term in (2.8) is bounded by

$$\int_0^{2\pi} [(\Phi, \mathbf{E}(\lambda + \delta)\Phi) - (\Phi, \mathbf{E}(\lambda - \delta)\Phi)] d(\Phi, \mathbf{E}(\lambda)\Phi).$$

Now the convergence

$$(\Phi, \mathbf{E}(\lambda + \delta)\Phi) - (\Phi, \mathbf{E}(\lambda - \delta)\Phi) \rightarrow 0$$

is uniform in $\lambda \in [0, 2\pi]$ by the uniform continuity of $\lambda \rightarrow (\Phi, \mathbf{E}(\lambda)\Phi)$. Therefore the first term goes to zero with $\delta \rightarrow 0$. The second term for every $\delta > 0$ can be made arbitrary small if n is large. This proves Lemma 2.4.

2.5. COROLLARY. — Let C be compact and $\Psi \in \mathcal{H}^{\text{cont}}(U(T, 0))$, then

$$\frac{1}{\tau} \int_0^\tau dt \|CU(t, 0)\Psi\| \leq f(\tau) \|\Psi\|$$

and

$$f(\tau) \rightarrow 0 \text{ for } |\tau| \rightarrow \infty.$$

Proof of Corollary 2.5. — By the Schwarz inequality

$$\begin{aligned} \left\{ \frac{1}{\tau} \int_0^\tau \|CU(t, 0)\Psi\| dt \right\}^2 &\leq \\ &\leq \frac{1}{\tau} \int_0^\tau \|CU(t, 0)\Psi\|^2 dt = \left(\Psi, \frac{1}{\tau} \int_0^\tau dt U^*(t, 0)C^*CU(t, 0)P^{\text{cont}}\Psi \right) \leq \\ &\leq \left\| \frac{1}{\tau} \int_0^\tau U^*(t, 0)C^*CU(t, 0)P^{\text{cont}} dt \right\| \|\Psi\|^2 = f^2(\tau) \|\Psi\|^2. \end{aligned}$$

By Lemma 2.4 $f(\tau) \rightarrow 0, |\tau| \rightarrow \infty$ which proves the corollary.

Proof of Theorem 2.3. — If Ψ is an eigenvector of $U(T, 0)$, i. e., if

$$U(T, 0)\Psi = e^{-i\alpha}\Psi,$$

then the trajectory $\{U(t, 0)\Psi; t \in \mathbb{R}\}$ is compact. Since $\mathcal{H}^{pp}(U(T, 0))$ is a closed subspace spanned by the eigenvectors; vectors, we obtain $\mathcal{H}^{pp}(U(T, 0)) \subseteq \mathcal{H}_\pm^p$. Let us now prove the opposite inclusion. For any $\Phi \in \mathcal{H}_+^p$ and any $\varepsilon > 0$ there exists a finite dimensional projection C such that

$$\sup_{t > 0} \|CU(t, 0)\Phi - U(t, 0)\Phi\| < \varepsilon.$$

By Corollary 2.5 there is a $\tau > 0$ such that for any normalized $\Psi \in \mathcal{H}^{\text{cont}}(U(T, 0))$

$$\frac{1}{\tau} \int_0^\tau \|CU(t, 0)\Psi\| dt < \varepsilon$$

holds. In particular there is a time $t_\varepsilon \leq \tau$ such that

$$\|CU(t_\varepsilon, 0)\Psi\| < \varepsilon.$$

Thus,

$$|(\Phi, \Psi)| = |(U(t_\varepsilon, 0)\Phi, U(t_\varepsilon, 0)\Psi)| \leq \|(1 - C)U(t_\varepsilon, 0)\Phi\| + \|CU(t_\varepsilon, 0)\Psi\| < 2\varepsilon.$$

Since ε is arbitrary \mathcal{H}_+^p (and similarly \mathcal{H}_-^p) is orthogonal to $\mathcal{H}^{\text{cont}}(U(T, 0))$. By $\mathcal{H}^{pp}(U(T, 0)) \subseteq \mathcal{H}_\pm^p$ it follows that

$$\mathcal{H}_+^p = \mathcal{H}_-^p = \mathcal{H}^{pp}(U(T, 0)). \quad \text{Q. E. D.}$$

The preceding theorem shows that our intrinsic definition of bound states reproduces the point spectral subspaces.

Note that the time independent case, in which

$$U(T, 0) = e^{-iT\mathbf{H}}$$

and

$$\mathcal{H}^{pp\text{cont}}(\mathbf{U}(T, 0)) = \mathcal{H}^{pp\text{cont}}(\mathbf{H}),$$

is also covered by Theorem 2.3.

A more general situation arises when a time evolution is asymptotically well approximated by a periodic one. In applications this happens typically if exterior fields are turned on or off.

Let us assume that for a propagator $\mathbf{U}(t, s)$ there exists a T_+ -periodic propagator $\mathbf{U}_+(t, s)$ and a T_- -periodic propagator $\mathbf{U}_-(t, s)$ such that the following isometric wave operators exist on \mathcal{H}

$$(2.9) \quad \begin{aligned} \Omega_+ &= s\text{-}\lim_{t \rightarrow +\infty} \mathbf{U}_+^*(t, 0)\mathbf{U}(t, 0), \\ \Omega_- &= s\text{-}\lim_{t \rightarrow -\infty} \mathbf{U}_-^*(t, 0)\mathbf{U}(t, 0). \end{aligned}$$

Then the range projection of Ω_{\pm} commutes with $\mathbf{U}_{\pm}(T_{\pm}, 0)$.

2.6. THEOREM. — Let the propagator $\mathbf{U}(t, s)$ be asymptotically periodic i. e. that the wave operators Ω_{\pm} (2.9) exist. Then

$$\begin{aligned} \mathcal{H}_+^p &= \Omega_+^* \mathcal{H}^{pp}(\mathbf{U}_+(T_+, 0)) = \mathcal{H}^{pp}(\Omega_+^* \mathbf{U}_+(T_+, 0)\Omega_+), \\ \mathcal{H}_-^p &= \Omega_-^* \mathcal{H}^{pp}(\mathbf{U}_-(T_-, 0)) = \mathcal{H}^{pp}(\Omega_-^* \mathbf{U}_-(T_-, 0)\Omega_-). \end{aligned}$$

Moreover for any $\Psi \in \mathcal{H}_+^{p\perp} = \Omega_+^* \mathcal{H}^{\text{cont}}(\mathbf{U}_+(T_+, 0))$ and any compact operator \mathbf{C}

$$(2.10) \quad \lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \int_0^\tau dt \|\mathbf{C}\mathbf{U}(t, 0)\Psi\| = 0$$

and similarly for negative times.

Proof. — By definition of Ω_+

$$\lim_{t \rightarrow \infty} \|\mathbf{U}(t, 0)\Psi - \mathbf{U}_+(t, 0)\Omega_+\Psi\| = 0.$$

Therefore (2.10) is implied by Corollary 2.5 and $\Psi \in \mathcal{H}_+^p$ if and only if $\Omega_+\Psi$ has a precompact trajectory w.r.t. \mathbf{U}_+ . By Theorem 2.3 the latter is equivalent to $\Omega_+\Psi \in \mathcal{H}^{pp}(\mathbf{U}_+(T_+, 0))$. Thus the remaining statements of the theorem follow. Q. E. D.

We have shown in this section that our Definition 2.1 of states with precompact trajectories is a proper generalization of the states in the closed span of the eigenvectors of a time independent Hamiltonian \mathbf{H} . For periodic or asymptotically periodic time evolutions these states can be equivalently characterized by point-spectral properties of suitably chosen operators. Moreover in this case the states in the orthogonal complement leave any compact subset of the Hilbert space in the time average.

III. GEOMETRICAL DEFINITION OF BOUND STATES AND PROPAGATING STATES

According to (1.10) for a general unitary propagator in $L^2(\mathbb{R}^v, d^v x)$ we define the set \mathcal{M}_{\pm}^{bd} of the « geometrically bound states » as the set of all Ψ for which

$$(3.1) \quad \limsup_{R \rightarrow \infty} \sup_{t \geq 0} \| F(|x| > R)U(t, 0)\Psi \| = 0.$$

Here (and in what follows) $F(\cdot)$ denotes the spectral projection belonging to the indicated operator and the indicated region. In this definition, too, only the behavior for large $|t|$ matters. It is easily verified that \mathcal{M}_{\pm}^{bd} are closed linear subspaces. The inclusion

$$(3.2) \quad \mathcal{H}_+^p \subset \mathcal{M}_+^{bd}, \quad \mathcal{H}_-^b \subset \mathcal{M}_-^{bd}$$

are simple consequences of the fact that $s\text{-}\lim_{R \rightarrow \infty} F(|x| > R) = 0$ and that for any precompact set $\mathcal{C} \subset \mathcal{H}$ we have

$$\limsup_{R \rightarrow \infty} \sup_{\Psi \in \mathcal{C}} \| F(|x| > R)\Psi \| = 0.$$

It is convenient to generalize the above definition to a family

$$(3.3) \quad \{ P_r \}_{r \geq 0}$$

of bounded operators (the Hilbert space \mathcal{H} may be arbitrary) having the property

$$(3.4) \quad \| P_r \| \leq 1, \quad s\text{-}\lim_{r \rightarrow \infty} P_r = \mathbb{1}.$$

Then, analogously, the spaces $\mathcal{M}_{\pm}^{bd}(P)$ defined by

$$(3.5) \quad \limsup_{r \rightarrow \infty} \sup_{t \geq 0} \| (1 - P_r)U(t, 0)\Psi \| = 0$$

are closed and satisfy

$$(3.6) \quad \mathcal{H}_+^p \subset \mathcal{M}_+^{bd}(P), \quad \mathcal{H}_-^p \subset \mathcal{M}_-^{bd}(P).$$

Of course, the main problem is whether in (3.2) or (3.6) the identities hold. A simple abstract condition guaranteeing this is a relative compactness property, generalizing the local compactness property (1.11). We say that the family (3.3) is *relatively compact with respect to U at $\pm \infty$* , if the set

$$\{ P_r U(t, 0)\Psi, t \geq 0 \}$$

is precompact in \mathcal{H} for any r and any $\Psi \in \mathcal{H}$. Obviously, it is enough to check the precompactness for a total set of Ψ 's.

It is immediately seen that in the time-independent case

$$U(t, 0) = \exp(-itH)$$

the \pm U-compactness is implied by the compactness of

$$P_r(i + H)^{-1}$$

for every $r > 0$. This covers $F(|x| < R)$ for all H satisfying the local compactness condition (1.11).

3.1. THEOREM. — Let U be a given propagator in \mathcal{H} and let a family P_r (3.3), (3.4) be U-compact at $\pm \infty$. Then

$$(3.7) \quad \mathcal{M}_{\pm}^{bd}(\mathbf{P}) = \mathcal{H}_{\pm}^p.$$

Proof. — Let $\Psi \in \mathcal{M}_{\pm}^{bd}(\mathbf{P})$; then

$$\{U(t, 0)\Psi, t \geq 0\} \subset \{P_r U(t, 0)\Psi, t \geq 0\} + \{(1 - P_r)U(t, 0)\Psi, t \geq 0\}.$$

Since $\Psi \in \mathcal{M}_{\pm}^{bd}$ the second summand on the right hand side is arbitrarily small if r is large enough, whereas the first summand is always precompact. Thus, the left hand side is precompact, too. Thus,

$$\mathcal{M}_{\pm}^{bd}(\mathbf{P}) \subset \mathcal{H}_{\pm}^p$$

which together with (3.6) gives (3.7). Q. E. D.

For the « propagating states » we have the analogous definitions of states which leave certain sets in the time average

$$(3.8) \quad \mathcal{M}_{\pm}^f(\mathbf{P}) = \left\{ \Psi, \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|P_r U(t, 0)\Psi\| dt = 0 \text{ for all } r \right\}.$$

We set

$$\mathcal{M}_{\pm}^f(\mathbf{P}) = \mathcal{M}_{\pm}^f$$

for the special case $\mathcal{H} = L^2(\mathbb{R}^v, d^v x)$, $P_r = F(|x| < r)$. These are the states which leave any bounded region in configuration space. Here again one proves easily that $\mathcal{M}_{\pm}^f(\mathbf{P})$ are closed linear subspaces and also that $\mathcal{M}_{\pm}^f(\mathbf{P})$ is orthogonal to $\mathcal{M}_{\pm}^{bd}(\mathbf{P})$ (cf. [1]).

3.2. THEOREM (An abstract RAGE-Theorem). — Let U be a unitary propagator in a Hilbert space \mathcal{H} which is T-periodic as in (1.18). Let a family P_r (3.3), (3.4), be U-compact at $+\infty$ and $-\infty$.

Then

$$(3.9) \quad \mathcal{M}_{+}^f(\mathbf{P}) = \mathcal{M}_{-}^f(\mathbf{P}) = \mathcal{H}^{\text{cont}}(U(T, 0)).$$

In particular

$$\begin{aligned} \mathcal{M}_{+}^f(\mathbf{P}) \oplus \mathcal{M}_{+}^{bd} &= \mathcal{H}, \\ \mathcal{M}_{-}^f(\mathbf{P}) \oplus \mathcal{M}_{-}^{bd} &= \mathcal{H}. \end{aligned}$$

Proof. — Let $\Psi \in \mathcal{H}^{\text{cont}}(U(T, 0))$. The set

$$\{P_r U(t, 0)\Psi; t \geq 0\}$$

is precompact and therefore for any $\varepsilon > 0$ there exists a finite dimensional orthogonal projection Q_ε such that

$$\sup_{t \geq 0} \|(1 - Q_\varepsilon)P_r U(t, 0)\Psi\| < \varepsilon/2.$$

Then

$$\frac{1}{\tau} \int_0^\tau \|P_r U(t, 0)\Psi\| \leq \frac{1}{\tau} \int_0^\tau \|Q_\varepsilon P_r U(t, 0)\Psi\| dt + \varepsilon/2.$$

Using Corollary 2.5 (with $C = Q_\varepsilon P_r$) the first summand can be made $< \varepsilon/2$ for τ large enough. This proves that $\Psi \in \mathcal{M}_\pm^f(P)$ and therefore also $\mathcal{H}^{\text{cont}}(U(T, 0)) \subset \mathcal{M}_\pm^f(P)$. Now (3.9) follows from Theorem 3.1 and the orthogonality of $\mathcal{M}_\pm^{bd}(P)$ and $\mathcal{M}_\pm^f(P)$. Q. E. D.

The condition of U-compactness at $+\infty$, as natural as it may appear, has proved, to be very difficult to check in concrete cases. As we already said, for time independent Hamiltonians ⁽¹⁾ this condition is implied by the local compactness condition (1.11) which is very mild and is fulfilled for all quantum mechanical potentials of interest.

Pearson [9] constructed a counterexample in which a complicated singularity of the potential at the origin produces a continuum state which is asymptotically free in the past but a part of it is trapped at the origin in the future and the RAGE-separation is violated. The trapping at the origin occurs at the cost of an infinite growth of the kinetic energy with time.

In the time-dependent case there is no automatic energy conservation and the energy can grow indefinitely even with well behaved potentials, as for instance the resonantly perturbed harmonic oscillator in §4 below. Such resonance phenomena, however, seem to be rather exceptions than a rule. This induces us to consider propagators with bounded energy.

We say that a propagator U has a *time-bounded energy* H_1 at $\pm\infty$, if there is a total set of Ψ 's for which

$$(3.10) \quad \limsup_{\lambda \rightarrow \infty} \sup_{t \geq 0} \|F(|H_1| > \lambda)U(t, 0)\Psi\| = 0$$

where H_1 is some self-adjoint operator satisfying the local compactness criterion (1.11). A typical energy operator will be the kinetic energy H_0 from (1.8).

3.3. LEMMA. — U has a time-bounded energy H_1 if and only if

$$(3.11) \quad M = \sup_{t \geq 0} \|f(H_1)U(t, 0)\Psi\| < \infty$$

for a total set of Ψ 's, where f is a real, nonnegative function, possibly depending on Ψ , such that

$$f(\lambda) \rightarrow \infty, \quad \text{for } \lambda \rightarrow \infty.$$

⁽¹⁾ And similarly for Hamiltonians which are asymptotically constant for $|t|$ large enough.

Proof. — If (3.11) holds then

$$M \geq \sup_{t \geq 0} \|f(H_1)F(|H_1| > \lambda)U(t, 0)\Psi\| \geq f(\lambda) \sup_{t \geq 0} \|F(|H_1| > \lambda)U(t, 0)\Psi\|.$$

Thus, (3.10) follows. Conversely, if (3.10) holds we can pick a sequence λ_n such that

$$\sup_{t \geq 0} \|F(|H_1| > \lambda_n)U(t, 0)\Psi\| \leq 2^{-n-1} \|\Psi\|.$$

Now define $f(\lambda) = n$ for $\lambda_{n-1} \leq \lambda \leq \lambda_n$. Then, using the Cauchy inequality, we obtain

$$\begin{aligned} \sup_{t \geq 0} \|f(H_1)U(t, 0)\Psi\|^2 &= \sup_{t \geq 0} \sum_{n=1}^{\infty} n^2 \|F(\lambda_{n-1} < |H_1| < \lambda_n)U(t, 0)\Psi\|^2 \\ &\leq \sum_n n^2 2^{-2n} < \infty \end{aligned}$$

so that (3.11) follows. Q. E. D.

3.4. COROLLARY. — Let a propagator U in $L^2(\mathbb{R}^v, d^v x)$ have time-bounded energy H_1 , such that H_1 satisfies the local compactness condition (1.11). Then the family $\{F(|x| < R)\}$ is U -compact at $\pm \infty$.

Proof. — Local compactness implies compactness of

$$C = F(|x| < R)(i + f(H_1))^{-1}.$$

For the total set of Ψ 's from Lemma 3.3

$$F(|x| < R)U(t, s)\Psi = C(i + f(H_1))U(t, 0)\Psi$$

where the compact operator C is applied to a family of vectors, bounded in t . Q. E. D.

Although we believe that a large class of time dependent Hamiltonians (1.8) will have bounded kinetic energy, it seems not easy to prove or disprove this fact. This is an open problem having its own interest.

The situation changes if instead of $\{F(|x| < R)\}$ another choice for P_r is made. This, of course, changes the underlying geometry. Taking

$$P_r = F(|x| < r)F(|p| < r),$$

which means a phase-space localization, we see that all P_r are already compact and therefore U -compact for *any* propagator U . Thus, for the family (3.12) Theorems 3.1 and 3.2 are trivially true in $\mathcal{H} = L^2(\mathbb{R}^v, d^v x)$ without any further restrictions on U !

Let us return to the T -periodic case (1.18) and use a Floquet form (1.20)

$$(3.12) \quad U(t, 0) = P(t) \exp(-iGt).$$

Obviously for any such G

$$\mathcal{H}^{pp}(U(T, 0)) = \mathcal{H}^{pp}(G).$$

If G can be chosen such that local compactness holds

$$(3.13) \quad F(|x| < R)(G + i)^{-1} \quad \text{is compact for all } R,$$

then one has some results about local decay.

3.5. PROPOSITION. — Let a T -periodic time evolution satisfy (3.12), (3.13), then

$$(3.14) \quad \mathcal{M}_{\pm}^{bd} = \mathcal{H}_{\pm}^p = \mathcal{H}^{pp}(G)$$

and for any $\Psi \in \mathcal{H}^{cont}(G)$ and any $R < \infty, n \in \mathbb{Z}$,

$$(3.15) \quad \lim_{|n| \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \| F(|x| < R)U(kT, 0)\Psi \| = 0.$$

Remark. — In applications to scattering theory usually only the following particular consequence of the RAGE-theorem has been used:

For $\Psi \in \mathcal{H}^{cont}(G)$ there exist sequences $t_n \rightarrow \pm \infty$ such that for any R

$$(3.16) \quad \| F(|x| < R)U(t_n, 0)\Psi \| \rightarrow 0.$$

Obviously this follows also from (3.15).

Proof. — As above (3.15) implies that $\mathcal{H}^{cont}(G)$ is orthogonal to \mathcal{M}_{\pm}^{bd} which shows (3.14). It is sufficient to consider a dense set of Ψ 's with

$$(3.17) \quad F(|G| < E)\Psi = \Psi$$

for some E . In the proof of Lemma 2.4 we have shown that

$$(3.18) \quad \lim_{|n| \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} U(kT, 0)^* F(|G| < E) F(|x| < R) F(|G| < E) \right. \\ \left. \times U(kT, 0) P^{cont}(G) \right\| = 0$$

using compactness (3.13). This implies (3.15) as in the proof of Corollary 2.5. Q. E. D.

If one wants to treat the continuous time average here as well one needs the stronger assumption:

The family:

$$(3.19) \quad F(|x| < R)P(t)(G + i)^{-1}$$

is a norm continuous compact operator-valued function of $t \in [0, T]$.

It is easy to give a norm continuous $P(t)$ such that (3.13) holds, but (3.19) and (3.21) are violated. Whether this can happen for time evolutions generated by reasonable Schrödinger operators is an interesting open problem.

It may be easy to check (3.19) in concrete situations as our example in Section VII shows.

3.6. PROPOSITION — Let a T-periodic time evolution satisfy (3.12) and (3.19), then the family $\{F(|x| < R) \mid R \in \mathbb{R}\}$ is U-compact at $\pm \infty$.

$$(3.20) \quad \mathcal{M}_{\pm}^{bd} = \mathcal{H}^{pp}(G),$$

and for any $\Psi \in \mathcal{H}^{cont}(G)$, $R < \infty$

$$(3.21) \quad \lim_{|T| \rightarrow \infty} \frac{1}{T} \int_0^T dt \|F(|x| < R)U(t, 0)\Psi\| = 0.$$

Proof. — We use the dense set of vectors (3.17). By (3.19) there is a finite dimensional projection Q for any E and R such that for $t \in \mathbb{R}$

$$\|(\mathbb{1} - Q)F(|x| < R)P(t)F(|G| < E)\| < \varepsilon.$$

Then

$$\begin{aligned} \|(\mathbb{1} - Q)F(|x| < R)U(t, 0)\Psi\| \\ = \|(\mathbb{1} - Q)F(|x| < R)P(t)F(|G| < E)e^{-iGt}\Psi\| < \varepsilon. \end{aligned}$$

This shows U-compactness at $\pm \infty$. The remaining statements follow from Theorems 3.1 and 3.2. Q. E. D.

It is obvious how these results generalize to the case of asymptotically periodic time evolutions as discussed at the end of Section II. We need not state them explicitly.

IV. HARMONIC OSCILLATOR. I. AN EXACTLY SOLVABLE CASE

We consider now the one dimensional harmonic oscillator of mass $m=1$ in $L^2(\mathbb{R}, dx)$ with

$$V(x, t) = \frac{\omega^2 x^2}{2} + f(t)x,$$

where the function $f(\cdot)$ is supposed to be continuous and bounded on \mathbb{R} . The propagator can be computed explicitly and reads

$$(4.1) \quad U(t, s) = e^{-i\varphi_1(t,s)x} e^{i\varphi_2(t,s)p/\omega} e^{-iH_{\omega}(t-s) - i\psi(t,s)}$$

where $p = -id/dx$ and

$$(4.2) \quad \varphi_1(t, s) = \int_s^t f(\tau) \cos \omega(\tau - t) d\tau,$$

$$(4.3) \quad \varphi_2(t, s) = - \int_s^t f(\tau) \sin \omega(\tau - t) d\tau,$$

$$(4.4) \quad \psi(t, s) = -\frac{1}{2} \int_s^t (\varphi_1(\tau, s)^2 - \varphi_2(\tau, s)^2) d\tau,$$

$$(4.5) \quad H_\omega = \frac{1}{2} p^2 + \frac{\omega^2 x^2}{2}.$$

In order to see that (4.1) holds it is sufficient to apply the right hand side of (4.1) to a Ψ from the Schwartz space \mathcal{S} (note that each factor on the right hand side of (4.1) leaves \mathcal{S} invariant) and then insert it into the Schrödinger equation which turns out to be identically satisfied.

Note that the functions $\varphi_{1,2}$ appear in the solution of the classical harmonic oscillator equation

$$(4.6) \quad \ddot{x} + \omega^2 x = -f(t), \quad x(0) = x, \quad \dot{x}(0) = \frac{p}{\omega}$$

which in this case exactly reproduces the operator solution of the Heisenberg operator equations. Thus, we have

$$(4.7) \quad x(t) = U(0, t)xU(t, 0) = x \cos \omega t + \frac{p}{\omega} \sin \omega t - \frac{1}{\omega} \varphi_2(t, 0),$$

$$(4.8) \quad p(t) = U(0, t)pU(t, 0) = -\omega x \sin \omega t + p \cos \omega t + \varphi_1(t, 0).$$

For $\Psi \in \mathcal{S}$ we have

$$(4.9) \quad \| p^n U(t, 0) \Psi \|^2 = (\Psi, p(t)^{2n} \Psi),$$

$$(4.10) \quad \| x^n U(t, 0) \Psi \|^2 = (\Psi, x(t)^{2n} \Psi).$$

By (4.7) and (4.8) we see that the boundedness of (4.9) and (4.10) in t is equivalent to the boundedness of φ_1 and φ_2 , respectively. On the other hand, it is easy to see that the boundedness of φ_1 in t is equivalent to the boundedness of φ_2 . Thus,

4.1. PROPOSITION. — The boundedness of any of $\varphi_1(t, 0)$, $\varphi_2(t, 0)$ at $\pm \infty$ is equivalent to the precompactness of every trajectory $\{ U(t, 0) \Psi \}$ at $\pm \infty$.

We continue our study with the more restricted class of periodic f . As a typical example we take the so called « AC Stark effect ».

$$(4.11) \quad f(t) = \sin \omega_0 t.$$

In this case we have

4.2. THEOREM. — For $f(t) = \sin \omega_0 t$ the following alternative holds

I. $\omega \neq \omega_0$. The monodromy operator $U\left(\frac{2\pi}{\omega}, 0\right)$ has a pure point spectrum and every trajectory is precompact at $+\infty$ and $-\infty$.

II. $\omega = \omega_0$. The monodromy operator has a purely absolutely continuous spectrum. The characterization (3.21) holds for every $\Psi \in L^2(\mathbb{R}, dx)$ but the time mean cannot be dropped.

Proof. — We have

$$(4.12) \quad \varphi_1(t, 0) = \begin{cases} \frac{\omega_0}{\omega_0^2 - \omega^2} (\cos \omega t - \cos \omega_0 t), & \omega \neq \omega_0 \\ \frac{t \sin \omega t}{2}, & \omega = \omega_0 \end{cases}$$

and

$$(4.13) \quad \varphi_2(t, 0) = \begin{cases} \frac{1}{\omega_0^2 - \omega^2} (\omega_0 \sin \omega t - \omega \sin \omega_0 t), & \omega \neq \omega_0 \\ -\frac{t \cos \omega t}{\omega} + \frac{\sin \omega t}{2\omega}, & \omega = \omega_0. \end{cases}$$

This settles the possibility I. For $\omega = \omega_0$ we have from (4.1) with ψ as in (4.4)

$$U\left(\frac{5\pi}{2\omega}, \frac{\pi}{2\omega}\right) = e^{-i\psi_0} e^{-i\frac{\pi}{\omega}x}$$

where $\psi_0 = \psi\left(\frac{5\pi}{2\omega}, \frac{\pi}{2\omega}\right)$. This is nothing else but the monodromy operator taken at $s = \pi/2\omega$ and it is purely absolutely continuous.

We see at once that

$$\|F(|x| < R)U(t, s)\Psi\| \rightarrow 0$$

can hold for no Ψ and no $R > 0$ since

$$U\left(\frac{(4n+1)\pi}{2\omega}, \frac{\pi}{2\omega}\right) = \left[U\left(\frac{5\pi}{2\omega}, \frac{\pi}{2\omega}\right)\right]^n$$

commutes with $F(|x| < R)$! We have, however,

$$(4.14) \quad \mathcal{M}_+^f = \mathcal{M}_-^f = L^2(\mathbb{R}, dx)$$

where the formula (3.21) has to be proved directly.

To see that we note first that

$$| [U(t, 0)\Psi](x) | = \left| \Psi_0\left(x + \frac{\varphi_2(t, 0)}{\omega}, t\right) \right|$$

where

$$\Psi_0(x, t) = (e^{-iH_0 t}\Psi)(x)$$

is the motion of the stationary harmonic oscillator. We take as a total set

$$\Psi = \Psi_a = \exp\left(-\frac{\omega(x-a)^2}{2}\right), \quad a \in \mathbb{R}.$$

Then, as it is well known,

$$|\Psi_0(x, t)|^2 = \exp(-\omega(x - a \cos \omega t)^2).$$

By (4.13) we have

$$\begin{aligned} h(t) &= \|F(|x| < R)U(t, 0)\Psi_a\|^2 = \\ &= \int_{|x| < R} \exp\left[-\omega\left(x - \frac{t}{2\omega^2} \cos \omega t + \frac{1}{2\omega^2} \sin \omega t - a \cos \omega t\right)^2\right] dx. \end{aligned}$$

Now it can be seen easily that

$$\frac{1}{s} \int_0^s h(t) dt \rightarrow 0, \quad s \rightarrow \pm \infty$$

is valid for any a . Since the set of all Ψ_a is total and \mathcal{M}_{\pm}^f is a closed subspace (4.14) holds. Q. E. D.

4.3. REMARK. — The case $\omega = \omega_0$ in the above theorem is an example in which the family $F(|x| < R)$ is not U-compact, and the kinetic energy is therefore not bounded. In fact, the set $\left\{ U\left(\frac{(4n+1)\pi}{2\omega}, 0\right)\Psi, n \in \mathbb{N} \right\}$ is not precompact for any Ψ , and Theorem 3.2 does not apply with the sequence $F(|x| < R)$. The fact that nevertheless all states leave any bounded region of configuration space in the time average depends on a particular physical mechanism which is discussed in Section VI.

V. HARMONIC OSCILLATOR. II. LOCAL PERTURBATIONS

We consider here the existence and stability of the bound states for time-periodic Hamiltonians. We pose the following

5.1. PROBLEM. — Let a T-periodic propagator U in a Hilbert space be given which has only bound trajectories. Does it preserve this property under a « reasonable » class of perturbations? How large is the class of U's having this stability property?

In the time-independent case there is one class of Hamiltonians whose behaviour is satisfactory — as long as the perturbation is also time-independent. These are the Hamiltonians with compact resolvent. However, as soon as the perturbation becomes time-dependent, the situation may change. Note that even in the case $\omega = \omega_0$ in Theorem 4.2 the perturbation $f(t)x$ was H_ω -compact for any t ! In the time-dependent case it is not the difference of resolvents but rather the difference of the propagators whose compactness matters.

We consider the Hamiltonian

$$(5.1) \quad \mathbf{H}(t) = \mathbf{H} + \mathbf{V}(t)$$

where $\mathbf{V}(t)$ is uniformly bounded, continuous in t and T -periodic, and we assume that a strongly differentiable propagator exists.

5.2. THEOREM. — Let $\mathbf{H}(t)$ be as above and let

$$(5.2) \quad \mathbf{W}(t, s) = \int_s^t e^{i\mathbf{H}\sigma} \mathbf{V}(\sigma) e^{-i\mathbf{H}\sigma} d\sigma$$

be compact for all s, t . Then

$$e^{-i\mathbf{H}t} - \mathbf{U}(t, 0)$$

is compact, where $\mathbf{U}(t, 0)$ is the propagator generated by $\mathbf{H}(t)$.

Proof. — We use the Duhamel formula for $\mathbf{U}(t, 0)$:

$$(5.3) \quad \mathbf{U}(t, s) - e^{-i\mathbf{H}(t-s)} = \frac{1}{i} e^{-i\mathbf{H}t} \int_s^t e^{i\mathbf{H}\sigma} \mathbf{V}(\sigma) \mathbf{U}(\sigma, s) d\sigma.$$

From this it follows uniformly in t

$$(5.4) \quad \|e^{-i\mathbf{H}s} - \mathbf{U}(t + s, t)\| \leq |s| \sup_{\sigma} \|\mathbf{V}(\sigma)\|.$$

For $h = t/n$ we have

$$\begin{aligned} ie^{i\mathbf{H}t}(\mathbf{U}(t, 0) - e^{-i\mathbf{H}t}) &= \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} e^{i\mathbf{H}\sigma} \mathbf{V}(\sigma) \mathbf{U}(\sigma, 0) d\sigma = \\ &= \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} e^{i\mathbf{H}\sigma} \mathbf{V}(\sigma) \mathbf{U}(\sigma, kh) \mathbf{U}(kh, 0) d\sigma = \\ &= \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} e^{i\mathbf{H}\sigma} \mathbf{V}(\sigma) e^{-i\mathbf{H}\sigma} d\sigma e^{i\mathbf{H}kh} \mathbf{U}(kh, 0) + \\ &+ \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} e^{i\mathbf{H}\sigma} \mathbf{V}(\sigma) [\mathbf{U}(\sigma, kh) - e^{-i\mathbf{H}(\sigma-kh)}] d\sigma \mathbf{U}(kh, 0). \end{aligned}$$

The first summand in the last expression is compact for any $h > 0$. The norm of the second summand is estimated by

$$nh \sup_{\sigma} \|\mathbf{V}(\sigma)\| \max_k \|\mathbf{U}(\sigma - kh + kh, kh) - e^{-i\mathbf{H}(\sigma-kh)}\| \leq th \sup_{\sigma} \|\mathbf{V}(\sigma)\|^2,$$

where we have used (5.4) and the equality $t = nh$. Thus, the left hand side of (5.3) is a sum of a compact term and an arbitrarily small term and must be compact. Q. E. D.

5.3. LEMMA. — If $V(\sigma) = f(\sigma)V$ with a scalar, piecewise continuous function f , then the compactness of

$$(5.5) \quad W_1(t, s) = \int_s^t e^{+iH\sigma} V e^{-iH\sigma} d\sigma$$

implies that of $W(t, s)$ in (5.2).

Proof. — The proof follows from the fact that under our conditions on f the integral (5.2) can be norm approximated by integrals in which f is approximated by step functions. Q. E. D.

5.4. LEMMA. — Let the operator H have a purely discrete spectrum

$$\lambda_1 < \lambda_2 < \dots$$

with the corresponding spectral projections

$$P_1, P_2, \dots, \quad \dim P_k < \infty.$$

Then the compactness of the matrix operator

$$(5.6) \quad \begin{cases} \frac{P_k V P_j}{\lambda_k - \lambda_j}, & k \neq j \\ P_k V P_k, & k = j \end{cases}$$

implies the compactness of $W_1(t, s)$ in (5.5).

Proof. — The assertion follows from

$$(5.7) \quad \begin{aligned} P_k W_1(t, s) P_j &= \left\{ \begin{array}{l} (-i) \frac{P_k V P_j}{\lambda_k - \lambda_j} (e^{-i(\lambda_j - \lambda_k)t} - e^{-i(\lambda_j - \lambda_k)s}) \\ (t - s) P_k V P_k \end{array} \right\} \\ &= (-i) [e^{+iHt} V_1 e^{-iHt} - e^{+iHs} V_1 e^{-iHs}] + (t - s) V_2, \end{aligned}$$

where V_2 and V_1 are the diagonal and the off-diagonal part of the matrix (5.6), respectively. Q. E. D.

5.5. COROLLARY. — If under the conditions of Theorem 5.2 the spectrum of $\exp(-iHT)$ happens to have at most a finite number of accumulation points, then the same is valid for $U(T, 0)$ as well and therefore $\mathcal{H}_+^p = \mathcal{H}_-^p = \mathcal{H}$.

Although the corollary above yields some means to prove the existence of bound states its power is rather limited since in most cases the spectrum of $\exp(-iHT)$, i. e. the set

$$\{ e^{-i\lambda_k T} \}_{k=1}^{\infty}$$

will lie densely on the unit circle. And, as shows the well-known counter-

example of von Neumann, compact perturbations of such operators need not preserve the completeness of the eigenvectors.

Consider again a concrete example in which $H = H_\omega$, where H_ω is the Hamiltonian of the stationary harmonic oscillator from (4.5). Then

$$\lambda_k = \omega \left(k - \frac{1}{2} \right), \quad k = 1, 2, \dots$$

and the spectrum of $\exp(-iH_\omega T)$ is dense on the unit circle, if $\omega T/2\pi$ is irrational. If, however, $\omega T/2\pi$ is rational then the spectrum of $\exp(-iH_\omega T)$ is a finite set of equidistant points. The applicability of Corollary 5.5 depends in this case on the compactness of the scalar matrices

$$(5.8) \quad \left\{ \frac{(\Psi_k, V\Psi_j)}{k-j} \right\}_{k \neq j}, \quad \{(\Psi_k, V\Psi_k \delta_{jk})\}$$

where Ψ_k are normalized eigenfunctions of the harmonic oscillator. Taking a bounded $V = V(x)$ with compact support we have

$$|(\Psi_k, V\Psi_j)| \leq \text{const} \left[\int_a^b |\Psi_k(x)|^2 dx \int_a^b |\Psi_j(x)|^2 dx \right]^{1/2}.$$

From (see [13], III/2, § 162)

$$\int_a^b |\Psi_k(x)|^2 dx \leq \frac{\text{const}}{\sqrt{k}}$$

the compactness of the matrices in (5.8) is easily seen. Thus, the system consisting of a harmonic oscillator H , perturbed by $f(t)V$, where f is periodic with a period commensurable with that of H , has only bound states provided that V is bounded and has compact support (or sufficiently rapid decay).

Comparing this result with that of § 4 looks almost paradoxical: There the only case where the bound states disappeared was that with $\omega T = 2\pi$. Here just in this case we can prove the existence of bound states! Of course, in Sec. III the potential V is much more singular. In any case new more powerful techniques seem to be needed in order to settle this problem satisfactorily.

There is one more example which can be treated by Corollary 5.5 namely the one where in (5.1) H is given by $-d^2/dx^2$ on a finite interval $[0, l]$ with, say, Dirichlet boundary conditions (infinite potential barrier). The normalized eigenfunctions are

$$\varphi_k = \sqrt{\frac{2}{l}} \sin \frac{k\pi x}{l}, \quad k = 1, 2, \dots,$$

and the eigenvalues are

$$\lambda_k = \frac{k^2 \pi^2}{l^2}, \quad k = 1, 2, \dots$$

so that $\exp(-iHT)$ will have a finite essential spectrum, if $T\pi/l^2$ is rational. However the diagonal part of the matrix (5.6) reads

$$\frac{2}{l} \int_0^l V(x) \sin^2 \frac{k\pi x}{l} dx = \frac{1}{l} \int_0^l V(x) dx - \frac{1}{l} \int_0^l V(x) \cos \frac{2k\pi x}{l} dx.$$

Since the second term above goes to zero the diagonal part of (5.6) will generally not represent a compact operator. In this case, however, one can lump a part of the diagonal term in the perturbation to the unperturbed operator (with which it commutes)! Then, as it is easily seen, the new « unperturbed » monodromy operator

$$e^{-iHT} e^{-i \int_0^T f(\tau) d\tau V_3}, \quad V_3 = \frac{1}{l} \int_0^l V(x) dx \delta_{kj}$$

has again a finite essential spectrum i. e. its spectrum has finitely many accumulation points. The remaining parts of (5.6) being obviously compact we again conclude that in this system all states are bound.

VI. LOCAL DECAY OF CONTINUUM STATES

In Section III we have given a geometric definition of the set \mathcal{M}_{\pm}^{bd} of bound states and the set \mathcal{M}_{\pm}^f of propagating states which move far out in the time mean. The abstract RAGE-theorem (Th. 3.2) connects these two geometric notions under a natural, but yet not easily verifiable condition of the U-compactness of the configuration space projections $F(|x| < R)$.

In this section we prove the RAGE-theorem for the class of potentials

$$H(t) = H_0 + V(t)$$

under the condition of the uniform boundedness

$$(6.1) \quad \sup_t \|V(t)\| < \infty$$

and without using the U-compactness of $F(|x| < R)$. The class of potentials (6.1) is not very large and probably (6.1) implies the mentioned U-compactness. It is hoped, however, that a similar technique could handle potentials V with some local singularities and not too rapid increase towards infinity, e. g.

$$(6.2) \quad |\nabla V(x, t)| \leq a + b|x|$$

where a, b are constant in space and time. This condition includes the resonant case in Section IV in which $F(|x| < R)$ are not U-compact but the RAGE-Theorem still holds.

Our proof relies on the following geometric consideration. A classical particle with high kinetic energy $mv^2/2$ will leave a region of diameter R in a time $2R/v$. Then the time average, over an interval of length T , of the probability to find the particle in this region will decrease as v^{-1} provided that T is small enough such that the particle cannot be deflected to return into the mentioned region. The condition (6.2) now guarantees that this return time is uniformly bounded from below. In the extreme case of a reflecting barrier the return time behaves as v^{-1} and it should be longer for any weaker potential. Therefore it should be possible with more technical effort to eliminate growth conditions at infinity completely. Note that the potentials here may be non-local, i. e. velocity-dependent.

6.1. THEOREM. — Let U be the propagator on $\mathcal{H} = L^2(\mathbb{R}^v, d^v x)$ generated by

$$H(t) = H_0 + V(t)$$

where H_0 is as in (1.8) and $V(t)$ is any continuous family of bounded selfadjoint operators with

$$V(t + T) = V(t) \quad \text{for all } t$$

and

$$\sup_t \|V(t)\| = M < \infty$$

Then

$$(6.3) \quad \begin{aligned} \mathcal{M}_+^{bd} &= \mathcal{M}_-^{bd} = \mathcal{H}^{pp}(U(T, 0)), \\ \mathcal{M}_+^f &= \mathcal{M}_-^f = \mathcal{H}^{\text{cont}}(U(T, 0)). \end{aligned}$$

Proof. — The family $H(t)$ satisfies the conditions of Theorem 1.1 and by the results of Sec. III it is sufficient to show that for a total set of $\Psi \in \mathcal{H}^{\text{cont}}(U(T, 0))$

$$(6.4) \quad \lim_{|T| \rightarrow \infty} \frac{1}{T} \int_0^T \|F(|x| < R)U(t, 0)\Psi\| dt = 0 \quad \text{for all } R.$$

We will treat the case $T \rightarrow \infty$. Pick a family of functions $g_v \in \mathcal{C}^\infty(\mathbb{R}_+)$, $v > 0$, such that

$$|g_v| \leq 1, \quad \text{supp } g_v \subset \{\lambda; \lambda > mv^2/2\}$$

and such that $1 - g_v$ has compact support. Then the states in the range of $g_v(H_0)$ have velocities $\geq v$. Writing

$$(6.5) \quad \begin{aligned} \|F(|x| < R)U(t, 0)\Psi\| &\leq \\ &\leq \|F(|x| < R)g_v(H_0)U(t, 0)\Psi\| + \|F(|x| < R)(1 - g_v(H_0))U(t, 0)\Psi\| \end{aligned}$$

and taking into account the compactness of $(F(|x| < R)(1 - g_v(H_0)))$ and Corollary 2.5 we conclude that the second summand goes to zero in the time mean.

For the treatment of the first term we use the boundedness of $V(t)$ to

estimate the real motion by the unperturbed one. We have as in (5.4) uniformly in t

$$\| e^{-iH_0s} - U(t + s, t) \| \leq |s| M.$$

Let now $\varepsilon > 0$. Then for $\sigma = \varepsilon/M$

$$(6.6) \quad \sup_{0 \leq s \leq \sigma} \| e^{-iH_0s} - U(t + s, t) \| \leq \varepsilon.$$

For integrals with positive integrands we apply the inequality

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \dots \leq \sup_{t \geq 0} \frac{1}{\sigma} \int_t^{t+\sigma} \dots$$

valid for any $\sigma > 0$ to the first summand in (6.5). For $0 \leq s \leq \sigma$ we obtain

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \| F(|x| < R) g_v(H_0) U(t, 0) \Psi \| dt &\leq \\ &\leq \sup_t \frac{1}{\sigma} \int_0^\sigma ds \| F(|x| < R) g_v(H_0) U(t + s, t) U(t, 0) \Psi \| \leq \\ &\leq \varepsilon + \frac{1}{\sigma} \int_0^\sigma \| F(|x| < R) g_v(H_0) e^{-iH_0s} U(t, s) \Psi \| ds. \end{aligned}$$

By the Schwarz inequality the convergence to zero as $v \rightarrow \infty$ of the second term above is implied by the zero-convergence of

$$(6.7) \quad \frac{1}{\sigma} \int_0^\sigma ds \| F(|x| < R) g_v(H_0) e^{-iH_0s} U(t, 0) \Psi \|^2 \leq \left\| \frac{1}{\sigma} \int_0^\sigma e^{iH_0s} g_v(H_0) F(|x| < R) g_v(H_0) e^{-iH_0s} ds \right\|.$$

To show this we estimate the square of (6.7) by

$$\left\| \frac{1}{\sigma^2} \int_0^\sigma ds \int_0^\sigma F(|x| < R) e^{-iH_0(s-s')} g_v^2(H_0) F(|x| < R) ds' \right\|.$$

Set $\delta = \varepsilon/2\sigma$. Then the integration over $|s - s'| < \delta$ gives a contribution bounded by ε . By the dominated convergence theorem it is sufficient to show that

$$(6.8) \quad \lim_{v \rightarrow \infty} \| F(|x| < R) e^{-iH_0t} g_v^2(H_0) F(|x| < R) \| = 0$$

for any $R < \infty, \quad 0 < \delta \leq |t| \leq \sigma.$

This follows from the well known estimate of the free Schrödinger wave propagation (see e. g. [3], Corollary 2.9). We have therefore proved that uniformly in T and Ψ for any R the time average of the first term in (6.5) goes to zero as $v \rightarrow \infty$. Q. E. D.

Remark. — A similar result has been obtained by Yajima and Kitada [16] with different methods. For potentials which decay towards infinity the local decay of high energy states has been studied recently by Kitada [17].

Notice that in the proof of the preceding theorem until the inequality (6.7) no use has been made of the special form of the operator H_0 . Substituting instead of H_0 the operator H_ω from (4.5) and using the notation (5.5)-(5.7) we have for (6.7)

$$\begin{aligned} g_v(H_\omega) \frac{1}{\sigma} \int_0^\sigma e^{iH_\omega s} F(|x| < R) e^{-iH_\omega s} ds g_v(H_\omega) &= g_v(H_\omega) \frac{1}{\sigma} \int_0^\sigma W(\sigma, 0) g_v(H_\omega) = \\ &= \frac{-i}{\sigma} (e^{iH_\omega \sigma} g_v(H_\omega) V_1 g_v(H_\omega) e^{-iH_\omega \sigma} - g_v(H_\omega) V_1 g_v(H_\omega)) + g_v(H_\omega) V_2 g_v(H_\omega), \end{aligned}$$

which converges to zero in norm as $v \rightarrow \infty$ because $g_v(H_\omega) \rightarrow 0$, $v \rightarrow \infty$ strongly and V_1 and V_2 are fixed compact operators as was shown in § 5 (see formula (5.8)). Thus, Theorem 6.1 holds, if instead of the free Hamiltonian H_0 the one-dimensional harmonic oscillator Hamiltonian H_ω is taken.

The result can be extended to the case of a v -dimensional harmonic oscillator

$$\begin{aligned} H(t) &= \sum_{k=1}^v H_{\omega,k} + V(t), \\ H_{\omega,k} &= \frac{p_k^2}{2} + \frac{\omega^2 x_k^2}{2}. \end{aligned}$$

Here starting from (6.7) the last line is bounded by

$$\varepsilon + \| g_v(H_{\omega,1}) \frac{1}{\sigma} \int_0^\sigma e^{iH_{\omega,1} s} F(|x_1| < R) e^{-iH_{\omega,1} s} ds g_v(H_{\omega,1}) \|$$

where the second summand goes to zero in norm as we showed before.

VII. PERTURBATION BY A CIRCULARLY POLARIZED ELECTROMAGNETIC WAVE

In this section we consider a model of a charged particle in a spherically symmetric potential (e. g. harmonic oscillator or hydrogen atom) exposed to an external radiation field. Owing to the high amplitude laser experiments this model has been given a considerable attention in the recent physical literature (cf. [4] [18] and the references cited there). We shall be interested in the existence of bound states, as well as in the RAGE characterization.

Let the plane monochromatic electromagnetic wave be given by the vector potential

$$(7.1) \quad A(x_3, t) = a(\cos(kx_3 - \omega t), \sin(kx_3 - \omega t), 0).$$

This is a circularly polarized wave with the x_3 -axis as direction of propagation. Note that

$$(7.2) \quad pA(x_3, t) = A(x_3, t) \cdot p$$

and that $A(x_3, t)$ commutes with the generator

$$L_3 = x_1 p_2 - x_2 p_1$$

of rotations around the x_3 -axis. This implies

$$(7.3) \quad \frac{1}{2m} (p - A(x_3, t))^2 = e^{i\omega L_3 t} \frac{1}{2m} (p - A(x_3, 0))^2 e^{-i\omega L_3 t}.$$

About the potential $V(x)$ we assume that it is spherically symmetric (in fact, only the x_3 -axial symmetry would do) and that it can be split into a sum

$$V(x) = V_1(x) + V_+(x),$$

where $V_1(x)$ is form-bounded relative to H_0 with bound smaller than one:

$$|(\Psi, V_1 \Psi)| \leq (\Psi, H_0 \Psi) + b \|\Psi\|^2,$$

for an $a < 1$ and all $\Psi \in Q(H_0)$, and $V_+(x)$ is a positiv operator with $Q(H_0) \cap Q(V_+)$ dense in \mathcal{H} . Then by the KLMN theorem

$$H = H_0 + V_1 + V_+$$

defines a closed symmetric quadratic form on $Q(H) = Q(H_0) \cap Q(V_+)$ which determines a self adjoint operator H with domain $\mathcal{D}(H)$. We have

$$(7.4) \quad \|(H_0 + i)^{1/2}(H + i)^{-1/2}\| < \infty,$$

which implies that $p_j, j = 1, 2, 3$ is H -bounded with the relative bound zero. Thus, $A(x_3, t)p$ and $A(x_3, t)^2$ are H -bounded perturbations with the relative bound zero for any field strength a . By the Kato-Rellich theorem

$$(7.5) \quad H(t) = \frac{1}{2m} (p - A(x_3, t))^2 + V$$

is selfadjoint and has a time-independent domain

$$\mathcal{D}(H(t)) = \mathcal{D}(H)$$

and Theorem 1.1 applies.

Let now $U(t, s)$ be the generated propagator. Observe that

$$(7.6) \quad e^{-i\omega L_3 s} U(s + \tau, s) e^{i\omega L_3 s} = U(\tau, 0)$$

for any s . Indeed, for $\tau = 0$ (7.6) is true and

$$(7.7) \quad i \frac{d}{d\tau} e^{-i\omega L_3 s} U(s + \tau, s) e^{i\omega L_3 s} = H(\tau) e^{-i\omega L_3 s} U(s + \tau, s) e^{i\omega L_3 s},$$

where we have used the identity

$$(7.8) \quad \mathbf{H}(t) = e^{i\omega L_3 t} \mathbf{H}(0) e^{-i\omega L_3 t}.$$

From (7.6) it follows that $e^{-i\omega L_3 t} \mathbf{U}(t, 0)$ is a strongly continuous one-parameter group, i. e.

$$(7.9) \quad e^{-i\omega L_3 t} \mathbf{U}(t, 0) = e^{-iGt}$$

where G is a uniquely determined self-adjoint operator. The operator G is a selfadjoint extension of

$$(7.10) \quad G_0 = \mathbf{H}(0) + \omega L_3$$

with the dense domain

$$(7.11) \quad \mathcal{D}(G_0) = \mathcal{D}(\mathbf{H}(0)) \cap \mathcal{D}(L_3) = \mathcal{D}(\mathbf{H}) \cap \mathcal{D}(L_3).$$

We have chosen to study circularly polarized waves because of the possibility to transform the system to a time-independent one. This is well known in the physical literature, see e. g. [12] and references therein.

The linearly polarized case, called « AC-Stark-effect » by Howland has been studied recently e. g. in [8] [15].

7.1. — LEMMA. — The operator

$$F(|x| < R)(G + i)^{-1}, \quad R > 0$$

is compact.

Proof. — For any $\Phi \in \mathcal{D}(L_3)$ we have

$$(7.12) \quad (G + i)^{-1} \Phi = (\mathbf{H} + i)^{-1} \Phi + (G + i)^{-1} (\omega L_3 - \mathcal{A})(\mathbf{H} + i)^{-1} \Phi.$$

where

$$\mathcal{A} = -2ap_1 \cos kx_3 - 2ap_2 \sin kx_3 + a^2 \quad (2).$$

To prove the compactness of $(G + i)^{-1} F(|x| < R)$ it is obviously enough to prove

$$(G + i)^{-1} f(x)$$

is compact for any spherically symmetric $f \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. By $f(x)L_3 = L_3 f(x)$ we have

$$(7.13) \quad (G + i)^{-1} f(x) = (\mathbf{H} + i)^{-1} f(x) + (G + i)^{-1} (\mathbf{H} + i)^{-1} \omega L_3 f(x) + (G + i)^{-1} \mathcal{A} (\mathbf{H} + i)^{-1/2} [(\mathbf{H} + i)^{-1/2} f(x)].$$

The third summand in (7.13) is compact by (7.4) and the compactness of $(\mathbf{H}_0 + i)^{-1/2} f(x)$. (Note that $\mathcal{A}(\mathbf{H} + i)^{-1/2}$ is bounded.) Thus it is the

(2) Note that L_3 and \mathbf{H} commute. The formula (7.12) can be taken as a definition of G as a kind of sum of $\mathbf{H}(0)$ and L_3 .

compactness of the second summand which has to be proved. We have

$$x_1 p_2 f(x)(H + i)^{-1} = p_2 (H + i)^{-1/2} \{ (H + i)^{-1/2} x_1 f(x) + (H + i)^{-1/2} [(H + i)x_1 f(x) - x_1 f(x)(H + i)](H + i)^{-1} \}.$$

The first summand in the curly bracket is compact. Observing that $x f(x)$ maps $Q(H_0) \cap Q(V_+)$ into itself the second summand is proportional to

$$(H + i)^{-1/2} [p^2, x_1 f(x)](H + i)^{-1} = (H + i)^{-1/2} [-\Delta(x_1 f(x)) - 2i\nabla(x_1 f(x)) \cdot p](H + i)^{-1}.$$

Here the compactness follows from the compact support of $f(\cdot)$. The way of treating $x_2 p_1$ is analogous. Thus,

$$L_3 f(x)(H + i)^{-1}$$

is compact. Taking adjoints and setting $i \rightarrow -i$ we obtain the compactness of

$$L_3 (H + i)^{-1} f(x)$$

and therefore of the second summand on the right hand side of (7.13). Here the commutativity of L_3 with H and f was used. Q. E. D.

In order to establish the RAGE decomposition in this situation note the particular Floquet form

$$(7.14) \quad U(t, 0) = P(t)e^{-iGt} = e^{-i\omega L_3 t} e^{-iGt}.$$

Now Lemma 7.1 implies norm continuity and compactness of

$$F(|x| < R)P(t)(G + i)^{-1} = P(t)F(|x| < R)(G + i)^{-1}.$$

By Proposition 3.6 we have

7.2. THEOREM. — Let for $U(t, s)$ the subspaces \mathcal{H}_\pm^p , \mathcal{M}_\pm^f and \mathcal{M}_\pm^{bd} be defined as in Sections II and III. Then

$$\begin{aligned} \mathcal{H}_\pm^p &= \mathcal{M}_\pm^{bd} = \mathcal{H}^{pp}(G), \\ \mathcal{M}_\pm^f &= \mathcal{H}^{cont}(G). \end{aligned}$$

In particular

$$\mathcal{M}_\pm^f \oplus \mathcal{M}_\pm^{bd} = L^2(\mathbb{R}^3, d^3x).$$

Note that the theorem above remains true for more general vector potentials $A(x, t)$, which satisfy (7.3) and map $\mathcal{D}(|p|)$ into itself.

A further problem is: does the system governed by $H(t)$ in (7.5) have any bound states? As we know this is equivalent to the problem of the existence of eigenvalues for the operator G .

It can be easily shown (see [4]) that G has a compact resolvent and is bounded from below if

$$(7.15) \quad V(x) \geq \frac{m\omega_1^2 x^2}{2} + \text{const}$$

and $\omega_1 > \omega$. For the potential

$$V(x) = \frac{m\omega^2 x^2}{2}$$

a resonant behaviour is expected. The special case $k = 0$ is exactly solvable and we have

7.3. THEOREM. — Let for $\omega_1 > 0$

$$(7.16) \quad G = \frac{1}{2m}(p - av)^2 + \omega L_3 + \frac{m\omega_1^2 x^2}{2}.$$

Then the following four cases are possible:

- i) $\omega_1 > \omega$; G is bounded from below and has a purely discrete spectrum. Therefore $\mathcal{H}_\pm^p = \mathcal{M}_\pm^{bd} = L^2(\mathbb{R}^3, d^3x)$.
- ii) $\omega_1 = \omega$; G has an absolutely continuous spectrum over $(-\infty, \infty)$.
- iii) $\omega_1 < \omega$, ω_1/ω rational; G has a pure point spectrum, its eigenvalues are equidistant and have infinite degeneracies.
- iv) $\omega_1 < \omega$, ω_1/ω irrational; G has a complete set of eigenvectors and the eigenvalues form a dense subset of $(-\infty, \infty)$.

Proof. — By the canonical transformation :

$$(7.17) \quad \begin{aligned} x_1 &= \frac{1}{2}(Q_+ + Q_-), & p_1 &= \frac{1}{2}(P_+ + P_-), \\ x_2 &= \frac{1}{2m\omega_1}(P_- - P_+), & p_2 &= \frac{m\omega_1}{2}(Q_+ - Q_-), \end{aligned}$$

we obtain

$$(7.18) \quad G = \frac{\omega_1 + \omega}{2\omega_1} \left(\frac{P_+^2}{2m} + \frac{m\omega_1^2}{2} Q_+^2 \right) + \frac{\omega_1 - \omega}{2\omega_1} \left(\frac{P_-^2}{2m} + \frac{m\omega_1^2}{2} Q_-^2 \right) - \frac{a}{2m}(P_+ + P_-) + \frac{p_3^2}{2m} + \frac{m\omega_1^2 x_3^2}{2}.$$

Now, if $\omega_1 = \omega$ then

$$(7.19) \quad e^{iQ-s} G e^{-iQ-s} = G + \frac{as}{2m}.$$

This means that G is absolutely continuous on \mathbb{R} [19].

Let now $\omega_1 \neq \omega$. Then

$$G = \frac{\omega_1 + \omega}{2\omega_1} \left[\frac{1}{2m} \left(P_+ - \frac{a\omega_1}{\omega + \omega_1} \right)^2 + \frac{m\omega_1^2}{2} Q_+^2 \right] + \frac{\omega_1 - \omega}{2\omega_1} \left[\frac{1}{2m} \left(P_- - \frac{a\omega_1}{\omega_1 - \omega} \right)^2 + \frac{m\omega_1^2}{2} Q_-^2 \right] + \frac{1}{2m} p_3^2 + \frac{m\omega_1^2}{2} x_3^2 + \text{const.}$$

Since Q_+ , Q_- , x_3 and $P_+ - a\omega_1/(\omega_1 + \omega)$, $P_- - a\omega_1/(\omega_1 - \omega)$, p_3 are

again canonical variables G has a complete set of eigenvectors. Their eigenvalues are given by

$$(7.20) \quad \frac{\omega_1 + \omega}{2} n_+ + \frac{\omega_1 - \omega}{2} n_- + \omega_1 n_3 + \text{const}$$

where n_+, n_-, n_3 are arbitrary positive integers. Now all the assertions of the theorem follow from (7.20) and a corresponding number-theoretic result (see [6], Th. 185). Q. E. D.

7.4. REMARKS. — *i*) The preceding result is similar to that of a perturbed one dimensional oscillator in Sec. IV (Note that in both models the Heisenberg equations are explicitly solvable). Moreover, the present model distinguishes between $\omega_1 < \omega$ and $\omega_1 > \omega$. In the first case the point spectrum is known to be stable under a large class of perturbations (bounded, etc.). In the second case, we have again to do with operators having dense point spectra whose stability is questionable.

ii) In the resonant case $\omega_1 = \omega$ in the formula (3.21) which is valid for any initial data Ψ the time mean can be dropped (in contrast to the one-dimensional oscillator).

Consider finally the case of a potential V which is spherically symmetric and vanishes at infinity (e. g. Coulomb potential), such that the unperturbed operator (7.5) with $a = 0$ has some bound states. Here in general no bound states are expected no matter how deep V and how small a is. An argument which indicates such a conjecture is as follows. Take $k=0$ and a spherically symmetric and bounded V with compact support and arbitrary depth.

Then

$$G = e^{\frac{ip_2a}{m\omega}} G e^{-\frac{ip_2a}{m\omega}} = \frac{p^2}{2m} + \omega L_3 + V_1, \quad V_1(x) = V\left(x + \frac{a}{m\omega} j\right).$$

Since V_1 has again a compact support we have

$$\| V_1 e^{-it\left(\frac{p^2}{2m} + \omega L_3\right)} \Psi \| \leq \| V_2 e^{-it\frac{p^2}{2m}} \Psi \|$$

where V_2 has again a compact support. By Cook's Lemma ([10], Theorem XI.24) the wave operators

$$\Omega_{\pm} \left(\frac{p^2}{2m} + \omega L_3 + V_1, \quad \frac{p^2}{2m} + \omega L_3 \right)$$

exist. Since $\frac{p^2}{2m} + \omega L_3$ is absolutely continuous over $(-\infty, \infty)$ the existence

of Ω_{\pm} implies that G possesses an invariant subspace in which it is absolutely continuous over $(-\infty, \infty)$. If G had eigenvalues then they would be imbedded into the continuum and are very likely not to exist at all. Note

that there is no symmetry which could preserve the eigenvalues. Exactly the same argument works for short range forces and with the standard modifications for Coulomb interactions.

Physically such a model describes a system like a hydrogen atom exposed to an electromagnetic field of frequency ω and amplitude proportional to a . If ω is larger than the distance of the bound state energy from the continuum (the binding energy) then second order perturbation theory ([10], Section XII.6 and Notes) shows that all bound states turn into resonances and the spectrum of G is purely continuous. For special systems one can calculate a few more orders corresponding to the absorption of several photons, but for very small ω this method becomes practically inapplicable. Experimentally photo-dissociation of atoms is well known for high enough frequency. For low frequencies it has been observed that large numbers of photons from high intensity laser beams have been absorbed simultaneously to ionize atoms (references in [18]). If the frequency and/or amplitude are too low this effect is too rare to be observable. The question is whether there are positive threshold values such that for sufficiently low frequency and weak fields the atom can follow the exterior field along a periodic trajectory (bound state). Physical intuition for this and several equivalent models supports the following.

CONJECTURE. — Let the periodic time evolution be generated by $H(t)$ (7.5) with a spherically symmetric potential V tending to zero as $|x| \rightarrow \infty$ (e. g. Coulomb). Then for any $a \neq 0$, $\omega \neq 0$ the system has no bound states.

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