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Section A :
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Euclidean φ_3^4 theory in an electromagnetic potential

by

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ABSTRACT. — We study the n-point functions of a φ_3^4 theory coupled minimally to an external electromagnetic potential with the help of a phase-space-cell expansions and a cluster expansion. The standard methods of φ_3^4 are shown to apply for a class of electromagnetic potentials by use of Kato's inequality and a new inequality of a similar type for the gradient of the covariance.

RÉSUMÉ. — On étudie les fonctions à n points d'une théorie couplée de façon minimale à un potentiel électromagnétique extérieur au moyen d'un développement en cellules dans l'espace de phase et d'un développement en clusters. On montre que les méthodes habituelles pour φ_3^4 s'appliquent à une certaine classe de potentiels électromagnétiques en utilisant l'inégalité de Kato et une nouvelle inégalité similaire pour le gradient de la covariance.

I. INTRODUCTION

Motivated by the increasing interested in gauge field theories, we study a Euclidean φ_3^4 theory in an external electromagnetic potential and consider this work to be a preparation of the corresponding gauge invariant

(*) Adress since 1-10-1980.

field theory, the « abelian Higgs model » in three dimensions, following the arguments of Schrader [11].

The theory has the formal Lagrangian density

$$\mathscr{L} = \varphi^*(x)(-\Delta_A + m_0^2)\varphi(x) + \lambda : (\varphi^*\varphi)^2 : (x) + \text{counterterms}$$

where φ is a complex massive boson field on R^3 and $\Delta_A = \sum_{\mu=1}^{3} (\partial_\mu + ieA_\mu)^2$. A_μ the electromagnetic potential.

The Wick ordering is taken w. r. t. $(-\Delta_A + m_0^2)^{-1}$, following Schrader [11].

We study the n-point functions of the corresponding doubly cutoff theory and prove two standard bounds in theorem III.1.9. By the methods of the proof of this theorem the ultraviolet-and infinite-volume-limit of the cutoff Schwingerfunctions may be taken by applying the arguments of [1] [2] [6] [9]; – we do not repeat the proof of the existence of the limits here. Taking these limits leads to our main results, theorems II.1.5 and II.1.6, stating existence of Schwinger functions, their « distribution property E_0 » nonzero mass gap and C^∞ property in the coupling constant.

The proof of theorem III.1.9 involves the standard techniques of φ_3^4 theory, namely phase-space-cell expansion (PSCE) and cluster-expansion (CE) and we show convergence of both. The Clue why these methods apply again for our case is Kato's inequality as it is presented e. g. in [5] [11] [12], which roughly says, that the contraction of two fields is bounded by the propagator without coupling to an electromagnetic potential.

We arrive at somewhat weaker results than Schrader [11], since the uniformity of the estimates w. r. t. A_μ is lost. The question wether the origin of this is of physical nature (the mass renormalization estimate relies on regularity properties of the propagator which depends on A_μ) or is merely a technical problem remains unanswered. Also, we confine ourselves to the case of an electromagnetic (i. e. Abelian) potential instead of a general Lie algebra valued Yang-Mills potential, since for the nonabelian case the validity of the heuristic Wiener representation

$$(-\Delta_A + m_0^2)^{-1}(x, y) = \int_0^\infty dt e^{-tm_0^2} \int dP_{xy}^t(\omega) \mathscr{P} \exp\left(ie \sum_{\mu=1}^{3} \int_0^t A_\mu(\omega(\tau)) d\omega_\mu(\tau)\right)$$

$$\mathscr{P} \exp\left(ie \sum_{\mu=1}^{3} \int_0^t A_\mu(\omega(\tau)) d\omega_\mu(\tau)\right) = \prod_{[0,t]}^{\uparrow} \exp\left[ie \sum_{\mu=1}^{3} A_\mu(\omega(\tau)) d\omega_\mu(\tau)\right]$$

is not quite clear ($\prod\uparrow$ denoting a product whose factors are ordered with increasing time to the left). On the other hand we were not able to dispense with the Wiener representation of the propagator. We hope to clear this point in a forthcoming paper.

Actually this work is a hybrid of the papers of Feldman and Osterwalder [2], Magnen and Sénéor [7] and Schrader [11]. Our notation stays as far as possible close to these papers and the reader's familiarity with these works as well as with the background works [3] and [4] is assumed. In order to take over the graph estimates of [2], we prove two inequalities for differences of covariances, which follow essentially from an estimate of the gradient of the covariance $(-\Delta_A + m_0^2)^{-1}(x, y)$ by the free one (cf. prop. III.3.2).

Also, we present a slightly simplified proof of the control of divergent P_α vertices (see [7] and below.

The organization is as follows. In section II we define the theory and state the main results. In section III.1 we prove the central estimate (lemma 1.13) with the help of the expansions. Sections III.2 and III.3 provide the techniqualities of this estimate: in III.2 it is reduced to graph estimates, white in III.3 it is shown how to apply the graph estimates of [2] to our case.

II. THE MODEL

In this section we shall first give the definitions needed to write down the interaction-measure of the theory and then state the main results. Then we proceed by discussing the modifications of graphs and their norms as they are given in [1] [2].

II.1. Definition of the model and main results.

DÉFINITION 1.1. — Define a class $\mathscr{A}$ of electromagnetic potentials

$$A = \{ A_\mu \}_{1 \leqslant \mu \leqslant 3}, \qquad A_\mu : \mathbb{R}^3 \to \mathbb{R},$$

by requiring

i) $$A_\mu \in L^4_{loc} \cap L^q$$

ii) $$\partial_\nu A_\mu \in L^p, \qquad 1 \leqslant \mu, \nu \leqslant 3$$

for any $q, p > 3$.

As in [11] we have:

THEOREM 1.2. — Let $\Delta_A = \sum_{\mu=1}^{3} \left(\frac{\partial}{\partial x_\mu} + ieA_\mu \right)^2$, e real fixed and $A \in \mathscr{A}$.

Then Δ_A is a nonpositive operator on $L^2(\mathbb{R}^3, \ )$, essentially selfadjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C})$. For $m_0 > 0$ the kernel of the resolvent $D_A = (-\Delta_A + m_0^2)^{-1}$ satisfies

$$(1.1) \qquad | D_A(x, y) | \leqslant C_\phi(x, y) \equiv (-\Delta + m_0^2)^{-1}(x, y)$$

(Kato's inequality), Δ the Laplacian on $\mathbb{R}^3$.

Remarks. — The proof of the first statement of Schrader/Schechter in [*10*] [*11*] applies again, since we use a subclass of potentials of the one of Schrader.

A general proof of the second statement is e. g. given in [*5*] or in [*12*].

As in [*11*], theorem 1.2 is the key for the definition of the « free theory », i. e. of a Gaussian measure over $\mathcal{S}'(\mathbb{R}^3, \mathbb{C})$ of mean zero and covariance D_A.

A very useful formula in view of the introduction of cutoffs and for other technical purposes is the Wiener representation of the kernel of the resolvent D_A [*11*] [*12*]:

$$(1.2) \qquad \begin{aligned} D_A(x, y) &= \int_0^\infty dte^{-tm_0^2} \int dP_{xy}^t(\omega) \exp\left(ie \sum_{\mu=1}^{3} \int_0^t A_\mu(\omega(\tau))d\omega_\mu(\tau) \right) \\ &\equiv \int_0^\infty dte^{-tm_0^2} \int dP_{xy}^t(\omega) \mathcal{G}^t(\omega) \end{aligned}$$

where dP_{xy}^t is the usual conditional Wiener measure and $\int_0^t A_\mu(\omega(\tau))d\omega_\mu(\tau)$ is a symmetric form of Itô's integral:

$$(1.3) \qquad \int_0^t A_\mu(\omega(\tau))d\omega_\mu(\tau)$$

$$= \underset{k\to\infty}{\text{l.i.m.}} \sum_{k=1}^{2^n} \frac{1}{2}\left(A_\mu\left(k\frac{t}{2^n}\right) + A_\mu\left((k-1)\frac{t}{2^n}\right)\right)\left(\omega_\mu\left(k\frac{t}{2^n}\right) - \omega_\mu\left((k-1)\frac{t}{2^n}\right)\right)$$

where $\underset{k\to\infty}{\text{l.i.m.}}$ denotes the limit $k \to \infty$ in $L^2(dP_{xy}^t(\omega))$-sense.

(The proof of the existence of this limit in [*12*; chap. 14] extends to this integral, using past – and future – independent increments of the Brownian bridge.)

(1.2) comes essentially from Trotter's product formula applied to the semigroup $\exp(-t(-\Delta_A + m_0^2))$.

From now on we consider a fixed $A \in \mathcal{A}$ and occasionaly drop the subscript « A ».

Euclidean cutoff-fields are introduced in the next

DEFINITION 1.3. — Let $\psi(t, x)$ denote complex the Gaussian process with mean zero and covariance

$$(1.4) \qquad \langle \psi^*(t, x)\psi(s, y) \rangle = \delta(t - s)G_{B,A}(t, x, y)$$

with

$$G_{B,A}(t, x, y) = e^{-tm_0^2} \int dP_{xy}^t(\omega)\mathcal{G}^t(\omega)B(\omega)$$

where $\langle \, . \, \rangle$ denotes expectation and B is some (measurable) function on paths.

The Euclidean (cutoff) field (in an electromagnetic potential A) is defined as

$$\varphi(x; t_m, t_M) = \int_{t_m}^{t_M} dt \psi(t, x) \tag{1.5}$$

Remarks. — $B(\omega)$ will be used to introduce boundary conditions. We shall use the notations:

$$\varphi(x; \kappa^{-2}, \infty) = \varphi_\kappa(x)$$

and the corresponding covariance

$$D_A(x, y; \kappa^{-2}, \infty) = D_\kappa(x, y)$$

for a universal upper cutoff κ.

The field in (1.5) is constructed such that in its corresponding covariance $D_A(x, y; t_m, t_M)$ momenta larger (smaller) than $t_m^{-1/2}$ ($t_M^{-1/2}$ resp.) are exponentially damped. This can easily be seen by first using Kato's inequality for the cutoff covariance (for which it also holds: either seen by the use of Kato's inequality for semigroups [5] or e. g. by inspection of the Wiener representation) and then considering the Fourier transform of the covariance $C_\phi(x, y; t_m, t_M, m_0^2)$ built as in [2].

DEFINITION 1.4. — Let $\langle \, . \, \rangle$ denote $\int \cdot \, d\nu_A$, $d\nu_A$ the Gaussian measure of mean zero and covariance $G_{B\equiv 1,A}(t, x, y)$. Let $g \in C_0^\infty(\mathbb{R}^3)$, $0 \leqslant g \leqslant 1$, $\kappa \in \mathbb{R}^+$ large and $\lambda \in \mathcal{D}_{\rho,\sigma}, \mathcal{D}_{\rho,\sigma} = \left\{ \lambda \in \mathbb{C} \mid |\lambda| \leqslant \rho < \infty, |\arg \lambda| \leqslant \frac{\pi}{2} - \sigma < \frac{\pi}{2} \right\}$.

The interaction of the cutoff $\lambda \mid \varphi \mid_3^4$-theory in an external electromagnetic potential A is given by:

$$\begin{aligned}
&i) && V(\lambda g, \kappa) = V_I(\lambda g, \kappa) + V_C(\lambda g, \kappa) \\
&ii) && V_I(\lambda g, \kappa) = \lambda \int_{\mathbb{R}^3} d^3x g(x) :(\varphi^*\varphi)^2:(x) \\
&iii) && V_C(\lambda g, \kappa) = \frac{1}{2}\langle V_I^2 \rangle - \frac{1}{6}\langle V_I^3 \rangle - \frac{1}{2} V_m \\
&iv) && V_m(\lambda g, \kappa) = \lambda^2 \int_{\mathbb{R}^3} d^3x g^2(x) \delta m^2(x, \kappa) :\varphi^*\varphi:(x) \\
&v) && \delta m^2(x, \kappa) = 4 \cdot 2 \int_{\mathbb{R}^3} d^3y \{ D_\kappa^2(x, y) D_\kappa(y, x) + D_\kappa^2(y, x) D_\kappa(x, y) \}
\end{aligned} \tag{1.6}$$

where Wick ordering is w. r. t. $D_\kappa(x, y)$.

Euclidean Green's functions (Schwinger functions) are defined in the usual way: for $f_1, \ldots, f_m, \ldots, f_{m+1} \ldots f_{n+m} \in \mathscr{S}(\mathbb{R}^3, \mathbb{C})$

$$S^{(m,n)}(\lambda g, \kappa; f_1, \ldots, f_m, f_{m+1}, \ldots, f_{n+m}) = Z^{-1}(\lambda g, \kappa) \left\langle \prod_{i=1}^{m} \varphi^*(f_i) \prod_{j=m+1}^{n+m} \varphi(f_j) e^{-V(\lambda g, \kappa)} \right\rangle$$

$$Z(\lambda g, \kappa) = \langle e^{-V(\lambda g, \kappa)} \rangle$$

and

$$\varphi^{(*)}(f_i) = \int d^3x \varphi^{(*)}(x) f_i(x)$$

Remark. — We could have incorporated Dirichlet boundary conditions in (1.6), as in [*2*], and all results hold for this case too, as it is easily seen upon inspecting the proofs.

From now on we will omit subscripts and arguments quite freely, unless there is danger of confusion.

THEOREM 1.5. — For suffiently weak coupling (i. e. ρ small, m_0^2 large enough) the limit

$$S^{(n,m)}(f_1, \ldots, f_{n+m}) = \lim_{g \to 1} \lim_{\kappa \to \infty} S^{(n,m)}(\lambda g, \kappa; f_1, \ldots, f_{n+m})$$

exists and obeys

$$| S^{(n,m)}(f_1, \ldots, f_{n+m}) | \leqslant (n + m)! \prod_{j=1}^{n+m} | f_j |$$

for some Schwartz-norm $| \, . \, |$.

THEOREM 1.6. — The Euclidean Green's functions have a nonzero lower mass gap and are C^∞ in $\lambda \in \mathscr{D}_{\rho,\sigma}$.

These theorems follow essentially from theorem III.1.9 below, which is the corresponding (slightly more general) result for the doubly cutoff theory. The theorems are then proved by taking the limits $\kappa \to \infty, g \to 1$, as in [2] [6] [9], using the methods of the proof of theorem III.1.9.

II.2. **Graphs and their norms.**

Our notion of a graph G is essentially the same as in the paper of Feldman and Osterwalder [*2*]. Since we have to modify slightly the definition of graph norms in order to take full advantage of Kato's inequality (1.1), we first give the general expression of G and then introduce the adapted norm.

Let V(G) be a set of vertices, E(G) a set of external legs (i. e. (cutoff) fields), E(v), for $v \in$ V(G), the set of external legs of the vertex v, I(G) a set of internal lines (i. e. contractions of two legs by D_A). Then a graph G is the following expression

$$(2.1)\quad G = \int \prod_{v\in V(G)} \left\{ d^3x_v h_v(x_v) : \prod_{\varphi^*,\varphi\in E(v)} \varphi^*(x_v)\varphi(x_v) : \right\} \prod_{l_{ij}\in I(G)} l_{ij}(x_i, x_j)$$

where $h_v(x_v)$ is a cutoff function, $\varphi^{(*)}(x_v)$ a cutoff field and

$$(2.2)\qquad l_{ij}(x_i, x_j) = \sum_{\Gamma\subset\mathcal{F}} \int_{t_m}^{t_M} dtf(t,\Gamma)\left(\int dP^t_{x_ix_j}(\omega)\mathcal{G}^t(\omega)\chi_\Gamma(\omega)\right)$$

with $f(t, \Gamma) \geqslant 0$, $\mathcal{F}$ denoting set of faces of unit cubes of a cover of $\mathbb{R}^3$, and

$$(2.3)\qquad \chi_\Gamma(\omega) = \begin{cases} 0 & \text{if } \omega(\tau)\in\Gamma, \quad 0 \leqslant \tau \leqslant t \\ 1 & \text{other wise} \end{cases}$$

Similarly as in [*1*] [*2*] we introduce a family of norms for G in

DEFINITION 2.1. — Let $\delta > 2\alpha > 0$. Define a norm $\|\,.\,\|_{\delta,\alpha}$ for G by

$$\| G \|_{\delta,\alpha} = \sup_{\delta'} \sup_{\mathcal{P}} \sup_{\mathcal{C}} \| E\mathcal{P}_\alpha\mathcal{C}_{\delta'} \,|\, G \,|\, \|_{H.S.}$$

where

1) $\delta' = (\delta'_1, \ldots, \delta'_{|E(G)|})$ and each $\delta'_i \in (2\alpha, \delta)$, $1 \leqslant i \leqslant |E(G)|$ and belongs to the i^{th} external leg of G.

2) $|\,.\,|$ replaces each cutoff function h_v by $|h_v|$ and each internal line $l_{ij}(x_i, x_j)$ by

$$\sum_{\Gamma\subset\mathcal{F}} \int_{t_m}^{t_M} dtf(t,\Gamma) \int dP^t_{x_ix_j}(\omega)\chi_\Gamma(\omega)$$

(mimicking Kato's inequality)

3) $\mathcal{C}_{\delta'}$ is a contraction scheme of G's external legs and contracts the external legs

$$\varphi(x_1)(\varphi^*(x_1)) \qquad \text{and} \qquad \varphi^*(x_2)(\varphi(x_2) \quad \text{resp.})$$

by

$$(2.4)\qquad K(\delta, \alpha) \int_0^\infty dtt^{-\delta'_{i_1}-\delta'_{i_2}} e^{-t/2}(8\pi t)^{-3/2} e^{-\frac{(x_1-x_2)^2}{8t}}$$

(also with a « built-in » Kato-inequality), $K(\delta, \alpha)$ an appropriate constant. Denote the propagator (2.4) $C_{-\delta'_{i_1}-\delta'_{i_2}}(x_1, x_2)$.

4) $\mathcal{P}_\alpha$ takes a collection of (cutoffs) vertices : $(\varphi^*\varphi)^2$: and connects them with the external legs of $(\mathcal{C}_{\delta'} \,|\, G \,|)$, with the exception that it does not contract such a vertex to a subgraph of G consisting in a single leg $\varphi^{(*)}(h)$. Contraction lines arising in this operation are given the propagator $C_\alpha(x_1, x_2)$.

5) E replaces the remaining external legs of G by the operator square root of $C_1(x_1, x_2)$.

6) Finally $\| \, . \, \|_{\text{H.S.}}$ denotes the Hilbert-Schmidt norm of the kernel $(E\mathscr{P}_\alpha\mathscr{C}_{\delta'} \mid G \mid)$.

Remark. — We have only changed the original definition of [*2*] in two respects: first we have in addition replaced all covariances (and legs viewed as their operator square roots) that would appear by those without coupling to A; second we have dropped the δ'-dependence of the $\mathscr{P}$ and E operation, possible because, using the form of the PSCE of Magnen and Sénéor [*7*], G is only squared once.

III. THE EXPANSIONS

The central result of this section is lemma 1.13, which, combined with two standard lemmas used for the CE, provides theorem 1.9, the cutoff-version of the main theorems.

Lemma 1.13 is proved in subsection III.1 by resummation of the expansions and a technical result, proposition 1.15, whose proof is given in subsections III.2 and III.3.

The basis of the proof of lemma 1.13 is the combination of the PSCE, which provides ultraviolet uniform bounds, and of the CE, which gives exponential clustering. These, in turn, are used to take the ultraviolet, resp. infinite volume limit.

The idea of the PSCE is to exploit the Wick bound of V_I, while the CE exploits the exponential falloff of the covariances; see also [*3*] [*4*].

III.1. **Definition of the expansions and resummations.**

We begin with the PSCE and follow Magnen and Sénéor [*7*] quite closely.

DEFINITION 1.1. — Given positive constants $t_1 < 1$, $\nu < \frac{1}{2}$, to be fixed later, define $\tau = -\ln t_1/\ln 4$ and two (infinite) sequences

$$i) \qquad \{ t_i \}_{i \geqslant 0} \; ; \qquad t_0 = \infty \; ; \quad t_{i+1} = t_i^{(1+\nu)} = t_1^{(1+\nu)^i}, \quad i \in \mathbb{Z}$$

$$ii) \qquad \{ | \Delta_i | \}_{i \geqslant 0} \; ; \qquad | \Delta_0 | = 1 \; ; \qquad | \Delta_{i+1} | = 8^{-[(1+\nu)^i\tau]}$$

where $[x]$ denotes the largest integer s. t. $[x] \leqslant x$.

The sequence $\{ t_i \}$ will serve as a sequence of momentum cutoffs, while

$\{ |\Delta_i| \}$ will be taken as cube sizes (of cubes Δ_i) of compatible covers $\mathscr{D}_i$ of $\mathbb{R}^3$ [1].
We will assume, without loss of generality, that :

i) κ^{-2} is an element in $\{ t_i \}$, which we suggestively denote t_κ

ii) supp $g \equiv \Lambda$ is exactly paved by $\mathscr{D}_0$, thereby exactly paved by all $\mathscr{D}_l$.
We denote

$$\Lambda_l = \bigcup_{\substack{\Delta_l \in \mathscr{D}_l \\ \Delta_l \subset \Lambda}} \Delta_l$$

During the PSCE one creates vertices which have certain momentum cutoffs t_i and which are located in cubes Δ_l. To prove convergence of the expansion, it is important to distinguish two cases for the relation of t_i and Δ_l.

DEFINITION 1.2. — A pair $(t_i, \Delta_l) \equiv (i, \Delta_l)$, $\Delta_l \in \Lambda_l$, is said to be of type α (resp. β), if $|\Delta_l| \geqslant t_i^{\nu/4}$ ($|\Delta_l| < t_i^{\nu/4}$ resp.).

Also a vertex indexed by (i, Δ_l) (s. below), will be called α- or β-type accordingly.

Also, for fixed i, let $\alpha(i)$ be the largest integer s. t. $(i, \Delta_{\alpha}(i))$ is of type α[2].

Finally, define an order relation between pairs (i, Δ_l) and (j, Δ_k) by:

$$(i, \Delta_l) \leqslant (j, \Delta_k) \begin{cases} \text{if} \quad |\Delta_l| = |\Delta_k| \quad \text{and} \quad i \geqslant j \\ \text{or if} \quad |\Delta_l| > |\Delta_k| \end{cases}$$

For later purposes, we collect some properties of the sequences in

PROPOSITION 1.3. — For ε_0, $0 < \varepsilon_0 < 1$, and for $\nu = \nu(\varepsilon_0)$ small enough, $\tau = \tau(\nu, \varepsilon_0)$ large enough (i. e. t_1 small enough), we have

a) *i)* $|\Delta_j|^{1/3} t_i^{-1/2} \geqslant 1$ for $0 \leqslant j \leqslant i$, $i \geqslant 1$

ii) $|\Delta_i| t_{i+1}^{-1} \leqslant 1$ for $i \geqslant 1$

b) *i)* $e \cdot |\Delta_{j-1}| \, |\Delta_j|^{-1} \leqslant |\Delta_j|^{-\varepsilon_0}$ for $j \geqslant 1$

ii) $e \cdot |\Delta_{j-1}| \, |\Delta_j|^{-1} \geqslant t_i^{-\varepsilon_0/2}$ for $i, j \geqslant 1$, (i, Δ_j) of type α

iii) $(t_i^{\varepsilon_0/2})^{n_i} \leqslant (n_i\,!)^{-1}$ for $1 \leqslant n_i \leqslant [t_i^{-\nu}]$, $i \geqslant 1$

Remarks. — A field $\varphi^{(*)}$ localized in Δ_j of lower cutoff t_i, satisfying *(a) i)* is said to be « property localized », in accordance with the uncertainty principle.

[1] Two covers are said to be compatible, if one is the refinement of the other.

[2] It is easily verified that $\alpha(i) = [i + (\ln \nu/6)/\ln (1 + \nu)]$.

If it has upper cutoff t_{i+1} and (*a*) *ii*) holds, it is said to « saturate the Wick bound », since then $(i = j)$

$$\left| \int_{\Delta_i} d^3x : (\varphi^*\varphi)^2 : (x) \right|$$

$$\begin{aligned} &\geqslant - 0(1) \mid \Delta_i \mid \mid D_A(x, x ; \infty, t_{i+1})^2 \mid \\ &\geqslant - 0(1) \mid \Delta_i \mid C_\phi(x, x ; \infty, t_{i+1})^2 \qquad \text{(Kato's ineq.)} \\ &\geqslant - 0(1) \mid \Delta_i \mid t_{i+1}^{-1} \geqslant - 0(1) \end{aligned}$$

Properties (*b*) ensure convergence of the PSCE.

The PSCE is generated as a « perturbation » series, by taking derivatives with respect to an interpolation variable introduced into the field via

DEFINITION 1.4. — To each t_i, $2 \leqslant i \leqslant \kappa$ and each $\Delta_l \in \Lambda_l$, $0 \leqslant l \leqslant \kappa - 1$ associate

i) an interpolation variable $s_{(i,\Delta_l)}$ with values in $[0, 1]$

ii) the function

$$A(i, \Delta_l) = \begin{cases} [t_i^{-\nu}] & \text{if } l = 0 \\ [t_i^{-\nu/2}] & \text{if } l \neq 0 \quad \text{and} \quad (i, \Delta_l) \text{ is of type } \alpha \\ 1 & \text{otherwise} \end{cases}$$

Furthermore, define the interpolating (cutoff) field

$$\varphi_\kappa(\underline{s}, x) = \sum_{i=1}^{\kappa} \sum_{\Delta \in \Lambda_{i-1}} \prod_{\substack{\Delta_l \subset \Delta \\ \Delta_l \in \Lambda_l}} (s_{(i,\Delta_l)} \chi_{\Delta_l}(x)) \varphi(x ; t_i, t_{i-1}) \tag{1.1}$$

where $\varphi(x ; t_i, t_{i-1})$ is the field defined in chapter II.

Remarks. — $A(i, \Delta_l)$ will be the number of vertices: $(\varphi^*\varphi)^2$: to be generated during the PSCE.

Since these vertices provide convergence factors (in some cases), this number has to be larger than 1 in case α in order to compensate the occuring divergences (s., e. g., III.1). We emphasize that $\varphi(\underline{s} ; x)$ has no s-dependence for its « improperly localized parts » (s. remark after prop. 1.3), i. e. for fixed i (lower cutoff t_{i-1}), there is no s-dependence in cubes smaller than $\mid \Delta_{i-1} \mid$ which is the smallest cube size of proper localization.

Consider now for a function $f = f(s)$ the identity

$$\text{(1.2)} \quad \begin{aligned} &i) \qquad f = (I + P)f, \quad \text{where} \\ &ii) \qquad (Pf)(s) = \int_0^s \left(\frac{d}{ds'} f(s') \right) ds' \\ &iii) \qquad (If)(s) = f(s = 0) \end{aligned}$$

and the contraction formula (s. e. g. [*4*]) (³):

$$(1.3) \quad \int :(\varphi^+)^n:(x)\mathrm{R}(\varphi^+)e^{-\mathrm{V}(\varphi^+)}dv_{\mathrm{A}}$$
$$= \int d^3y \langle \varphi^+(x)\varphi^{+*}(y) \rangle \int :(\varphi^+)^{n-1}:(x)\left\{ \frac{\delta \mathrm{R}}{\delta\varphi^{+*}(y)} - \frac{\delta \mathrm{V}}{\delta\varphi^{+*}(y)} \right\} e^{-\mathrm{V}(\varphi^+)}$$

where R is a Wick monomial in the fields.

For the first term in { } in (1.3) we say that φ^+ has contracted to an old vertex, while for the second we say that it has created a C-vertex.

DEFINITION 1.5. — Let $\mathrm{V}(\lambda g, \underline{s}) = \mathrm{V}(\Lambda, \underline{s}) \equiv \mathrm{V}_\Lambda$. The PSCE of the expression $\int \mathrm{G}e^{-\mathrm{V}_\Lambda}dv_{\mathrm{A}}$ is defined by

$$(1.4) \qquad \int \mathrm{G} \prod_{(i,\Delta_l)} ((\mathrm{I} + \mathrm{C} \circ \mathrm{P})_{(i,\Delta_l)})^{\mathrm{A}(i,\Delta_l)} e^{-\mathrm{V}_\Lambda} dv_{\mathrm{A}}$$

where

i) $(\mathrm{I}+\mathrm{C} \circ \mathrm{P})_{(i,\Delta_l)}$ acts on the variable $s_{(i,\Delta_l)}$ and I, P act on the exponent only

ii) the product is taken over all pairs (i, Δ_l) in decreasing order

iii) C is a contraction operation via (1.3), given as the next (separate) definition.

A vertex created by a $\mathrm{P}_{(i,\Delta_l)}$-operation is called a $\mathrm{P}_{(i,\Delta_l)}$-vertex.

DEFINITION 1.6. — The contraction operation $\mathrm{C}_{(i,\Delta_l)}$ is given by:

i) if (i, Δ_l) is of type α, then $\mathrm{C}_{(i,\Delta_l)}$ contracts all legs of the $\mathrm{P}_{(i,\Delta_l)}$-vertex and furthermore all legs of all C-vertices created by this procedure except

1) if the $\mathrm{P}_{(i,\Delta_l)}$-vertex is a mass counter term, then one does not contract the legs of the (first) new C-vertex,

2) if the $\mathrm{P}_{(i,\Delta_l)}$-vertex contracts three times to one new C-vertex and once to another new C-vertex, then only the fourth leg of the first new C-vertex is contracted

3) if the $\mathrm{P}_{(i,\Delta_l)}$-vertex contracts three times to old vertices and once to a new C-vertex, then we reverse the last contraction, using (1.3) in reverse.

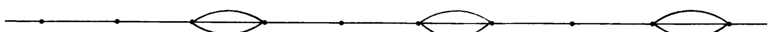
ii) If (i, Δ_l) is of type β, then we decompose each of the legs of the $\mathrm{P}_{(i,\Delta_l)}$-vertex as

$$(1.5) \qquad \begin{aligned} \varphi_{\Delta_l}(x) &= \varphi(x)\chi_{\Delta_l}(x) \\ &= \varphi^l_{\Delta_l}(x) + \varphi^h_{\Delta_l}(x) \end{aligned}$$

($\varphi^*_{\Delta_l}$ analogously), where $\varphi^l_{\Delta_l}$ is the improperly localized part of the field w. r. t. Δ_l. $\mathrm{C}_{(i,\Delta_l)}$ now contracts all $\varphi^h_{\Delta_l}$ legs.

(³) Let φ^+ denote either φ or φ^* and let $\varphi^{**} = \varphi$.

Remark. — In case α C serves to cancel the vacuum energy counter terms exactly. Exceptions 1) and 2) provide the combination of the mass counter terms with the graphs  , while exception 3) prevents the formation of chains P C .



P C C P C C P C C P

In case β only the high momentum part of the second order energy counter terms is renormalized. All other divergencies (which are only logarhythmic) are controlled by the cube size.

This completes the definition of the PSCE. We now introduce the cluster expansion. (Actually we have not changed its definition as given in [*4*], but include it here for convenience of the reader and to set up some notation. As in section II, let $\mathscr{F}$ denote the set of faces of the unit cubes $\Delta \in \mathscr{D}_0$, $\mathscr{D}_0$ a cover of $\mathbf{R}^3$, and associate with each $f \in \mathscr{F}$ the interpolation variable σ_f, taking values in [*0*, *1*], and denote by $\underline{\sigma} = \{ \sigma_f \}$ the collection of all these variables.

DEFINITION 1.7. — First define the conditioning function $B_{\underline{\sigma}}(\omega)$ on paths ω which, say, travel from x to y in time t:

$$B_{\underline{\sigma}}(\omega) = \sum_{\Gamma \subset \mathscr{F}} \prod_{f \in \Gamma} \sigma_f \prod_{f' \in \Gamma^c} (1 - \sigma_{f'}) \chi_{\Gamma^c}(\omega) \tag{1.6}$$

with $\chi_{\Gamma^c}(\omega)$ as in eq. (II.2.3).

The cutoff covariance with Dirichlet boundary conditions (as given by $\underline{\sigma}$) is then defined as:

$$D_{\underline{\sigma}}(x, y ; t_m, t_M) = \int_{t_m}^{t_M} dt e^{-tm_0^2} G_{B_{\sigma}}(t, x, y) \tag{1.7}$$

G_{B_σ} as in definition II.1.3, suppressing the A-dependence.

Remarks. — We note that Kato's inequality (II.1.1) holds for $D_{\underline{\sigma}}(x, y ; t_m, t_M)$ as well, i. e.:

$$| D_{\underline{\sigma}}(x, y ; t_m, t_M) | \leqslant C_{\underline{\sigma}}(x, y ; t_m, t_M)$$

as can be easily seen from the Wiener representation of $D_{\underline{\sigma}}(x, y; t_m, t_M)$.

$D_{\underline{\sigma}}$ is interpreted as a covariance which interpolates between complete coupling and complete decoupling of certain regions of $\mathbb{R}^3$. Furthermore we emphasize that all appearing covariances (e. g. in Wick ordering) are from now on taken to be $D_{\underline{\sigma}}(x, y ; t_m, t_M)$ unless otherwise stated.

Then as in the work of Glimm, Jaffe, Spencer [*4*], we make the

DEFINITION 1.8. — The cluster expansion of $\int Ge^{-V_\Lambda}dv_A$ is given by:

$$(1.8)\quad \int Ge^{-V_\Lambda}dv_A = \sum_{X,\Gamma}\left(\int_0^1 \partial^\Gamma \int dv_{\underline{\sigma}} G_{\underline{\sigma}} e^{-V_{\Lambda\cap X(\sigma)}} d\underline{\sigma}(\Gamma)\right)\frac{Z_{\partial X}(\Lambda \sim X)}{Z(\Lambda)}$$

and

i) the sum over X ranges over all finite unions of closed cubes, while Γ ranges over finite subsets of $\mathscr{F}$ so that $(X \sim \Gamma^c) \cap \{ X_v \mid v \in V(G) \} \neq \emptyset$ and $\Gamma \subset \operatorname{Int} X$

ii)
$$\partial^\Gamma = \prod_{f\in\Gamma}\frac{d}{d\sigma_f}; \qquad \underline{\sigma}(\Gamma) = \{ \sigma_f(\Gamma) \}_{f\in\mathscr{F}}$$

$$\sigma_f(\Gamma) = \begin{cases} \sigma_f, & f\in\Gamma \\ 0, & f\notin\Gamma \end{cases}$$

iii)
$$Z_{\partial X}(\Lambda \sim X) = \int dv_{\partial X} e^{-V_{\Lambda\sim X}}$$

$dv_{\partial X}$ the Gaussian measure of mean zero and covariance

$$D_{\partial X} = (-\Delta_{A,\partial X} + m_0^2)^{-1}, \qquad \Delta_{A,\partial X}$$

having zero Dirichlet b. c. on ∂X.

Remark. — The evaluation of the derivatives $\frac{d}{d\sigma_f}$ in (1.8) can be done as in [2] [9] to yield a number of vertices, which we call E vertices. By some Wick reordering it is possible to cancel the linear divergent second order vaccuum energy counter terms arising in this evaluation, so that such a derivative generates a number of at most logarhythmic divergent vertices. (We avoid to give the evaluation of these derivatives here, since this would require the introduction of diagrams, to keep the expressions at reasonable length, which were only needed in this step of this paper.)

THEOREM 1.9. — Suppose we have sufficiently weak coupling (s. a.) and let G and G′ have finite $\| \cdot \|_{\delta,\alpha}$ norms. Let G (G′) have $n(\Delta)$ ($n'(\Delta)$) legs in Δ, $\Delta \in \mathscr{D}_0$, and assume that the supports of the functions h and h' in G, G′ resp., are separated by two parallel planes a distance d apart. Then

i) $$| Z(\lambda g, \kappa)^{-1} \langle Ge^{-V(\lambda g,\kappa)} \rangle | \leqslant \prod_{\Delta\in\mathscr{D}_0} n(\Delta)\,!\, \| G \|_{\delta,\alpha}$$

ii) $$| Z(\lambda g, \kappa)^{-1} \langle GG'e^{-V(\lambda g,\kappa)} \rangle - Z(\lambda g, \kappa)^{-2} \langle Ge^{-V(\lambda g,\kappa)} \rangle \langle G'e^{-V(\lambda g,\kappa)} \rangle | \leqslant \prod_{\Delta\in\mathscr{D}_0} n(\Delta)\,!\, n'(\Delta)\,!\, \| G \|_{\delta,\alpha}\, \| G' \|_{\delta,\alpha} e^{-md}$$

for some $m > 0$, independent of the cutoffs.

As in [2] [4], the theorem is a consequence of the following lemmas:

LEMMA 1.10. — Consider the CE in (1.8) and let $|X|$ be fixed. There exists a constant K_1 such that the number of terms in the sum is bounded by $\exp K_1 |X|$.

Assume now sufficiently weak coupling.

LEMMA 1.11. — There exists a constant K_2 (independent of λ, m_0 and the cutoffs) such that

$$| Z_{\partial X}(\Lambda \sim X)/Z(\Lambda) | \leqslant \exp K_2 |X|$$

COROLLARY 1.12.

$$| Z(\Lambda) | \geqslant \exp(-K_2 |X|)$$

LEMMA 1.13. — There exists a constant K_3 such that for any $K_4 > 0$

$$\left| \int \partial^\Gamma \int G_{\underline{\sigma}} e^{-V_\sigma(\Lambda \cap X)} dv_{\underline{\sigma}} d\underline{\sigma}(\Gamma) \right| \leqslant \prod_{\Delta \in \mathscr{D}_0} n(\Delta)! \, \| G \|_{\delta,\alpha} e^{-K_4|\Gamma| + K_3|\Lambda \cap X|}$$

for a graph G with finite norm $\| G \|_{\delta,\alpha}$ and $n(\Delta)$ legs in $\Delta \in \mathscr{D}_0$. K_3 is independent of the cutoffs.

Lemmas 1.10 and 1.11 follow as in [4] and we concentrate on lemma 1.13.

Proof of lemma 1.13. — Consider

$$(1.9) \qquad \partial^\Gamma \langle G e^{-V(\Lambda \cap X)} \rangle$$

We proceed in three steps.

First we apply the PSCE in unit cubes (formula 1.5 for $l = 0$), then evaluate the derivatives ∂^Γ (s. remark after def. 1.8), and finally complete the PSCE.

Consider one term in the sum after the first step, which is formally given by

$$(1.10) \qquad \partial^\Gamma \langle R_{\underline{a}} e^{-V_{\underline{a}}} \rangle$$

where $\underline{a}$ denotes a collection of variables $a_{(i,\Delta_0)}$, $0 \leqslant a_{(i,\Delta_0)} \leqslant A(i, \Delta_0)$, which denotes that the corresponding term has undergone $a_{(i,\Delta_0)}$ applications of the $P_{(i,\Delta_0)}$ operation; $R_{\underline{a}}$ includes a factor G, P and C vertices (at this stage).

The evaluation of the derivatives in (1.10) in the second step yields by Leibniz' rule (s. also [2] [6] [9]):

$$(1.11)\quad \partial^{\Gamma}\langle R_{\underline{a}} e^{-V_{\underline{a}}}\rangle = \sum_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=(\gamma_1,\dots,\gamma_K)}} \left\langle \int\prod_{i=1}^{2K} dx_i \partial^{\gamma_1}D(x_1,x_2)\dots\partial^{\gamma_K}D(x_{2K-1},x_{2K})R'_{\underline{a},\pi}e^{-V_{\underline{a}}}\right\rangle$$

where $\mathscr{P}(\Gamma)$ is the set of all partitions π of Γ and $R'_{\underline{a},\pi}$ now also includes E-vertices.

After completion of the PSCE a generic term of the final sum looks as the r. h. s. of (1.11) with $\underline{a}$ now a collection of $a_{(i,\Delta_l)}$, $0\leqslant a_{(i,\Delta_l)}\leqslant A_{(i,\Delta_l)}$, $l\geqslant 0$.

The resummation over the partitions (of one generic term) is controlled by

PROPOSITION 1.14. — For sufficiently large m_0 there exist constants $K_5(\gamma)$ and K_6 (independent of m_0) such that

$$(1.12)\qquad \sum_{\pi\in\mathscr{P}(\Gamma)}\prod_{\gamma_i\in\pi}K_5(\gamma_i) = e^{K_6|\Gamma|}$$

(The proof of this proposition is as in [4] or [2].) Thus a final term is bounded as

$$e^{K_6|\Gamma|}\sup_{\substack{\pi\in\mathscr{P}(\Gamma)\\ \pi=(\gamma_1,\dots,\gamma_K)}} \left\langle\int\prod_{i=1}^{2K}dx_i(K_5^{-1}(\gamma_1))\partial^{\gamma_1}D(x_1,x_2)\dots K_5^{-1}(\gamma_K)\partial^{\gamma_K}D(x_{2K-1},x_{2K})\cdot R'_{\underline{a},\pi}e^{-V_{\underline{a}}}\right\rangle$$
$$\equiv e^{K_6|\Gamma|}\sup_{\pi\in\mathscr{P}(\Gamma)}\langle R''_{\underline{a},\pi}e^{-V_{\underline{a}}}\rangle$$

and each $(\partial^{\gamma}D)$-line in R″ has now a factor $K_5^{-1}(\gamma)$. We will drop the double prime in sequel, for convenience, and consider $\langle R_{\underline{a},\pi}e^{-V_{\underline{a}}}\rangle$, for which we use

PROPOSITION 1.15. — For sufficiently weak coupling there exist constants K_7, K_8 (independent of $\underline{a}$, π and the cutoffs) such that, for t_1 small enough

$$(1.13)\quad |\langle R_{\underline{a},\pi}e^{-V_{\underline{a}}}\rangle| \leqslant \prod_{\Delta} n(\Delta)!\,\|G\|_{\delta,\alpha}e^{-K_7|\Gamma|+K_8|\Lambda\cap X|}\prod_{\substack{\mathscr{P}_{(i,\Delta)}\text{ of}\\ \text{type }\alpha}}(t_i^{\varepsilon_0})^{a_{(i,\Delta)}}\prod_{\substack{\mathscr{P}_{(i,\Delta)}\text{ of}\\ \text{type }\beta}}(|\Delta|^{\varepsilon_0})^{a_{(i,\Delta)}}$$

for some $\varepsilon_0>0$. K_7 can be taken as large as we want, provided m_0 is large enough.

This proposition will be proved in subsections 2 and 3. We continue the proof of lemma 1.13:

By definition 1.4 and 1.5 and proposition 1.15 we now have

$$(1.14)\quad |\partial^{\Gamma}\langle Ge^{-V(\lambda g,K)}\rangle| \leqslant \prod_{\Delta} n(\Delta)!\,\|G\|_{\delta,\alpha}e^{-K_4|\Gamma|+K_8|\Delta\cap X|}$$

$$\cdot\prod_{i=1}^{\kappa-1}\Big\{\prod_{\Delta_0\in\Lambda_0\cap X}(1+t_i^{\varepsilon_0}+\ldots+(t_i^{\varepsilon_0})^{A(i,\Delta_0)}\prod_{\Delta_1\subset\Delta_0}(1+t_i^{\varepsilon_0}+\ldots+(t_i^{\varepsilon_0})^{A(i,\Delta_1)}\prod_{\Delta_2\subset\Delta_1}(1+\ldots$$

$$\cdot\prod_{\Delta_{\alpha(i)}\subset\Delta_{\alpha(i)-1}}(1+t_i^{\varepsilon_0}+\ldots+(t_i^{\varepsilon_0})^{A(i,\Delta_{\alpha(i)})}\prod_{\Delta_{\alpha(i)+1}\subset\Delta_{\alpha(i)}}(1+|\Delta_{\alpha(i)+1}|^{\varepsilon_0}$$

$$\prod_{\Delta_{\alpha(i)+2}\subset\Delta_{\alpha(i)+1}}(1+|\Delta_{\alpha(i)+2}|^{\varepsilon_0}\prod(1+\ldots$$

with $K_4 = K_7 - K_6 > 0$, choosing K_7 large enough.

Proceeding now as Magnen and Sénéor [7], we bound this, using proposition 1.3 *b*), by

$$(1.15)\qquad \prod_{\Delta} n(\Delta)!\,\|G\|_{\delta,\alpha}e^{-K_4|\Gamma|+(K_8+K_9)|\Lambda\cap X|}$$

where $K_9 = \sum_{i=1}^{\infty} t_i^{\varepsilon_0/2} < \infty$ and the lemma is proved by taking $K_3 = K_8 + K_9$.

III.2. The reduction to graph estimates.

Again, in this section we follow Magnen and Sénéor [7]. For the sequel, assume that t_1 is small enough. We begin with two definitions.

DEFINITION 2.1. — For a field $\varphi_\Delta(\underline{s}\,;x) \equiv \varphi(\underline{s},x)\ \chi_\Delta(x)$ its averaged field is given by

$$(2.1)\qquad \overline{\varphi}_\Delta(\underline{s}\,;x) = \overline{\varphi}_\Delta(\underline{s})\chi_\Delta(x) = \frac{\chi_\Delta(x)}{|\Delta|}\int\varphi_\Delta(\underline{s}\,;x)d^3x$$

and its fluctuation piece $\delta\varphi_\Delta(\underline{s}\,;x)$ by

$$(2.2)\qquad \varphi_\Delta(\underline{s}\,;x) = \overline{\varphi}_\Delta(\underline{s}\,;x) + \delta\varphi_\Delta(\underline{s}\,;x)$$

Remark. — A field φ_Δ in a β-vertex (P_β or C_β), $\Delta\in\Lambda_l$, has then a decomposition (cf. def. 1.6, eq. (1.5)):

$$(2.3)\quad \varphi_\Delta(\underline{s}\,;x) = \varphi_\Delta^h(\underline{s}\,;x) + \overline{\varphi}_\Delta(\underline{s}\,;x) - \overline{\varphi}_\Delta^h(\underline{s}\,;x) + \sum_{i=1}^{l}\delta\varphi_{\Delta,i}(\underline{s}\,;x)$$

$\delta\varphi_{\Delta,i}$ has momentum cutoffs $t_{i-1}^{-1/2}$ and $t_i^{-1/2}$ and the corresponding sum for each P_β-vertex and the C_β-vertices generated by it has a number of terms bounded by $|\Delta|^{-\varepsilon_1}$, $\varepsilon_1 > 0$ as small as we like. Later, taking the supremum over all summands arising in this decomposition, we associate the factor $|\Delta|^{-\varepsilon_1}$ with the P_β-vertex.

DÉFINITION 2.2. — Let dist (Σ_1, Σ_2) denote the Euclidean distance between the sets Σ_1 and Σ_2. Let

$d_B(\Delta') = \text{dist}(\Delta', \partial\Delta)$ the distance from the cube Δ' to the boundary $\partial\Delta$ of the unit cube Δ containing it

$d_E(\Delta') = \text{dist}(\Delta', E(\Delta))$ the distance from a cube Δ' to the set $E(\Delta)$ of edges of faces of the unit cube Δ containing it

$d(\Delta, \Delta', \gamma) = \max_{b \in \gamma} (\text{dist}(\Delta, b) + \text{dist}(\Delta', b))$

For the contraction line of a field in Δ and a field in Δ', not both $\delta\varphi$ fields and $t_M^{-1/2}$ the lower cutoff of this line, let

$$d_s(\Delta, \Delta') = \max \{ 1, m_0 \, \text{dist}(\Delta, \Delta'), \ t_M^{-1/2} \, \text{dist}(\Delta, \Delta') \}$$

be the scaled distance.

For the contraction line of $\delta\varphi_\Delta$ to $\delta\varphi_{\Delta'}$, with lower cutoff $t_M^{-1/2}$, define

$$b(\Delta, \Delta') = \begin{cases} (1 + t_M^{-1/2} \, \text{dist}(\Delta, \Delta'))^6 t_M^{(37/52)(1+\nu)} (|\Delta| \, |\Delta'|)^{-10/52} \left(\dfrac{d_B(\Delta) d_B(\Delta')}{d_E(\Delta) d_E(\Delta')} \right)^{-10/52} \\ \qquad \cdot \left(d_B^{-1}(\Delta) + t_M^{-(1+\nu)/2} \right)^{-30/52} \left(d_B^{-1}(\Delta') + t_M^{-(1+\nu)/2} \right)^{-30/52} \\ \qquad \text{if} \quad d_B(\Delta) \neq 0, \quad d_B(\Delta') \neq 0 \\ d_s^4(\Delta, \Delta') \qquad \text{otherwise} \end{cases}$$

Remark. — For the scaled distance $d_s(\Delta, \Delta')$ we have for $n \geqslant 4$:

$$\sum_{\Delta' \in \mathscr{D}_{l'}} d_s(\Delta, \Delta')^{-n} \leqslant 0(1) \tag{2.4}$$

if $|\Delta'| \geqslant |\Delta|$ and one of the fields is properly localized. Property (2.4) will be crucial for the control of the final Gaussian integration. A similar relation holds for $b(\Delta, \Delta')^{-1}$, see [2]: p. 98.

The main result of this section is:

LEMMA 2.3. — For sufficiently weak coupling there exist (pos.) constants ε_2, K_8, K_{10}, n_1 (independent of $\underline{a}$, $\pi \in \mathscr{P}(\Gamma)$ and the cutoffs) such that

$$|\langle R_{\underline{a},\pi} e^{-\dot{V}_{\underline{a}}} \rangle| \leqslant e^{K_8|\Lambda \cap X| + K_{10}|\Gamma|} \prod_{\Delta} n(\Delta)^{n(\Delta)} \sup_{R_v} \{ C(R_v) R_v \}^{1/2}$$

where R_v is a vacuum graph (i. e. $E(R_v) = \emptyset$) arising as a term in the sum after decomposition (2.3) in $R_{\underline{a},\pi}$, leaving out $\overline{\varphi}_\Delta$ legs, squaring each summand and contracting each term to a vacuum diagram (s. proposition below). The factor $C(R_v)$ is bounded by the following factors:

$t_i^{-\varepsilon_2}$	per P_α-vertex of index (i, Δ)
$\lvert \Delta \rvert^{-\varepsilon_2}$	per P_β-vertex of index (i, Δ)
$d_s^{n_1}(\Delta, \Delta')$	per contraction of a P or C vertex in Δ to a P or C vertex in Δ', not both legs of type $\delta\varphi$
$\text{dist}^{n_1}(\Delta, \Delta')$	per contraction of a leg in Δ to an E-vertex or to a G leg in Δ'
$b(\Delta, \Delta')$	per contraction of $\delta\varphi_\Delta$ to $\delta\varphi_{\Delta'}$
$e^{K_{11} d(\Delta, \Delta', \gamma)}$	per contraction line $\partial^\gamma D$ joining Δ and Δ'
$\lvert \Delta \rvert^{-1+r/4}$	per β-vertex (in Δ) with r legs not of type $\overline{\varphi}_\Delta$

ε_2 can be taken as small as we want.

The proof of the lemma, which we sketch below, consists in three steps, given as propositions 2.4-2.5, the proof of which are easy generalizations of sections III.1-III.4 in [7], s. [9].

Sketch of the proof. — We consider all vertices (hence all legs) localized in cubes, i. e. decompose E vertices into vertices localized in unit cubes, C vertices into vertices localized in cubes belonging to the same cover as the cube of the generating P-vertex. Let $R_{\underline{a},\pi} = \sum_{R'} R'$ denote the sum corresponding to decomposition (2.3) and let R'' denote the graph given by R' leaving out its $\overline{\varphi}_\Delta$-legs.

Then we use

PROPOSITION 2.4. — (Domination of Improperly Localized Legs).

$$(2.5) \quad \lvert \langle R_{\underline{a},\pi} e^{-V_{\underline{a}}} \rangle \rvert \leqslant \sum_{R''} \lvert \langle (R'')^2 e^{-2\mathrm{Re}\, V_{\underline{a}} + \mathrm{Re}\,\lambda \int_{\Lambda \cap X} (\varphi^*_{\underline{a}} \varphi_{\underline{a}})^2(x) d^3x} \rangle \rvert^{1/2}$$

with additional factors

$\lvert \Delta \rvert^{-1+r/4}$	per β_Δ-vertex with r, $1 \leqslant r \leqslant 4$, legs in R''
$\lvert \Delta \rvert^{-\varepsilon_3}$	per $P_{\beta,\Delta}$-vertex, $\varepsilon_3 > 0$
$d_s(\Delta, \Delta')$	per contraction of a $C_{\beta,\Delta}$-vertex to its generating $P_{\beta,\Delta'}$-vertex.

included in R''.

Roughly, proposition 2.4 is used to dominate the $\overline{\varphi}_\Delta$ legs, — for which there is no combinatoric factor to control the final Gaussian integration (s. below) — by an exponential factor $\exp\left(\mathrm{Re}\,\lambda \int_{\Lambda \cap X} \lvert \varphi_{\underline{a}} \rvert^4\right)$.

The exponential on the r. h. s. of (2.5) is now estimated by

PROPOSITION 2.5. — (Wick bound) (4):

There exists K_8, $\varepsilon_4 > 0$ such that

$$(2.7)\quad e^{-2\mathrm{Re}\,V_{\underline{a}}+\mathrm{Re}\,\lambda\int_{\Lambda\cap X}(\varphi^*_{\underline{a}}\varphi_{\underline{a}})^2(x)d^3x} \leqslant e^{K_8|\Lambda\cap X|}\prod_{\substack{P_{(i,\Delta)}\text{-}\alpha\text{-vert.}\\ \text{in } R''}} t_i^{-\varepsilon_4}\prod_{\substack{P_{(i,\Delta)}\text{-}\beta\text{-vert.}\\ \text{in } R''}}|\Delta|^{-\varepsilon_4}$$

It remains to estimate a Gaussian integral. In order use the graph estimates given in section III.3, we have to control the sums arising in the contraction procedure properly (5).

PROPOSITION 2.6 (Combinatoric factors (6)). — The following are bounds on the combinatoric factors for the sum arising in the final contraction:

$n(\Delta)^{n(\Delta)}$	per unit cube
$t_i^{-\varepsilon_5}$	per P_α-vertex of index (i, Δ)
$\|\Delta\|^{-\varepsilon_5}$	per P_β-vertex of index (i, Δ)
$d_s^{n_2}(\Delta, \Delta')$	per contraction (other than $\delta\varphi_\Delta$ to $\delta\varphi_{\Delta'}$) between a P/C vertex in Δ to a P/C vertex in Δ'
$b(\Delta, \Delta')$	per contraction of $\delta\varphi_\Delta$ to $\delta\varphi_{\Delta'}$
$\mathrm{dist}^{n_2}(\Delta, \Delta')$	per contraction of a field in Δ to an E vertex in Δ' or a G-leg in Δ'
$e^{K_{11}d(\Delta,\Delta',\gamma)}$	per $\partial^\gamma D(x_i, x_j)$ line, $x_i \in \Delta$, $x_j \in \Delta'$
$e^{K_{10}\|\Gamma\|}$	for the whole graph;

for (pos.) constants K_{10}, K_{11}, n_2, ε_6 ; ε_6 as small as we like.

The lemma follows now by taking $\varepsilon_2=\varepsilon_1+\varepsilon_5+2(\varepsilon_3+\varepsilon_4)$, $n_1=n_2+2$.

Remark. — The factor $|\Delta|^{-1+\nu/4}$ is considered as remaining at the β_Δ-vertex and is taken into account in the graph estimates for β-vertices (s. lemma 5.2.*i* of [*2*]).

III.3. **Graph estimates.**

We conclude the paper by showing how to estimate R_v, thereby proving proposition II.1.15.

After giving the estimate, we state two propositions, which, together

(4) See remark after prop. 1.3.

(5) The contractions of the final Gaussian integration are performed beginning in smallest cubes, going on successively to larger cube sizes. In unit cubes we contract E legs before P/C legs, those before G legs.

(6) C_n is said to be a combinatoric factor for the sum $\sum_n A_n$ if $\sum_n C_n^{-1} \leqslant 0(1)$ since then $\sum_n A_n \leqslant 0(1)\sup_n C_nA_n$.

with Kato's inequality, allow to use the graph estimates of Feldman and Osterwalder [*2*, lemma 5.2]. The main result of this section is

LEMMA 3.1. — For v and $t_1 = t_1(v)$ small enough, some δ, α, $\delta > 2\alpha > 0$ and m_0 large enough there exist pos. constants ε_6, n_3, K_{12} such that R_v (s. lemma 2.1) is bounded by the following factors:

$\| G \|_{\delta,\alpha}$	for each copy of G
$t_i^{\varepsilon_6}$	per P_α-vertex of index (i, Δ)
$\| \Delta \|^{\varepsilon_6}$	per P_β-vertex of index (i, Δ)
$d_s^{-n_3}(\Delta, \Delta')$	per contraction of a P/C vertex in Δ to a P/C vertex in Δ', not both legs of type $\delta\varphi$
$\mathrm{dist}^{-n_3}(\Delta, \Delta')$	per contraction of a leg in Δ to a leg in Δ' belonging to G or an E-vertex
$b^{-1}(\Delta, \Delta')$	per contraction of $\delta\varphi_\Delta$ to $\delta\varphi_{\Delta'}$
$K_5(\gamma) e^{-K_{11} d(\Delta, \Delta', \gamma) - K_{12} \|\gamma\|}$	per $\partial^\gamma D$ contraction line between Δ and Δ';

where n_3 and K_{12} can be taken as large as we want.

Proposition II.1.15 now follows readily from lemma 2.3 and lemma 3.1, taking $0 < \varepsilon_0 < \varepsilon_6 - \varepsilon_2$, $n_3 > n_1$, $K_7 = K_{12} - K_{10}$ and noting that $n^n \leqslant e^n n!$

Next we need

PROPOSITION 3.2. — Denote $D_\Gamma(x, y) = D_{\Gamma,A}(x, y; t_m, t_M, m_0^2)$. Then for $A \in \mathscr{A}$ (7)

$$| D_\Gamma(x, y) - D_\Gamma(x, y') | \leqslant 0(1) | y - y' | (t_m^{-1/2} + d_B^{-1}(y) + d_B^{-1}(y')) \cdot \left[C_\phi\left(x, y; t_m, 2t_M, \frac{m_0^2}{4}\right) + C_\phi\left(x, y'; t_m, 2t_M, \frac{m_0^2}{4}\right) \right]$$

Proof. — All our covariances are convex combinations of those of the form

$$(3.1) \quad \int_{t_m}^{t_M} dt e^{-tm_0^2} \int dP_{xy}^t(\omega) \mathscr{G}^t(\omega) \prod_{f \in \Gamma} \chi_f(\omega) = D_\phi(x, y) - \int_{t_m}^{t_M} dt e^{-tm_0^2} \int dP_{xy}^t(\omega) \mathscr{G}^t(\omega) \left(1 - \prod_{f \in \Gamma} \chi_f(\omega)\right)$$

and we prove the result for these two terms. Let $\eta(s) = sy' + (1 - s)y$ and $p^A(t; x, y) = \int dP_{xy}^t(\omega) \mathscr{G}^t(\omega)$.

(7) Remember $C_\phi(x, y; \alpha, \beta, \mu^2) = \int_\alpha^\beta dt e^{-t\mu^2} \int dP_{xy}^t(\omega)$; $d_B(y) = \mathrm{dist}(y, \Delta)$, $y \in \Delta \in \mathscr{D}_0$.

i) By the mean-value theorem we bound

$$| D_\phi(x, y) - D_\phi(x, y') | \leqslant \sup_{0 \leqslant s \leqslant 1} | y - y' | \, | \nabla_\eta D_\phi(x, \eta) |$$

and compute the derivative using (s. e. g. [*12*])

$$(3.2) \quad \omega(\tau) = \left(1 - \frac{\tau}{t}\right)x + \frac{\tau}{t}y + \sqrt{t}\alpha\left(\frac{\tau}{t}\right) \text{ (eq. in sense of prob. distributions)}$$

where $\alpha(\sigma)$, $0 \leqslant \sigma \leqslant 1$, is a Gaussian process with values in $\mathbb{R}^3$ of mean zero and covariance

$$E_\alpha(\alpha_\mu(\sigma)\alpha_\nu(\tau)) = \delta_{\mu\nu}\sigma(1 - \tau), \quad 0 \leqslant \sigma \leqslant \tau \leqslant 1, \quad \underbrace{1 \leqslant \mu, \nu \leqslant 3}$$

Thus

$$| \nabla_\eta p^A(t; x, \eta) |$$

$$\overset{(8)}{=} \left| \nabla_\eta E_\alpha \left\{ \exp\left(ie \int_0^t A\left(\left(1 - \frac{\tau}{t}\right)x + \frac{\tau}{t}\eta + \sqrt{t}\alpha\left(\frac{\tau}{t}\right)\right) \cdot \left(\frac{\eta - x}{t} d\tau + \sqrt{t} d\alpha\left(\frac{\tau}{t}\right)\right)\right) \right\} p(t; x, \eta) \right|$$

$$\overset{(9)}{\leqslant} p(t, x, \eta)\left(\frac{| x - \eta |}{2t} + \left| \nabla_\eta E_\alpha \left\{ \exp\left(ie \int_0^1 A(\beta) \cdot ((\eta - x)d\sigma + \sqrt{t} d\alpha(\sigma))\right) \right\} \right| \right.$$

$$\leqslant p(t, x, \eta)\left(\frac{| x - \eta |}{2t} + \sqrt{t}\left(E_\alpha\left(\left| \nabla_\eta \int_0^1 A(\beta) \cdot d\alpha(\sigma) \right|^2 \right)\right)^{1/2} \right.$$

$$(3.3) \qquad \left. + E_\alpha\left(\left| \nabla_\eta \int_0^1 A(\beta) \, . \, (\eta - x) d\sigma \right| \right)\right)$$

using $| \mathscr{G}^t(\cdot) | = 1$ and Schwarz' inequality for $E_\alpha(\cdot)$. Now

$$(3.4) \quad \left(E_\alpha\left(\left| \nabla_{\eta,\mu} \int_0^1 A(\beta) \cdot d\alpha(\sigma) \right|^2 \right)\right)^{1/2}$$

$$\leqslant \left(E_\alpha\left(\int_0^1 | (\partial_\mu A)(\beta) |^2 d\sigma \right)\right)^{1/2}$$

$$+ \left(E_\alpha\left(\left(\int_0^1 \frac{\sigma}{2}\left(\frac{1}{\sigma} + \frac{1}{1 - \sigma}\right) | (\partial_\mu A)(\beta) | \, | \alpha(\sigma) | \, d\sigma \right)^2 \right)\right)^{1/2}$$

by a modification of an inequality given by Simon [*12*; chap. 14] for the symmetrized Ito-integral.

---

(8) $p(t, x, \eta) = (4\pi t)^{-3/2} \exp\left(- \frac{| x - \eta |^2}{4t} \right)$.

(9) Let $\beta := (1 - \sigma)x + \sigma\eta + \sqrt{t}\alpha(\sigma)$, $0 \leqslant \sigma \leqslant 1$.

The expectations on the r. h. s. of 3.4 can be easily computed and are estimated, using Hölder's inequality by $0(1)t^{-3/2p} \| \partial_\mu A \|_p$, $p > 3$.

Using the fact that a. e. α is continuous, one extends the bound to the case $\partial_\mu A_\nu \in L^p_{\mathrm{loc}}$, $p > \nu$.

The third term on the r. h. s. of (3.3) is bounded using

$$
\begin{aligned}
(3.5)\quad & E_\alpha\left(\left| \nabla_{\eta,\mu} \int_0^1 A(\beta)\cdot(\eta - x)d\sigma \right| \right) \\
& \leqslant \sum_{\nu=1}^{3} E_\alpha\left(\left| \int_0^1 d\sigma\sigma(\partial_\mu A_\nu)(\beta)(\eta - x)_\nu \right| \right) + E_\alpha\left(\left| \int_0^1 d\sigma A_\mu(\beta) \right| \right) \\
& \leqslant 0(1) \sup_\nu \left\{ |\eta - x| \int_0^1 d\sigma\sigma E_\alpha(|\partial_\mu A_\nu(\sqrt{t}\alpha(\sigma))|) + \int_0^1 d\sigma E_\alpha(|A_\mu(\sqrt{t}\alpha(\sigma))|) \right\}
\end{aligned}
$$

and the explicit expressions for the expectations together with Hölder's inequality give the bound

$$0(1)(|\eta - x| t^{-3/2p} + t^{-3/2q})$$

for $A \in \mathscr{A}$, $p > 3$, $q > 3$.

Thus

$$
\begin{aligned}
(3.6)\quad & |\nabla_\eta p^A(t, x, \eta)| \\
& \leqslant 0(1)(t^{-3/2q} + t^{1/2-3/2p} + t^{-3/2p}|\eta - x| + t^{-1}|\eta - x|)p(t, x, \eta) \\
& \leqslant 0(1)(t^{-3/q} + t^{-1/2} + t^{1/2-3/2p})p\left(\frac{4}{3}t, x, \eta\right) \\
& \leqslant 0(1)t_m^{-1/2}(1 + t^{1/2})p\left(\frac{4}{3}t, x, n\right)
\end{aligned}
$$

since $t \in [t_m, t_M]$, $0 < t_m < 1$, and $x^{\alpha/2}e^{-\frac{1}{4}x2} \leqslant 0(1)$, $0 \leqslant \alpha \leqslant 1$. Finally

$$
\begin{aligned}
(3.7)\quad |\nabla_\eta D_\phi(x, \eta)| & \leqslant 0(1)t_m^{-1/2} \int_{t_m}^{t_M} dt e^{-tm_{\frac{0}{2}}^2}(1 + t^{1/2})p\left(\frac{4}{3}t, x, \eta\right) \\
& \leqslant 0(1)t_m^{-1/2} \int_{t_m}^{t_M} dt e^{-t\frac{m_0}{2}} p\left(\frac{4}{3}t, x, \eta\right) \\
& \leqslant 0(1)t_m^{-1/2} C_\phi\left(x, \eta\,; t_m, 2t_M, \frac{m_0^2}{4}\right)
\end{aligned}
$$

which proves the result for the first term on the r. h. s. of (3.1).

ii) Consider the second term in (3.1).

Due to the Markoff-property of the Wiener process and the factorization of $\mathscr{G}^t(\omega)$, we may use the method of first hitting times, as in e. g. [*2*] [*13*].

We write

$$(3.8)\quad \left| \nabla_\eta \int dP^t_{x\eta}(\omega)\mathcal{G}^t(\omega)(1 - \chi_\Gamma(\omega) \right| = \left| \int_{0 \leq t_1 \leq t} E_x \{ \chi(\tau_\Gamma \in dt_1)\mathcal{G}^{t_1}(\omega)\nabla_\eta p^A(t - t_1, \omega(t_1), \eta) \} \right|$$

where τ_Γ is the first hitting time and E_x the usual Wiener expectation of paths starting at x. Using the inequality (3.6) the r. h. s. of (3.8) is bounded by

$$0(1)\left| \int E_x \{ \chi(\tau_\Gamma \in dt_1)((t - t_1)^{-3/2q} + (t - t_1)^{-1/2} + (t - t_1)^{1/2-3/2p}) \} p\left(\frac{4}{3}(t - t_1, \omega(t_1), \eta)\right) \right|$$

$$\leqslant 0(1)\left| \int E_x \{ \chi(\tau_\Gamma \in dt_1)(\mathrm{dist}^{-3/2q}(\eta, \Gamma) + \mathrm{dist}^{-1}(\eta, \Gamma) + t^{1/2-1/2p}) \} p(2(t - t_1), \omega(t_1), \eta)) \right|$$

$$\leqslant 0(1)\left| \int E_x \{ \chi(\tau_\Gamma \in dt_1)(d_B^{-1}(\eta) + t^{1/2-3/2p})p(2(t - t_1), \omega(t_1), \eta) \} \right|$$

since $d_B(\eta) < 1$.

Thus

$$\left| \nabla_\eta D_{\Gamma^c}(x, \eta) \right| \leqslant 0(1) \left| \int_{t_m}^{t_M} dt e^{-tm_0^2}(d_B^{-1}(\eta) + t^{1/2-3/2q}) \cdot \int_{0 \leqslant t_1 \leqslant t} E_x \{ \chi(\tau_\Gamma \in dt_1)p(2(t - t_1), \omega(t_1), \eta) \} \right|$$

$$\leqslant 0(1)(d_B^{-1}(\eta) + 1) \cdot \left| \int_{t_m}^{t_M} dt e^{-t\frac{m_0^2}{2}} \int E_x \{ \chi(\tau_\Gamma \in dt_1)p(2(t - t_1), \omega(t_1), \eta) \} \right|$$

It is easy to see that

$$\left| \int_{t_m}^{t_M} dt e^{-tm_0^2} \int_{0 \leqslant t_1 \leqslant t} E_x \{ \chi(\tau_\Gamma \in dt_1)p(2(t - t_1), \omega(t_1), \eta) \} \right| \leqslant \left\| \int_{t_m}^{2t_M} dt e^{-t\frac{m_0^2}{4}} \int dP^t_{xy}(\omega)(1 - \chi_\Gamma(\omega) \right|$$

so that finally $(t_m^{-1/2} > 1)$

$$(3.9)\quad | \nabla_\eta D_{\Gamma^c}(x, \eta) | \leqslant 0(1)(t_m^{-1/2} + d_B(y)^{-1} + d_B^{-1}(y')) \cdot \left[C_\phi\left(x, y ; t_m, 2t_M, \frac{m_0^2}{4}\right) + C_\phi\left(x, y' ; t_m, 2t_M, \frac{m_0^2}{4}\right) \right]$$

proving the proposition.

With essentially the same methods one proves

PROPOSITION 3.3. —

$$| D_\Gamma(x, y) - D_\Gamma(x', y) - D_\Gamma(x, y') + D_\Gamma(x', y') | \leqslant 0(1) | y - y' | | x - x' | \max_{\substack{u \in \{x,x'\} \\ v \in \{y,y'\}}} \left\{ (t_m^{-1/2} + d_B^{-1}(u))(t_m^{-1/2} + d_B^{-1}(v)) \cdot C_\phi\left(u, v; t_m, 2t_M, \frac{m_0^2}{4}\right) \right\}$$

provided $A \in \mathscr{A}$.

Remark. — The estimate of proposition 3.2 is needed to control the mass renormalization cancellations, while proposition 3.3 is needed for the contraction of two $\delta\varphi$ legs; see lemma 5.2 of [2].

One now proceeds in the standard way [1] [2] [3] [7]. First one defines « divergent » and « convergent » vertices (subscript D, C resp.) and associates divergent with convergent vertices as in [7] (10). Then one decomposes R_v into subgraphs according to the association and finally decomposes the subgraphs into elementary subgraphs as in [1] and [3].

Then by Kato's inequality and propositions 3.2 and 3.3 all our elementary subgraphs are estimated by lemma 5.2 of Feldman and Osterwalder yielding.

PROPOSITION 3.4. — For v sufficiently small and m_0 large enough, there exist $\varepsilon_7 > 0$, such that for t_1 small enough, R_v is bounded by a product of the following factors:

$\| G \|_{\delta,\alpha}$	per copy of G, for some $\delta, \alpha > 0$
$t_i^{\varepsilon_7}$	per $P_{\alpha,C}$, or $P_{\alpha,D}$ vertex associated with G (of index (i, Δ))
$\| \Delta \|^{\varepsilon_7}$	per $P_{\beta,C}$ vertex (of index (i, Δ))
$\ln t_i^{-1}$	per $P_{\alpha,D}$, $P_{I,D}$ (not ass. with G), C_D or E vertex of high momentum t_i
$d_s^{-n_4}(\Delta, \Delta')$	per contraction from P/C in Δ to P/C in Δ', not both legs of type $\delta\varphi$
$\text{dist}^{-n_4}(\Delta, \Delta')$	per contraction of a field in Δ to a field in Δ' belonging to an E vertex or to G
$b^{-1}(\Delta, \Delta')$	per contraction of $\delta\varphi_\Delta$ to $\delta\varphi'_{\Delta'}$
$e^{-2K_{11}d(\Delta,\Delta',\gamma) - K_{12}\|\gamma\|}$	per $\partial^\gamma D$ line joining Δ and Δ'

n_4 can be taken as large as we want (for t_1 small enough).

Lemma 3.1 is now proved by showing how to compensate the logarhythmic divergencies. We confine ourselves to the case of $P_{\alpha,D}$ vertices, for the other cases see [2] [7] [9].

Consider a $P_{\alpha,D}$ of index (i, Δ), $\Delta \in \mathscr{D}_l$, with divergence $\ln t_j^{-1}$, $j > i$.

(10) We remind that there are at most $72 P_{\alpha,D}$ associated with one $P_{\alpha,C}$.

We replace divergencies of this type by a divergent factor $(\ln t_j^{-1})^j$ per P_α vertex of index (j, Δ) [7]. An arbitrarity small fraction of the convergence factor $t_j^{\varepsilon_7}$ of $P_{\alpha,C}$ vertices dominates this factor, leaving $P_{\alpha,C}$ vertices with a factor $t_j^{\varepsilon_8}$, $\varepsilon_7 > \varepsilon_8 > 0$.

Now we replace the divergence $(\ln t_j^{-1})^j$ per $P_{\alpha,D}$ by $t_j^{-\varepsilon_6}$ (s. lemma 3.1 above) and choose $\varepsilon_6 < \varepsilon_8\,(2\cdot 13\cdot 72)^{-1}$ for t_1 small enough, and consider two cases:

a) $1 = 0$

Each $P_{\alpha,D}$ is associated with a $P_{\alpha,C}$ vertex, of index (k, Δ'), say, $\Delta' \in \mathscr{D}_0$, and there are at most 72 divergent vertices associated with it. By definition we have $k \geqslant j$. We use $t_k^{\varepsilon_8/2}$ of the convergence factor of the $P_{\alpha,C}$ to compensate the divergencies by

$$(t_j^{-\varepsilon_6})^{72} t_k^{\varepsilon_8/2} \leqslant (t_j^{12\varepsilon_6})^{72}$$

thereby leaving each P vertex in a unit cube with a factor bounded by $t_j^{12.\varepsilon_6}$.

b) $l \neq 0$

We bound the number of P_α « contained » in each unit cube. By construction, in each cube $\Delta_l \in \Lambda_l \cap X$, $l \geqslant 1$ and (j, Δ) of type α, there are at most $A(j, \Delta_l) \leqslant t_j^{-\nu/2} P_\alpha$ vertices. The number of cubes in Δ_0, $|\Delta_0| = 1$, of smallest size so that (j, Δ) is of type α, is bounded by $|\Delta_{\alpha(j)}|^{-1} \leqslant t_j^{-\nu/4}$. Furthermore the number of contributing covers is bounded as

$$\sum_{k:|\Delta_k|^{-1} \leqslant t_j^{-\nu/4}} 1 \leqslant \sum_{k:t_k^{-1} \leqslant t_j^{-\nu/4}} 1 \leqslant j \leqslant t_j^{-\nu/4}$$

Thus the total number of P_α vertices localized in Δ_l, $l \geqslant 1$ contained in Δ_0 is bounded by $t_j^{-\nu}$. Thus we use $t_j^{6\varepsilon_6}$ of the $t_j^{-\nu}$ generated P_α vertices of index (j, Δ_0) to compensate the divergencies, leaving each P_α vertex with a convergence factor $t_j^{5\varepsilon_6}$.

One uses $t_j^{4\varepsilon_6}$ of this factor for the other cases.

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