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Soliton Equations and Hyperbolic Maps

by

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ABSTRACT. — A solution of the AKNS scattering equation [*6*] associated to a non-linear evolution equation determines an isometry from $(\mathbb{R}^2, g)$ to the hyperbolic plane H, where g is the metric of curvature -1 defined by the scattering equation. This correspondence is (locally) 2-1 from solutions to isometries. For the modified K*d*V and sin-Gordon equations, the scattering equation can be seen as a flow on the space of constant-speed curves in H, with a simply-described curvature function. A geometrical interpretation of the Bäcklund transformation is given, together with a « soliton » example.

RÉSUMÉ. — Une solution de l'équation de diffusion AKNS [*6*] associée à une équation d'évolution non-linéaire donne une isométrie de $(\mathbb{R}^2, g)$ dans le plan hyperbolique H, g étant la métrique de courbure -1 que définit l'équation de diffusion; cette correspondance des solutions aux isométries est (localement) 2-1. Pour les équations K*d*V modifiée et sinus-Gordon, l'équation de diffusion sera alors un flot sur l'espace des courbes à vitesse constante dans H, et la formule pour la courbure est simple. On donne une interprétation géométrique de la transformation de Bäcklund, ainsi qu'un exemple de type « soliton ».

1. INTRODUCTION

It has been recognized for some time that the non-linear partial differential equations which admit « soliton » type solutions are closely related to the group SL(2, $\mathbb{R}$) and its geometry (see in particular [*1*]-[*4*]); going somewhat further, Sasaki and Bullough [5] described an explicit relation between the AKNS scattering scheme and metrics of constant curvature -1 on $\mathbb{R}^2$. It turns out that an even simpler way of looking at the theory arises when we bring in the standard space of curvature -1, i. e., the upper half plane $H = SL(2, \mathbb{R})/SO(2)$ with the hyperbolic metric. We find

(1) that solutions of the scattering equations correspond almost exactly to isometries from $\mathbb{R}^2$ (with the metric of [5]) to H;

(2) that other features such as Bäcklund transformations have geometrical descriptions in terms of such isometries;

(3) that for the sin-Gordon and modified K*d*V equations the basic functions are natural geometrical ones.

The idea (which I shall only use as a suggestion here) is the following. A scattering scheme defines an SL(2, $\mathbb{R}$) connection on $\mathbb{R}^2$ which is *integrable* iff the associated non-linear equation is satisfied [4]. Its integrability means that $\mathbb{R}^2$ can be isometrically « developed » on H, and such a development is the isometry we are looking for.

This note is concerned only with the general theory and not with the (multi) soliton solutions; but they also seem likely to correspond to objects with a geometrical meaning.

2. THE CORRESPONDENCE

We begin with the scattering equations themselves, in a formalism which is a mixture of [4] and [5] adapted for the present purposes. Let σ^1, σ^2, ω be 1-forms on $\mathbb{R}^2$, defining a metric of constant curvature -1,

$$g = (\sigma^1)^2 + (\sigma^2)^2 \tag{1}$$

via the structure equations

$$d\sigma^1 = \omega \wedge \sigma^2, \qquad d\sigma^2 = -\,\omega \wedge \sigma^1, \qquad d\omega = \sigma^1 \wedge \sigma^2 \tag{2}$$

Then if Ω is the SL(2, $\mathbb{R}$)-valued 1-form on $\mathbb{R}^2$

$$\Omega = \begin{pmatrix} \frac{1}{2}\sigma^2 & \frac{1}{2}(-\,\omega + \sigma^1) \\ \frac{1}{2}(\omega + \sigma^1) & -\frac{1}{2}\sigma^2 \end{pmatrix} \tag{3}$$

a *solution* of the scattering equation is a map $G: \mathbb{R}^2 \to SL(2, \mathbb{R})$ such that

$$G^{-1}dG = \Omega \tag{4}$$

or $\qquad G^*(\omega^1) = \sigma^1, \qquad G^*(\omega^2) = \sigma^2, \qquad G^*(\omega^3) = \omega$

where ω^i $(i = 1, 2, 3)$ are Maurer-Cartan forms corresponding to the basis of the Lie algebra defined by (3).

Locally such solutions exist provided that Ω satisfies the integrability condition

$$d\Omega = \Omega \wedge \Omega \tag{5}$$

which in particular cases defines the non-linear equation in question. And if G, G′ are two solutions defined on the same (connected) subset of $\mathbb{R}^2$, they are related by $G'(x,t) = Q \cdot G(x,t)$, where $Q \in SL(2,\mathbb{R})$ is constant.

Note that our way of writing the scattering equation (4) is that of [*4*], although the basis of forms is different. The forms σ^1, σ^2, ω are essentially those of [*5*], given that the equation $dv = \Omega v$ has been replaced by its adjoint $dv^t = v^t\Omega^t$; our Ω is therefore Ω^t in the more usual formalism.

Let $\pi: SL(2,\mathbb{R}) \to SL(2,\mathbb{R})/SO(2) = H$ be the canonical projection:

$$\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{ai + b}{ci + d}. \tag{6}$$

Write ξ, η for the coordinates on H. We choose for H the metric defined by the 1-forms $\sigma_H^1 = \frac{1}{\eta} d\xi$, $\sigma_H^2 = \frac{1}{\eta} d\eta$, $\omega_H = \frac{1}{\eta} d\xi$;

$$g_H = (\sigma_H^1)^2 + (\sigma_H^2)^2 = \frac{1}{\eta^2}(d\xi^2 + d\eta^2). \tag{7}$$

Our first observation is that *if* G *is a solution of* (4), *then* $f = \pi \circ G$ *is an isometry from* $(\mathbb{R}^2, g)$ *to* (H, g_H).

In fact, because $SL(2,\mathbb{R})$ acts by isometries on H, $\pi^*(g_H)$ is a left invariant symmetric 2-form on $SL(2,\mathbb{R})$; by looking at the derivative of π at the identity this can be identified with $(\omega^1)^2 + (\omega^2)^2$. Hence

$$f^*(g_H) = G^*((\omega^1)^2 + (\omega^2)^2) = g.$$

Similarly, f^* takes the standard volume form $\sigma_H^1 \wedge \sigma_H^2 = \frac{1}{\eta^2} d\xi \wedge d\eta$ on H to $\sigma^1 \wedge \sigma^2$ on $\mathbb{R}^2$. Hence f is orientation preserving from the orientation of $\mathbb{R}^2$ defined by $\sigma^1 \wedge \sigma^2$ (which may or may not be the standard one) to H.

In (one version of) the explicit AKNS scattering scheme we have [*6*]

$$\Omega = \begin{pmatrix} \lambda & r \\ q & -\lambda \end{pmatrix} dx + \begin{pmatrix} A & C \\ B & -A \end{pmatrix} dt. \tag{8}$$

(Recall that our Ω corresponds to the usual Ω^t.) To keep everything in $SL(2,\mathbb{R})$, we specify that $\lambda (= -i\zeta)$ is a real constant, q, r are real-valued functions of x, t, and A, B, C are expressions involving λ and q, r and their derivatives, also real. (There is a corresponding theory for *complex* Ω and maps into $SL(2,\mathbb{C})$, which is certainly important — e. g., when λ is complex — but which we shall not deal with here.)

From (3) and (8) we have

(9) $\sigma^1 = (q + r)dx + (\mathrm{B} + \mathrm{C})dt, \qquad \sigma^2 = 2(\lambda dx + \mathrm{A}dt),$

$$\omega = (q - r)dx + (\mathrm{B} - \mathrm{C})dt.$$

The « volume » form $\sigma^1 \wedge \sigma^2$ is $2(\mathrm{A}(q + r) - \lambda(\mathrm{B} + \mathrm{C}))dxdt$. Where it vanishes — in general a 1-dimensional subset of $\mathbb{R}^2$ — the map f is singular. For example, in the sin-Gordon case [6], we can take $q = -r = -\frac{1}{2}u_x$, $\mathrm{B} = \mathrm{C} = \frac{1}{4\lambda}\sin u$; so $\sigma^1 \wedge \sigma^2 = -\sin u dxdt$. The orientation is determined by the sign of $\sin u$, while f is singular on the subset $\sin u = 0$.

3. THE INVERSE CORRESPONDENCE : LIFTING ISOMETRIES

We have seen that a solution G of (4) determines an isometry

$$f = \pi \circ \mathrm{G} : \mathbb{R}^2 \to \mathrm{H}.$$

(The « isometry » ceases to be a genuine isometry precisely when g ceases to be a proper metric on $\mathbb{R}^2$, i. e., becomes indefinite.) Suppose now that we are given a map $f: \mathbb{R}^2 \to \mathrm{H}$ satisfying

(10) $$f^*(g_{\mathrm{H}}) = g, \qquad f^*(\sigma^1_{\mathrm{H}} \wedge \sigma^2_{\mathrm{H}}) = \sigma^1 \wedge \sigma^2$$

— that is, an oriented isometry in the general sense. By topological considerations, f has a number of lifts to maps $\mathrm{G}: \mathbb{R}^2 \to \mathrm{SL}(2, \mathbb{R})$ such that $\pi \circ \mathrm{G} = f$. It is a remarkable fact that we can specify geometrically those lifts which are solutions of the scattering problem — and that they are all but unique.

We can do this by looking at the tangents to x-parameter curves in H. From the formula

$$d\mathrm{G}(\partial_x(x, t)) = \mathrm{G}(x, t) \cdot \Omega(x, t)(\partial_x(x, t))$$

we find that if $f = \pi \circ \mathrm{G}$ *and* G *satisfies* (4),

(11) $$df(\partial_x(x, t)) = \mathrm{G}(x, t) \cdot (i, \sigma^1(\partial_x) + i\sigma^2(\partial_x))$$

Here $(i, \sigma^1(\partial_x) + i\sigma^2(\partial_x)) \in \mathrm{T}_i(\mathrm{H}) = \{ i \} \times \mathbb{C}$, and $\mathrm{G}(x, t)$ acts as an isometry on H and so also on its tangent vectors. The effect of isometries on tangent vectors at $i \in \mathrm{H}$ is not complicated: we find that if

$$\mathrm{G}(x, t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(12) $$df(\partial_x(x, t)) = \left(\frac{ai + b}{ci + d}, \frac{\sigma^1(\partial_x) + i\sigma^2(\partial_x)}{(ci + d)^2} \right).$$

Hence except in the « special » singular case where both σ^1 and σ^2 vanish on $\partial_x(x, t)$, we can find both $\dfrac{ai+b}{ci+d}$ and $\dfrac{1}{(ci+d)^2}$ from f by (12). (This case is explicitly excluded for sin-Gordon, where $\lambda \neq 0$, but could give trouble elsewhere.)

By a simple calculation, these two complex numbers determine

$$G(x, t) \in SL(2, \mathbb{R})$$

up to a factor ± 1 — which is the most we could hope for, given that -1 acts trivially on H. Now if f is *any* isometry satisfying (10), define $G: \mathbb{R}^2 \to SL(2, \mathbb{R})$ by (12) (we also, of course require G to be continuous). Then G is unique up to ± 1; we call the two maps the *canonical lifts* of f with respect to Ω. The essential fact is that *the canonical lifts of an isometry are solutions of the scattering equation* (4). To see this we first check from (12) that when G is a canonical lift, $G^*(\omega^i)(\partial_x(x, t)) = \sigma^i(\partial_x(x, t))$ for $i = 1, 2$; and then use the fact that

$$G^*((\omega^1)^2 + (\omega^2)^2) = (\sigma^1)^2 + (\sigma^2)^2, \qquad G^*(\omega^1 \wedge \omega^2) = \sigma^1 \wedge \sigma^2$$

(since f is an isometry and G is a lift of f) to show that $G^*(\omega^i)$ and σ^i also agree on ∂_t for $i = 1, 2$. Now $G^*(\omega^3) = \omega$ follows from (2) and the Maurer-Cartan equations.

Schematically therefore we have a 2-1 correspondence

$$\begin{pmatrix}\text{solutions G of the}\\ \text{scattering equation}\end{pmatrix} \underset{\text{canonical lift}}{\overset{\text{compose with } \pi}{\rightleftarrows}} \begin{pmatrix}\text{isometries}\\ f: (\mathbb{R}^2, g) \to (H, g_H)\end{pmatrix}$$

Note 1. — If G were taken as mapping into the group of isometries of H, the projective group $PL(2, \mathbb{R}) = SL(2, \mathbb{R})/(\pm 1)$, we'd have a $1-1$ correspondence; but it would be no easier to write down, so it seems best to stay in $SL(2, \mathbb{R})$.

Note 2. — We can in fact define a canonical lift except where σ^1, σ^2 are identically zero. For if they are zero on ∂_x but not on ∂_t we can replace the procedure above by one involving the t-curves; the same argument works.

4. SPEED AND CURVATURE

We now specialize to the case where Ω is defined by (8) and $q+r=0$; this will work for the sin-Gordon equation

$$u_{xt} = \sin u \left(q = -\frac{1}{2} u_x \right) \tag{13}$$

and for the modified KdV equation in the form

(14) $$q_t + 6q^2 q_x + q_{xxx} = 0\,.$$

(See [6]). Then $\sigma^1(\partial_x) = 0$ and $\sigma^2(\partial_x) = 2\lambda$. So *the x-parameter curves in* $(\mathbb{R}^2, g)$ *have constant speed* $|2\lambda|$ — and hence so also do their images under f, the x-parameter curves in H. The scattering equation can therefore be regarded as a flow on the space of curves of speed $|2\lambda|$ in H.

Next, we have a very simple description of the canonical lift, from (12). In fact $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ takes the standard tangent vector $(i, i) \in \mathrm{T}_i(\mathrm{H})$ to $\left(\dfrac{ai + b}{ci + d}, \dfrac{i}{(ci + d)^2}\right)$. Hence if $f : \mathbb{R}^2 \to \mathrm{H}$ is an isometry, the canonical lift $\mathrm{G}(x, t)$ is the unique (up to ± 1) isometry of H which takes (i, i) to $\dfrac{1}{2\lambda} df(\partial_x(x, t)) = \left(f(x, t), \dfrac{1}{2\lambda} f_x(x, t)\right) \in \mathrm{T}_{f(x,t)}(\mathrm{H})$. Note that this definition works precisely when $\lambda \neq 0$, which corresponds to the non-singular case.

Geometrically, $\mathrm{G}(x, t)$ *maps the standard unit tangent vector* (i, i) *to the unit vector along the x-curve in* H *at* $f(x, t)$ *pointing forwards* (*backwards*) *if* λ *is positive* (*negative*).

The function q in its turn is described in terms of curvature. To see this, consider the standard basis vector fields $\underline{e}_1$, $\underline{e}_2$ on $\mathbb{R}^2$ corresponding to the forms σ^1, σ^2 [5],

(15) $$\underline{e}_1 = -\frac{\mathrm{A}}{\lambda(\mathrm{B} + \mathrm{C})}\partial_x + \frac{1}{(\mathrm{B} + \mathrm{C})}\partial_t; \qquad \underline{e}_2 = \frac{1}{2\lambda}\partial_x\,.$$

From $\omega = 2q dx + (\mathrm{B} - \mathrm{C})dt$ we deduce

(16) $$\nabla_{\partial_x}(\underline{e}_2) = -2q\underline{e}_1\,;$$

In other words, $2q$ *is the covariant « rate of change of angle » along an x-parameter curve.* To find the geodesic curvature κ_g of the curve we compute $\nabla_{\underline{e}_2}\underline{e}_2$(for $\lambda > 0$) or $\nabla_{(-\underline{e}_2)}(-\underline{e}_2)$ (for $\lambda < 0$), and find $-\dfrac{q}{\lambda}\underline{e}_1$ in each case. Since $(\underline{e}_1, \underline{e}_2)$ *are positively oriented this gives in general* $\kappa_g = q/_{|\lambda|}$.

Again, since f is an isometry, the same is true for the x-parameter curves in H.

To make clear what is meant by describing $2q$ as the covariant rate of change of angle, suppose q derived from a potential function u by the formula

(17) $$q = -\frac{1}{2}u_x.$$

(This is standard for the sin-Gordon equation, of course.) Define the vector field $\underline{v}$ by

(18) $$\underline{v} = \underline{e}_1 \sin u + \underline{e}_2 \cos u\,.$$

A simple calculation then shows that $\underline{v}$ is *parallel* along the x-parameter curves; while u is the *clockwise* angle of rotation from $\underline{e}_2$ to $\underline{v}$. Hence the *anticlockwise* angle from $\underline{v}$ to ∂_x is $-u$ (for $\lambda > 0$) and $\pi - u$ (for $\lambda < 0$); its rate of change is $-u_x = 2q$.

5. THE SIN-GORDON EQUATION

Here the situation is particularly simple — corresponding to the classical geometrical problem which the equation describes [*7*]. The equation is given by (13), and we have [6] [*8*].

$$B = C = \frac{1}{4\lambda}\sin u, \qquad A = \frac{1}{4\lambda}\cos u \tag{19}$$

whence using (15), (18),

$$\partial_t = \frac{1}{2\lambda}(\underline{e}_1 \sin u + \underline{e}_2 \cos u) = \frac{1}{2\lambda}\underline{v}. \tag{20}$$

It follows that the t-curves have constant speed $\frac{1}{|2\lambda|}$ and that ∂_t is parallel along the x-curves; u is the clockwise angle of rotation from ∂_x to ∂_t whatever the sign of λ. We can state:

A solution of the scattering problem for sin-Gordon with given function $u(x, t)$ and parameter λ is (the canonical lift of) a map $f: \mathbb{R}^2 \to H$ such that in H

i) the x-curves have constant speed $|2\lambda|$ and the t-curves have constant speed $1/_{|2\lambda|}$.

ii) the clockwise angle from $f_x(x, t)$ to $f_t(x, t)$ is $u(x, t)$.

6. BACKLUND TRANSFORMATIONS

Crampin in [*4*] gives a nice geometric description of a BT which corresponds to the « usual » one for particular choices of gauge. We shall investigate this only in the sin-Gordon case; unfortunately here as he points out his Ω differs from that of AKNS (and so from ours) by a gauge transformation. But this in itself deserves attention.

Let $P(x, t)$ be the matrix $\begin{pmatrix} \cos u/4 & -\sin u/4 \\ \sin u/4 & \cos u/4 \end{pmatrix} \in SL(2, \mathbb{R})$. Then $P(x, t)$ leaves $i \in H$ fixed and induces a rotation through $-u/2$ on $T_i(H)$. Hence if G is the *canonical* lift of f, $GP: \mathbb{R}^2 \to SL(2, \mathbb{R})$ where

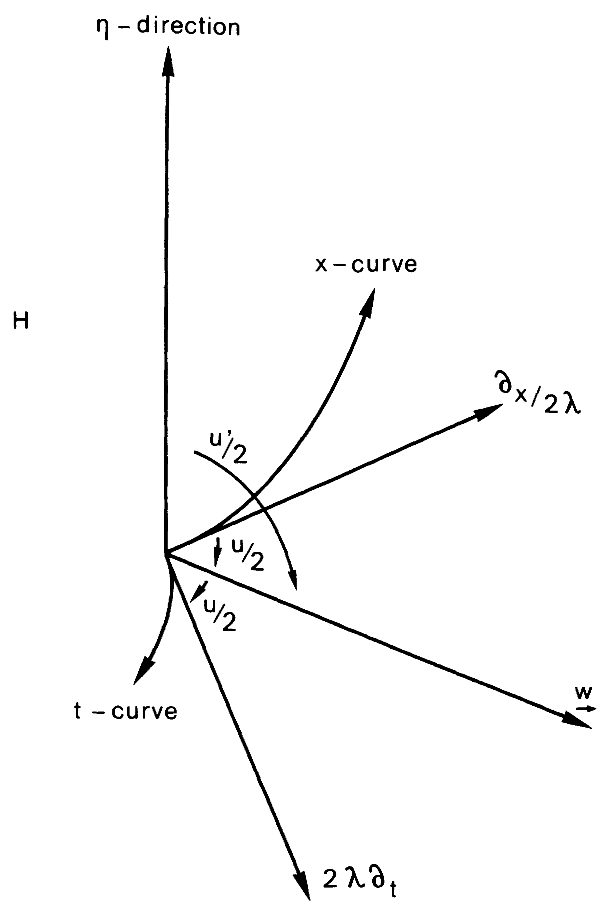
$$GP(x, t) = G(x, t) \cdot P(x, t) \tag{21}$$

is another lift of f related by a gauge transformation; and

$$(GP)^{-1}d(GP) = P^{-1}dP + P^{-1}\Omega P \tag{22}$$

Comparing with Crampin's formula we see that his G is our GP, and his form Θ is given by (22).

The geometric meaning of this is as follows. G maps (i, i) to $\frac{1}{2\lambda} df(\partial_x(x, t))$; P rotates through $-u(x, t)/2$. Since the angle from $\frac{1}{2\lambda} df(\partial_x)$ to $2\lambda df(\partial_t)$ is $-u$, GP maps (i, i) *to the unit bisector of the angle between the two*. And it is this lift that gives rise to the form Θ of [*4*] for scattering in the sin-Gordon equation.



The relations between u, u' etc. for $\lambda > 0$.

Write GP = G′; the method of [*4*] is to write

$$G' = TR^{-1} \tag{23}$$

where T is upper triangular and R is rotation. Then if R is rotation through $u'/2$, u' is a BT of u, and the equations can be derived in their standard form.

(Note that there is an error in the matrix representation of R in [*4*], which should, like P, contain *quarter* angles to give the BT as we shall see.)

Now T (a dilation + translation) does not change angles in the tangent space. So if $\underline{w}(x, t)$ is the unit bisector of the angle between $\frac{1}{2\lambda} df(\partial_x)$ and $2\lambda df(\partial_t)$, $u'/2$ is just *the clockwise rotation from the vertical* (the direction of (i, i)) *to* $\underline{w}(x, t)$. The diagram will perhaps make this relation clearer, as well as the geometrical nature of the angle u'.

Now we derive the formula for the BT in essentially the same way as [*4*] (not surprisingly). We have

$$G = G'P^{-1} = TR^{-1}P^{-1}: \tag{24}$$

where

$$R'^{-1} = R^{-1}P^{-1} = \begin{pmatrix} \cos\frac{u'-u}{4} & -\sin\frac{u'-u}{4} \\ \sin\frac{u'-u}{4} & \cos\frac{u'-u}{4} \end{pmatrix}$$

and we require that

$$R'^{-1}dR' + R'^{-1}\Omega R' \tag{25}$$

should be upper triangular. The lower left corner of (25) is

$$\frac{du - du'}{4} + \frac{1}{2}\omega + \frac{1}{2}\left(\sigma^1 \cos\frac{u'-u}{2} + \sigma^2 \sin\frac{u'-u}{2}\right) \tag{26}$$

giving, when the values of ω, σ^1, σ^2 are substituted in,

$$\frac{u'_x + u_x}{2} = 2\lambda \sin\frac{u'-u}{2}$$

$$\frac{u'_t - u_t}{2} = \frac{1}{2\lambda}\sin\frac{u'+u}{2}$$

which is a standard form of the BT.

Conversely, let u' be a function satisfying (27). Then if R′ is defined by (25), it is easy to see that GR′ = ST, where T is upper triangular and S is *constant*. Hence, u' is a function which has the above geometric description for the solution $S^{-1}G$ of the scattering equation. So all Bäcklund transforms of u can be obtained geometrically; and the group of isometries of H acts on them (in a rather complicated way).

To end with a very simple example, set

$$f(x, t) = x - t + i \cosh (x + t) . \tag{28}$$

It is easy to check that the x and t curves in H described by (28) have speed 1, so can be related to a sin-Gordon scattering problem with $\lambda = \frac{1}{2}$. To find $u(x, t)$, we have

$$f_x(x, t) = 1 + i \sinh (x + t), \quad f_t(x, t) = -1 + i \sinh (x + t) . \tag{29}$$

So the clockwise angle u is given by

$$e^{iu(x,t)} = \frac{1 + i \sinh (x + t)}{-1 + i \sinh (x + t)} = (-\tanh (x + t) + i \operatorname{sech} (x + t))^2 . \tag{30}$$

From this we can deduce that u is a simple soliton, $u(x, t) = 4 \tan^{-1} e^{x+t}$. The singular locus is $\sin u = 0$, which is simply $x + t = 0$ if we take $0 < u < 2\pi$.

It is immediate from (29) that f_x and f_t are symmetrical with respect to the imaginary axis in H. Hence the corresponding BT u', using the geometrical definition, is trivial: $u'/2$ is $(2n + 1)\pi$ and

$$u' = (4n + 2)\pi . \tag{31}$$

However, non-trivial BT's can be obtained by applying an isometry to (28) and evaluating the corresponding angle u'.

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