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A low temperature expansion for the pseudoscalar Yukawa model of quantum fields in two space-time dimensions

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Section A :
Physique théorique.

A low temperature expansion for the pseudoscalar Yukawa model of quantum fields in two space-time dimensions

by

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ABSTRACT. — It is shown that the pseudoscalar Yukawa model $\lambda\bar{\psi}i\gamma_5\psi\varphi$ of quantum field theory in two space-time dimensions possesses a massive phase with spontaneous breakdown of the $\varphi \to -\varphi$ symmetry. The phase is analyzed in the Euclidean region by means of combined convergent Peierls and cluster expansions.

RÉSUMÉ. — On démontre que le modèle de Yukawa $\lambda\bar{\psi}i\gamma_5\psi\varphi$ de théorie quantique des champs en deux dimensions d'espace-temps possède une phase massive avec brisure spontanée de la symmétrie $\varphi \to -\varphi$. La phase est analysée dans la région euclidienne par un développement convergent qui combine le développement de Peierls avec un développement en « clusters ».

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CHAPTER I

INTRODUCTION

The recent years have witnessed a fast development of the theory of superrenormalizable quantum field models in two and three space-time dimensions. We have learnt not only how to construct models but also how to investigate their basic properties. Here special attention was attracted by the phenomenon of phase transition [*12*] [*13*] [*4*] [*5*] [*6*] [*7*] [*2*] [*1*] [*8*] [*20*]. In the present paper we study the problem of phase transitions for pseudo-scalar two-dimensional Yukawa model. This model, but not the scalar one, possesses the boson field $\varphi \to -\varphi$ symmetry. Existence of a phase transition accompanied by the spontaneous breakdown of this symmetry was noticed in [*4*]. Here, by the appropriate choice of boundary conditions, we construct two different phases for the model, coexisting in the region of broken symmetry. All Wightman axioms are proven for each of the phases. Our work is a continuation of the main stream of papers applying Euclidean methods to the Yukawa model [*23*] [*24*] [*25*] [*26*] [*17*] [*18*] [*21*] which culminated in the high temperature cluster expansion providing a rich information about the one-phase region [*3*] [*16*] [*22*]. Inspired by the paper [*13*] by Glimm-Jaffe-Spencer on the two-phase region $(\varphi^4)_2$ model we extend their low temperature expansion to the present case.

On the formal level the Schwinger functions for the pseudoscalar Yukawa model are given by the Matthews-Salam formula with the fermion field integrated out

$$S((f_i)_{i=1}^{I}, (g_j)_{j=1}^{J}, (h_j)_{j=1}^{J}) = \frac{1}{Z}\int \prod_i \varphi(f_i) \det_{j,k}(g_j \mid (1 - \lambda S\Gamma\varphi)^{-1} Sh_k)_{L^2} \cdot \det(1 - \lambda S\Gamma\varphi) e^{\lambda \mathrm{Tr}(S\Gamma\varphi) + \frac{1}{2}(\mu^2 + \delta\mu^2)\int\varphi^2 - \frac{1}{2}\int(\nabla\varphi)^2} \prod_x d\varphi(x), \quad (1)$$

where Z is the normalization factor,

$$S = (\not{P} + m)(P^2 + m^2)^{-1}, \qquad P = \frac{1}{i}\frac{\partial}{\partial x}, \quad (2)$$

$$\delta\mu^2 = -\frac{2}{(2\pi)^2}\int \frac{dp}{p^2 + m^2} \quad (3)$$

is the infinite boson mass counterterm, $\lambda \mathrm{Tr}(S\Gamma\varphi)$ is the fermion Wick ordering counterterm (it vanishes in fact), $\Gamma = i\gamma_5$, $\mu^2 > 0$, compare [*23*].

To get the insight into the problem of phase transitions we compute the effective potential V in the one-loop approximation. Explicit computation gives

$$V(\varphi) = \frac{m^2}{4\pi}\left[\left(1 + \frac{\lambda^2\varphi^2}{m^2}\right)\ln\left(1 + \frac{\lambda^2\varphi^2}{m^2}\right) - \frac{\lambda^2\varphi^2}{m^2}\right] - \frac{1}{2}\mu^2\varphi^2 . \quad (4)$$

The shape of V is sketched on fig. 1

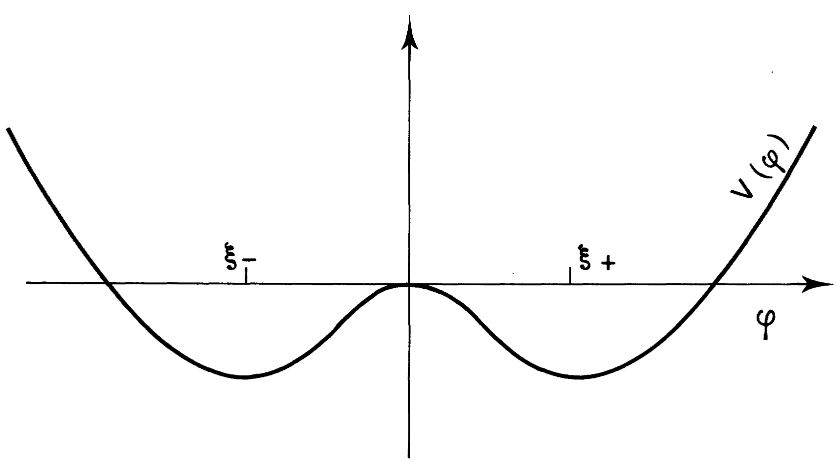

FIG. 1.

V has two symmetric minima at $\xi_{\pm} = \pm \frac{m}{\lambda}\left(e^{\frac{2\pi\mu^2}{\lambda^2}} - 1\right)^{\frac{1}{2}}$ separated by a barrier whose height is given by $|V(\xi_{\pm})|$, where

$$V(\xi_{\pm}) = -\frac{m^2}{4\pi}\left(e^{\frac{2\pi\mu^2}{\lambda^2}} - 1 - \frac{2\pi\mu^2}{\lambda^2}\right).$$

The minima get deeper and more separated when $m \to \infty$. The curvature at the minima (classical mass squared) $m_c^2 := V''(\xi_{\pm}) = \frac{\lambda^2}{\pi}\left(1 - e^{-\frac{2\pi\mu^2}{\lambda^2}}\right)$ does not depend on m. The conventional wisdom about phase transitions suggests existence of two pure phases interrelated by the $\varphi \to -\varphi$ symmetry for large m. Our method allows for showing this to be true provided that additionally λ is taken large and $m \geqslant m_0(\lambda)$. The restriction on λ seems to be a technical one connected with the way in which we do the low-temperature expansion.

Before a rigorous form of (1) is stated, let us perform some heuristic transformations expanding the action in (1) around the minimal value $\varphi = \xi_+$. For $\mathscr{K}$ being a linear mapping from functions to operators write

$$\mathscr{V}(\varphi) = -\ln \det_2 (1 - \lambda\mathscr{K}(\varphi)) - \frac{\lambda^2}{2}\int \mathscr{B}^{\xi_+}\varphi^2 , \tag{5}$$

where

$$\mathscr{B}^{\xi_+}(x) := \frac{1}{2}\frac{\delta}{\delta\varphi(x)} \operatorname{Tr}\left(\mathscr{K}(\varphi)(1 - \lambda\mathscr{K}(\xi_+))^{-1}\mathscr{K}(\varphi)\right)\restriction_{\varphi=1} \tag{6}$$

and

$$\det_n (1 - A) = \det\left((1 - A)e^{\sum_{k=1}^{n=1}\frac{1}{k}A_k}\right). \tag{7}$$

If we choose $\mathscr{K}(\varphi) = S\Gamma\varphi$ then it is easy to check that formally

$$\lambda^2\mathscr{B}^{\xi_+}(x) = \mu^2 + \delta\mu^2$$

and

$$\mathscr{V}(\varphi) = -\ln\det(1 - xS\Gamma\varphi) - \lambda\operatorname{Tr}(S\Gamma\varphi) - \frac{1}{2}(\mu^2 + \delta\mu^2)\int\varphi^2 \tag{8}$$

is the effective boson field action for (1).

We have

$$\begin{aligned}\mathscr{V}(\varphi) &= -\ln\det(1 - \lambda\mathscr{K}(\varphi)) - \lambda\operatorname{Tr}\mathscr{K}(\varphi) - \frac{\lambda^2}{2}\int\mathscr{B}^{\xi_+}\varphi^2\\ &= -\ln\det(1 - \lambda\mathscr{K}(\xi_+)) - \ln\det(1 - \lambda\mathscr{K}^{\xi_+}(\varphi - \xi_+)) - \lambda\operatorname{Tr}\mathscr{K}(\varphi)\\ &\qquad - \frac{\lambda^2}{2}\int\mathscr{B}^{\xi_+}\varphi^2 ,\end{aligned} \tag{9}$$

where

$$\mathscr{K}^{\xi_+}(f) := (1 - \lambda\mathscr{K}(\xi_+))^{-1}\mathscr{K}(f). \tag{10}$$

Thus

$$\begin{aligned}\mathscr{V}_0(\varphi) &:= \mathscr{V}(\varphi) - \mathscr{V}(\xi_+)\\ &= -\ln\det(1 - \lambda\mathscr{K}^{\xi_+}(\varphi - \xi_+)) - \lambda\,\mathrm{Tr}(\mathscr{K}(\varphi - \xi_+))\\ &\quad - \frac{\lambda^2}{2}\int \mathscr{B}^{\xi_+}(\varphi - \xi_+)^2 - \lambda^2\int \mathscr{B}^{\xi_+}\xi_+(\varphi - \xi_+)\\ &= -\ln\det{}_2(1 - \lambda\mathscr{K}^{\xi_+}(\varphi - \xi_+)) - \frac{\lambda^2}{2}\int \mathscr{B}^{\xi_+}(\varphi - \xi_+)^2 \qquad (11)\end{aligned}$$

since

$$\lambda\,\mathrm{Tr}\,\mathscr{K}^{\xi_+}(\varphi - \xi_+) - \lambda\,\mathrm{Tr}\,\mathscr{K}(\varphi - \xi_+) - \lambda^2\int \mathscr{B}^{\xi_+}\xi_+(\varphi - \xi_+) = 0.$$

Now let $\mathrm{D} := (\mathrm{P}^2 + m^2)^{\frac{1}{2}}$. With $\mathscr{K}(\varphi) = \mathrm{D}^{\frac{1}{2}}\mathrm{S}\Gamma\varphi\mathrm{D}^{-\frac{1}{2}} =: \mathrm{K}(\varphi)$ rather than $\mathrm{S}\Gamma\varphi$, which is equivalent to considering the latter as an operator in the Sobolev space $\mathrm{H}^{\frac{1}{2}}$, not in L^2, we shall use $\mathscr{V}_0(\varphi)$ rather than $\mathscr{V}(\varphi)$ as the action in (1) (subtraction of the classical energy of the ground state). To make things well defined we introduce a volume cut-off substituting $\varphi \to \varphi\Lambda + \xi_+(1 - \Lambda)$.

Λ is the volume cut-off function. The $\xi_+(1 - \Lambda)$ tail will agree with the boundary condition which will press φ to stay near the ξ_+ minimum outside the support of Λ. For the sake of definiteness we choose Λ to be of the form

$$\Lambda(x) = \int_{\tilde{\Lambda}} dy k(x - y) \qquad (12)$$

where $k \in \mathrm{C}_0^\infty(\mathbb{R}^2)$, $0 \leqslant k \leqslant 1$, $\int k = 1$, is a fixed spherically symmetric function and $\tilde{\Lambda}$ is any arbitrarily situated square in $\mathbb{R}^2$.

After the formal operations described above our effective action becomes

$$-\ln\det{}_2(1 - \lambda\mathrm{K}^{\xi_+}((\varphi - \xi_+)\Lambda)) - \frac{\lambda^2}{2}\int \mathrm{B}^{\xi_+}((\varphi - \xi_+)\Lambda)^2 \qquad (13)$$

where

$$\mathrm{B}^{\xi_+}(x) = \frac{1}{2}\frac{\delta}{\delta\varphi(x)}\mathrm{Tr}(\mathrm{K}(\varphi)\mathrm{K}^{\xi_+}(\varphi)) \upharpoonright_{\varphi=1}. \qquad (14)$$

Exposing the second order terms in φ, we may represent (13) as

$$\begin{aligned}&-\ln\det{}_3(1 - \lambda\mathrm{K}^{\xi_+}((\varphi - \xi_+)\Lambda))\\ &\quad + \frac{\lambda^2}{2}\mathrm{Tr}(\mathrm{K}^{\xi_+}((\varphi - \xi_+)\Lambda)^2) - \frac{\lambda^2}{2}\int \mathrm{B}^{\xi_+}((\varphi - \xi_+)\Lambda)^2. \qquad (15)\end{aligned}$$

Next using the notation of [3] we write

$$\det{}_{j,k}(g_j \mid (1 - \lambda\mathrm{S}\Gamma(\varphi\Lambda + \xi_+(1 - \Lambda)))^{-1}\mathrm{S}h_k)_{\mathrm{L}^2}$$

as

$$\det_{j,k}\left(D^{-\frac{1}{2}}g_j \,|\, (1 - \lambda K^{\xi_+}((\varphi-\xi_+)\Lambda))^{-1}(1-\lambda K(\xi_+))^{-1}D^{\frac{1}{2}}Sh_k\right)$$
$$= \tau_J\left(\Lambda^J(1 - \lambda K^{\xi_+}((\varphi - \xi_+)\Lambda))^{-1} \bigwedge_{j=1}^{J}(1 - \lambda K(\xi_+))^{-1}P_j\right) \quad (16)$$

where

$$\tau_J := J\,!\ \mathrm{Tr}_{\Lambda^J L^2(\mathbb{R}^2)} \quad (17)$$

and

$$P_j := |\, D^{\frac{1}{2}}Sh_j \rangle \langle D^{-\frac{1}{2}}g_j \,| . \quad (18)$$

The appropriate boundary condition which, as mentioned above, makes φ stay near the ξ_+ minimum outside the support of Λ, is chosen as in [13] by using

$$e^{\frac{1}{2}m_c^2\int((\varphi-\xi_+)\Lambda)^2}d\mu_{m_c}^{\xi_+} \quad (19)$$

instead of $\exp\left[-\frac{1}{2}\int(\nabla\varphi)^2\right]\prod_x d\varphi(x)$. $d\mu_{m_c}^{\xi_+}$ is the Gaussian measure with mean ξ_+ and covariance $(-\Delta + m_c^2)^{-1}$.

The final step in our formal derivation of the formula defining the volume cut-off Schwinger functions of the considered model is the Wick ordering with respect to square mass m_c^2 of the terms quadratic in φ appearing under the exponential function in the version of (1) obtained by performing all the transformations described previously. This way we arrive at the following expression for the unnormalized Schwinger functions

$$ZS_\Lambda((f_i)_{i=1}^{I}, (g_j)_{j=1}^{J}, (h_j)_{j=1}^{J})$$
$$= \int\prod_i \varphi(f_i)\tau_J(\Lambda^J(1-\lambda K^{\xi_+}((\varphi-\xi_+)\Lambda))^{-1}\bigwedge_{j=1}^{J}(1-\lambda K(\xi_+))^{-1}P_j\Big).$$
$$\cdot \det_3(1 - \lambda K^{\xi_+}((\varphi - \xi_+)\Lambda)\exp\left[-\frac{\lambda^2}{2}:\mathrm{Tr}\,K^{\xi_+}((\varphi-\xi_+)\Lambda)^2:\right].$$
$$\cdot\exp\left[\frac{\lambda^2}{2}\int B^{\xi_+}:((\varphi-\xi_+)\Lambda)^2: + \frac{1}{2}m_c^2\int:((\varphi-\xi_+)\Lambda)^2:\right]d\mu_{m_c}^{\xi_+}. \quad (20)$$

This formal expression may be easily given a precise meaning essentially the same way as in [17] [18] [23] [24] [25].

$K^{\xi_+}((\varphi - \xi_+)\Lambda)$ may be considered as a random variable with values in operators in $L^2(\mathbb{R}^2)$ possesing trace with the power $2 + \varepsilon$, see the estimates of Appendices I and II. Thus the first line under the integral in (20) is a well defined random variable (we consider τ_J and $\det_3$ jointly, see [27]). The Wick ordered quadratic form in the exponent is again a well defined random variable, when considered jointly. The existence of the integral in (20) (together with detailed bounds on it) follows from the estimates of Chapter V. In the future we shall be slightly careless in transforming the Wick ordered quadratic terms of the action. These transformations may be easily substantiated once all the quadratic terms are put together.

Our main result is contained in the following.

THEOREM I.1. — There exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ and all $m \geqslant m_0(\lambda)$ the finite volume Schwinger functions $S_\Lambda := ZS_\Lambda/Z_\Lambda$ converge in the space D′ when $\Lambda \to 1$ and the limiting infinite volume Schwinger functions satisfy all the Osterwalder-Schrader axioms [*19*] with exponential clustering included.

As we mentioned above this result is obtained by a low temperature expansion patterned on that worked out by Glimm, Jaffe and Spencer in [*13*]. The main difficulty in applying their method to our case is the non-locality of the effective action $\mathscr{V}(\varphi)$, which makes the separation of low-momentum part of the action from the high-momentum fluctuations technically more difficult.

The phase we construct developes a non-zero expectation of the boson field φ which is easily seen from the estimates we prove. Thus a spontaneous breakdown of the $\varphi \to -\varphi$ symmetry is realized. The other phase is obtained by replacing ξ_+ by ξ_- in (20). For definiteness we consider the case of ξ_+ only. Moreover we assume λ and m to be positive and bounded away from zero.

The paper consists of six chapters. After Introduction we describe a general formalism of the low temperature expansion patterned on [*13*], but adapted to our model, with improvements along the ideas of Kunz and Souillard [*15*]. Chapter II contains also statements of the main theorems. Their proofs are reduced to proof of estimates for a general term of the low temperature expansion. Chapter III contains a combinatoric analysis of such a general term. There it is shown that the needed estimates result from three technical theorems stating bounds which we call a gaussian integration estimate, a lower linear bound and an upper bound respectively. These theorems are proven in the last three chapters. Some technical estimates used throughout the body of the paper were gathered in two Appendices.

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CHAPTER II

THE EXPANSION

The method applied to prove Theorem I.1 is a combination of the low temperature expansion in phase separation contours and of the high temperature cluster expansion, as used by Glimm-Jaffe-Spencer (G-J-S) in [13] to prove similar statements for $\left(\lambda\varphi^4 - \frac{1}{4}\varphi^2 + (64\lambda)^{-1}\right)$ field which also possesses the $\varphi \to -\varphi$ symmetry. Let us remind the main ideas of the G-J-S expansion. The classical potential $\lambda\varphi^4 - \frac{1}{4}\varphi^2 + (64\lambda)^{-1}$ had two minima at $\xi_\pm = \pm(8\lambda)^{-\frac{1}{2}}$ with value zero and a maximum at zero with value $(64\lambda)^{-1}$, hence had two symmetric wells getting deeper and more separated when $\lambda \to 0$, with the curvature at the minima (classical mass squared) being constant. The first step in the expansion for unnormalized Schwinger functions in volume Λ ZS_Λ was to insert into the interacting Euclidean measure a partition of unity. Each function of the partition restricted the averages of the field over squares of the unit lattice to stay within a definite ($+$ or $-$) potential well. This way ZS_Λ was expressed as a sum of terms $ZS_{\Lambda,\Sigma}$, each connected with an Ising configuration Σ. Σ assigned to each square of the unit lattice inside Λ plus or minus sign and plus sign to each square outside Λ. Each Σ could also be labeled by the contour made up of lattice bounds separating the plus and minus sign regions. Within the plus or minus sea the term of the partitioned interacting measure was close to a Gaussian of positive mass centered around $\varphi = \xi_\pm$, with deviations getting small when λ went to zero. Across the contour the field was forced to change its average by an amount of order $2\xi_+$ and the non-local gradient term $\exp\left(-\frac{1}{2}\int(\nabla\varphi)^2\right)$ together with the local potential term providing the barrier between the minima conspired producing a damping factor $\exp(-C\cdot\text{contour length})$, with C growing to infinity for λ getting small. This damping effect was exhibited via translation of φ by its approximate mean g being close to $\xi_\pm$ within the $\pm$ region of the Ising configuration. After translation each term of the interacting measure became close to a mean zero positive mass Gaussian in the regions far from the phase separation contours and in the next step its relatively mild non-locality, all the time due to the gradient term, was coped by use of an expansion step-wise introducing Dirichlet bonds similarly as in the standard high temperature cluster expansion of G-J-S [11]. Across a closed Dirichlet line introduced into a measure term the cluster expansion for $ZS_{\Lambda,\Sigma}$ factored, each factor depending only on the Ising variables within its respective region. Thus for a fixed closed Dirichlet line the sum over the Ising configurations also factored across the line. This was very important since it enabled a partial resummation of the expansions and finally the proof of uniform in Λ convergence of the resulting expansion for the normalized Schwinger functions.

We would like to stress that the G-J-S method was based on an interplay between the shape of local potential and the short range (gradient type) non-locality provided by the kinetic term of the action. The field liked to sit around one of the minima of the potential. The non-locality damped strongly the possibility that it fluctuated much between the two minima having the term where φ was sitting in the potential well determined by the boundary condition dominant. Once the field was made to stay in prescribed wells and the damping effect was taken into account other effects of non-locality were small and could be handled by the standard cluster expansion.

The situation we encounter in the pseudo-scalar $(Y)_2$ model is very similar to that of the model considered by G-J-S. The one-loop potential has a similar shape to the classical potential of the pure boson case: two wells get deeper and more separated when $m \to \infty$, the curvature at the minima being $0(1)$. There is also a gradient non-local term in the measure coming from the kinetic boson term of the action. A new factor is an additional non-locality in the effective action which moreover has a long distance tail.

The first step in the expansion is similar as in [13]. Write

$$ZS_\Lambda = \sum_\Sigma ZS_{\Lambda,\Sigma}, \tag{1}$$

where Σ maps the set of Δ squares of a lattice whose diameter d will be chosen later independently of λ and m into $\{+, -\}$, $\Sigma = +$ for squares not meeting the support of Λ

$$\begin{aligned} ZS_{\Lambda,\Sigma} &= \int \prod_\Delta \chi_{\Sigma(\Delta)}(\varphi_\Delta^\Lambda) \prod_{i=1}^{I} \varphi(f_i)\tau_J\Big(\Lambda^J(1-\lambda K^{\xi_+}((\varphi-\xi_+)\Lambda))^{-1} \cdot \bigwedge_{j=1}^{J}(1-\lambda K(\xi_+))^{-1}P_j\Big) \\ &\cdot \det_3(1-\lambda K^{\xi_+}((\varphi-\xi_+)\Lambda))e^{-\frac{\lambda^2}{2}:\mathrm{Tr}\,K^{\xi_+}((\varphi-\xi_+)\Lambda)^2:} \\ &\cdot e^{\frac{\lambda^2}{2}\int B^{\xi_+}:((\varphi-\xi_+)\Lambda)^2: + \frac{1}{2}m_c^2\int\Lambda:((\varphi-\xi_+)\Lambda)^2:} d\mu_{m_c}^{\xi_+} \end{aligned} \tag{2}$$

$\chi_\pm$ are smeared indicator functions of $[0, \pm\infty[$ respectively, whose exact shape (differing from that chosen in [13]) will be specified later.

$$\varphi_\Delta^\Lambda := \frac{1}{|\Delta|}\int_\Delta (\varphi\Lambda + \xi_+(1-\Lambda)). \tag{3}$$

We shall also assume that each of functions f_i, g_j, h_j is localized in a d-lattice square.

Let us have a more careful look at the nature of the effective action non-locality. Note that (formally)

$$\mathscr{V}(\varphi) = 0 + \frac{1}{2}\int \frac{\delta^2\mathscr{V}(0)}{\delta\varphi(x)\delta\varphi(y)}\,\varphi(x)\varphi(y)dxdy + 0(\varphi^4). \tag{4}$$

It is easy to see that

$$\begin{aligned}\frac{\delta^2\mathscr{V}(0)}{\delta\varphi(x)\delta\varphi(y)} &= \lambda^2(\mathrm{Tr}\,\mathrm{K}(\delta_x)\mathrm{K}(\delta_y) - \mathrm{B}^{\xi+}(x)\delta(x-y)) \\ &= \frac{\lambda^2}{(2\pi)^2}\,\tilde{\mathrm{F}}_{\mathrm{reg}}(x-y) - \mu^2\delta(x-y),\end{aligned} \tag{5}$$

where $\tilde{\mathrm{F}}_{\mathrm{reg}}$ is the inverse Fourier transform of $\mathrm{F}_{\mathrm{reg}}$. The latter is computed in [*23*], formula (A.4) and satisfies

$$0 < \mathrm{F}_{\mathrm{reg}}(k) \leqslant 0(1)\,\frac{k^2}{m^2}. \tag{6}$$

Thus the non-local term in $\mathscr{V}(\varphi)$ up to the second order is positive and can be estimated from above by $0(1)\dfrac{\lambda^2}{m^2}\displaystyle\int(\nabla\varphi)^2$. If $\dfrac{\lambda^2}{m^2}$ is small this is dominated by the $\dfrac{1}{2}\displaystyle\int(\nabla\varphi)^2$ term and we should hope that the damping of long contours is uneffected by the fermion determinant non-locality. Thus it seems that the contour expansion of the interacting measure and the translation of each term of the latter into the mean zero regime should work as in the pure boson case, the next step being a cluster expansion within the $+$ and $-$ regions. This is where we shall be confronted with the effective action non-locality for the second time. Breaking step-wise this non-locality we shall have to make sure that

I. the fully decoupled terms depend on spin variables only within their respective regions,

so that not only the cluster expansion but also the contour one factors across the decoupling line (this was almost automatic in the pure boson case since the gradient non-locality was of —infinitesimally— short range). Moreover we should make sure that

II. the fermionic action decoupling does not break the $\varphi \to -\varphi$ symmetry, i. e. that for each fully decoupled term of the interacting measure (inside the volume of interaction) there exists a term corresponding to the opposite values of the spin variables and related by the change of sign of the non-translated φ.

In order to fulfil I and II we shall introduce the fermion decoupling *before* the translation of the measure terms into the mean zero regime. This will be done by changing $\mathscr{V}(\varphi)$ functional (I.8) to $\mathscr{V}(s, \varphi)$, where s parametrizes the degree of decoupling. If $\mathscr{V}(s, \varphi) = \mathscr{V}(s, -\varphi)$ then I and II will hold. We take $\mathscr{V}(s, \varphi)$ to be defined by (I.5) but with $\mathscr{K}(\varphi)$ equal to $K(s, \varphi)$, which is a partially decoupled version of $K(\varphi)$. Because the source of non-locality of $\mathscr{V}(\varphi)$ is the non-locality of the fermion propagator in $K(\varphi)$, which gets weak when $m \to \infty$ it is not difficult to choose $K(s, \varphi)$ so that $\mathscr{V}(\varphi) - \mathscr{V}(s, \varphi)$ gets small for m large and $0(1)$ φ, see [*3*] [*16*]. However φ sits around $\xi_\pm$ and after translation by the (decoupling independent) function g, $g \simeq \xi_\pm = 0(m)$ inside the $\pm$ regions, $\mathscr{V}(g + \varphi) - \mathscr{V}(s, g + \varphi)$ could be large if no special care were taken. Our choice of $\mathscr{V}(s, \hat{\varphi})$ minimizes this un-wanted effect since in the Taylor expansion of $\mathscr{V}(\varphi) - \mathscr{V}(s, \varphi)$ around $\varphi \equiv \xi_+$ there is no first order term and the zeroth order term disappears when we pass from $\mathscr{V}$ to $\mathscr{V}_0$ subtracting the values at $\varphi \equiv \xi_+$. Because of that there is a bigger chance that $\mathscr{V}_0(\xi_\pm + \varphi) - \mathscr{V}_0(s, \xi_\pm + \varphi)$ is small for $0(1)$ φ. Then $\mathscr{V}_0(g + \varphi) - \mathscr{V}_0(s, g + \varphi)$ should also be small except for the contributions from the regions where g jumps between ξ_+ and ξ_-, which hopefully will be over-powered by the damping factor connected with the phase-separation contours.

Repeating the transformations leading to (I.11) we get

$$\mathscr{V}_0(s, \varphi) = -\ln \det_2 (1 - K^{\xi_+}(s, \varphi - \xi_+)) - \frac{\lambda^2}{2}\int B^{\xi_+}(s)(\varphi - \xi_+)^2 \quad (7)$$

where

$$K^{\xi_+}(s, \varphi) = (1 - \lambda K(s, \xi_+))^{-1} K(s, \varphi) \quad (8)$$

and

$$B^{\xi_+}(s, x) = \frac{1}{2}\frac{\delta}{\delta\varphi(x)} \operatorname{Tr}(K(s, \varphi) K^{\xi_+}(s, \varphi)) \restriction_{\varphi = 1}. \quad (9)$$

For the difference $\mathscr{V}(\varphi) - \mathscr{V}_0(s, \varphi)$ to be small for φ around the minimal values it is necessary that $K^{\xi_+}(s, \varphi)$ be close to $K^{\xi_+}(\varphi)$ in appropriate sense. Exact estimates of Chapter IV will show that for this to hold we shall need not only $\lambda K(s, \varphi)$ to be close to $\lambda K(\varphi)$ in some trace norm for $0(1)$ φ but also that $(1 - \lambda K(s, \xi_+))^{-1}$ be close to identity in the operator norm. However $\| \lambda K(s, \xi_+) \| = 0\left(\frac{\lambda\xi_+}{m}\right)$ and if this is smaller than, say, $\frac{1}{2}$ then $\| (1 - \lambda K(s, \xi_+))^{-1} - 1 \| = 0\left(\frac{\lambda\xi_+}{m}\right) = 0\left(\left(e^{\frac{2\pi\mu^2}{\lambda^2}} - 1\right)^{\frac{1}{2}}\right)$. Hence the requirement that m be large must be supplemented with that for λ being large, if the expansion is to converge. μ^2 will be held constant since then the classical mass square $m_c^2 \simeq 2\mu^2$ is $0(1)$.

Now we shall specify the form of $K(s, \varphi)$. We shall follow [*16*] with slight modifications. For a prescribed value Σ of the Ising variables denote by $\mathscr{B}(\Sigma)$ the set of l-lattice bonds within distance L of which Σ is constant,

$l = 0\left((\ln \lambda)^{\frac{1}{2}}\right)$, $L = 0((\ln m)^2)$ ($L \gg l$ in the regime we shall consider). Let $s = (s_b)_{b \in \mathscr{B}(\Sigma)}$, $s_b \in [0, 1]$ be interpolating parameters. Let A be an operator on $L^2(\mathbb{R}^2)$. Define

$$A_s := \sum_{\tilde{\Delta}} \chi_{\tilde{\Delta}} A \chi_{\tilde{\Delta}} + \sum_{\tilde{\Delta} \neq \tilde{\Delta}'} H(s, \tilde{\Delta}, \tilde{\Delta}') \chi_{\tilde{\Delta}} A \chi_{\tilde{\Delta}'} \tag{10}$$

where for l-lattice squares $\tilde{\Delta}$, $\tilde{\Delta}'$

$$H(s, \tilde{\Delta}, \tilde{\Delta}') = \sum_{\gamma \subset \mathscr{B}(\Sigma), \gamma \text{ finite}} \prod_{b \in \gamma} s_b \prod_{b \notin \gamma} (1 - s_b) \frac{\overline{C}_{m_c}^{\mathscr{B}(\Sigma) \sim \gamma}(\tilde{\Delta}, \tilde{\Delta}')}{\overline{C}_{m_c}(\tilde{\Delta}, \tilde{\Delta}')}, \tag{11}$$

$$\overline{C}_{m_c}^{\gamma}(\tilde{\Delta}, \tilde{\Delta}') = \int_{\tilde{\Delta}} dx \int_{\tilde{\Delta}'} dy\, C_{m_c}^{\gamma}(x, y), \tag{12}$$

$$C_{m_c}^{\gamma} = (-\Delta_{\gamma}^{D} + m_c^2)^{-1} \tag{13}$$

and Δ_{γ}^{D} is the Laplace operator with Dirichlet boundary conditions on γ [*11*]. Notice that A_s decouples across any closed line on which $s = 0$.

Put

$$K(s, \varphi) := ((\not{P} + m)D^{-1})_s (D^{-\frac{1}{2}})_s \Gamma \varphi (D^{-\frac{1}{2}})_s . \tag{14}$$

Notice that

$$K(s, \varphi) = A(s, \varphi) B(s, \varphi) \tag{15}$$

where

$$A(s, \varphi) = ((\not{P} + m)D^{-1})_s \Gamma, \qquad B(s, \varphi) = (D^{-\frac{1}{2}})_s \varphi (D^{-\frac{1}{2}})_s . \tag{16}$$

$A(s, \varphi)$ is skew-adjoint and $B(s, \varphi)$ is self-adjoint and there exists an anti-unitary operator U such that

$$UA(s, \varphi)U^{-1} = A(s, \varphi) \tag{17}$$

and

$$UB(s, \varphi)U^{-1} = B(s, \varphi) . \tag{18}$$

U may be chosen the same way as in [*23*], pages 167-168. Now

$$\operatorname{Tr} K(s, \varphi)^k = \overline{\operatorname{Tr} (UK(s, \varphi)U^{-1})^k} = \overline{\operatorname{Tr} K(s, \varphi)^k} = \operatorname{Tr} (K(s, \varphi)^*)^k = (-1)^k \operatorname{Tr} K(s, \varphi)^k .$$

Hence

$$\operatorname{Tr} K(s, \varphi)^{2n+1} = 0 , \tag{19}$$

which is the source of the $\varphi \to -\varphi$ symmetry

$$\mathscr{V}(s, \varphi) = \mathscr{V}(s, -\varphi) . \tag{20}$$

With the use of $K(s, \varphi)$ instead of $K(\varphi)$ and of

$$P_j(s) := |((\not{P} + m)D^{-\frac{3}{2}})_s h_j \rangle \langle (D^{-\frac{1}{2}})_s g^j | \tag{21}$$

instead of P_j we arrive at the expression for $Z_{\Lambda,\Sigma,s}$ which is given by the same formula as (2) except that $K(.)$, $K^{\xi+}(.)$, $B^{\xi+}$ and P_j are changed for $K(s, .)$, $K^{\xi+}(s, .)$, $B^{\xi+}(s)$ and $P_j(s)$ respectively.

Having introduced the partial decoupling of the fermionic non-locality we proceed with the translation of the measure by g which is an appropriately smeared version of the function h,

$$h(x) := \xi_\pm \quad \text{if} \quad \Sigma(\Delta) = \pm \quad \text{for} \quad x \in \Delta . \tag{22}$$

The exact form of g is copied from [13] (formula (I.44)). After this operation we end up with the following expression:

$$\begin{aligned} ZS_{\Lambda,\Sigma,s} &= \int d\mu_{m_c} \prod_\Delta \chi_{\Sigma(\Delta)}((\varphi + g)^\Lambda_\Delta) \prod_{i=1}^{I} (\varphi + g)(f_i) \\ &\cdot \tau_J(\Lambda^J(1 - \lambda K^{\xi+}(s, (\varphi + g - \xi_+)\Lambda))^{-1} \\ &\cdot \bigwedge_{j=1}^{J} (1 - \lambda K(s, \xi_+))^{-1} P_j(s)) \det_3 (1 - \lambda K^{\xi+}(s, (\varphi + g - \xi_+)\Lambda)) . \\ &\cdot \exp \Big[- \frac{\lambda^2}{2} : \operatorname{Tr} K^{\xi+}(s, (\varphi + g - \xi_+)\Lambda)^2 : \\ &+ \frac{\lambda^2}{2} \int B^{\xi+}(s) : ((\varphi + g - \xi_+)\Lambda)^2 : + \frac{1}{2} m_c^2 \int : ((\varphi + g - \xi_+)\Lambda)^2 : \\ &- \int \varphi(-\Delta + m_c^2)(g - \xi_+) - \frac{1}{2} \int (g - \xi_+)(-\Delta + m_c^2)(g - \xi_+) \Big] \end{aligned} \tag{23}$$

where $d\mu_{m_c}$ is the Gaussian measure with mean zero and covariance $(-\Delta + m_c^2)^{-1}$.

The partial decoupling of the Gaussian measure $d\mu_{m_c}$ is introduced in a standard way [11] [13] via step-wice insertion of the Dirichlet boundary conditions along bonds of the l-lattice. Thus for the set $\tau = (\tau_b)_{b \in \mathscr{B}(\Sigma)}$, $\tau_b \in [0, 1]$ of the interpolating parameters put

$$C_{m_c}(\tau) = \sum_{\gamma \subset \mathscr{B}(\Sigma),\, \gamma \text{ finite}} \prod_{b \in \gamma} \tau_b \prod_{b \notin \gamma} (1 - \tau_b) C_{m_c}^{\mathscr{B}(\Sigma) \sim \gamma} . \tag{24}$$

Denote by $d\mu_{m_c}(\tau)$ the Gaussian measure with mean zero and covariance $C_{m_c}(\tau)$ and by $: \;:_\tau$ the Wick ordering with respect to this measure. Changing in (23) $d\mu_{m_c}$ to $d\mu_{m_c}(\tau)$ and the Wick ordering $: \;:$ to $: \;:_\tau$ we obtain the expression for $ZS_{\Lambda,\Sigma,s,\tau}$.

Note that $ZS_{\Lambda,\Sigma,s,\tau}$ factors across an $s, \tau = 0$ closed line (composed of l-lattice bonds):

$$ZS_{\Lambda,\Sigma,s,\tau} = \pm ZS_{\Lambda,\Sigma,s,\tau,Z} ZS_{\Lambda,\Sigma,s,\tau,\sim Z} , \tag{25}$$

where Z is the region encircled by the line and

$$ZS_{\Lambda,\Sigma,s,\tau,Z}$$

$$:= \int d\mu_{m_c}(\tau) \prod_{\Delta\subset Z} \chi_{\Sigma(\Delta)}((\varphi + g)^{\Lambda}_{\Delta}) \prod_{\alpha=1}^{I_Z} (\varphi + g)(f_{i_\alpha})$$

$$\cdot \tau_{J_Z}\left(\Lambda^{J_Z}(1-\lambda K^{\xi_+}(s, (\varphi+g-\xi_+)\Lambda\chi_Z))^{-1} \cdot \bigwedge_{\beta=1}^{J_Z} (1-\lambda K(s, \xi_+))^{-1} P_{Z,\beta}(s) \right)$$

$$\cdot \det{}_3 (1-\lambda K^{\xi_+}(s, (\varphi+g-\xi_+)\Lambda\chi_Z)) \exp\left[-\frac{\lambda^2}{2} : \mathrm{Tr}\, K^{\xi_+}(s, (\varphi+g-\xi_+)\Lambda\chi_Z)^2 :_\tau \right.$$

$$+ \frac{\lambda^2}{2} \int_Z B^{\xi_+}(s) : ((\varphi + g - \xi_+)\Lambda)^2 :_\tau + \frac{1}{2} m_c^2 \int_Z : ((\varphi + g - \xi_+)\Lambda)^2 :_\tau$$

$$\left. - \int_Z \varphi(-\Delta + m_c^2)(g - \xi_+) - \frac{1}{2} \int_Z (g - \xi_+)(-\Delta + m_c^2)(g - \xi_+) \right]. \quad (26)$$

In the above formula i_α are those $i \in I$ for which suppt $f_i \subset \overline{Z}$, I_Z being their number. $P_{Z,\beta}(s) = 0$ if the numbers of $h_j - s$ and of $g_j - s$ with support in $\overline{Z}$ are not equal,

$$P_{Z,\beta}(s) := | ((\not{P} + m)D^{-\frac{3}{2}})_s h_{j_\beta} \rangle \langle (D^{-\frac{1}{2}})_s g_{j'_\beta} | \quad (27)$$

if the latter numbers are equal ($= J_Z$), h_{j_β} and $g_{j'_\beta}$ being just those $h_j - s$ and $g_j - s$ supported in $\overline{Z}$. $\tau_0(.)$ is always taken to be 1. The $\pm$ sign in (25) must be included because functions g_j and h_j can appear on the right hand side in different effective order than on the left hand side.

$ZS_{\Lambda,\Sigma,s,\tau,Z}$ depends only on Σ, s and τ restricted to $\overline{Z}$, so that (25) gives the desired decoupling. Of course $ZS_{\Lambda,\Sigma,s,\tau} = ZS_{\Lambda,\Sigma,s,\tau,\mathbb{R}^2}$.

The cluster expansion consists in writing (see [*11*])

$$ZS_{\Lambda,\Sigma} = \sum_{\Gamma,\Pi\subset\mathscr{B}(\Sigma)} \int ds_\Gamma d\tau_\Pi \partial_\tau^\Pi \partial_s^\Gamma ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi}, \quad (28)$$

where Γ and Π are finite,

$$(s_\Gamma)_b = \begin{cases} s_b & \text{if } b \in \Gamma \\ 0 & \text{if } b \notin \Gamma, \end{cases} \quad (29)$$

$$\partial_s^\Gamma = \prod_{b\in\Gamma} \partial_{s_b} \quad (30)$$

and similarly for τ_Π and ∂_τ^Π.

(28) holds provided that $ZS_{\Lambda,\Sigma,s,\tau}$ is regular at infinity (see [11]), i. e. that

$$\lim_{\Gamma,\Pi \text{ finite}} ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi} = ZS_{\Lambda,\Sigma,s,\tau} \tag{31}$$

which easily follows from the estimates we shall prove.

For given Σ, Γ, Π call $b \in \mathscr{B}(\Sigma)$ a decoupling bond if $b \notin \Gamma \cup \Pi$. Label by $Z_\varkappa$ the closures of the connected components of $\mathbb{R}^2$ with the decoupling bonds taken out.

$$ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi} = \pm \prod_\varkappa ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z_\varkappa} \tag{32}$$

and each $ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z_\varkappa}$ depends only on the restriction of Σ, s_Γ and τ_Π to $Z_\varkappa$ denoted by $\Sigma_\varkappa$, $s_{\Gamma_\varkappa}$, $\tau_{\Pi_\varkappa}$ respectively. Hence we can write

$$\begin{aligned} ZS_\Lambda &= \sum_\Sigma ZS_{\Lambda,\Sigma} \\ &= \sum_\Sigma \sum_{\Gamma,\Pi \subset \mathscr{B}(\Sigma)} (\pm 1) \prod_\varkappa \int ds_{\Gamma_\varkappa} d\tau_{\Pi_\varkappa} \partial_\tau^{\Pi_\varkappa} \partial_s^{\Gamma_\varkappa} ZS_{\Lambda,\Sigma_\varkappa,s_{\Gamma_\varkappa},\tau_{\Pi_\varkappa},Z_\varkappa}. \end{aligned} \tag{33}$$

Denote by $\partial Z_\varkappa^\pm$ the set of l-lattice lines of $\partial Z_\varkappa$ which run inside the $\pm$ regions of the Ising configuration specified by Σ. We shall reorder the sum $\sum_\Sigma \sum_{\Gamma,\Pi \subset \mathscr{B}(\Sigma)}$ by first fixing $(Z_\varkappa, \partial Z_\varkappa^+, \partial Z_\varkappa^-) := \mathbb{Z}_\varkappa$ and then summing over all Σ, Γ and Π which lead to the set $\{ \mathbb{Z}_\varkappa \}$. This yields

$$\sum_\Sigma \sum_{\Gamma,\Pi \subset \mathscr{B}(\Sigma)} = \sum_{\{\mathbb{Z}_\varkappa\} \text{ admissible}} \prod_\varkappa \sum_{\Sigma_\varkappa} \sum_{\substack{\Gamma_\varkappa,\Pi_\varkappa \subset \mathscr{B}(\Sigma_\varkappa,Z_\varkappa) \\ \text{constrained}}} \tag{34}$$

The set $\{ \mathbb{Z}_\varkappa \}$ is admissible if it corresponds to at least one (Σ, Γ, Π). $\Sigma_\varkappa$ run over all mappings of the d-lattice squares in $Z_\varkappa$ into $\{ +, - \}$ taking the constant value $\pm$ within distance L of $\partial Z_\varkappa^\pm$ and the value $+$ on squares outside the support of Λ.

$\mathscr{B}(\Sigma_\varkappa, Z_\varkappa)$ is composed of the l-lattice bonds in $Z_\varkappa$ within distance L of which $\Sigma_\varkappa$ is constant. The sums $\sum_{\Gamma_\varkappa,\Pi_\varkappa \subset \mathscr{B}(\Sigma_\varkappa,Z_\varkappa)}$ are constrained by the requirements that

i) $\Gamma_\varkappa$ and $\Pi_\varkappa$ do not contain the bonds of $\partial Z_\varkappa$ and *ii*) no proper subset of $Z_\varkappa$ with boundary composed of decoupling bonds exists.

Inserting (34) into (33) we obtain

$$ZS_\Lambda = \sum_{\{\mathbb{Z}_\varkappa\}\ \text{admissible}} (\pm 1) \prod_\varkappa \rho_\Lambda(\mathbb{Z}_\varkappa), \tag{35}$$

where

$$\rho_\Lambda(\mathbb{Z}_\varkappa) := \sum_{\Sigma_\varkappa} \sum_{\substack{\Gamma_\varkappa, \Pi_\varkappa \subset \mathscr{B}(\Sigma_\varkappa, Z_\varkappa) \\ \text{constrained}}} \int ds_{\Gamma_\varkappa} d\tau_{\Pi_\varkappa} \cdot \partial_\tau^{\Pi_\varkappa} \partial_s^{\Gamma_\varkappa} ZS_{\Lambda, \Sigma_\varkappa, s_{\Gamma_\varkappa}, \tau_{\Pi_\varkappa}, Z_\varkappa} \tag{36}$$

($s_{\Gamma_\varkappa}$ and $\tau_{\Pi_\varkappa}$ are viewed as defined for bonds of $\mathscr{B}(\Sigma_\varkappa, Z_\varkappa)$).

We still have to perform a partial resummation in (35). We shall do this following an elegant method of Kunz and Souillard [*15*].

Since Γ and Π in (33) are finite, all $Z_\varkappa$ except a finite number are l-lattice squares which do not contain supports of functions f_i, g_j and h_j. Call such squares vacuum ones. We shall eliminate their contributions to (35). Denote by ρ_Λ^0 the ρ_Λ function for the case $I = J = 0$, that is for the case of the pure partition function. Of course for a vacuum square

$$\rho_\Lambda(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing) = \rho_\Lambda^0(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing) \tag{37}$$

and also

$$\rho_\Lambda(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing) = \rho_\Lambda(\tilde{\Delta}, \varnothing, \partial\tilde{\Delta}) \tag{38}$$

whenever the latter appears in (35), by virtue of the $\varphi \to -\varphi$ symmetry. Moreover

$$\rho_\Lambda^0(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing) = 1 \quad \text{if} \quad \tilde{\Delta} \cap \operatorname{suppt} \Lambda = \varnothing . \tag{39}$$

We shall also see that $\rho_\Lambda^0(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing) \neq 0$. Define

$$\widetilde{ZS}_\Lambda := ZS_\Lambda \prod_{\tilde{\Delta}} \rho_\Lambda^0(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing)^{-1} \tag{40}$$

From (35) we get

$$\widetilde{ZS}_\Lambda = \sum_{\{\mathbb{Z}_\sigma\}}{}' (\pm 1) \prod_\sigma \tilde{\rho}_\Lambda(\mathbb{Z}_\sigma), \tag{41}$$

where

$$\tilde{\rho}_\Lambda(\mathbb{Z}_\sigma) := \rho_\Lambda(\mathbb{Z}_\sigma) \prod_{\Delta \subset Z_\sigma} \rho_\Lambda^0(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing)^{-1} \tag{42}$$

In $\sum_{\{\mathbb{Z}_\sigma\}}{}'$ we sum over the sets of those $\mathbb{Z}_\sigma$ which are neither $(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing)$ nor $(\tilde{\Delta}, \varnothing, \partial\tilde{\Delta})$ with vacuum $\tilde{\Delta}$ but which can be supplemented by these to form an admissible set. Hence in an allowed set $\{ \mathbb{Z}_\sigma \}$ each Z_σ is different from single vacuum square, closed set built out of a finite number of l-lattice squares and cannot be devided into two nonempty parts by taking a finite numbers of points out of it. ∂Z_σ is composed of closed loops some of them entering ∂Z_σ^+, the other ones ∂Z_σ^-. Call any such set with specified signs of the boundary loops a cluster. Thus each Z_σ is a cluster and different $Z_\sigma - s$ can at most touch (cannot overlap). Their boundary signs must agree on the loops being not separated by other $Z_\sigma - s$. Each non-vaccum square must sit in one of $Z_\sigma - s$. We can allow *all* such sets of clusters in $\sum_{\{\mathbb{Z}_\sigma\}}'$, provided that for clusters $\mathbb{Z}_\sigma$ for which there are no $\Sigma_\sigma - s$ which enter the sum in (36) we put $\rho_\Lambda(\mathbb{Z}_\sigma) = \tilde{\rho}_\Lambda(\mathbb{Z}_\sigma) = 0$. Since the notion of a vacuum square depends on the particular Schwinger function we consider, so do the notions of a cluster and of an allowed set of clusters.

Call a boundary loop of $\mathbb{Z}_\sigma$ inner if it is of other sign than that of the external loop of $\mathbb{Z}_\sigma$.

Notice that for a set $\{ \mathbb{Z}_\sigma \}$ leading to a non-zero term of (41) we can recollect the boundary signs if we know only the relative signs of the boundary loops, since the most external loops must have the + sign. Call $\mathbb{Z}_\sigma$ positive if its external boundary loop is positive. Call $\mathbb{Z}_\sigma$ a vacuum cluster if it is built of vacuum squares only. Label non-vacuum $\mathbb{Z}_\sigma - s$ as $\mathbb{X}_1, \ldots, \mathbb{X}_k$ and the other ones as $\mathbb{Y}_1, \ldots, \mathbb{Y}_l$.

Following Kunz and Souillard [*15*] we shall rewrite (41) putting most of the compatibility conditions among $\mathbb{X} - s$ and $\mathbb{Y} - s$ into the summand. To this end introduce operations U acting on functions of clusters $\mathbb{X}_r - s$ and $\mathbb{Y}_s - s$.

$$U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) := \begin{cases} 0 \text{ if } X_{r_1} \text{ and } X_{r_2} \text{ overlap, changes } (X_{r_1}, X_{r_1}^+, X_{r_1}^-) \text{ to} \\ (X_{r_1}, X_{r_1}^+, X_{r_1}^-) \text{ if } X_{r_1} \text{ and } X_{r_2} \text{ do not overlap and an} \\ \text{inner boundary loop of } \mathbb{X}_{r_2} \text{ surrounds } X_{r_1}, \text{ changes} \\ (X_{r_2}, \partial X_{r_2}^+, \partial X_{r_2}^-) \text{ to } (X_{r_2}, \partial X_{r_2}^-, \partial X_{r_2}^+) \text{ if } X_{r_1} \text{ and } X_{r_2} \text{ do not} \\ \text{overlap and an inner boundary loop of } \mathbb{X}_{r_1} \text{ surrounds} \\ \mathbb{X}_{r_2}, \\ 1 \text{ otherwise,} \end{cases} \tag{43}$$

$$U(\mathbb{X}_r, \mathbb{Y}_s) := \begin{cases} 0 \text{ if } X_r \text{ and } Y_s \text{ overlap, changes } (X_r, \partial X_r^+, \partial X_r^-) \text{ to} \\ (X_r, \partial X_r^-, \partial X_r^+) \text{ if } X_r \text{ and } Y_s \text{ do not overlap and an} \\ \text{inner boundary loop of } \mathbb{Y}_s \text{ surrounds } X_r, \\ 1 \text{ otherwise,} \end{cases} \tag{44}$$

$$U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}) := \begin{cases} 0 \text{ if } Y_{s_1} \text{ and } Y_{s_2} \text{ overlap,} \\ 1 \text{ otherwise.} \end{cases} \tag{45}$$

Now we can rewrite (41) as

$$\widetilde{ZS}_\Lambda = \sum_k \sum_{\{\mathbb{X}_1,\dots,\mathbb{X}_k\}\text{ positive}} (\pm 1) \sum_{l=0}^{\infty} \frac{1}{l!} \cdot \sum_{(\mathbb{Y}_1,\dots,\mathbb{Y}_l)\text{ positive}} \prod_{1\le r_1<r_2\le k} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) \prod_{r=1}^{k}\prod_{s=1}^{l} U(\mathbb{X}_r, \mathbb{Y}_s) \cdot \prod_{1\le s_1<s_2\le l} U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}) \prod_{r=1}^{k} \tilde\rho_\Lambda(\mathbb{X}_r) \prod_{s=1}^{l} \tilde\rho_\Lambda(\mathbb{Y}_s). \tag{46}$$

In (46) $\mathbb{X}_r - s$ are positive non-vacuum clusters and $\bigcup_{r=1}^{k} X_r$ must contain all non-vacuum squares. $\mathbb{Y}_s - s$ are vacuum clusters with no further restrictions except positivity. $\frac{1}{l!}$ comes from the fact that we sum over ordered families of $\mathbb{Y}_s - s$. In arriving at (46) we used the fact that whenever a vacuum cluster $\mathbb{Y}_r$ in a non-zero term of (41) is surrounded by a negative boundary loop of another cluster then in virtue of the $\varphi \to -\varphi$ symmetry

$$\tilde\rho_\Lambda(Y_r, \partial Y_r^+, \partial Y_r^-) = \tilde\rho_\Lambda(Y_r, \partial Y_r^-, \partial Y_r^+) \tag{47}$$

since Y_r is inside the region where $\Lambda = 1$ and does not feel the (non-symmetric) boundary conditions.

Since each vacuum cluster for a Schwinger function is also a vacuum cluster for the partition function we can relax the conditions for $\mathbb{Y}_s - s$ in (46) taking all partition function clusters and replacing

$$\tilde\rho_\Lambda(\mathbb{Y}_r) \quad \text{by} \quad \tilde\rho^0_\Lambda(\mathbb{Y}_r) := \rho^0_\Lambda(\mathbb{Y}_r) \prod_{\tilde\Delta \subset Y_r} \rho^0_\Lambda(\tilde\Delta, \partial\tilde\Delta, \varnothing)^{-1}.$$

This does not change the value of the right hand side of (46) since the extra terms give only zero contributions. In the sequel we assume that this modification of (46) has been done.

Now we shall partly rewrite (46) in terms of operations $A = U - 1$ using

$$\prod_{\alpha\in\mathscr{A}} (1 + A_\alpha) = \sum_{G\subset\mathscr{A}} \prod_{\alpha\in G} A_\alpha. \tag{48}$$

This yields

$$\widetilde{ZS}_\Lambda = \sum_k \sum_{\{\mathbb{X}_1,\dots,\mathbb{X}_k\}} (\pm 1) \sum_l \frac{1}{l!} \sum_{(\mathbb{Y}_1,\dots,\mathbb{Y}_l)} \sum_G \cdot \prod_{r_1<r_2} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) \prod_{\mathscr{L}\in G} A(\mathscr{L}) \prod_r \tilde\rho_\Lambda(\mathbb{X}_r) \prod_s \tilde\rho^0_\Lambda(\mathbb{Y}_s), \tag{49}$$

where G (a Mayer graph) is any set of unordered pairs $\{ \mathbb{X}_r, \mathbb{Y}_s \}$ or $\{ \mathbb{Y}_{s_1}, \mathbb{Y}_{s_2} \}$ (called lines $\mathscr{L}$). Each graph G contains a part G_c composed of the lines directly or indirectly connected to one of $\mathbb{X}_r - s$. The disconnected lines form a complementary graph G_0. G is said to be connected with respect to $\{ \mathbb{X}_1, \ldots, \mathbb{X}_k \}$ if $G_0 = \varnothing$.

Label all $\mathbb{Y}_r - s$ entering in the lines of G_c as $\mathbb{Y}'_1, \ldots, \mathbb{Y}'_{l_c}$ and all the other ones as $\mathbb{Y}''_1, \ldots, \mathbb{Y}''_{l_0}$, $l_0 + l_c = l$. The sum $\sum_l \frac{1}{l!} \sum_{(\mathbb{Y}_1, \ldots, \mathbb{Y}_l)} \sum_G$ in (49) can be rewritten as

$$\sum_{l_c} \frac{1}{l_c!} \sum_{(\mathbb{Y}'_1, \ldots, \mathbb{Y}'_{l_c})} \sum_{G_c} \sum_{l_0} \frac{1}{l_0!} \sum_{(\mathbb{Y}''_1, \ldots, \mathbb{Y}''_{l_0})} \sum_{G_0}$$

where G_c is any graph connected w. r. t. $\{ \mathbb{X}_1, \ldots, \mathbb{X}_k \}$ composed of lines between $\mathbb{X}_r - s$ and $\mathbb{Y}'_s - s$ and between $\mathbb{Y}'_s - s$ with all $\mathbb{Y}'_s - s$ involved. G_0 is any graph composed of lines between $\mathbb{Y}''_s - s$. Now (49) becomes

$$\widetilde{ZS}_\Lambda = \Big(\sum_k \sum_{\{\mathbb{X}_1, \ldots, \mathbb{X}_k\}} (\pm 1) \sum_{l_c} \frac{1}{l_c!} \sum_{(\mathbb{Y}'_1, \ldots, \mathbb{Y}'_{l_c})} \cdot \sum_{G_c} \prod_{r_1 < r_2} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_r \tilde{\rho}_\Lambda(\mathbb{X}_r) \prod_s \tilde{\rho}^0_\Lambda(\mathbb{Y}'_s) \Big) \cdot \Big(\sum_{l_0} \frac{1}{l_0!} \sum_{(\mathbb{Y}''_1, \ldots, \mathbb{Y}''_{l_0})} \sum_{G_0} \prod_{\mathscr{L} \in G_0} A(\mathscr{L}) \prod_s \tilde{\rho}^0_\Lambda(\mathbb{Y}''_s) \Big). \tag{50}$$

By (49) the second factor on the right hand side of (50) corresponds to $\widetilde{ZS}_\Lambda$ with no non-vacuum cluster $\mathbb{X}_r$, i. e. to the partition function $\tilde{Z}_\Lambda$. Extraction of full $\tilde{Z}_\Lambda$ out of the expansion for $\widetilde{ZS}_\Lambda$ is the main virtue of the Kunz-Souillard method of resummation.

Dividing by $\tilde{Z}_\Lambda$ we get expressions for normalized volume cut-off Schwinger functions S_Λ:

$$S_\Lambda = \sum_k \sum_{\{\mathbb{X}_1, \ldots, \mathbb{X}_k\}} (\pm 1) \sum_l \frac{1}{l!} \sum_{(\mathbb{Y}_1, \ldots, \mathbb{Y}_l)} \sum_{G_c} \cdot \prod_{r_1 < r_2} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_r \tilde{\rho}_\Lambda(\mathbb{X}_r) \prod_r \tilde{\rho}^0_\Lambda(\mathbb{Y}_s). \tag{51}$$

This is the final form of our expansion. We recall that the sum over $\{ \mathbb{X}_1, \ldots, \mathbb{X}_k \}$ is over the sets of (non-overlapping) positive non-vacuum clusters such that $\bigcup_{r=1}^{k} X_r$ contains supports of functions f_i, g_j and h_j entering the Schwinger function S_Λ. The sign ($\pm$ 1) is determined by the set $\{\mathbb{X}_1, \ldots, \mathbb{X}_k\}$. $(\mathbb{Y}_1, \ldots, \mathbb{Y}_l)$ runs through all ordered families of positive partition-function-clusters and G_c through all graphs connected w. r. t. $\{\mathbb{X}_1, \ldots, \mathbb{X}_k\}$ involving all $\mathbb{Y}_j - s$.

It should be mentioned that going from (28) to (51) we were slightly carelessly transforming infinite sums about whose convergence we do not know much yet. However if we introduced another cut-off restricting ourselves in (28) to the sum over Γ and Π inside a large volume V (well bigger than suppt Λ) then all the sums would be finite, the resulting summation in (51) being over clusters $\mathbb{X}_r - s$ and $\mathbb{Y}_s - s$ in V. When we remove this cut-off the Schwinger functions go to their original versions in virtue of the « regularity at infinity ». Since on the right hand side of (51) the first four sums converge absolutely, as we show beneath, on the right hand side of (51) we recover infinite sums when V tends to infinity.

Now we shall state the main estimates on $\rho_\Lambda(\mathbb{X}_r)$ and $\rho_\Lambda^0(\mathbb{Y}_s)$ yielding the convergence of (51) and consequently Theorem I.1. For a d-lattice square Δ denote by $I(\Delta)$ the number of i, $i = 1, \ldots, I$, such that Δ supports f_i.

PROPOSITION II.1. — For each $C > 0$ there exists $\lambda_0 > 0$ and constants $0(1)$ such that for all $\lambda \geqslant \lambda_0$, $m \geqslant m_0(\lambda)$ and arbitrary Λ

$$| \rho_\Lambda(\mathbb{X}_r) | \leqslant \prod_{\alpha=1}^{I_{X_r}} (0(1)m \| f_{i_\alpha} \|_{L^2}) \prod_{\beta=1}^{J_{X_r}} (0(1)^l \| g_{j_\beta} \|_{H^{-\frac{1}{2}}} \cdot \| h_{j'_\beta} \|_{H^{-1}}) \prod_{\Delta \subset X_r} (I(\Delta)\,!)^{\frac{1}{2}} e^{-Cl^{-2}|X_r|}, \tag{52}$$

$$| \rho_\Lambda^0(\mathbb{Y}_s) | \leqslant e^{-Cl^{-2}|Y_s|}, \tag{53}$$

where $\| f \|_{H^{-\frac{1}{2}}} := \| D^{-\frac{1}{2}} f \|_{L^2}$.

PROPOSITION II.2. — There exists $\nu > 0$, $\lambda_0 > 0$ and a constant $0(1)$ such that for all $\lambda \geqslant \lambda_0$, $m \geqslant m_0(\lambda)$ and arbitrary Λ

$$\rho_\Lambda^0(\tilde{\Delta}, \partial\tilde{\Delta}, \varnothing) \geqslant e^{-0(1)m^{-\nu}} \tag{54}$$

for each l-lattice square $\tilde{\Delta}$.

Propositions II.1 and 2 will be proven in the subsequent chapters. They immediately give

COROLLARY II.3. — Under the assumptions of Proposition II.1 (52) and (53) hold also for $\tilde{\rho}_\Lambda(\mathbb{X}_r)$ and $\tilde{\rho}_\Lambda^0(\mathbb{Y}_s)$.

Thus we see that $\tilde{\rho}_\Lambda(\mathbb{X}_r)$ and $\tilde{\rho}_\Lambda^0(\mathbb{Y}_s)$ become exponentially small for large $|X_r|$ or $|Y_s|$. This is one of the sources of convergence of (51). The other one is that the $\prod_{\mathscr{L} \in G_c} A(\mathscr{L})$ operations in (51) eliminate the terms in which clusters $\mathbb{Y}_s$ are not « grouped » around clusters $\mathbb{X}_r$.

To prove the convergence of (51) with use of Corollary II.3 we shall use a Kirkwood-Salzburg type equations elaborated by Kunz and Souillard. We shall obtain them now.

Let for $k \geqslant 1$, $l \geqslant 0$, $\mathbb{Z}_1, \ldots, \mathbb{Z}_k$ be any non-vacuum clusters for the given Schwinger function or partition function clusters and let $\mathbb{Y}_1, \ldots, \mathbb{Y}_l$ be any partition function clusters. Define

$$\varphi_\Lambda(\mathbb{Z}_1, \ldots, \mathbb{Z}_k ; \mathbb{Y}_1, \ldots, \mathbb{Y}_l) := \sum_{G_c} \prod_{r_1 < r_2} U(\mathbb{Z}_{r_1}, \mathbb{Z}_{r_2}) \cdot \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_{r=1}^{k} \tilde{\rho}_\Lambda^{(0)}(\mathbb{Z}_r) \prod_{s=1}^{l} \tilde{\rho}_\Lambda^{0}(\mathbb{Y}_s) \quad (55)$$

so that

$$S_\Lambda = \sum_k \sum_{\{\mathbb{X}_1, \ldots, \mathbb{X}_k\}} (\pm 1) \sum_l \frac{1}{l!} \sum_{(\mathbb{Y}_1, \ldots, \mathbb{Y}_l)} \varphi_\Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_k ; \mathbb{Y}_1, \ldots, \mathbb{Y}_l). \quad (56)$$

Given G_c in (55) for $k \geqslant 2$, let Ω be the set of $s \in \{1, \ldots, l\}$ such that $\{\mathbb{Z}_1, \mathbb{Y}_s\} \in G_c$. Denote by G_c' the part of G_c composed of lines $\{\mathbb{Y}_s, \mathbb{Y}_{s'}\}$ and $\{\mathbb{Z}_r, \mathbb{Y}_s\}$, $s, s' \in \Omega$, $r = 2, \ldots, k$. Let G_c'' be composed of all other lines except those of the form $\{\mathbb{Z}_1, \mathbb{Y}_s\}$. We shall reorder the sum over G_c in (55) first fixing Ω and summing over G_c' and G_c'' and then summing over Ω. It is easy to see that for fixed Ω G_c' is arbitrary but G_c'' must be a graph connected w. r. t. $\{\mathbb{Z}_2, \ldots, \mathbb{Z}_k, (\mathbb{Y}_s)_{s \in \Omega}\}$. Using moreover the relation

$$\sum_{G_c'} \prod_{\mathscr{L} \in G_c'} A(\mathscr{L}) = \prod_{r=2}^{k} \prod_{s \in \Omega} U(\mathbb{Z}_r, \mathbb{Y}_s) \prod_{s_1 < s_2, s_1, s_2 \in \Omega} U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}) \quad (57)$$

we can rewrite (55) as

$$\begin{aligned} \varphi_\Lambda(\mathbb{Z}_1, \ldots, \mathbb{Z}_k ; \mathbb{Y}_1, \ldots, \mathbb{Y}_l) &= \prod_{r=2}^{k} U(\mathbb{Z}_1, \mathbb{Z}_r) \\ &\cdot \sum_{\Omega \subset \{1, \ldots, l\}} \prod_{s \in \Omega} A(\mathbb{Z}_1, \mathbb{Y}_s) \tilde{\rho}_\Lambda^{(0)}(\mathbb{Z}_1) \sum_{G_c''} \prod_{2 \leqslant r_1 < r_2 \leqslant k} U(\mathbb{Z}_{r_1}, \mathbb{Z}_{r_2}) \\ &\cdot \prod_{r=2}^{k} \prod_{s \in \Omega} U(\mathbb{Z}_r, \mathbb{Y}_s) \prod_{s_1 < s_2, s_1, s_2 \in \Omega} U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}) \prod_{\mathscr{L} \in G_c''} A(\mathscr{L}) \\ &\cdot \prod_{r=2}^{k} \tilde{\rho}_\Lambda^{(0)}(\mathbb{Z}_r) \prod_{s=1}^{l} \tilde{\rho}_\Lambda^{0}(\mathbb{Y}_s). \end{aligned} \quad (58)$$

Thus
$$\varphi_\Lambda(\mathbb{Z}_1, \ldots, \mathbb{Z}_k ; \mathbb{Y}_1, \ldots, \mathbb{Y}_l) = \sum_\Omega \prod_{r=2}^{k} \mathrm{U}(\mathbb{Z}_1, \mathbb{Z}_r) \cdot \prod_{s\in\Omega} \mathrm{A}(\mathbb{Z}_1, \mathbb{Y}_s)\tilde{\rho}_\Lambda^{(0)}(\mathbb{Z}_1)\varphi_\Lambda(\mathbb{Z}_2, \ldots, \mathbb{Z}_k, (\mathbb{Y}_s)_{s\in\Omega} ; (\mathbb{Y}_s)_{s\notin\Omega}) . \quad (59)$$

This is the Kirkwood-Salzburg type equation we searched for. It also holds for $k = 1$ if we put $\varphi_\Lambda(\varnothing; \mathbb{Y}_1, \ldots, \mathbb{Y}_l) := 0$.

Now
$$|\varphi_\Lambda(\mathbb{Z}_1, \ldots, \mathbb{Z}_k ; \varnothing)| = \left| \prod_{r_1<r_2} \mathrm{U}(\mathbb{Z}_{r_1}, \mathbb{Z}_{r_2}) \cdot \prod_r \tilde{\rho}_\Lambda^{(0)}(\mathbb{Z}_r) \right| \leqslant \prod_r \mathrm{C}(\mathrm{Z}_r) , \quad (60)$$

where by $\mathrm{C}(\mathrm{Z}_r)$ we have denoted the right hand side of (52) or of (53). Consider the $l > 0$ case.

LEMMA II.4. — For each $\mathrm{C} > 0$ there exists $\lambda_0 > 0$ and constants $0(1)$ such that for all $\lambda \geqslant \lambda_0$, $m \geqslant m_0(\lambda)$, arbitrary Λ and $\mathrm{N} \geqslant 1$
$$\sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_l),\ \sum_s |\mathrm{Y}_s| = l^2\mathrm{N}} |\varphi_\Lambda(\mathbb{Z}_1, \ldots, \mathbb{Z}_k ; \mathbb{Y}_1, \ldots, \mathbb{Y}_l)| \leqslant l! \prod_r \mathrm{C}(\mathrm{Z}_r) e^{2l^{-2}\sum_r |\mathrm{Z}_r| - \frac{1}{4}\mathrm{C}(l+\mathrm{N})} . \quad (61)$$

Proof. — We proceed by induction over $k + l$. For $k + l = 2$ we have $k = l = 1$. For a function f of clusters $\mathbb{Z}_i$ put
$$f(\mathrm{Z}_i) := \max_{(\mathbb{Z}_i') : \mathrm{Z}_i' = \mathrm{Z}_i} |f(\mathbb{Z}_i')|$$
$$\cdot \sum_{\mathbb{Y}_1, |\mathrm{Y}_1| = l^2\mathrm{N}} |\varphi_\Lambda(\mathbb{Z}_1 ; \mathbb{Y}_1)| \leqslant \sum_{\substack{\mathbb{Y}_1, |\mathrm{Y}_1| = l^2\mathrm{N}, \\ \mathrm{Y}_1 \text{ overlaps or surrounds } \mathrm{Z}_1}} 2\tilde{\rho}_\Lambda^{(0)}(\mathrm{Z}_1) |\tilde{\rho}_\Lambda^{0}(\mathbb{Y}_1)|$$
$$\leqslant 2l^{-2} |\mathrm{Z}_1| \mathrm{C}(\mathrm{Z}_1) e^{(0(1)-\mathrm{C})\mathrm{N}} \leqslant \mathrm{C}(\mathrm{Z}_1) e^{2l^{-2}|\mathrm{Z}_1| - \frac{1}{4}\mathrm{C}(1+\mathrm{N})} , \quad (62)$$

since $\mathrm{A}(\mathbb{Z}_1, \mathbb{Y}_1)$ which appears in $\varphi_\Lambda(\mathbb{Z}_1, \mathbb{Y}_1)$ is zero if $\mathbb{Y}_1$ does not overlap nor surround Z_1 and the number of clusters $\mathbb{Y}_1$, $|\mathrm{Y}_1| = l^2\mathrm{N}$, overlapping or surrounding Z_1 is bounded by $l^{-2} |\mathrm{Z}_1| e^{0(1)\mathrm{N}}$. Now for any $k + l \geqslant 2$, $k \geqslant 1$, $l \geqslant 1$ (59) gives
$$\sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_l),\ \sum_s |\mathrm{Y}_s| = l^2\mathrm{N}} |\varphi_\Lambda(\mathbb{Z}_1, \ldots, \mathbb{Z}_k ; \mathbb{Y}_1, \ldots, \mathbb{Y}_l)|$$
$$\leqslant \sum_\Omega \sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_l),\ \sum_s |\mathrm{Y}_s| = l^2\mathrm{N}} 2^{|\Omega|} \tilde{\rho}_\Lambda^{(0)}(\mathrm{Z}_1) \varphi_\Lambda(\mathrm{Z}_2, \ldots, \mathrm{Z}_k, (\mathbb{Y}_s)_{s\in\Omega} ; (\mathbb{Y}_s)_{s\notin\Omega}) \quad (63)$$

and on the right hand side we restrict $\mathbb{Y}_s$, $s \in \Omega$ to clusters overlapping or intersecting Z_1. Using the inductive hypothesis we obtain

$$\sum_{(\mathbb{Y}_1,\dots,\mathbb{Y}_l),\ \sum_s |Y_s| = l^2 N} | \varphi_\Lambda(\mathbb{Z}_1, \dots, \mathbb{Z}_k ; \mathbb{Y}_1, \dots, \mathbb{Y}_l) |$$

$$\leqslant \sum_{\Omega \subsetneq \{1,\dots,l\},\ \Omega \neq \varnothing} \sum_{k=1}^{N-1} \sum_{\substack{(\mathbb{Y}_s)_{s\in\Omega},\ \sum_s |Y_s| = l^2 k, \\ Y_s \text{ overlaps or surrounds } Z_1}} 2^{|\Omega|} \tilde{\rho}_\Lambda^{(0)}(Z_1) \sum_{(\mathbb{Y}_s)_{s\notin\Omega},\ \sum_s |Y_s| = l^2(N-k)}$$

$$\cdot\, \varphi_\Lambda(Z_2, \dots, Z_k, (\mathbb{Y}_s)_{s\in\Omega} ; (\mathbb{Y}_s)_{s\notin\Omega})$$

$$+ \sum_{\substack{(\mathbb{Y}_1,\dots,\mathbb{Y}_l),\ \sum_s |Y_s| = l^2 N, \\ Y_s \text{ overlaps or surrounds } Z_1}} 2^l \tilde{\rho}_\Lambda(Z_1) \varphi_\Lambda(Z_2, \dots, Z_k, \mathbb{Y}_1, \dots, \mathbb{Y}_l ; \varnothing)$$

$$+ \tilde{\rho}_\Lambda(Z_1) \sum_{(\mathbb{Y}_1,\dots,\mathbb{Y}_l),\ \sum_s |Y_s| = l^2 N} \varphi_\Lambda(Z_2, \dots, Z_k ; \mathbb{Y}_1, \dots, \mathbb{Y}_l)$$

$$\leqslant l! \prod_{r=1}^{k} C(Z_r) e^{2l^{-2}\sum_{r=2}^{k} |Z_r|} \left[\sum_{|\Omega|=1}^{l-1} \frac{1}{|\Omega|!} \sum_{k=1}^{N-1} \sum_{\sigma_s \geqslant 1,\, s\in\Omega,\, \sum_s \sigma_s = k} (2l^{-2} | Z_1 |)^{|\Omega|} \right.$$

$$\cdot\, e^{(O(1)+2-C)k - \frac{1}{4}C(l - |\Omega| + N - k)} + \sum_{\sigma_s \geqslant 1,\, s=1,\dots,l,\, \sum_s \sigma_s = N} \frac{1}{l!} (2l^{-2} | Z_1 |)^l e^{(O(1)+2-C)N}$$

$$\left. + e^{-\frac{1}{4}C(l+N)} \right] \leqslant l! \prod_{r=1}^{k} C(Z_r) e^{2l^{-2}\sum_{r=2}^{k} |Z_r| - \frac{1}{4}CN}$$

$$\cdot \left[\sum_{|\Omega|=1}^{l} \frac{1}{|\Omega|!} \sum_{\sigma_s \geqslant 1,\, s\in\Omega} (2l^{-2} | Z_1 |)^{|\Omega|} e^{-\frac{1}{3}C\sum_{s\in\Omega}\sigma_s} e^{-\frac{1}{4}C(l - |\Omega|)} + e^{-\frac{1}{4}Cl} \right]$$

$$\leqslant l! \prod_{r=1}^{k} C(Z_r) e^{2l^{-2}\sum_{r=2}^{k} |Z_r| - \frac{1}{4}C(l+N)} \sum_{|\Omega|=0}^{l} \frac{1}{|\Omega|!} (2l^{-2} | Z_1 |)^{|\Omega|}$$

$$\leqslant l! \sum_{r=1}^{k} C(Z_r) e^{2l^{-2}\sum_{r=1}^{k} |Z_r| - \frac{1}{4}C(l+N)} .$$

(56), (60) and (61) give

$$|S_\Lambda| \leqslant \sum_k \sum_{\{\mathbb{X}_1,\dots,\mathbb{X}_k\}} \sum_l \frac{1}{l!} \sum_{(\mathbb{Y}_1,\dots,\mathbb{Y}_l)} |\varphi_\Lambda(\mathbb{X}_1,\dots,\mathbb{X}_k;\mathbb{Y}_1,\dots,\mathbb{Y}_l)|$$

$$\leqslant 0(1) \sum_k \sum_{\{\mathbb{X}_1,\dots,\mathbb{X}_k\}} \prod_{r=1}^{k} C(X_r) e^{2l^{-2}\sum_{r=1}^{k}|X_r|} . \quad (64)$$

Fix for each $\mathbb{X}_r$ a non-vacuum square Δ_r in X_r. With Δ_r fixed the number of choices of $\mathbb{X}_r - s$ of volume $l^{-2}\sigma_r$ is bounded by $\exp\left(0(1)\sum_r \sigma_r\right)$. But since $X_r - s$ are non-ovelapping for non-zero terms of (56), the number of choices of k $\Delta_r - s$ is bounded by $\binom{I+2J}{k}$. Using also definitions of $C(X_r)$ we obtain

$$|S_\Lambda| \leqslant 0(1) \prod_{i=1}^{I} (0(1)m \|f_i\|_{L^2}) \prod_{j=1}^{J} (0(1)^l \|g_j\|_{H^{-\frac{1}{2}}} \cdot \|h_j\|_{H^{-\frac{1}{2}}})$$

$$\prod_\Delta (I(\Delta)!)^{\frac{1}{2}} \sum_{k=1}^{I+2J} \binom{I+2J}{k} \left(\sum_{\sigma=1}^{\infty} e^{(0(1)+2-C)\sigma} \right)^k$$

$$\leqslant \prod_{i=1}^{I} (0(1)m \|f_i\|_{L^2}) \prod_{j=1}^{J} (0(1)^l \|g_j\|_{H^{-\frac{1}{2}}} \|h_j\|_{H^{-\frac{1}{2}}}) \cdot \prod_\Delta (I(\Delta)!)^{\frac{1}{2}} . \quad (65)$$

This way we have proven.

Lemma II.5. — There exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$ and $m \geqslant m_0(\lambda)$ the first four sums on the right hand side of (51) converge absolutely uniformely in Λ. Moreover (65) holds with constant 0(1) independent of λ, m and Λ.

Let us notice that $\tilde{\rho}_\Lambda(\mathbb{X})$ and $\tilde{\rho}^0_\Lambda(\mathbb{Y})$ are independent of Λ if supp Λ is large enough. Introduce

$$\begin{aligned} \rho(\mathbb{X}) &:= \lim_\Lambda \rho_\Lambda(\mathbb{X}) , \\ \rho^0(\mathbb{Y}) &:= \lim_\Lambda \rho^0_\Lambda(\mathbb{Y}) , \end{aligned} \quad (66)$$

and the same for $\tilde{\rho}$ and $\tilde{\rho}^0$. Just each term on the right hand side of (51) converges in Λ. Together with Lemma 5 this gives.

Proposition II.6. — There exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$

and $m \geqslant m_0(\lambda)$ S_Λ converge with $\Lambda \to 1$ to S given by

$$S = \sum_k \sum_{\{\mathbb{X}_1,\dots,\mathbb{X}_k\}} (\pm 1) \sum_l \frac{1}{l!} \sum_{(\mathbb{Y}_1,\dots,\mathbb{Y}_l)} \sum_{G_c} \cdot \prod_{r_1<r_2} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) \prod_{\mathscr{L}\in G_c} A(\mathscr{L}) \prod_r \tilde{\rho}(\mathbb{X}_r) \prod_s \tilde{\rho}^0(\mathbb{Y}_s) \quad (67)$$

with the first four sums converging absolutely. Moreover

$$| S | \leqslant \prod_{i=1}^{I} (0(1)m \| f_i \|_{L^2}) \prod_{j=1}^{J} (0(1)^l \| g_j \|_{H^{-\frac{1}{2}}} \| h_j \|_{H^{-\frac{1}{2}}}) \prod_{\Delta} (I(\Delta)\,!)^{\frac{1}{2}} . \quad (68)$$

From Proposition 6 it follows at once that volume cut-off Schwinger functions converge in $\mathscr{D}'$ and that their limits are tempered distributions satisfying all Osterwalder-Schrader axioms [*19*] but clustering. Showing clustering we shall complete Proof of Theorem I.1.

PROPOSITION II.7. — Consider a Schwinger function S dependent on two groups of functions supported in disjoint regions. Let S′ and S″ be the Schwinger functions dependent on the first group and the second group of functions respectively. For each $C > 0$ there exists λ_0 such that for all $\lambda \geqslant \lambda_0$, $m \geqslant m_0(\lambda)$ and test functions f_i, g_j and h_j supported in d-lattice squares

$$| S - S'S'' | \leqslant \prod_{i=1}^{I} (0(1)m \| f_i \|_{L^2}) \prod_{j=1}^{J} (0(1)^l \| g_j \|_{H^{-\frac{1}{2}}} \cdot \| h_j \|_{H^{-\frac{1}{2}}}) \prod_{\Delta} (I(\Delta)\,!)^{\frac{1}{2}} e^{-Cl^{-1}d} , \quad (69)$$

where d is the shortest distance between supports of functions of the two groups.

Proof. — Notice that the result of multiplication of the expansions (67) for S′ and S″ may be written as

$$S'S'' = \sum_{k'<k} \sum_{\{\mathbb{X}_1,\dots,\mathbb{X}_{k'},\mathbb{X}_{k'+1},\dots,\mathbb{X}_k\}} (\pm 1) \sum_l \frac{1}{l!} \sum_{(\mathbb{Y}_1,\dots,\mathbb{Y}_l)} \cdot {\sum_{G_c}}' \prod_{1\leqslant r_1<r_2\leqslant k' \text{ or } k'+1\leqslant r_1<r_2\leqslant k} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) \prod_{\mathscr{L}\in G_c} \prod_r \tilde{\rho}(\mathbb{X}_r) \cdot \prod_s \tilde{\rho}^0(\mathbb{Y}_s), \quad (70)$$

where $\mathbb{X}_1, \dots, \mathbb{X}_{k'}(\mathbb{X}_{k'+1}, \dots, \mathbb{X}_k)$ are non-vacuum clusters for S′(S″) containing all supports of the first (second) group of the test functions and in ${\sum_{G_c}}'$ we sum over graphs connected with respect to $\{ \mathbb{X}_1, \dots, \mathbb{X}_k \}$

but for which no path built up of graph lines joins $\{ \mathbb{X}_1, \ldots, \mathbb{X}_{k'} \}$ with $\{ \mathbb{X}_{k'+1}, \ldots, \mathbb{X}_k \}$.

Comparing (67) and (70) it is easy to see that the terms of both with $l^{-2}\left(\sum_{r=1}^{k} | X_r | + \sum_{s=1}^{l} | Y_s | \right) \leqslant l^{-1}d$ coincide. Hence we must estimate the sum of terms of (67) and of (70) with $l^{-2}\left(\sum_r | X_r | + \sum_s | Y_s | \right) \geqslant l^{-1}d$.

Let for $N \geqslant 1$

$$S^N := \sum_k \sum_l \frac{1}{l!} \sum_{\substack{\{\mathbb{X}_1,\ldots,\mathbb{X}_k\},(\mathbb{Y}_1,\ldots,\mathbb{Y}_l),\\ l^{-2}(\sum_r |X_r| + \sum_s |Y_s|) = N}} \varphi(\mathbb{X}_1, \ldots, \mathbb{X}_k ; \mathbb{Y}_T, \ldots, \mathbb{Y}_l), \quad (71)$$

where $\varphi = \lim_\Lambda \varphi_\Lambda$ is given by (55) provided we erase the subscript Λ at $\tilde{\rho}_\Lambda$ and $\tilde{\rho}^0_\Lambda$. Of course φ satisfies (59) and (61) provided the same change is done. Using (61) and repeating the transformations which led us to (65) we obtain

$$| S^N | \leqslant \sum_k \sum_{\{X_1,\ldots,X_k\}, l^{-2}\sum_r |X_r| = N} | \varphi(\mathbb{X}_1, \ldots, \mathbb{X}_k ; \varnothing) |$$
$$+ \sum_{K=1}^{N-1} \sum_k \sum_{\{\mathbb{X}_1,\ldots,\mathbb{X}_k\}, l^{-2}\sum_r |X_r| = K} \sum_{l=1}^{\infty} \frac{1}{l!} \cdot$$
$$\cdot \sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_l), l^2\sum_s |Y_s| = N-K} | \varphi(\mathbb{X}_1, \ldots, \mathbb{X}_k ; \mathbb{Y}_1, \ldots, \mathbb{Y}_l)$$
$$\leqslant 0(1) \prod_{i=1}^{I} (0(1)m \, \| f_i \|_{L^2}) \prod_{j=1}^{J} (0(1)^l \, \| g_j \|_{H^{-\frac{1}{2}}} \, \| h_j \|_{H^{-\frac{1}{2}}})$$
$$\cdot \prod_\Delta (I(\Delta)\,!)^{\frac{1}{2}} \sum_{k=1}^{I+2J} \binom{I+2J}{k} \Bigg[\sum_{\sigma_s \geqslant 1, s=1,\ldots,k, \sum_s \sigma_s = N} e^{(0(1)-C)N}$$
$$+ \sum_{K=1}^{N-1} \sum_{\sigma_s \geqslant 1, s=1,\ldots,k, \sum_s \sigma_s = K} e^{(0(1)+2-C)K - \frac{1}{4}C(N-K)} \Bigg] \leqslant \prod_{i=1}^{I} (0(1)m \, \| f_i \|_{L^2})$$
$$\cdot \prod_{j=1}^{J} (0(1)^l \, \| g_j \|_{H^{-\frac{1}{2}}} \, \| h_j \|_{H^{-\frac{1}{2}}}) \prod_\Delta (I(\Delta)\,!)^{\frac{1}{2}} e^{-\frac{1}{4}CN}. \quad (72)$$

Now with the use of (72) we obtain

$$| S - S'S'' | \leqslant \sum_{N \geqslant l^{-1}d} | S^N | + \sum_{N \geqslant l^{-1}d} \sum_{K=1}^{N-1} | S'^K S''^{N-K} |$$

$$\leqslant \prod_{i=1}^{I} (0(1)m \| f_i \|_{L^2}) \prod_{j=1}^{J} (0(1)^l \| g_j \|_{H^{-\frac{1}{2}}} \| h_j \|_{H^{-\frac{1}{2}}}) \prod_{\Delta} (I(\Delta)\,!)^{\frac{1}{2}} e^{-\frac{1}{5}Cl^{-1}d}, \quad (73)$$

which ends the proof.

PROPOSITION II.8. — Implies immediately the last Osterwalder-Schrader axiom: clustering. Thus the proof of Theorem I.1, with Propositions II.1 and 2 assumed, has been completed. □

CHAPTER III

THE COMBINATORICS

In this chapter we shall perform an analysis of a general term occuring in the cluster expansion and reduce the proof of fundamental Proposition II.1 to the proof of a « linear lower bound » for the interaction measure and to estimates of gaussian integrals.

The general term is $\partial_\tau^\Pi \partial_s^\Gamma ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z}$, where $ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z}$ is given by (II.26). It is necessary to rewrite (II.26) expanding the effective potential in each d-lattice square Δ around $\xi_\pm$ according to the value of $\Sigma(\Delta)$. It means that we make the next shift of the external field in the fermion propagators with the help of the formula

$$(1 - \lambda K^{\xi_+}(s, (\varphi + g - \xi_+)\Lambda\chi_Z))^{-1}(1 - \lambda K(s, \xi_+\chi_Z))^{-1}$$
$$= (1 - \lambda K^h(s, \psi))^{-1}(1 - \lambda K(s, h\chi_Z))^{-1}, \quad (1)$$

where

$$\psi = ((\varphi + g - h)\Lambda + (\xi_+ - h)(1 - \Lambda))\chi_Z, \quad (2)$$

$$K^h(s, \psi) = (1 - \lambda K(s, h))^{-1} K(s, \psi), \quad (3)$$

and h is given by (II.22).

After lengthy but simple transformations we get the formula

$$\partial_\tau^\Pi \partial_s^\Gamma Z S_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z} = \partial_\tau^\Pi \partial_s^\Gamma \int d\mu_{m_c}(\tau_\Pi) \prod_{\Delta\subset Z} \chi_{\Sigma(\Delta)}((\varphi + g)_\Delta^\Lambda)$$

$$\cdot \prod_{\alpha=1}^{I_Z} (\varphi + g)(f_{i_\alpha})\tau_{J_Z}\Big(\Lambda^{J_Z}(1 - \lambda K^h(s_\Gamma, \psi))^{-1} \cdot \bigwedge_{\beta=1}^{J_Z} P_{Z,\beta}(h, s_\Gamma)\Big)$$

$$\cdot \det{}_3 (1 - \lambda K^h(s_\Gamma, \psi)) \exp\Big[-\frac{\lambda^2}{2} : \mathrm{Tr}\, K^h(s_\Gamma, \psi)^2 :_{\tau_\Pi}$$

$$+ \frac{\lambda^2}{2}\int B^{\xi_+}(s_\Gamma) : \psi^2 :_{\tau_\Pi} + \frac{\eta}{2}\int : \psi^2 :_{\tau_\Pi} + \frac{m_c^2 - \eta}{2}\int_Z : (\varphi\Lambda)^2 :_{\tau_\Pi}\Big]$$

$$\cdot \exp\Big[\frac{\lambda^2}{2}\int d\mu_{m_c}(\tau_\Pi)\, \mathrm{Tr}\,(K^{\xi_+}(s_\Gamma, \varphi\Lambda\chi_Z)^2 - K^h(s_\Gamma, \varphi\Lambda\chi_Z)^2)\Big]$$

$$\cdot \exp[-E_1(Z, \Sigma)] \exp[-E_2'(Z, \Sigma)] \exp[-E_2''(Z, \Sigma)] \exp[-F(Z, \Sigma)], \quad (4)$$

where

$$P_{Z,\beta}(h, s_\Gamma) = (1 - \lambda K(s_\Gamma, h\chi_Z))^{-1} P_{Z,\beta}(s_\Gamma), \quad (5)$$

$$E_1(Z, \Sigma) = \lambda^2\, \mathrm{Tr}\, K(s_\Gamma, h) K^h(s_\Gamma, \psi) - \lambda^2 \int B^{\xi_+}(s_\Gamma) h\psi, \quad (6)$$

$$E_2'(Z, \Sigma) = \int_0^\lambda d\lambda' \lambda'^2\, \mathrm{Tr}\,(K^{\lambda'\lambda^{-1}h}(s_\Gamma, h\chi_Z) K(s_\Gamma, h\chi_Z)^2 - K^{\lambda'\lambda^{-1}\xi_+}(s_\Gamma, \xi_+\chi_Z) K(s_\Gamma, \xi_+\chi_Z)^2), \quad (7)$$

$$E_2''(Z, \Sigma) = \frac{\lambda^2}{2}\, \mathrm{Tr}\,(K(s_\Gamma, h\chi_Z)^2 - K(s_\Gamma, \xi_+\chi_Z)^2), \quad (8)$$

$$F(Z, \Sigma) = \sum_{i=1}^4 F_i(Z, \Sigma), \quad (9)$$

$$F_1(Z, \Sigma) = \frac{1}{2}\eta \int_Z (g - h)^2 + \frac{1}{2}\int_Z (\nabla g)^2, \quad (10)$$

$$F_2(Z, \Sigma) = \frac{1}{2}\eta \int_Z (1 - \Lambda)(g - \xi_+)^2 - \frac{1}{2}\eta \int_Z (1 - \Lambda)(g - h)^2 + \frac{1}{2}\eta \int_Z (1 - \Lambda)(h - \xi_+)^2 + \frac{1}{2}(m_c^2 - \eta)\int_Z (1 - \Lambda^2)(g - \xi_+)^2, \quad (11)$$

$$F_3(Z, \Sigma) = \eta \int_Z \varphi(g - h) + \int_Z \varphi(-\Delta)(g - \xi_+) = \int_Z \varphi(-\Delta + \eta)(g - g_c), \quad (12)$$

$$F_4(Z, \Sigma) = \eta \int_Z (1 - \Lambda)\varphi(h - \xi_+) + (m_c^2 - \eta)\int_Z (1 - \Lambda^2)\varphi(g - \xi_+) \quad (13)$$

ang g_c (as well as g) is copied from [13] (formula (4.3)). η is a number between 0 and m_c^2, and will be fixed later.

Now we shall analyze the effect of differentiations in (4). Together with this analysis we shall count the number of terms appearing during these differentiations. This counting will be done by the method of « combinatoric coefficients » (see [9] [10]).

Is is simply an estimate of the form

$$\left|\sum_{i\in I} A(i)\right| \leqslant \sup_{i\in I} c(i)\,|\,A(i)\,|\,, \qquad \text{if} \qquad \sum_{i\in I} c(i)^{-1} \leqslant 1\,. \tag{14}$$

It means that instead of estimating a sum, we have to estimate « a general term » in the sum multiplied by a proper coefficient. We will loosely speak that we fix a term in the sum by the choice of a combinatoric coefficient.

Let us now perform the differentiations. We fix some, arbitrary, order in the set of bonds $b \subset Z$ and we do the differentiations in this order.

In the sequel we shall use widely the notations and results of the paper [3] by A. Cooper and L. Rosen, so our presentation will be sometimes sketchy.

At first let us consider the differentiations with respect to variables s_Γ. They occur in fermion propagators only. Denoting an operator

$$(1 - \lambda K^h(s_\Gamma, \psi))^{-1} \qquad \text{by} \qquad R(s_\Gamma)$$

and the determinant with the exponential functions in (4) by $\rho(s_\Gamma)$, we have a formula:

$$\begin{aligned}
&\frac{\partial}{\partial s_b}\,[G(s_\Gamma)\tau_M(\Lambda^M R(s_\Gamma)\cdot Q(s_\Gamma))\rho(s_\Gamma)] \\
&= \Bigg[\left(\frac{\partial}{\partial s_b} G(s_\Gamma)\right)\tau_M(\Lambda^M R(s_\Gamma)\cdot Q(s_\Gamma)) - G(s_\Gamma)\tau_{M+1}(\Lambda^{M+1}R(s_\Gamma)\cdot Q(s_\Gamma)\wedge A(s_\Gamma, b)) \\
&+ G(s_\Gamma)\tau_M(\Lambda^M R(s_\Gamma)\cdot Q(s_\Gamma) d\Lambda^M E(s_\Gamma, b)) \\
&+ G(s_\Gamma)\tau_M\left(\Lambda^M R(s_\Gamma)\cdot\frac{\partial}{\partial s_b} Q(s_\Gamma)\right) - G(s_\Gamma)\frac{\partial}{\partial s_b} B(s_\Gamma)\tau_M(\Lambda^M R(s_\Gamma)\cdot Q(s_\Gamma))\Bigg]\rho(s_\Gamma)\,,
\end{aligned} \tag{15}$$

where

$$A(s_\Gamma, b) = \lambda^2 K^h(s_\Gamma, \psi)^2 \frac{\partial}{\partial s_b} \lambda K^h(s_\Gamma, \psi)\,, \tag{16}$$

$$E(s_\Gamma, b) = (1 + \lambda K^h(s_\Gamma, \psi))\frac{\partial}{\partial s_b}\lambda K^h(s_\Gamma, \psi)\,, \tag{17}$$

and $B(s_\Gamma)$ is an expression occuring in the exponents of the exponential functions in (4). We write it as a sum of five elementary expressions (corresponding to different exponents). A choice of one of the terms on the right hand side of (15), respecting the decomposition of $B(s_\Gamma)$ into five terms, the decomposition of $E(s_\Gamma, b)$ into two terms, and a choice of one term in a decomposition

$$K^h(s_\Gamma, \psi) = K^h(s_\Gamma, \varphi\Lambda\chi_Z) + K^h(s_\Gamma, ((g - h)\Lambda + (\xi_+ - h)(1 - \Lambda))\chi_Z)$$

in A and E terms, leads to a combinatoric factor $0(1)$, so it leads to a combinatoric factor $0(1)^{|\Gamma|}$ after all differentiations.

We shall call an elementary expression each term occuring in (16), (17), $B(s_\Gamma)$ and $P_{Z,\beta}(h, s_\Gamma)$, also after the transformations described further in this Chapter.

The next operation will be a preliminary localization of the expressions of type A or E (given by (16) or (17)) appearing after a differentiation $\frac{\partial}{\partial s_b}$. We localize in the squares of the d-lattice, it means we have

$$A(s_\Gamma, b) = \sum_{\Delta, \Delta'} \chi_\Delta A(s_\Gamma, b)\chi_{\Delta'} . \tag{18}$$

A localization is fixed according to (14), that is a pair of squares Δ, Δ' is chosen with the help of the combinatoric factor $0(l^4)e^{\varepsilon(d(b,\tilde{\Delta}) + d(b,\tilde{\Delta}'))}$ attached to the considered expression.

By $\tilde{\Delta}$ we denote the square of the l-lattice containing Δ. The inequality

$$\sum_{\tilde{\Delta}} e^{-\varepsilon d(b,\tilde{\Delta})} \leqslant 1 \qquad \text{for} \qquad \varepsilon l \geqslant 1 \tag{19}$$

was used in this fixing, as well as the fact that a square of l-lattice contains $0(l^2)$ squares of the d-lattice. We shall also assume that $\varepsilon \leqslant 1$. We localize the operators $P_{Z,\beta}(h, s_\Gamma)$ also in the d-squares, and we fix a localization $\chi_{\bar{\Delta}_\beta} P_{Z,\beta}(h, s_\Gamma)\chi_{\bar{\Delta}'_\beta}$ with the help of the factor $0(1)e^{d(\bar{\Delta}_\beta,\Delta_\beta)}0(1)e^{d(\Delta'_\beta,\bar{\Delta}'_\beta)}$, where Δ_β, Δ'_β are the squares of the localizations of the functions h_{j_β}, g_{j_β}, see (5) and (I.27). Here the inequality

$$\sum_{\Delta'} e^{-d(\Delta,\Delta')} \leqslant 0(1) \tag{20}$$

was used.

The choice of the next combinatoric factor is connected with the choice of terms in the sums $Q(s_\Gamma)d\Lambda^M E(s_\Gamma, b)$, which occur in the final expression obtained after all fermion differentiations.

We have
$$Q(s_\Gamma) = \bigwedge_{j=1}^{M} Q_j(s_\Gamma)$$

and

$$Q(s_\Gamma)d\Lambda^M E(s_\Gamma, b) = \sum_{j=1}^{M} Q_1(s_\Gamma) \wedge \ldots \wedge Q_j(s_\Gamma)E(s_\Gamma, b) \wedge \ldots \wedge Q_M(s_\Gamma). \tag{21}$$

Every operator $Q_j(s_\Gamma)$ is a product of the operators $\partial_s^{\Gamma_j} P_{Z,\beta}(h, s_\Gamma)$, $\partial_s^{\Gamma_j'} A(s_\Gamma, b')$, $\partial_s^{\Gamma_j''} E(s_\Gamma, b'')$. Let us denote the number of terms of type E left-localized in d-square Δ by $e_L(\Delta)$. When distributed these terms can meet terms of type P, A or E right localized in Δ. Denote respective numbers of such terms by $J_R(\Delta)$, $a_R(\Delta)$ and $e_R(\Delta)$. The number of possible distributions is bounded by

$$\prod_\Delta \binom{J_R(\Delta) + a_R(\Delta) + e_R(\Delta)}{e_L(\Delta)} e_L(\Delta)\,! \leqslant \prod_\Delta 2^{J_R(\Delta)+a_R(\Delta)+e_R(\Delta)} e_L(\Delta)\,! \leqslant 2^{J_Z+|\Gamma|} \prod_\Delta e_L(\Delta)\,! \tag{22}$$

For an l-lattice square $\tilde{\Delta}$ put $e_L(\tilde{\Delta}) = \sum_{\Delta\subset\tilde{\Delta}} e_L(\Delta)$. We have

$$\prod_\Delta e_L(\Delta)\,! \leqslant \prod_{\tilde{\Delta}} e_L(\tilde{\Delta})\,! \leqslant 0(1)^{|\Gamma|} \prod_k e^{\varepsilon d(b_k, \tilde{\Delta}_k)}, \tag{23}$$

where the product $\prod_k$ is the product over differentiations producing E-terms, and Δ_k is the left localization square of $E(s_\Gamma, b_k)$. The second inequality of (23) follows as in Proof of Lemma 10.2 of [*11*]. Thus we can fix one term produced by the distribution of E factors attaching to each such factor the coefficient $e^{\varepsilon d(b_k, \tilde{\Delta}_k)}$ and adding a global coefficient $0(1)^{J_Z+|\Gamma|}$. Consider now the derivatives acting on the Q-term according to (15). Fix those acting on $P_{Z,\beta}(h, s_\Gamma)$ terms, undifferentiated so far, by means of the overall combinatoric factor $2^{|\Gamma|}$. Suppose there are r such derivatives ($r \leqslant J_Z$). The considered derivatives act on arbitrary set R of r P-terms and in arbitrary order. We shall fix the set by the combinatoric factor $0(1)^{J_Z}$. Factor of the same type determines whether the bra or the ket part of each P-term of R is differentiated. Now we shall fix the order in which the derivatives act on the chosen set of the P-terms. To this end consider

$$\sum_\pi \prod_{k=1}^{r} e^{-\varepsilon d(b_k, \tilde{\Delta}_{\pi(k)})}, \tag{24}$$

where π is a one-to-one mapping from $\{1, \ldots, r\}$ to R and $\tilde{\Delta}_{\pi(k)}$ is either right (bra) or left (ket) localization of the $\pi(k)$ term of R, according to the choice made before. Denote by $r(\tilde{\Delta})$ the number of P-terms in R with localization Δ in $\tilde{\Delta}$. Then

$$(24) = \sum_{\substack{\text{partitions } \{R(\tilde{\Delta})\} \text{ or } \{1,\dots,r\} \\ |R(\tilde{\Delta})| = r(\tilde{\Delta})}} \prod_{\tilde{\Delta}} r(\tilde{\Delta})\,! \prod_{k \in R(\tilde{\Delta})} e^{-\varepsilon d(b_k, \tilde{\Delta})}. \tag{25}$$

Copying the argument of Proof of Lemma 10.3 [11] we can estimate

$$\prod_{\tilde{\Delta}} r(\tilde{\Delta})! \prod_{k \in R(\tilde{\Delta})} e^{-\frac{1}{2}\varepsilon d(b_k, \tilde{\Delta})} \leqslant 0(1)^r, \tag{26}$$

hence

$$(24) \leqslant 0(1)^r \sum_{\text{all partitions}} \prod_{\tilde{\Delta}} \prod_{k \in R(\tilde{\Delta})} e^{-\frac{1}{2} d(b_k, \tilde{\Delta})}$$

$$\leqslant 0(1)^r \prod_{k=1}^{r} \left(\sum_{\tilde{\Delta}} e^{-\frac{1}{2}\varepsilon d(b_k, \tilde{\Delta})} \right) = 0(1)^r. \tag{27}$$

Thus we can fix the first derivatives b_k acting on the P-terms together with the order of action by combinatoric factors $e^{\varepsilon d(b_k, \tilde{\tilde{\Delta}})}$ or $e^{\varepsilon d(b_k, \tilde{\tilde{\Delta}}')}$ and an overall factor $0(1)^{J_Z + |\Gamma|}$.

The results of action of all the other derivatives can be described as a sum of terms in which these derivatives act on elementary expressions produced earlier. This sum can be written as a sum over a set $\mathscr{P}'(\Gamma)$ of partitions of Γ. Each element of the partition groups the bonds corresponding to the derivatives acting on one elementary expression.

Thus we get an estimate:

$$\left| \partial_\tau^\Pi \partial_s^\Gamma Z S_{\Lambda, \Sigma, s_\Gamma, \tau_\Pi, Z} \right|$$

$$\leqslant \operatorname{Sup} 0(1)^{J_Z + |\Gamma|} \sum_{\{\Gamma_j\} \in \mathscr{P}'(\Gamma)} \prod_{\beta=1}^{J_Z} e^{0(1)(d(\Delta_\beta, \overline{\Delta}_\beta) + d(\Delta'_\beta, \overline{\Delta}'_\beta))}$$

$$\cdot \prod_{\substack{b:\ \text{first derivatives} \\ \text{of P-terms}}} e^{\varepsilon d(b, \tilde{\tilde{\Delta}} \text{ or } \tilde{\tilde{\Delta}}')} \prod_{\substack{b:\ \text{derivatives} \\ \text{producing A- or E-terms}}} 0(l^4) e^{0(1)\varepsilon(d(b, \tilde{\Delta}) + d(b, \tilde{\Delta}'))}$$

$$\cdot \left| \partial_\tau^\Pi \int d\mu_{m_c}(\tau_\Pi) \prod_{\Delta} \chi_{\Sigma(\Delta)}((\varphi + g)^{\Lambda}_{\Delta}) \prod_{\alpha=1}^{I_Z} (\varphi + g)(f_{i_\alpha}) \right.$$

$$\left. \cdot\, G(s_\Gamma) \tau_K(\Lambda^K R(s_\Gamma) \cdot Q(s_\Gamma)) \rho(s_\Gamma) \right|, \tag{28}$$

where

$$G(s_\Gamma) = \prod_i \partial_s^{\Gamma_i} B_i(s_\Gamma), \tag{29}$$

$$Q(s_\Gamma) = \bigwedge_{j=1}^{K} Q_j(s_\Gamma) \tag{30}$$

and each operator $Q_j(s_\Gamma)$ is a product of the operators $\partial_s^{\Gamma_i} P_{Z,\beta}(h, s_\Gamma)$ or $\partial_s^{\Gamma_j \sim \{b_j\}} A(s_\Gamma, b_j)$, and some number of the operators $\partial_s^{\Gamma_j \sim \{b_j\}} E(s_\Gamma, b_j)$. The lowest upper bound in (28) is taken with respect to all choices of the type of term for each differentiation (or bond $b \in \Gamma$), of the places occupied by the elementary expressions of the type E, of the localizations and of places for first derivatives of the P-terms.

Now we shall analyze the elementary expressions $\partial_s^{\Gamma_j \sim \{b_j\}} A(s_\Gamma, b_j), \ldots$, occuring in $G(s_\Gamma)$ or $Q(s_\Gamma)$ more exactly.

Let us fix the convention that by « a fermion propagator » we mean any of the operators $((\not{P} + m)D^{-1})_s$, $(D^{-1})_s$, $(1 - \lambda K(s_\Gamma, h))^{-1}$ or $(1 - \lambda K(s_\Gamma, \xi_+))^{-1}$ occuring in the definition of $K^h(s_\Gamma, \psi)$ or $K^{\xi_+}(s_\Gamma, \psi)$. In each elementary expression there are at most 0(1) fermion propagators and each differentiation acts on one of them. It can be chosen with the help of a combinatoric factor 0(1), and this choice made for every differentiation gives us a factor $0(1)^{|\Gamma_j|}$. Thus we fix a partition $\Gamma_j = \Gamma_j' \cup \Gamma_j'' \cup \ldots$ and now every differentiation $\partial_s^{\Gamma_j'}, \partial_s^{\Gamma_j''}, \ldots$, acts on a fixed propagator. It acts on a factor $H(s, \tilde{\Delta}, \tilde{\Delta}')$ with the exception of the propagator $(1 - \lambda K(s_\Gamma, h))^{-1}$. For this we have

$$\partial_{s_b}(1 - \lambda K(s_\Gamma, h))^{-1} = (1 - \lambda K(s_\Gamma, h))^{-1} \partial_{s_b} \lambda K(s_\Gamma, h)(1 - \lambda K(s_\Gamma, h))^{-1} \tag{31}$$

and iterating the above formula and introducing the localizations we get

$$\partial_s^{\Gamma'}(1 - \lambda K(s_\Gamma, h))^{-1} = \sum_{\{\gamma'_k\} \in \mathscr{P}(\Gamma')} \sum_{\text{localizations}} \sum_{\text{permutations } \pi \text{ of } \{\gamma'_k\}} \chi_\Delta \cdot \left(\prod_k (1 - \lambda K(s_\Gamma, h))^{-1} \chi_{\Delta_k} \partial_s^{\gamma'_{\pi(k)}} \lambda K(s_\Gamma, h) \chi_{\Delta'_k} \right) (1 - \lambda K(s_\Gamma, h))^{-1} \chi_{\Delta'} . \tag{32}$$

Arguing as when proving (27) we fix one term in the sum over permutations by a combinatoric factor

$$\prod_k 0(1) e^{\varepsilon d(b_{\pi(k)}, \tilde{\Delta}_k)} \tag{33}$$

where $b_k \in \gamma'_k$.

Finally, each differentiation in $\partial_s^{\gamma'_k}$ acts on one of the three propagators occuring in $K(s_\Gamma, h)$ and it can be chosen by the factor 3, so we can fix a partition of γ'_k into at most three subsets, corresponding to the differentiations of propagator, with the help of the factor $3^{|\gamma'_k|}$.

Summation over the partitions of Γ_j considered above, together with summation over the partitions $\{ \Gamma_j \} \in \mathscr{P}'(\Gamma)$, give us the sum over a certain class of partitions $\{ \gamma_j \}$ of Γ. We will denote this class by $\mathscr{P}'(\Gamma)$ again. Now each differentiation $\partial_s^{\gamma_j}$ acts on a definite fermion propagator, obviously with the exception of the propagators $(1 - \lambda K(s_\Gamma, f))^{-1}, f = h$ or ξ_+.

The last step will consist of complete localization of the under-integral expression, that is of localization of every fermion propagator. It is done with the help of combinatoric factors $0(1)e^{d(\Delta,\Delta')}$.

We decompose the combinatoric factors attached to some elementary expression into the factors attached to propagators and vertices occuring in this expression. We do this the following way: if a square of localization Δ is connected with a propagator differentiated with respect to s_b by a chain of propagators localized successively in $\Delta, \Delta_1, \ldots, \Delta_k$, then

$$\begin{aligned} e^{0(l)}e^{\varepsilon d(\tilde{\Delta},b)} &\leqslant e^{0(l)}e^{d(\tilde{\Delta},\tilde{\Delta}_1)} \cdot \ldots \cdot e^{0(l)}e^{d(\tilde{\Delta}_{k-1},\tilde{\Delta}_k)}e^{0(l)}e^{\varepsilon d(\tilde{\Delta}_k,b)} \\ &\leqslant e^{0(l)}e^{d(\Delta,\Delta_1)} \cdot \ldots \cdot e^{0(l)}e^{d(\Delta_{k-1},\Delta_k)}e^{0(l)}e^{\varepsilon d(\tilde{\Delta}_k,b)} \end{aligned} \tag{34}$$

and we attach the localization factors $e^{d(\Delta_i,\Delta_{i+1})}$, $e^{\varepsilon d(\tilde{\Delta}_k,b)}$ to the corresponding propagators. We will have at most $0(1)$ factors of this type for one propagator. The factors $e^{0(l)}$ will be attached to the Yukawa vertices $\lambda\Gamma$.

This way the following inequality is obtained

$$\begin{aligned} | \partial_\tau^\Pi \partial_s^\Gamma Z S_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z} | &\leqslant \operatorname{Sup} 0(1)^{J_Z + |\Gamma|} \sum_{\{\gamma_j\}\in\mathscr{P}'(\Gamma)} \prod_{\text{fermion propagators}} e^{0(1)d(\Delta,\Delta')} \prod_{\text{vertices}} e^{0(l)} \\ &\prod_{\substack{\text{differentiated} \\ \text{fermion propagators}}} e^{0(1)\varepsilon(d(b,\tilde{\Delta}) + d(b,\tilde{\Delta}'))} \left| \partial_\tau^\Pi \int d\mu_{m_c}(\tau_\Pi) \prod_{\Delta} \chi_{\Sigma(\Delta)((\varphi+g)_\Delta^\wedge)} \right. \\ &\left. \cdot \prod_{\alpha=1}^{I_Z} (\varphi + g)(f_{i_\alpha}) G(s_\Gamma) \tau_K(\Lambda^K R(s_\Gamma) \cdot Q(s_\Gamma)) \rho(s_\Gamma) \right|, \end{aligned} \tag{35}$$

where G, R, Q and ρ have the same structure as previously, only the elementary expressions have a more complicated one. Nevertheless we shall keep denoting them by the same symbols as $\partial_s^{\Gamma_j \sim \{b_j\}} A(s_\Gamma, b_j)$ and so on. Since the number of factors $e^{0(l)}$, $0(l^4)$ and $0(1)$ per single vertex was bounded, we have gathered all of them into a single $e^{0(l)}$ coefficient.

Now let us analyze the effect of differentiations ∂_τ^Π. We have the formula

$$\frac{\partial}{\partial\tau_b}\int d\mu_{m_c}(\tau)\mathrm{R}(\tau) = \int d\mu_{m_c}(\tau)\left[\frac{\partial}{\partial\tau_b}\mathrm{R}(\tau) + \frac{1}{2}\int dxdy\,\frac{\partial}{\partial\tau_b}\mathrm{C}_{m_c}(\tau\,;\,x,y)\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}\mathrm{R}(\tau)\right]. \tag{36}$$

At first let us notice that applying the operation in the square bracket on the right hand side of (36) to a Wick polynomial of the second order in the field φ we get 0. Thus the only new term appearing in $\frac{\partial}{\partial\tau_b}\mathrm{R}(\tau)$ is

$$\frac{\lambda^2}{2}\frac{\partial}{\partial\tau_b}\int d\mu_{m_c}(\tau)\,\mathrm{Tr}\,(\mathrm{K}^{\xi+}(s_\Gamma,\,\varphi\Lambda\chi_Z)^2 - \mathrm{K}^h(s_\Gamma,\,\varphi\Lambda\chi_Z)^2)\,, \tag{37}$$

and the differentiation $\frac{\partial}{\partial\tau_b}$ can act also on the terms having appeared earlier, so either on the term of the form

$$\frac{\lambda^2}{2}\,\partial_\tau^{\Pi_j}\partial_s^{\Gamma_i}\int d\mu_{m_c}(\tau)\,\mathrm{Tr}\,(\mathrm{K}^{\xi+}(s_\Gamma,\,\varphi\Lambda\chi_Z)^2 - \mathrm{K}^h(s_\Gamma,\,\varphi\Lambda\chi_Z)^2)\,, \tag{38}$$

or on the propagator $\partial_\tau^{\Pi_j}\mathrm{C}_{m_c}(\tau)$ coming from the second part of the operation in (36).

Hence the effect of differentiation $\partial_\tau^{\Pi_j}$ can be written in the form

$$\partial_\tau^\Pi\int d\mu_{m_c}(\tau)\mathrm{R}(\tau) = \sum_{\{\pi_j\}\in\mathscr{P}^0(\Pi)}\int d\mu_{m_c}(\tau)\prod_j\frac{1}{2}\int dxdy\,\partial_\tau^{\Pi_j}\mathrm{C}_{m_c}(\tau;x,y)\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}\,\partial_\tau^{\Pi\sim\mathrm{U}\pi_j}\mathrm{R}(\tau)\,, \tag{39}$$

where $\mathscr{P}^0(\Pi)$ denotes the set of incomplete partitions, it means that $\mathrm{U}\pi_j \subset \Pi$ only, an operator of functional differentiation of the second order for a given j does not act on Wick polynomials of second order (obviously it can act with one differentiation on each such polynomial), and a differentiation ∂_{τ_b} for $b\in\Pi\sim\mathrm{U}\pi_j$ either generates a new term of the form (37), or acts on the old term of the form (38).

A combinatoric factor $3^{|\Pi|}$ fixes a partition of Π into three subsets: $\Pi = \Pi'\cup\Pi''\cup\Pi'''$, where $\Pi' = \mathrm{U}\pi_j$, Π'' corresponds to differentiations acting on the new terms (37), and Π''' to differentiations acting on the old terms (38) which were produced by fermion differentiations. The effect of the differentiation $\partial_\tau^{\Pi''}$ can be written in the form

$$\sum_{\{\pi_i\}\in\mathscr{P}(\Pi'')}\prod_i\frac{\lambda^2}{2}\,\partial_\tau^{\Pi_i}\int d\mu_{m_c}(\tau)\,\mathrm{Tr}\,(\mathrm{K}^{\xi+}(s_\Gamma,\,\varphi\Lambda\chi_Z)^2 - \mathrm{K}^h(s_\Gamma,\,\varphi\Lambda\chi_Z)^2)\,. \tag{40}$$

The differentiation $\partial_\tau^{\Pi'''}$ generates a slightly more complicated situation. Together with the sum over partitions $\{ \pi_i \}$ of Π''' there is a sum over all choices of the differentiated terms, or, which is the same, over all choices of the corresponding $\{ \Gamma_i \}$. We can fix one such choice. If $b_i \in \pi_i$ and $b_i' \in \Gamma_i$, then we make a choice with the help of a combinatoric factor

$$\prod_i 0(1)e^{\varepsilon d(b_i,b_i')}, \tag{41}$$

which is next divided similarly as in (34), into factors attached to propagators and vertices of the corresponding expression. Thus we are left with the summation over partitions of Π'''. We will write these three summations over the partitions of the sets Π', Π'', Π''' as summation over the corresponding set $\mathscr{P}'(\Pi)$ of partitions of Π. For a functional differentiation $\dfrac{\partial}{\delta\varphi(x)}$ we have the formula analogous to (15), (16) and (17), so the types and number of terms appearing after the differentiations is controlled by similar combinatoric factors as in (28). Thus we localize the propagators $\partial_\tau^\Pi C_{m_c}(\tau_\Pi)$ in the squares $\tilde{\Delta}$, $\tilde{\Delta}'$ of the l-lattice by a factor $0(1)e^{\varepsilon(d(b,\tilde{\Delta})+d(b,\tilde{\Delta}'))}$, $b \in \pi$. Next, if the differentiation $\dfrac{\delta}{\delta\varphi(x)}$ produces a new elementary expression of the type A, E or B, then we localize the propagator additionally in the squares of the d-lattice with the help of a factor $0(l^4)$. We localize also the whole expression (every propagator in it) by the localization factors $0(1)e^{d(\Delta,\Delta')}$, with the exception of the expressions of type B obtained by the differentiation of the Wick polynomial of second order in the exponential (the term in the first exponential). The localization for this expression will be analyzed in Chapter IV. Further, the type of a term is fixed by a factor $0(1)^{|\Pi|}$. Applying similar combinatoric factors as for fermion differentiations we fix the position of E-operators as well as elementary expression differentiated by $\dfrac{\delta}{\delta\varphi(x)}$ in the case when it acts on C or G. As in (34) we divide these factors into the corresponding factors attached to propagators and vertices. Each elementary expression is a polynomial of an order at most 3 with respect to φ, so the above procedure will enlarge a power of the combinatoric factors by 3. The only new types of terms are these, in which differentiations act on

$$\prod_\Delta \chi_{\Sigma(\Delta)}\big((\varphi + g)^\wedge_\Delta\big) \prod_{\alpha=1}^{I_Z} (\varphi + g)(f_{i_\alpha}),$$

or on $e^{-F(Z,\Sigma)}$. A choice of the differentiated function $\chi_{\Sigma(\Delta)}\big((\varphi + g)^\wedge_\Delta\big)$ or

$(\varphi + g)(f_{i_\alpha})$ is made by a factor $0(l^2)e^{\varepsilon d(b,\tilde{\Delta})}$ with an additional overall factor $0(1)^{Iz}$, by arguments similar to that used in (22)-(27).

In order to write the underintegral expression after all differentiations in a compact form, and for the estimation of it, it is convenient to introduce integrations with respect to additional gaussian measures instead of propagators $\partial_\tau^\Pi C_{m_c}(\tau)$. The problem is that these propagators, for $|\pi| > 1$, are not positive operators. We will proceed further in the exactly same manner as L. Rosen, A. Cooper in their paper [3], and will use extensively their results concerning propagators. Thus a propagator $\partial_\tau^\pi C_{m_c}(\tau)$ may be written in two ways as a linear combination of positive operators. The first representation is connected with the necessity to obtain estimates with exponential localization factors. We have

$$\partial_\tau^\pi C_{m_c}(\tau) = \partial_{\tau_b}\partial_\tau^{\pi\sim\{b\}}C_{m_c}(\tau) = \sum_{\rho\subset\pi\sim\{b\}} (-1)^{|\pi|-|\rho|-1}C_{m_c}^{\pi,\rho,b}(\tau)\,, \tag{42}$$

where $C_{m_c}^{\pi,\rho,b}(\tau) = \partial_{\tau_b}C_{m_c}(\tau)\upharpoonright_{\tau=1 \text{ on } \rho,\ \tau=0 \text{ on } \pi\sim(\rho\cup\{b\})}$. Each $C_{m_c}^{\pi,\rho,b}(\tau)$ is a positive operator, hence if defines a Gaussian measure, and we can write an expression containing $\partial_\tau^\pi C_{m_c}(\tau)$ as a sum of $2^{|\pi|-1}$ terms, in which the corresponding propagators are replaced by the integrals

$$\partial_\tau^\pi C_{m_c}(\tau\,; x, y) = \sum_{\rho\subset\pi\sim\{b\}} (-1)^{|\pi|-|\rho|-1}\int d\mu_{m_c}^{\pi,\rho,b}(\tau)\varphi^{\pi,\rho,b}(x)\varphi^{\pi,\rho,b}(y)\,. \tag{43}$$

A bond b can be chosen in such a way, that

$$d(b, \tilde{\Delta}) + d(b, \tilde{\Delta}') = \max_{b'\in\pi}\,(d(b', \tilde{\Delta}) + d(b', \tilde{\Delta}')) =: d(\pi, \tilde{\Delta}, \tilde{\Delta}')\,, \tag{44}$$

and

$$\|\, \chi_{\tilde{\Delta}} C_{m_c}^{\pi,\rho,b}(\tau)\chi_{\tilde{\Delta}'}\, \|_{L^q(\mathbb{R}^4)} \leqslant 0(l^{\frac{4}{q}})e^{-\delta m_c d(\pi,\tilde{\Delta},\tilde{\Delta}')} \tag{45}$$

for some $\delta > 0$ (which can be made arbitrarily close to 1). A second representation will be used for containing the estimates with exponentially sentation will be used for obtaining the estimates with exponentially small factors for $|\pi|$ large. Precisely: for each $q < +\infty$ there exists two non-negative operators $C_{m_c}^{\pi,\pm}(\tau)$ on the space $L^2(\mathbb{R}^2)$, such that

$$\partial_\tau^\pi C_{m_c}(\tau) = C_{m_c}^{\pi,+}(\tau) - C_{m_c}^{\pi,-}(\tau)\,, \tag{46}$$

$$\|\, C_{m_c}^{\pi,\pm}(\tau)\, \|_{L^q(\mathbb{R}^4)} \leqslant 0(l^{\frac{4}{q}})0(1)^{|\pi|}G_1(\pi, \delta) \tag{47}$$

for some $\delta > 0$, where G_1 has the property

$$G_1(\pi, \delta) \leqslant G_1(\pi, \delta_1)e^{-\delta_2 l|\pi|} \qquad \text{for} \qquad |\pi| > 7,\ \delta_1 + 14\delta_2 = \delta\,, \tag{48}$$

$$\sum_{\{\pi_j\}\in\mathscr{P}(\Pi)}\prod_j G_1(\pi_j, \delta') \leqslant 0(1)^{|\Pi|} \qquad \text{for arbitrary } \delta' > 0\,. \tag{49}$$

The decomposition (46) can be constructed by the method described in Section VI.8 of [3] and the above estimates can be obtained from Corollary VI.2 of this paper by scaling. The factor G_1 and its properties are described in Section IV of [3], (IV. 23-26).

The operators $C^{\pi,\pm}_{m_c}(\tau)$ define Gaussian measures and we have a representation similar to (43):

$$\partial^{\pi}_{\tau} C_{m_c}(\tau\,; x, y) = \int d\mu^{\pi,+}_{m_c}(\tau)\varphi^{\pi,+}(x)\varphi^{\pi,+}(y) - \int d\mu^{\pi,-}_{m_c}(\tau)\varphi^{\pi,-}(x)\varphi^{\pi,-}(y)\,. \quad (50)$$

We can now replace all propagators $\partial^{\pi}_{\tau} C_{m_c}(\tau)$ appearing after the differentiations by the right hand sides of the formulae (43) and (50). This way we get two expressions depending on the formula used. In the first case there is an integral with respect to the measure

$$\sum_{\{\rho_j\},\, \rho_j \subset \pi_j \sim \{b_j\}} \bigotimes_j d\mu^{\pi_j,\rho_j,b_j}_{m_c}(\tau) \quad (51)$$

in the second case with respect to the measure

$$\sum_{\{\varepsilon_j\},\, \varepsilon_j = \pm} \bigotimes_j d\mu^{\pi_j,\varepsilon_j}_{m_c}(\tau)\,. \quad (52)$$

A term in the sum (51) is fixed by a combinatoric factor $\prod_j 2^{|\pi_j|-1} \leqslant 2^{|\Pi|}$.

Similarly a term in (52) is fixed by $\prod_j 2 \leqslant \prod_j 2^{|\pi_j|} \leqslant 2^{|\Pi|}$.

Each of the two expressions obtained this way will be estimated separately and the final estimation will be the geometric mean of both.

Denote any of the measures $d\mu^{\pi_j,\rho_j,b_j}_{m_c}(\tau)$ and $d\mu^{\pi_j,\varepsilon_j}_{m_c}(\tau)$ by $d\mu^{\pi_j}_{m_c}(\tau)$, similarly for the fields $\varphi^{\pi_j,\rho_j,b_j}$ and $\varphi^{\pi_j,\varepsilon_j}$.

Thus we obtain the inequality:

$$\begin{aligned}
&| \partial^{\Pi}_{\tau}\partial^{\Gamma}_{s} ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z} | \\
&\quad \leqslant \text{Sup}\ 0(1)^{I_Z + J_Z + |\Gamma| + |\Pi|} \sum_{\{\gamma_j\}\in\mathscr{P}'(\Gamma),\, \{\pi_j\}\in\mathscr{P}'(\Pi)} \prod_{\Delta} 0(l^2)^{n_\Delta} \\
&\quad \cdot \prod_{\text{fermion propagators}} e^{0(1)d(\Delta,\Delta')} \prod_{\substack{\text{fermion propagators}\\ \text{differentiated}}} e^{0(1)\varepsilon(d(b,\tilde{\Delta}) + d(b,\tilde{\Delta}'))} \prod_{\text{the fields } \varphi^{\pi_j}} e^{0(1)\varepsilon d(b,\tilde{\Delta})} \\
&\quad \cdot \prod_{\text{vertices}} e^{0(l)} \prod_{i'} 0(1) \left| \int d\mu_{m_c}(\tau) \otimes \bigotimes d\mu^{\pi_j}_{m_c}(\tau) \prod_{\Delta} \chi^{(n_\Delta)}_{\Sigma(\Delta)}((\varphi + g)^{\Lambda}_{\Delta}) \right. \\
&\quad \cdot {\prod_{\alpha}}' (\varphi + g)(f_{i_\alpha}) {\prod_{\alpha}}'' \varphi^{\pi_\alpha}(f_{i_\alpha}) \prod_{\Delta} {\prod_{i=1}^{n_\Delta}}''' (\varphi^{\pi_i}\Lambda)_\Delta \prod_{i'} F_{3,4}(\tilde{\Delta}_{i'}, \Sigma, \varphi^{\pi_{i'}}) \\
&\quad \cdot G(s_\Gamma, \tau_\Pi)\tau_K(\Lambda^K R(s_\Gamma) \cdot Q(s_\Gamma))\rho(s_\Gamma, \tau_\Pi) \Big| \,, \qquad (53)
\end{aligned}$$

where

$$F_{3,4}(\tilde{\Delta}, \Sigma, \varphi^{\pi}) = \int_{\tilde{\Delta}} \varphi^{\pi}(-\Delta + \eta)(g - g_c) + \eta \int_{\tilde{\Delta}} \varphi^{\pi}(1 - \Lambda)(h - \xi_+)$$
$$+ (m_c^2 - \eta) \int_{\tilde{\Delta}} \varphi^{\pi}(1 - \Lambda^2)(g - \xi_+) . \quad (54)$$

The meaning of the remaining symbols in (53) should be clear from the previous discussion.

The combinatoric factors in (53) are naturally divided into groups corresponding to elementary expressions (except the overall ones).

Now we can formulate the inequalities necessary for proof of Proposition II.1. We estimate at first the integral on the right hand side of (53) using the Hölder inequality:

$$\left| \int d\mu_{m_c}(\tau) \otimes \bigotimes_j d\mu_{m_c}^{\pi_j}(\tau) \prod_{\Delta} \chi_{\Sigma(\Delta)}^{(n_\Delta)}((\varphi + g)_{\Delta}^{\Lambda}) \right.$$
$$\cdot \prod_{\alpha}{}' (\varphi + g)(f_{i_\alpha}) \prod_{\alpha}{}'' \varphi^{\pi_\alpha}(f_{i_\alpha}) \prod_{\Delta} \prod_{i}{}''' (\varphi^{\pi_i}\Lambda)_{\Delta}$$
$$\left. \cdot \prod_{i'} F_{3,4}(\tilde{\Delta}_{i'}, \Sigma, \varphi^{\pi_{i'}}) G(s_\Gamma, \tau_\Pi) \tau_K(\Lambda^K R(s_\Gamma) \cdot Q(s_\Gamma)) \rho(s_\Gamma, \tau_\Pi) \right|$$
$$\leqslant \left\| \prod_{\Delta} \chi_{\Sigma(\Delta)}^{(n_\Delta)}((\varphi + g)_{\Delta}^{\Lambda}) \tau_K(\Lambda^K R(s_\Gamma) \cdot Q(s_\Gamma)) \rho(s_\Gamma, \tau_\Pi) \right\|_{L^p}$$
$$\cdot \left\| \prod_{\Delta}{}' (\varphi + g)(f_{i_\alpha}) \right\|_{L^q} \left\| \prod_{\alpha}{}'' \varphi^{\pi_\alpha}(f_{i_\alpha}) \right\|_{L^q} \left\| \prod_{\Delta} \prod_{i}{}''' (\varphi^{\pi_i}\Lambda)_{\Delta} \right\|_{L^q}$$
$$\cdot \left\| \prod_{i'} F_{3,4}(\tilde{\Delta}, \Sigma, \varphi^{\pi_{i'}}) \right\|_{L^q} \| G(s_\Gamma, \tau_\Pi) \|_{L^q} \quad (55)$$

for some $p > 1$ and $\frac{1}{p} + \frac{5}{q} \leqslant 1$.

This inequality will be used next for p close to 1 and then q can be chosen as an integer divisible by 4.

The central technical results of the paper are contained in two theorems below.

THEOREM III.1. — (« Linear lower bound »). If the diameter d of the d-lattice is sufficiently small then there exists $p > 1$ such that

$$\left\| \prod_{\Delta} \chi_{\Sigma(\Delta)}^{(n_\Delta)}((\varphi + g)_{\Delta}^{\Lambda})\tau_{\mathrm{K}}\left(\Lambda^{\mathrm{K}}\mathrm{R}(s_\Gamma)\cdot \bigwedge_{j=1}^{\mathrm{K}} \mathrm{Q}_j\right)\rho(s_\Gamma, \tau_\Pi) \right\|_{\mathrm{L}^p}$$

$$\leqslant 0(1)^{\mathrm{K}} \prod_{\Delta} e^{0(1)n_\Delta}(n_\Delta\,!)^{0(1)} \exp\left[-0(1)m^{2-\alpha}\,|\,\mathrm{Z}^0 \cup \mathrm{Z}'\,|\right.$$

$$\left. + \, 0(1)m^{-\nu}\,|\,\mathrm{Z}\,|\,\right] \left\| \prod_{j=1}^{\mathrm{K}} \|\,\mathrm{Q}_j\,\|_1 \right\|_{\mathrm{L}^q} \tag{56}$$

for some $\nu > 0$ and every $\alpha > 0$, provided $m \geqslant m_0(\lambda, \alpha)$, with constants $0(1)$ independent of λ and m.

Z^0 is the set of d-squares in Z such that with in distance L Σ is not constant.

Z' is the set of d-squares Δ in Z such that $\chi_{\Sigma(\Delta)}$ is differentiated at least once.

THEOREM III.2. — (« Gaussian integration estimate »). For every expression of the class to which the lowest upper bound in (53) applies, the following inequality holds:

$$0(1)^{\mathrm{I_Z}+\mathrm{J_Z}+|\Gamma|+|\Pi|} \sum_{\{\gamma_j\}\in\mathscr{P}'(\Gamma),\{\pi_j\}\in\mathscr{P}'(\Pi)} \prod_{\Delta} 0(l^2)^n\,(n_\Delta\,!)^{0(1)}$$

$$\cdot \prod_{\text{fermion propagators}} e^{0(1)d(\Delta,\Delta')} \prod_{\text{fermion propagators differentiated}} e^{0(1)\varepsilon(d(b,\tilde{\Delta})+d(b,\tilde{\Delta}'))}$$

$$\cdot \prod_{\text{the fields } \varphi^{\pi_j}} e^{0(1)\varepsilon d(b,\tilde{\Delta})} \prod_{\text{vertices}} e^{0(l)} \prod_{i'} 0(1) \left\| \prod_{\alpha}{}' (\varphi + g)(f_{i_\alpha}) \right\|_{\mathrm{L}^q}$$

$$\cdot \left\| \prod_{\alpha}{}'' \varphi^{\pi_\alpha}(f_{i_\alpha}) \right\|_{\mathrm{L}^q}^{*} \left\| \prod_{\Delta} \prod_{i}{}''' (\varphi^{\pi_i}\Lambda)_\Delta \right\|_{\mathrm{L}^q}^{*}$$

$$\cdot \left\| \prod_{i'} \mathrm{F}_{3,4}(\tilde{\Delta}_{i'}, \Sigma, \varphi^{\pi_i'}) \right\|_{\mathrm{L}^q}^{*} \left\| \prod_{j=1}^{\mathrm{K}} \|\,\mathrm{Q}_j\,\|_1 \right\|_{\mathrm{L}^q}^{*} \|\,\mathrm{G}(s_\Gamma, \tau_\Pi)\,\|_{\mathrm{L}^q}^{*}$$

$$\leqslant 0(\xi_+)^{\mathrm{I_Z}} e^{0(l)(\mathrm{I_Z}+\mathrm{J_Z})} \left(\prod_{\Delta\subset \mathrm{Z}} \mathrm{I}(\Delta)\,! \right)^{\frac{1}{2}} e^{0(l)|\mathrm{Z}'|}$$

$$\cdot \, e^{-\delta_0 l(|\Gamma|+|\Pi|)} \prod_{\alpha} \|\,f_{i_\alpha}\,\|_{\mathrm{L}^2} \prod_{\beta} \|\,g_{j_\beta}\,\|_{\mathrm{H}^{-\frac{1}{2}}} \,\|\,h_{j_\beta'}\,\|_{\mathrm{H}^{-\frac{1}{2}}} \tag{57}$$

for some $\delta_0 > 0$, $\varepsilon > 0$ and for all $\lambda \geqslant \lambda_0$ and $m \geqslant m_0(\lambda)$.

$\| \cdot \|^{*}_{L^q}$ denotes the geometric mean of the corresponding norms for two possible choices of the measures (51), (52).

These two theorems and the inequalities (54), (55) imply

$$| \partial_\tau^\Pi \partial_s^\Gamma ZS_{\Lambda,\Sigma,s_\Gamma,\tau_\Pi,Z} | \leqslant 0(\xi_+)^{I_Z} e^{0(l)(I_Z+J_Z)}$$

$$\cdot \Big(\prod_{\Delta \subset Z} I(\Delta)\,! \Big)^{\frac{1}{2}} \exp\left[-0(1)m^{2-\alpha}|Z^0|+0(1)m^{-\nu}|Z|-\delta_0 l(|\Gamma|+|\Pi|)\right]$$

$$\prod_\alpha \| f_{i_\alpha} \|_{L^2} \prod_\beta \| g_{j_\beta} \|_{H^{-\frac{1}{2}}} \| h_{j'_\beta} \|_{H^{-\frac{1}{2}}} \tag{58}$$

Our aim is estimation of $\rho_\Lambda(\mathbb{Z})$ given by (II.36). For given Σ the derivation bonds of Γ and Π are « dense » in $Z \sim Z^0$ which results in the estimate

$$l^{-2}|Z \sim Z^0| \leqslant 0(1)(|\Gamma| + |\Pi| + I_Z + J_Z). \tag{59}$$

Thus the sum over Γ and Π of $\exp\,[-0(1)m^{2-\alpha}|Z^0| - \delta_0 l(|\Gamma|+|\Pi|)]$ may be estimated by $\exp\left[-0(1)m^{2-\alpha}|Z^0|+\frac{1}{2}\delta_0 l(I_Z+J_Z)-\delta_1 l^{-1}|Z\sim Z^0|\right]$ which after summation over choices of Z^0 and of $\Sigma \restriction_{Z_0}$ (Z^0 together with $\Sigma \restriction_{Z_0}$ determine Σ) is bounded in turn by $\exp\left[\frac{1}{2}\delta_0 l(I_Z+J_Z)-\delta_2 l^{-1}|Z|\right]$. This together with (II.36) and (58) yields Proposition II.1 if we notice that $0(\xi_+)e^{0(l)} \leqslant 0(m)$.

CHAPTER IV

GAUSSIAN INTEGRATION ESTIMATE

In this chapter we shall prove Theorem III.2. The proof will proceed in several steps.

1. At first let us estimate the norms of the products of fields φ and φ^{π_i} occuring on the left hand side of (III.57). The estimate of the first factor is standard. We use the checkerboard estimate for $d\mu_{m_c}(\tau)$ (uniform in τ) which can be easily obtained by adapting the version proven in [*14*], the Hölder inequality and the hypercontractivity:

$$\Big\| \prod_\alpha{}' (\varphi+g)(f_{i_\alpha}) \Big\|_{L^q} \leqslant \prod_\Delta \Big\| \prod_{\alpha=1}^{I(\Delta)} (\varphi+g)(f_{i_\alpha}) \Big\|_{L^{q'}} \leqslant \prod_\Delta \prod_{\alpha=1}^{I(\Delta)} \|(\varphi+g)(f_{i_\alpha})\|_{L^{q'I(\Delta)}}$$

$$\leqslant \prod_\Delta \prod_{\alpha=1}^{I(\Delta)} (0(I(\Delta))^{\frac{1}{2}} \| \varphi(f_{i_\alpha}) \|_{L^2} + |g(f_{i_\alpha})|) \leqslant \prod_\Delta (I(\Delta)\,!)^{\frac{1}{2}} 0(\xi_+)^{I(\Delta)} \prod_{\alpha=1}^{I(\Delta)} \| f_{i_\alpha} \|_{L^2(\mathbb{R}^2)} \tag{1}$$

In the future the following remark will be very useful: for arbitrary π_j the field φ^{π_j} occurs only twice in the whole expression in (III.53).

Hence every such field occurs at most twice in the second, third and fourth norms in (III.57), so making use of the Fubini theorem and Schwartz inequality we can factorize completely these norms:

$$\left\| \prod_\alpha{}'' \varphi^{\pi_\alpha}(f_{i_\alpha}) \right\|_{L^q}^* \leqslant \prod_\alpha{}'' \| \varphi^{\pi_\alpha}(f_{i_\alpha}) \|_{L^{2q}}^*$$

$$\leqslant \prod_\alpha{}'' \| f_{i_\alpha} \|_{L^2} e^{0(l)} e^{-\frac{1}{4}\delta m_c d(\pi_\alpha, \tilde{\Delta}_{i_\alpha}, \tilde{\Delta}_{i_\alpha})} G_1\left(\pi_\alpha, \frac{1}{4}\delta \right) 0(1)^{|\pi_\alpha|} \quad (2)$$

The following inequality holds for G_1-factors:

$$G_1(\pi, \delta) \leqslant G_1(\pi, \delta_1) e^{-\delta_2 l |\pi|} e^{0(l)}, \qquad \delta = \delta_1 + 14\delta_2 . \quad (3)$$

It follows from (III.48) and will be systematically used. From (1), (2) and (3) we obtain

$$\left\| \prod_\alpha{}' (\varphi + g)(f_{i_\alpha}) \right\|_{L^q} \left\| \prod_\alpha{}'' \varphi^{\pi_\alpha}(f_{i_\alpha}) \right\|_{L^q}^* \leqslant 0(\xi_+)^{I_Z} e^{0(l) I_Z}$$

$$\cdot \left(\prod_{\Delta \subset Z} I(\Delta)! \right)^{\frac{1}{2}} \prod_\alpha{}' \| f_{i_\alpha} \|_{L^2} \prod_\alpha{}'' 0(1)^{|\pi_\alpha|} e^{-\frac{1}{4}\delta m_c d(\pi_\alpha, \tilde{\Delta}_{i_\alpha}, \tilde{\Delta}_{i_\alpha})} G_1(\pi_\alpha, \delta_1) e^{-\delta_2 l |\pi_\alpha|} \quad (4)$$

A similar estimation of the third factor gives us

$$\left\| \prod_\Delta \prod_i{}''' (\varphi^{\pi_i}\Lambda)_\Delta \right\|_{L^q}^* \leqslant \prod_\Delta \prod_i{}''' \left\| (\varphi^{\pi_i}\Lambda)_\Delta \right\|_{L^{2q}}^*$$

$$\leqslant \prod_\Delta e^{0(l) n_\Delta} \prod_i{}''' 0(1)^{|\pi_i|} e^{-\frac{1}{4}\delta m_c d(\pi_i, \tilde{\Delta}, \tilde{\Delta})} G_1(\pi_i, \delta_1) e^{-\delta_2 l |\pi_i|} \quad (5)$$

Let us denote further $d(\pi, \tilde{\Delta}) := d(\pi, \tilde{\Delta}, \tilde{\Delta})$.

The fourth norm will be estimated together with the factors $0(1)$:

$$\prod_{i'} 0(1) \left\| \prod_{i'} F_{3,4}(\tilde{\Delta}_{i'}, \Sigma, \varphi^{\pi_{i'}}) \right\|_{L^q}^* \leqslant \prod_{i'} 0(1) \| F_{3,4}(\tilde{\Delta}_{i'}, \Sigma, \varphi^{\pi_{i'}}) \|_{L^{2q}}^*$$

$$\leqslant \prod_{i'} \| \chi_{\tilde{\Delta}_{i'}}(-\Delta + \eta)(g - g_c) + \chi_{\tilde{\Delta}_{i'}} \eta (1 - \Lambda)(h - \xi_+) + \chi_{\tilde{\Delta}_{i'}} (m_c^2 - \eta)(1 - \Lambda^2) \quad (6)$$

$$\cdot (g - \xi_+) \|_{L^2} 0(l) 0(1)^{|\pi_{i'}|} e^{-\frac{1}{4}\delta m_c d(\pi_{i'}, \tilde{\Delta}_{i'})} G_1(\pi_{i'}, \delta_1) e^{-\delta_2 l |\pi_{i'}|} .$$

The functions under the norm $\|\cdot\|_{L^2}$ are not identically equal to 0 only if dist $(\tilde{\Delta}_{i'}, Z \sim Z^0) \geqslant \frac{1}{2}L$. Then $d(b_{i'}, \tilde{\Delta}_{i'}) \geqslant \frac{1}{2}L$ for $b_{i'} \in \pi_{i'}$ and

$$e^{-\varepsilon d(b_{i'}, \tilde{\Delta}_{i'})} \leqslant e^{-\frac{1}{2}\varepsilon L} \leqslant m^{-K}$$

for arbitrary K and m sufficiently large. Because the norm $\| \dots \|_{L^2} \leqslant 0(l\xi_+)$, so for m large we have

$$\prod_{i'} 0(1) \left\| \prod_{i'} F_{3,4}(\tilde{\Delta}_{i'}, \Sigma, \varphi^{\pi_{i'}}) \right\|^*_{L^q}$$
$$\leqslant \prod_{i'} (m^{-\nu} 0(1)^{|\pi_{i'}|} \cdot e^{\varepsilon d(b_{i'}, \tilde{\Delta}_{i'})} e^{-\delta m_c d(\pi_{i'}, \tilde{\Delta}_{i'})} G_1(\pi_{i'}, \delta_1) e^{-\delta_2 l |\pi_{i'}|}). \quad (7)$$

The number $\frac{1}{4}\delta$ was replaced by δ in this inequality, and we will also do that in the future. The only important thing are some generic properties of the constants and their inter-relation.

2. Next we will estimate the coefficients with n_Δ in (III.57) (5). We will use the localization factors in a manner similar to that used in (III.24)-(III.27). As in Proof of Lemma 10.2 of [*11*] one shows that

$$\prod_i{}''' e^{-\varepsilon d(b_i, \tilde{\Delta})} \leqslant e^{-\varepsilon \gamma_0 l n_\Delta^{\frac{3}{2}}}, \qquad \text{if} \qquad n_\Delta \geqslant 0(1), \quad (8)$$

$b_i \in \pi_{i'}$. Hence

$$\prod_\Delta (n_\Delta !)^{0(1)} 0(l^2)^{n_\Delta} e^{0(l) n_\Delta} \cdot \prod_\Delta \prod_i{}''' e^{-\varepsilon d(b_i, \tilde{\Delta})}$$
$$\leqslant \prod_{\Delta : n_\Delta \geqslant 0(1)} e^{n_\Delta (0(1) \log n_\Delta + 0(l) - \varepsilon \gamma_0 l n_\Delta^{\frac{1}{2}})} \prod_{\Delta : 0 < n_\Delta < 0(1)} e^{0(l)} \leqslant \prod_{\Delta : n_\Delta > 0} e^{0(l)} = e^{0(l)|Z'|}, \quad (9)$$

and finally we have

$$\prod_\Delta (n_\Delta !)^{0(1)} 0(l^2)^{n_\Delta} e^{0(l) n_\Delta} \leqslant \prod_\Delta \prod_i{}''' e^{\varepsilon d(b_i, \tilde{\Delta})} e^{0(l)|Z'|}. \quad (10)$$

Let us notice also that obviously

$$0(1)^K \leqslant 0(1)^{J_Z + |\Gamma| + |\Pi|}. \quad (11)$$

3. Thus we need to estimate the last two norms on the left hand side of (III.57). The operators $Q_j(s_\Gamma)$ and the polynomial $G(s_\Gamma, \tau_\Pi)$ occuring under these norms are built of the fermion elementary expressions only.

The coefficients $H(s, \tilde{\Delta}, \tilde{\Delta}')$ and $\partial_s^\gamma H(s, \tilde{\Delta}, \tilde{\Delta}')$, connected with the localized fermion propagators, occur multiplicatively in these expressions, so we can exclude them from the norms and estimate using the inequalities:

$$0 \leqslant H(s, \tilde{\Delta}, \tilde{\Delta}') \leqslant 1\,, \tag{12}$$

$$|\,\partial_s^\gamma H(s, \tilde{\Delta}, \tilde{\Delta}')\,| \leqslant e^{0(l)} e^{2m_c d(\tilde{\Delta}, \tilde{\Delta}')} e^{-\delta m_c d(\gamma, \tilde{\Delta}, \tilde{\Delta}')} 0(1)^{|\gamma|} G_1(\gamma, \delta_1) e^{-\delta_2 l |\gamma|}\,. \tag{13}$$

For each vertex there are at most 3 factors $e^{0(l)}$, so we can attach them to this vertex, with the exception of factors connected with the operators P_j. If Δ, Δ' are the localization squares of the corresponding fermion propagator, then of course $d(\tilde{\Delta}, \tilde{\Delta}') \leqslant d(\Delta, \Delta')$. It follows that for every partition $\{\gamma_j\} \in \mathscr{P}'(\Gamma)$ we have

$$\prod_j |\,\partial_s^{\gamma_j} H(s, \tilde{\Delta}_j, \tilde{\Delta}_j')\,| \leqslant e^{0(l) J_Z} 0(1)^{|\Gamma|} e^{-\delta_2 l |\Gamma|} \prod_{\text{vertices}} e^{0(l)}$$
$$\cdot \prod_{\text{fermion propagators}} e^{2m_c d(\Delta, \Delta')} \prod_{\substack{\text{differentiated} \\ \text{fermion propagators}}} e^{-\delta m_c d(\gamma, \tilde{\Delta}, \tilde{\Delta}')} \prod_j G_1(\gamma_j, \delta_1)\,. \tag{14}$$

Let us denote by " $Q_j(s_\Gamma)$ ", " $G(s_\Gamma, \tau_\Pi)$ ", " $\partial_s^{\gamma_j \sim \{b_j\}} A(s_\Gamma, b_j)$ ", ..., the expressions obtained from $Q_j(s_\Gamma)$, $G(s_\Gamma, \tau_\Pi)$, $\partial_s^{\gamma_j \sim \{b_j\}} A(s_\Gamma, b_j)$, ..., after the exclusion of the coefficients $H(s, \tilde{\Delta}, \tilde{\Delta}')$. We shall formulate the missing inequality for the norms of " $Q_j(s_\Gamma)$ " and " $G(s_\Gamma, \tau_\Pi)$ ".

Proposition IV.1. — The following inequality

$$\left\|\prod_{j=1}^{K} \|\text{“}Q_j(s_\Gamma)\text{”}\|_1 \right\|_{L^q}^{*} \|\text{“}G(s_\Gamma, \tau_\Pi)\text{”}\|_{L^q}^{*}$$
$$\leqslant 0(1)^{J_Z} \prod_{\substack{\text{fermion} \\ \text{propagators}}} e^{0(1) d(\Delta, \Delta')} \prod_{\substack{\text{differentiated} \\ \text{fermion propagators}}} e^{0(1)\varepsilon d(\gamma, \tilde{\Delta}, \tilde{\Delta}')}$$
$$\prod_{\text{fields } \varphi^\pi} e^{0(1)\varepsilon d(b, \tilde{\Delta})} \prod_{\text{vertices}} e^{0(l)} \lambda^{\frac{1}{2}} \prod_{\text{fermion propagators}} e^{-\varepsilon_0 m d(\Delta, \Delta')}$$
$$\cdot \prod_{\text{fields } \varphi^\pi} e^{-\delta m_c d(\pi, \tilde{\Delta})} 0(1)^{|\pi|} G_1(\pi, \delta_1) e^{-\delta_2 l |\pi|} \prod_{\text{vertices}} \lambda^{-1} \prod_{\beta=1}^{J_Z} \|g_{j_\beta}\|_{H^{-\frac{1}{2}}} \|h_{j_\beta}\|_{H^{-\frac{1}{2}}} \tag{15}$$

holds for $\lambda \geqslant \lambda_0$ and $m \geqslant m_0(\lambda)$.

Now we can finish Proof of Theorem III.2. If the inequalities (4), (5), (7), (10), (11), (14), (15) are combined, the left hand side of (III.57) can be estimated by its right hand side times the expression

$$\sum_{\{\gamma_j\}\in\mathscr{P}'(\Gamma)} \sum_{\{\pi_k\}\in\mathscr{P}'(\Pi)} 0(1)^{|\Gamma|+|\Pi|} e^{-(\delta_2-\delta_0)l(|\Gamma|+|\Pi|)}$$
$$\cdot \prod_{\text{fermion propagators}} e^{(0(1)-\varepsilon_0 m)d(\Delta,\Delta')} \prod_j e^{(0(1)\varepsilon-\delta m_c)d(\gamma_j,\tilde{\Delta}_j,\tilde{\Delta}_j')} G_1(\gamma_j,\delta_1)$$
$$\cdot \prod_k e^{(0(1)\varepsilon-\delta m_c)d(\pi_k,\tilde{\Delta}_k,\tilde{\Delta}_k')} G_1(\pi_k,\delta_1) \prod_{\text{vertices}} e^{0(l)}\lambda^{-\frac{1}{2}}. \tag{16}$$

Taking m so large, that $0(1) - \varepsilon_0 m \leqslant 0$, ε fixing so small, that $0(1)\varepsilon - \delta m_c \leqslant 0$, and using (III.49) for the sums over partitions, we can estimate (16) by

$$0(1)^{|\Gamma|+|\Pi|} e^{-(\delta_2-\delta_0)l(|\Gamma|+|\Pi|)} \prod_{\text{vertices}} e^{0(l)}\lambda^{-\frac{1}{2}}. \tag{17}$$

We fix $\delta_0 > 0$ such that $\delta_2 - \delta_0 > 0$. Then for l sufficiently large, what means λ large, the first two factors in (17) can be estimated by 1. Finally, because $l = 0(\sqrt{\log \lambda})$,

$$e^{0(l)} \leqslant \lambda^{0\left(\frac{1}{\sqrt{\log \lambda}}\right)} \leqslant \lambda \qquad \text{for } \lambda \text{ large and} \quad \prod_{\text{vertices}} e^{0(l)}\lambda^{-\frac{1}{2}} \leqslant 1\,.$$

Thus Theorem III.2 is proven.

In the rest of the Chapter we shall be occupied with the proof of Proposition IV.1. Let us make some comments about it. The products on the right hand side of (15) can be factorized in a natural way into products corresponding to the elementary expressions occuring under the norms on the left hand side, so Proof of Proposition IV.1 can be obtained as a consequence of a set of sub-propositions: each elementary expression on the left hand side of (15) gives rise, after all estimations are made, to the corresponding product on the right hand side.

Unfortunately it is impossible to factorize the left hand side into the product of the corresponding norms of the elementary expressions, because the norms are defined by the gaussian integration with respect to the field φ, which can occur in all expressions. Nevertheless in the proof which follows we make some preliminary analysis and estimation for each elementary expression, and we obtain this way some factors on the right hand side of (15). The final estimate is produced by considering the Gaussian integrals of some simple expressions in the fields φ, φ^π.

Some general remarks can be made concerning the structure of arbitrary elementary expressions except the first of the B-terms, which will be analyzed later. These are operators, or traces of operators, each consisting of at most 3 operators of the form

$$\lambda K(\Delta, \Delta', \Delta'', f) := \lambda\chi_\Delta(\not{P} + m)D^{-1}\chi_{\Delta'}D^{-\frac{1}{2}}\Gamma\chi_{\Delta''}fD^{-\frac{1}{2}}, \qquad (19)$$

where f is equal to $\varphi\Lambda$, $\varphi^\pi\Lambda$ or $(g-h)\Lambda + (\xi_+ - h)(1-\Lambda)$, and also of at most 3 chains of operators of the form

$$\left[\prod_{i=1}^{r}(\chi_{\Delta_i}(1-\lambda K(s_\Gamma, h))^{-1}\lambda K(\Delta_i', \Delta_i'', \Delta_i''', h))\right]\chi_{\Delta_{r+1}}(1-\lambda K(s_\Gamma, h))^{-1}\chi_{\Delta'_{r+1}}, \qquad (20)$$

or of the same form but with h replaced by ξ_+ ($h \to \xi_+$). These chains appeared after the differentiations of the operators $(1-\lambda K(s_\Gamma, h))^{-1}$ or $(1-\lambda K(s_\Gamma, \xi_+))^{-1}$ according to the formula (III.32). In all considerations in the sequel we shall isolate and estimate chains of the operators (20) at first and then consider the remaining operators of the type (19).

4. Let us now consider the norm $\left\| \prod_{j=1}^{K} \| \text{“} Q_j(s_\Gamma) \text{”} \|_1 \right\|_{L^q}^*$. At the begining we estimate the norms $\| \text{“} Q_j(s_\Gamma) \text{”} \|_1$. According to the structure of the operators “ $Q_j(s_\Gamma)$ ” we estimate $\| \text{“} Q_j(s_\Gamma) \text{”} \|_1$ by the product of the norm $\| \text{“} \partial_s^{\Gamma_j \sim \{b_j\}} A(s_\Gamma, b_j) \text{”} \|_1$, or $\| \text{“} \partial_s^{\Gamma_j} P_{Z,\beta}(h, s_\Gamma) \text{”} \|_1$ and of the norms

$$\| \text{“} \partial_s^{\Gamma_j \sim \{b_j\}} E(s_\Gamma, b_j) \text{”} \|_\infty$$

of all operators of the type E.

Using the operator Hölder inequality the norm $\|\cdot\|_1$ of an expression of type A can be estimated by the product of three norms

$$\| \lambda K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_3$$

and a product of the norms $\|\cdot\|_\infty$ of the chains (20). The norm $\|\cdot\|_\infty$ of an expression of type E is estimated by the product of the same norm of the operators (19) and of the chains (20). Next, using the inequality $\|\cdot\|_\infty \leqslant \|\cdot\|_p$, $1 \leqslant p \leqslant \infty$, the norm $\|\cdot\|_\infty$ of the operators (19) is bounded again by the norm $\|\cdot\|_3$. Finally, an expression “ $\partial_s^{\Gamma_j} P_{Z,\beta}(h, s_\Gamma)$ ” is the product of one chain (20) and of an operator $P_{Z,\beta}(s_\Gamma)$ and we estimate the norm $\|\cdot\|_1$ by the norm $\|\cdot\|_\infty$ of the chain and $\| P_{Z,\beta}(s_\Gamma) \|_1$. For the last norm we have

$$\| P_{Z,\beta}(s_\Gamma) \|_1 \leqslant 0(1) \prod_{\substack{\text{fermion propagators} \\ \text{in } P_{Z,\beta}}} e^{-\varepsilon_0 m d(\Delta,\Delta')} \| g_j \|_{H^{-\frac{1}{2}}} \| h_j \|_{H^{-\frac{1}{2}}}. \qquad (21)$$

Now the norm $\|\cdot\|_\infty$ of a chain (20) will be estimated. Using Corollary A.I.1 and Proposition A.I.1 and $\| \lambda\chi_\Delta h \|_{L^\infty} = \lambda\xi_+$ we obtain

$$\| \text{chain } (20) \|_\infty \leqslant 0(1) \prod_{\text{propagators in (20)}} e^{-\varepsilon_0 m d(\Delta,\Delta')} \prod_{i=1}^{r} 0(1) \frac{\lambda\xi_+}{m} . \tag{22}$$

Because $\frac{\lambda\xi_+}{m} = \sqrt{e^{\frac{2\pi\mu^2}{\lambda^2}} - 1} = 0(\lambda^{-1})$, we have

$$\| \text{chain } (20) \|_\infty \leqslant 0(1) \prod_{\text{vertices}} 0(1) \prod_{\text{propagators in (20)}} e^{-\varepsilon_0 m d(\Delta,\Delta')} \prod_{\text{vertices in (20)}} \lambda^{-1} \tag{23}$$

and it is easily seen that this estimate is of the form required for (15), with the exception of the overall factor 0(1). This factor is attached to an operator of the form (19) occuring after the chain in a given expression, or to the operator $P_{Z,\beta}$ and then occurs on the right side of (15) in the factor $0(1)^{J_Z}$.

Thus after the estimations of $\prod_{j=1}^{K} \| \text{“ } Q_j(s_\Gamma) \text{ ”} \|_1$, performed above we are left with an expression

$$\prod_k 0(1) \| \lambda K(\Delta_k, \Delta_k', \Delta_k'', f_k)\chi_{\Delta_k'''} \|_3 , \tag{24}$$

where f_k are equal to $\varphi\Lambda$, $\varphi^\pi\Lambda$ or $(g - h)\Lambda + (\xi_+ - h)(1 - \Lambda)$. The norms $\|\cdot\|_3$ in (24) can be estimated further using Proposition A.I.1 which gives for some $v_1 > 0$ and $\beta > 0$

$$\| \lambda K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_3 \leqslant 0(1)\lambda m^{-v_1} e^{-\varepsilon_0 m(d(\Delta,\Delta') + d(\Delta',\Delta'') + d(\Delta'',\Delta'''))} S^\beta(f\chi_{\Delta_k''}) \tag{25}$$

where

$$S^\beta(f\chi_{\Delta''}) := \| D^{-\beta} f \chi_{\Delta''} D^{-\frac{1}{2}} \|_4 . \tag{26}$$

This gives us for $m \geqslant m_0(\lambda)$

$$(24) \leqslant \prod_{\text{vertices in (24)}} 0(1) \prod_{\substack{\text{fermion propagators} \\ \text{in (24)}}} e^{-\varepsilon_0 m d(\Delta,\Delta')} . \prod_{\text{vertices in (24)}} \lambda^{-1} \prod_k S^\beta(f_k\chi_{\Delta_k''}) . \tag{27}$$

The inequalities (21), (23) and (27) give all localization factors $e^{-\varepsilon_0 m d(\Delta,\Delta')}$ for fermion propagators and the factors λ^{-1} for vertices, needed in (15) and coming from “ $Q(s_\Gamma)$ ”.

Let us now consider the terms in the product over k on the right hand side of (27), for which $f_k = (g - h)\Lambda + (\xi_+ - h)(1 - \Lambda)$. If such a term is not equal to 0, then dist $(\Delta_k'', Z \sim Z^0) \geqslant \frac{1}{2}L$ and

$$S^{\beta}(f_k \chi_{\Delta_k'}) \leqslant 0(1) m^{-\nu_1} \xi_+ . \tag{28}$$

This term occurs in some elementary expression appearing after a fermion or boson differentiation corresponding to some bond b. Hence there is a sequence of fermion propagators in this expression localized in the squares $\Delta_k'' = \Delta_1, \Delta_2, \ldots, \Delta_r$, and the square Δ_r is connected with the bond b either by a boson propagator, or by a differentiation of a fermion propagator with respect to s_b. In both cases we have the factors $e^{-d(\Delta_i, \Delta_{i+1})}$, $i = 1, \ldots, r-1$, $e^{-\varepsilon d(b, \tilde{\Delta}_r)}$ to our disposal. Of course $d(b, \Delta_k'') = d(b, \Delta_1) \geqslant \frac{1}{2}L$ and

$$\begin{aligned} d(b, \Delta_1) &\leqslant d(\Delta_1, \Delta_2) + \sqrt{2} + d(\Delta_2, \Delta_3) + \sqrt{2} + \ldots + d(\Delta_{r-1}, \Delta_r) + \sqrt{2} + d(\Delta_r, b) \\ &\leqslant \sum_{i=1}^{r-1} d(\Delta_i, \Delta_{i+1}) + d(b, \tilde{\Delta}_r) + \sqrt{2}r + \sqrt{2}l . \end{aligned}$$

Let us remind that $l = 0(\sqrt{\log \lambda})$ and $L = 0((\log m)^2)$, so for m large l is much smaller than L, and we can assume for example $\sqrt{2}l \leqslant \frac{1}{10}L$. We will consider now two cases depending on the magnitude of r. 1° If $r < \frac{1}{20}L$, then $\frac{1}{4}\varepsilon L \leqslant \varepsilon d(b, \tilde{\Delta}_r) + \sum_{i=1}^{r-1} d(\Delta_i, \Delta_{i+1})$, and

$$e^{-\varepsilon d(b, \tilde{\Delta}_r)} \prod_{i=1}^{r-1} e^{-d(\Delta_i, \Delta_{i+1})} \leqslant e^{-\frac{1}{4}\varepsilon L} \leqslant m^{-\frac{1}{4}\varepsilon \log m} \leqslant m^{-N} .$$

for arbitrary N and m sufficiently large. We will need at most $N = 3$. If $r \geqslant \frac{1}{20}L$, then in the elementary expression there are at least $\frac{1}{4}r \geqslant \frac{1}{80}L$ vertices and the factors $\lambda^{-\frac{1}{2}}$ can be used:

$$(\lambda^{-\frac{1}{2}})^{\frac{1}{4}r} \leqslant \lambda^{-\frac{1}{160}L} = e^{-\frac{1}{160}L \log \lambda} \leqslant m^{-\frac{1}{160}\log m \log \lambda} \leqslant m^{-N}$$

for arbitrary N and m large.

It is obvious that the above reasoning is quite general and it leads to the following.

LEMMA IV.1. — If an elementary expression (*e. e.*) has two localization squares, or a localization square and a bond of differentiation, with a distance of order L at least, then for arbitrary N

$$1 \leqslant \prod_{\text{fermion propagators in e. e.}} e^{d(\Delta,\Delta')} \prod_{\substack{\text{differentiated fermion}\\ \text{propagators in e. e.}}} e^{\varepsilon d(\gamma,\tilde{\Delta},\tilde{\Delta}')} \cdot \prod_{\text{fields } \varphi^{\pi} \text{ in e. e.}} e^{\varepsilon d(\pi,\tilde{\Delta},\tilde{\Delta}')} \prod_{\text{vertices in e. e.}} \lambda^{\frac{1}{2}} \cdot m^{-\mathrm{N}}, \quad (29')$$

for m sufficiently large. □

This lemma will be used frequently in the future. The localization factors and the power of $\lambda^{\frac{1}{2}}$ are included to the factors on the right hand side of (15). The additional power $m^{-\mathrm{N}}$ is used to cancel some powers of m appearing in the estimation of the elementary expression.

In our case we have at most 3 expressions of the type occuring in (28) in one elementary expression, so at most $\xi_+^3 \leqslant 0(m^3)$ and it is sufficient to take $\mathrm{N} = 3$ to obtain the cancelation. Finally we are left with the norm

$$\left\| \prod_k \mathrm{S}^{\beta}(f_k \chi_{\Delta_k'}) \right\|_{\mathrm{L}^q}^{*}, \quad (30)$$

where for the all k, $f_k = \varphi$ or φ^{π}.

Using the remark from the beginning of this chapter about fields φ^{π} and the Hölder inequality we can factorize (30) into a product of similar expressions, but depending on one fixed field φ or φ^{π}.

We have to estimate two types of integrals:

$$\| \mathrm{S}^{\beta}(\varphi^{\pi} \Lambda \chi_{\Delta}) \|_{\mathrm{L}^q}^{*}, \quad (31)$$

$$\left\| \prod_{k=1}^{\mathrm{K}} \mathrm{S}^{\beta}(\varphi \Lambda \chi_{\Delta_k}) \right\|_{\mathrm{L}^q}. \quad (32)$$

Let us notice that

$$\mathrm{S}^{\beta}(f\chi_{\Delta})^4 = \int \prod_{i=1}^{4} d\chi_i w(\chi_1, \chi_2, \chi_3, \chi_4) \prod_{i=1}^{4} f(\chi_i), \quad (33)$$

and, as it follows from Proof of Lemma A.II.2

$$\| w \|_{\mathrm{L}^{p_1}(\mathbb{R}^8)} \leqslant 0(1) m^{-\nu_2} \qquad \text{for some} \qquad p_1 > 1,\ \nu_2 > 0. \quad (34)$$

We have the following lemmas giving the estimates of (31), (32)

LEMMA IV.2.

$$(31) \leqslant e^{0(l)} 0(1)^{|\pi|} e^{-\delta m_c d(\pi,\tilde{\Delta})} \mathrm{G}_1(\pi, \delta_1) e^{-\delta_2 l |\pi|}. \quad (35)$$

Proof. — For q divisible by 4 we have

$$(31)q \leqslant \int \prod_{j=1}^{q} d\chi_j \prod_{i=1}^{q/4} w(x_{4i-3}, \ldots, x_{4i}) \sum_{\substack{\text{partitions of } \{1, \ldots, q\} \\ \text{into pairs } k = \{k+, k-\}}} \prod_k C^{\pi}_{m_c}(\tau, x_{k+}, x_{k-}) . \tag{36}$$

We estimate this integral, using the Hölder inequality, by

$$\| w \|^{q/4}_{L^{p_1}(\mathbb{R}^8)} q^q \| \chi_\Delta C^{\pi}_{m_c}(\tau) \chi_\Delta \|^{q/2}_{L^{q_1}(\mathbb{R}^4)} \leqslant \left(0(1) m^{-\frac{\nu_2}{4}} q\right)^q \cdot \left(e^{0(l)} 0(1)^{|\pi|} e^{-\delta m_c d(\pi, \tilde{\Delta})} G_1(\pi, \delta_1) e^{-\delta_2 l |\pi|}\right)^{q/2} \tag{37}$$

and (35) follows. □

LEMMA IV.3.

$$(32) \leqslant \prod_{\text{fermion propagators}}' e^{3d(\Delta,\Delta')} \prod_{\substack{\text{differentiated} \\ \text{fermion propagators}}}' e^{3\varepsilon d(\gamma, \tilde{\Delta}, \tilde{\Delta}')} \cdot \prod_{\text{fields } \varphi^\pi}' e^{3\varepsilon d(\pi, \tilde{\Delta})} \prod_{\text{vertices}}' e^{0(l)} , \tag{38}$$

the products Π' are over the factors attached to the elementary expressions, which gave rise to $S^\beta(\varphi \Lambda \chi_{\Delta_k})$.

Proof. — An elementary expression giving rise to $S^\beta(\varphi \Lambda \chi_{\Delta_k})$ appears after the differentiation with respect to a bond b_k. Let $r(\Delta)$ be a number of indices k such that $\Delta_k = \Delta$, and let $R(\Delta)$ be the set of the corresponding bonds b_k. The checkerboard estimate and Lemma A.II.2 give us

$$\begin{aligned} \left\| \prod_{k=1}^{K} S^\beta(\varphi \Lambda \chi_{\Delta_k}) \right\|_{L^q} &= \left\| \prod_{\Delta} S^\beta(\varphi \Lambda \chi_\Delta)^{r(\Delta)} \right\|_{L^q} \leqslant \prod_{\Delta} \| S^\beta(\varphi \Lambda \chi_\Delta) \|^{r(\Delta)}_{L^{r(\Delta) q'}} \\ &\leqslant \prod_{\Delta} (0(1) r(\Delta) q')^{r(\Delta)} m^{-r(\Delta)\nu_3} \\ &= \prod_{k=1}^{K} e^{\varepsilon d(b_k, \tilde{\Delta}_k)} \cdot \prod_{\tilde{\Delta}} \prod_{\Delta \subset \tilde{\Delta}} e^{-\varepsilon \sum_{b \in R(\Delta)} d(b, \tilde{\Delta})} (0(1) r(\Delta) q')^{r(\Delta)} m^{-r(\Delta)\nu_3} . \end{aligned} \tag{39}$$

Estimating now the last expression the same way as in (9)-(11) we get

$$(32) \leqslant 0(1) \prod_{k=1}^{K} e^{\varepsilon d(b_k, \tilde{\Delta}_k)} . \tag{40}$$

The factors $e^{\varepsilon d(b_k, \tilde{\Delta}_k)}$ can be divided into the combinatoric factors attached to the given elementary expressions according to (III.34) and we obtain finally (38). We used the fact that a given elementary expression gives rise to at most 3 expressions S^β. The inequalities (35) and (38) terminate the estimation of the norm $\left\| \prod_{j=1}^{K} \| \text{“} Q_j(s_\Gamma) \text{”} \|_1 \right\|_{L^q}^* . \quad \square$

Now we pass to the estimation of $\| \text{“} G(s_\Gamma, \tau_\Pi) \text{”} \|_{L^q}^*$. The functional “ $G(s_\Gamma, \tau_\Pi)$ ” is a product of the elementary expressions of type B coming from the differentiations of the exponentials in (III.4). There are 5 types of such expressions. Some of them do not depend on the fields φ, φ^π and can be excluded from the norm $\| \cdot \|_{L^q}^*$. We will consider them at the beginning and next we will consider the norm of the remaining expressions depending on φ, φ^π.

5. The first expression we shall consider is

$$\text{“} \frac{\lambda^2}{2} \partial_s^{\Gamma_j} \operatorname{Tr} (K(s_\Gamma, \chi_Z h)^2 - K(s_\Gamma, \chi_Z \xi_+)^2) \text{”} . \tag{41}$$

Localisation, differentiation and exclusion of the coefficients $H(s, \tilde{\Delta}, \tilde{\Delta}')$ leads to an expression of the form

$$\frac{\lambda^2}{2} \operatorname{Tr} [(K(\Delta_1, \Delta_2, \Delta_3, h) K(\Delta_4, \Delta_5, \Delta_6, h) \chi_{\Delta_1}) - (h \to \xi_+)] . \tag{42}$$

It vanishes if $\Sigma(\Delta_3) = \Sigma(\Delta_6)$, and if $\Sigma(\Delta_3) \neq \Sigma(\Delta_6)$, then the term $-(h \to \xi_+)$ is equal to the first term. Thus we have to consider only the first term in (42) with $\Sigma(\Delta_3) \neq \Sigma(\Delta_6)$, which implies of course $\Delta_3 \neq \Delta_6$. Thus one of the operators under the trace (42) has two different localization squares. Let it be the first operator. Then the trace (42) can be bounded by

$$\lambda^2 \| K(\Delta_1, \Delta_2, \Delta_3, h) \chi_{\Delta_4} \|_{2-\varkappa_1} \| K(\Delta_4, \Delta_5, \Delta_6, h) \chi_{\Delta_1} \|_{2+\varkappa_2} , \tag{43}$$

where $\varkappa_1$, $\varkappa_2$ are positive and sufficiently small, $\dfrac{1}{2 - \varkappa_1} + \dfrac{1}{2 + \varkappa_2} = 1.$
(43) can be estimated further, using Proposition A.I.1, by

$$\prod_{\text{fermion propagators in (42)}} e^{-\varepsilon_0 m d(\Delta, \Delta')} \prod_{\text{vertices in (42)}} 0(1) \lambda m^{-\nu_1} \xi_+ , \tag{44}$$

and the product over vertices by $\left(\prod_{\text{vertices}} \lambda^{-1}\right)\xi_+^2$ for m large. The estimation has the required form to fit (15), except for the factor ξ_+^2. We have to analyze now the condition $\Sigma(\Delta_3) \neq \Sigma(\Delta_6)$ more exactly. The following cases can occur:

1° dist $(\Delta_i, Z \sim Z^0) \leqslant \frac{1}{2}L$ for $i = 3,6$ and Δ_3, Δ_6 belong to different domains $Z^\pm$, where $Z^\pm = \bigcup_{\Sigma(\Delta)=\pm} \Delta$, 2° dist $(\Delta_i, Z \sim Z^0) > \frac{1}{2}L$ for one or both i.

In all these cases either the distance between Δ_3 and Δ_6 is larger than L, or the distance between Δ_3 and a bond of differentiation $b \in \Gamma_j$ is larger than $\frac{1}{2}L$, and we can apply Lemma IV.1 and get an additional power m^{-K} together with additional localization and other factors on the right hand side of (29). This finishes the estimation of (42).

6. Now let us consider a term obtained by the differentiation of

$$\int_0^\lambda d\lambda'\lambda'^2\,[\mathrm{Tr}\,(K^{\lambda'\lambda^{-1}h}(s_\Gamma, \chi_Z h)K(s_\Gamma, \chi_Z h)^2) - (h \to \xi_+)]\,. \tag{45}$$

To see the cancellations which make this term small we have to transform the above expression. The set Z can be written as a union of three sets $\tilde{Z}^+$, $\tilde{Z}^-$, $\tilde{Z}^0$. The sets $\tilde{Z}^+$ and $\tilde{Z}^-$ are composed of d-squares $\Delta \subset Z$ with $\Sigma(\Delta) = +$ and $\Sigma(\Delta) = -$ respectively and with distances to $Z \sim Z^0$ smaller than $\frac{1}{2}L$. $\tilde{Z}^0$ is defined as $Z \sim (\tilde{Z}^+ \cup \tilde{Z}^-)$. Obviously $\tilde{Z}^0 \subset Z^0$.

(45) can be written now as a sum of three expressions

$$\sum_{\varepsilon=+,0,-} \int_0^\lambda d\lambda'\lambda'^2\,[\mathrm{Tr}\,(K^{\lambda'\lambda^{-1}h}(s_\Gamma, \chi_{\tilde{Z}^\varepsilon} h)K(s_\Gamma, \chi_Z h)^2) - (h \to \xi_+)] \tag{46}$$

and now we shall analyze them separately. For the expression with $\tilde{Z}^+$ we have

$$\begin{aligned}&\int_0^\lambda d\lambda'\lambda'^2\,\mathrm{Tr}\,[(1 - \lambda' K(s_\Gamma, h))^{-1}\lambda' K(s_\Gamma, h - \xi_+)(1 - \lambda' K(s_\Gamma, \xi_+))^{-1}\\ &\quad\cdot K(s_\Gamma, \chi_{\tilde{Z}^+}\xi_+)K(s_\Gamma, \chi_Z h)^2] + \int_0^\lambda d\lambda'\lambda'^2\,\mathrm{Tr}\,[K^{\lambda'\lambda^{-1}\xi_+}(s_\Gamma, \chi_{\tilde{Z}^+}\xi_+)\\ &\quad\cdot (K(s_\Gamma, \chi_Z h)^2 - K(s_\Gamma, \chi_Z \xi_+)^2)]\,.\end{aligned} \tag{47}$$

We choose one of the above expressions with the overall factor $2^{|\Gamma|}$.

Differentiation, localization and exclusion of the coefficients $H(s, \tilde{\Delta}, \tilde{\Delta}')$ (or their derivatives) gives

$$\int_0^\lambda d\lambda'\lambda'^2 \operatorname{Tr} [(\text{chain (20) but with } \lambda \to \lambda') \cdot \lambda' K(\Delta, \Delta_1, \Delta_2, h - \xi_+) (\text{second chain (20) with } h \to \xi_+, \lambda \to \lambda') \cdot K(\Delta', \Delta_1', \Delta_2', \xi_+) \prod_{i=1}^{2} K(\Delta_{3i}, \Delta_{3i+1}, \Delta_{3i+2}, h)\Big], \quad (48)$$

or

$$\int_0^\lambda d\lambda'\lambda'^2 \operatorname{Tr} [(\text{chain (20) with } h \to \xi_+, \lambda \to \lambda') K(\Delta, \Delta_1, \Delta_2, \xi_+) \cdot \left(\prod_{i=1}^{2} K(\Delta_{3i}, \Delta_{3i+1}, \Delta_{3i+2}, h) - (h \to \xi_+)\right)\Big]. \quad (49)$$

In (48) $\Delta_2' \subset \tilde{Z}^+$, and if this term is not equal 0, then $\Sigma(\Delta_2) = -$ and by the definition of $\tilde{Z}^+$ $d(\Delta_1, \Delta_2') \geqslant \frac{1}{2} L$. Similarly in (49) $\Delta_2 \subset \tilde{Z}^+$ and, by the same reasons as for (42), $\Sigma(\Delta_5) \neq \Sigma(\Delta_8)$, so one of the squares Δ_5, Δ_8, let it be for example Δ_5, has $\Sigma(\Delta_5) = -$. Then $d(\Delta_2, \Delta_5) \geqslant \frac{1}{2} L$ again. Thus for both elementary expressions (48), (49) there are two localization squares with $d(\Delta, \Delta') \geqslant \frac{1}{2} L$ and we may apply Lemma IV.1. The expressions (48), (49) can be estimated as usually using the Hölder inequality for traces, with norm $\| \cdot \|_\infty$ for the chains and $\| \cdot \|_3$ for the operators K. Next, using (23), (25), (27) we estimate these norms by the corresponding factors on the right side of (15) times ξ_+^4. We use Lemma IV.1 to cancel this additional factor.

Let us now consider a term in the sum (46) with $\tilde{Z}^-$. At first notice that

$$\operatorname{Tr} (K^{\lambda'\lambda^{-1}h}(s_\Gamma, \chi_{\tilde{Z}^-} h) K(s_\Gamma, \chi_Z h)^2) = \operatorname{Tr} (K^{\lambda'\lambda^{-1}(-h)}(s_\Gamma, \chi_{\tilde{Z}^-}(-h)) K(s_\Gamma, \chi_Z(-h))^2). \quad (50)$$

This equality is a consequence of the equality

$$\operatorname{Tr} (K(s_\Gamma, \chi_Z h)^{n+2} K(s_\Gamma, \chi_{\tilde{Z}^-} h)) = 0 \qquad \text{for } n \text{ even} \quad (51)$$

which follows as the $\varphi \to -\varphi$ symmetry, see (II.19). The equality (50) implies that the term with $\tilde{Z}^-$ can be written as

$$\int_0^\lambda d\lambda'\lambda'^2 [\operatorname{Tr} (K^{\lambda'\lambda^{-1}(-h)}(s_\Gamma, \chi_{\tilde{Z}^-}(-h)) K(s_\Gamma, \chi_Z(-h))^2) - ((-h) \to \xi_+)] \quad (52)$$

and now the set $\tilde{Z}^-$ plays the same role for the function $(-h)$ as the set $\tilde{Z}^+$ for h, so all further transformations and estimation are the same as in the case of $\tilde{Z}^+$.

The term in (46) with $\tilde{Z}^0$ gives rise to an expression

$$\int_0^\lambda d\lambda'\lambda'^2\left[\mathrm{Tr}\left((\text{chain}\,(20)\,\text{with}\,\lambda \to \lambda')\prod_{i=1}^{3}\mathrm{K}(\Delta_{3i-2},\Delta_{3i-1},\Delta_{3i})\right)-(h\to\xi_+)\right] \tag{53}$$

with $\Delta_3 \subset \tilde{Z}^0$. This expression appears after some differentiation with respect to a bond $b \in \Gamma_j$.

According to the definition of $\tilde{Z}^0$, $d(b, \Delta) \geqslant \frac{1}{2}\mathrm{L}$ and we can apply Lemma IV.1 again. Estimating (53) similarly as in previous cases we get the required part of the inequality (15).

7. Now we consider a term obtained by the differentiation of

$$\frac{\lambda^2}{2}\int d\mu_{m_c}(\tau_\Pi)\,\mathrm{Tr}\,(\mathrm{K}^{\xi+}(s_\Gamma, \chi_Z\varphi\Lambda)^2 - \mathrm{K}^h(s_\Gamma, \chi_Z\varphi\Lambda)^2) \tag{54}$$

with respect to Γ_j, π_j or both.

The trace occuring in (54) can be transformed into the following form:

$$\sum_{\varepsilon=+,0,-}\mathrm{Tr}\,(\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^\varepsilon}f)^2 - \mathrm{K}^h(s_\Gamma, \chi_{\tilde{Z}^\varepsilon}f)^2)$$
$$+\sum_{\substack{\varepsilon_1,\varepsilon_2=+,0,-\\ \varepsilon_1\neq\varepsilon_2}}\mathrm{Tr}\,((\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_1}}f)\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_2}}f)) - (\xi_+ \to h)), \tag{55}$$

where the same definitions of $\tilde{Z}^\pm$, $\tilde{Z}^0$ are used as in point 6) and f is equal $\varphi\Lambda$ or $\varphi^{\pi_j}\Lambda$.

We have an equality regularizing the trace above

$$\begin{aligned}\mathrm{Tr}\,(\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^-}f)^2 - \mathrm{K}^h(s_\Gamma, \chi_{\tilde{Z}^-}f)^2) &= [2\lambda\,\mathrm{Tr}\,(\mathrm{K}(s_\Gamma, \xi_+)\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^-}f)\mathrm{K}(s_\Gamma, \chi_{\tilde{Z}^-}f)) \\ &+ \lambda^2\,\mathrm{Tr}\,(\mathrm{K}(s_\Gamma, \xi_+)\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^-}f))^2] - [\xi_+ \to h].\end{aligned} \tag{56}$$

By the same reasons as for (50) we have moreover

$$\begin{aligned}\mathrm{Tr}\,(\mathrm{K}(s_\Gamma, \xi_+)\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^-}f)\mathrm{K}(s_\Gamma, \chi_{\tilde{Z}^-}f)) \\ = \mathrm{Tr}\,(\mathrm{K}(s_\Gamma, -\xi_+)\mathrm{K}^{\xi-}(s_\Gamma, \chi_{\tilde{Z}^-}f)\mathrm{K}(s_\Gamma, \chi_{\tilde{Z}^-}f)),\end{aligned} \tag{57}$$

and an analogous equality for the second trace in (56). This implies

$$\begin{aligned}\mathrm{Tr}\,(\mathrm{K}^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^-}f)^2 - \mathrm{K}^h(s_\Gamma, \chi_{\tilde{Z}^-}f)^2) \\ = \mathrm{Tr}\,(\mathrm{K}^{\xi-}(s_\Gamma, \chi_{\tilde{Z}^-}f)^2 - \mathrm{K}^h(s_\Gamma, \chi_{\tilde{Z}^-}f)^2).\end{aligned} \tag{58}$$

Using (58) we obtain after some transformations

$$(55) = \sum_{\varepsilon=+,-} \operatorname{Tr} \{ [(1 - \lambda K(s_\Gamma, \xi_\varepsilon))^{-1} \lambda K(s_\Gamma, \xi_\varepsilon - h)$$
$$\cdot (1 - \lambda K(s_\Gamma, h))^{-1}] K(s_\Gamma, \chi_{\tilde{Z}^\varepsilon} f) [(1 - \lambda K(s_\Gamma, \xi_\varepsilon))^{-1} + (1 - \lambda K(s_\Gamma, h))^{-1}]$$
$$\cdot K(s_\Gamma, \chi_{\tilde{Z}^\varepsilon} f) \} + \sum_{\substack{\varepsilon_1,\varepsilon_2=+,0,- \\ (\varepsilon_1,\varepsilon_2)\neq(\pm,\pm)}} \Big\{ \Big[2\lambda \operatorname{Tr} (K(s_\Gamma, \xi_+) K^{\xi_+}(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_1}} f)$$
$$\cdot K(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_2}} f)) + \lambda^2 \operatorname{Tr} \Big(\prod_{i=1}^{2} K(s_\Gamma, \xi_+) K^{\xi_+}(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_i}} f) \Big) \Big] - [\xi_+ \to h] \Big\} \quad (59)$$

Differentiation of (59) with respect to s, localization and exclusion of H's produces either a term of the form

$$\operatorname{Tr} \{ [(\text{chain (20) with } h \to \xi_\varepsilon) \lambda K(\Delta_0, \Delta_0', \Delta_0'', \xi_\varepsilon - h)$$
$$\cdot (\text{second chain (20)})] K(\Delta_1, \Delta_1', \Delta_1'', f) (\text{chain (20)},$$
$$\text{with } h \to \xi_\varepsilon \text{ eventually}) K(\Delta_2, \Delta_2', \Delta_2'', f) \}, \quad (60)$$

where $\Delta_1'', \Delta_2'' \subset \tilde{Z}^\varepsilon$, $\varepsilon = \pm$ or a term

$$\operatorname{Tr} \{ \lambda K(\Delta_0, \Delta_0', \Delta_0'', \xi_+ \text{ or } h)(\text{chain (20) with } h \to \xi_+ \text{ eventually})$$
$$\cdot K(\Delta_1, \Delta_1', \Delta_1'', f) [\lambda K(\Delta_3, \Delta_3', \Delta_3'', \xi_+$$
$$\text{or } h)(\text{chain (20) with } h \to \xi_+ \text{ eventually})$$
$$\text{these factors may be absent}] K(\Delta_2, \Delta_2', \Delta_2'', f) \}, \quad (61)$$

where $\Delta_1'' \subset \tilde{Z}^\varepsilon$ or $\tilde{Z}^0$, $\Delta_2'' \subset \tilde{Z}^{-\varepsilon}$ or $\tilde{Z}^0$.

In the first case (60) we have $\Sigma(\Delta_0'') = -\varepsilon$, so of course $d(\Delta_0'', \Delta_1'') \geq \frac{1}{2}L$.

In the second case (61) either $\Delta_1'' \subset \tilde{Z}^\varepsilon$, $\Delta_2'' \subset \tilde{Z}^{-\varepsilon}$, and then $d(\Delta_1'', \Delta_2'') \geq L$, or one of these squares, for example Δ_1'', is contained in $\tilde{Z}^0$, and then $d(b, \Delta_1'') \geq \frac{1}{2}L$, where $b \in \Gamma_j$ or $b \in \pi_j$. In both cases we can apply Lemma IV.1.

The expressions (60), (61) contain at least 3 operators K, so we can estimate them in the same way, as in point 6). Using Lemma IV.1 and the inequality (25) we obtain all factors needed in (15), with possible exception of boson factors, and a product

$$\prod_{i=1}^{2} S^\beta(f\chi_{\Delta_i'}) \quad (62)$$

Thus finally we need to bound the integral of (62) with respect to $d\mu_{m_c}(\tau)$ or $d\mu_{m_c}^{\pi_j}(\tau)$. Using the Hölder inequality and Lemma IV.3, or Lemma IV.2, the estimation is finished.

8. We pass now to an analysis of the most important terms of type B:

$$-\frac{\lambda^2}{2} : \mathrm{Tr}\, \mathrm{K}^h(s_\Gamma, \psi)^2 :_{\tau_\Pi} + \frac{\lambda^2}{2}\int \mathrm{B}^{\xi_+}(s_\Gamma) : \psi^2 :_{\tau_\Pi}$$
$$+ \frac{1}{2}\eta \int : \psi^2 :_{\tau_\Pi} + \frac{1}{2}(m_c^2 - \eta)\int : (\varphi\Lambda)^2 :_{\tau_\Pi}, \quad (63)$$

$$- \lambda^2 \,\mathrm{Tr}\,(\mathrm{K}(s_\Gamma, h)\mathrm{K}^h(s_\Gamma, \psi)) + \lambda^2 \int \mathrm{B}^{\xi_+}(s_\Gamma)(h\psi). \quad (64)$$

Let us remind the definition of $\mathrm{B}^{\xi_+}(s_\Gamma, \chi)$:

$$\mathrm{B}^{\xi_+}(s_\Gamma, \chi) = \frac{1}{2}\frac{\delta}{\delta\varphi(\chi)}\mathrm{Tr}\,(\mathrm{K}(s_\Gamma, \varphi)(1 - \lambda \mathrm{K}(s_\Gamma, \xi_+))^{-1}\mathrm{K}(s_\Gamma, \varphi))\,|_{\varphi=1}. \quad (65)$$

Every elementary expression obtained from (63) by differentiation and localization belongs to one of the following types of expressions, in which f, f', f'' denote, as usually, one of the fields $\varphi\Lambda$, $\varphi^{\pi_j}\Lambda$ or

$$(g - h)\Lambda + (\xi_+ - h)(1 - \Lambda):$$

$$-\frac{\lambda^2}{2}\partial_s^{\Gamma_j} : \mathrm{Tr}\,(\mathrm{K}^h(s_\Gamma, \chi_{\Delta'} f')\mathrm{K}^h(s_\Gamma, \chi_{\Delta''} f'')) :_{\tau_\Pi}, \quad \Gamma_j \neq \emptyset, \text{ or } \Delta' \neq \Delta'', \quad (66)$$

$$\lambda^2 \partial_s^{\Gamma_j}\int \mathrm{B}^{\xi_+}(s_\Gamma)\chi_\Delta : f'f'' :_{\tau_\Pi}, \quad \Gamma_j \neq \emptyset, \quad (67)$$

$$-\frac{\lambda^2}{2}\mathrm{Tr}\,(\mathrm{K}^h(s_\Gamma, \chi_\Delta f')\mathrm{K}^h(s_\Gamma, \chi_\Delta f'')) + \frac{\lambda^2}{2}\int \mathrm{B}^{\xi_+}(s_\Gamma)\chi_\Delta f'f''$$
$$+ \frac{1}{2}m_c^2 \int \chi_\Delta f'f'', \quad (68)$$

$$\frac{1}{2}(m_c^2 - \eta)\int \chi_\Delta f'f''. \quad (69)$$

At least one of the functions f', f'' is equal $\varphi^{\pi_j}\Lambda$ in (68) and is equal $(g - h)\Lambda + (\xi_+ - h)(1 - \Lambda)$ in (69). It is convenient to get rid of the Wick ordering in (66). This term can be written as a difference of two terms

$$-\frac{\lambda^2}{2}\partial_s^{\Gamma_j}\,\mathrm{Tr}\,(\mathrm{K}^h(s_\Gamma, \chi_{\Delta'} f')\mathrm{K}^h(s_\Gamma, \chi_{\Delta''} f'')) \quad (70)$$

and an integral of (69) with respect to $d\mu_{m_c}(\tau)$. A combinatoric factor 2, thus at most $2^{|\Gamma|}$ after all operations, allows a choice of one of these terms. If it is the integral, then it can be excluded from the norm $\|\cdot\|^*_{L^q}$ and, by the Hölder inequality, estimated by $\|(70)\|_{L^q}$. This norm is of the same type as $\|$ " $G(s_\Gamma, \tau_\Pi)$ " $\|^*_{L^q}$ so an estimate of the latter will produce a bound for it. We have the Wick ordering in (67).

(68) is not localized further, i. e. its propagators are not localized.

Let us analyze the expressions coming from (70) and (67). (70) gives rise to

$$-\frac{\lambda^2}{2}\mathrm{Tr}$$
$$[(\text{chain }(20))\mathrm{K}(\Delta'_1, \Delta'_2, \Delta', f')(\text{second chain }(20))\mathrm{K}(\Delta''_1, \Delta''_2, \Delta'', f'')]\,. \quad (71)$$

We will consider two cases. A simpler one is if there are two different squares in one of the sequences $(\Delta'_1, \Delta'_2, \Delta')$, $(\Delta''_1, \Delta''_2, \Delta'')$. Let the first sequence have two different squares. Then we estimate, similarly as in point 5., the trace (70) by the norm $\|\cdot\|_\infty$ of the chains and by

$$\frac{\lambda^2}{2}\,\|\,\mathrm{K}(\Delta'_1, \Delta'_2, \Delta', f')\,\|_{2-\varkappa_1}\,\|\,\mathrm{K}(\Delta''_1, \Delta''_2, \Delta'', f'')\,\|_{2+\varkappa_2} \quad (72)$$

with $\varkappa_1, \varkappa_2$ positive and sufficiently small, $\frac{1}{2-\varkappa_1} + \frac{1}{2+\varkappa_2} = 1$. Using (23) and Proposition A.I.1 we can estimate these norms by the corresponding factors in (15) and by

$$\mathrm{S}^\beta(\chi_{\Delta'}f')\mathrm{S}^\beta(\chi_{\Delta''}f'')\,. \quad (73)$$

The second case is a little bit more complicated. There is $\Delta'_1 = \Delta'_2 = \Delta'$, $\Delta''_1 = \Delta''_2 = \Delta''$ and either $\Delta' \neq \Delta''$, or $\Gamma_j \neq \emptyset$ and then for some $\gamma_j \neq \emptyset$ the differentiation $\partial_s^{\gamma_j}$ acts on a propagator in (70) with a localization Δ_1, Δ_2. Because $\mathrm{H}(s, \tilde{\Delta}, \tilde{\Delta}) = 1$ and $\partial_s^{\gamma_j}\mathrm{H}(s, \tilde{\Delta}, \tilde{\Delta}) = 0$, so $\Delta_1 \neq \Delta_2$ and the propagator occurs in one of the chains in (71). Thus in this case there is a propagator with different squares of localization in one of the chains. It occurs either in $\mathrm{K}(\Delta'_i, \Delta''_i, \Delta'''_i, h)\chi_{\Delta_{i+1}}$ or in $\chi_{\Delta_1}(1 - \lambda\mathrm{K}(s_\Gamma, h))^{-1}\chi_{\Delta_2}$. In this case we estimate (71) using $\|\cdot\|_3$ for the operators

$$\mathrm{K}(\Delta'_1, \Delta'_2, \Delta', f')\,, \quad \mathrm{K}(\Delta''_1, \Delta''_2, \Delta'', f'')\,, \quad \mathrm{K}(\Delta'_i, \Delta''_i, \Delta'''_i, h)\chi_{\Delta_{i+1}}$$

or

$$\chi_{\Delta_1}(1 - \lambda\mathrm{K}(s_\Gamma, h))^{-1}\chi_{\Delta_2}$$

and $\|\cdot\|_\infty$ for all the other ones. The use of Proposition A.I.1 (point 2.*b*) for $\|\,\mathrm{K}(\Delta'_i, \Delta''_i, \Delta'''_i, h)\chi_{\Delta_{i+1}}\,\|_3$ and of Proposition A.I.2 for

$$\|\,\chi_{\Delta_1}(1 - \lambda\mathrm{K}(s_\Gamma, h))^{-1}\chi_{\Delta_2}\,\|_3$$

produces the same estimate as in the previous case.

In the case when one of the functions f', f'' is equal

$$(g - h)\Lambda + (\xi_+ - h)(1 - \Lambda)$$

we use (28) and Lemma IV.1.

For the expression (67) the situation is very similar, because it can be written in the form

$$\frac{\lambda^2}{2}\partial_s^{\Gamma_j}\,\mathrm{Tr}\,(\mathrm{K}(s_\Gamma, 1)\mathrm{K}^{\xi_+}(s_\Gamma, \chi_\Delta : f'f'' :_{\tau_\Pi}))\,. \quad (74)$$

After differentiation, localization, and so on we obtain

$$\frac{\lambda^2}{2}\,\mathrm{Tr}\,[\mathrm{K}(\Delta_1, \Delta_2, \Delta_3, 1)(\text{chain (20) with } h \to \xi_+) \cdot \mathrm{K}(\Delta', \Delta'', \Delta, : f'f'' :_{\tau_\Pi})]\,. \quad (75)$$

Reasoning the same way as for (71) we obtain the factors of the right hand side of (15) and

$$\mathrm{S}^\beta(: f'f'' :_{\tau_\Pi}\chi_\Delta)\,. \quad (76)$$

Now let us consider the expression (68). We have

$$\begin{aligned}(68) = &-\frac{\lambda^2}{2}\,\mathrm{Tr}\,[(\mathrm{K}^h(s_\Gamma, \chi_\Delta f') - \mathrm{K}^{\xi\pm}(\chi_\Delta f'))(\mathrm{K}^h(s_\Gamma, \chi_\Delta f'') \\ &- \mathrm{K}^{\xi\pm}(\chi_\Delta f''))] - \frac{\lambda^2}{2}\,\mathrm{Tr}\,[(\mathrm{K}^h(s_\Gamma, \chi_\Delta f') - \mathrm{K}^{\xi\pm}(\chi_\Delta f')) \\ &\cdot \mathrm{K}^{\xi\pm}(\chi_\Delta f'')] - \frac{\lambda^2}{2}\,\mathrm{Tr}\,[\mathrm{K}^{\xi\pm}(\chi_\Delta f')(\mathrm{K}^h(s_\Gamma, \chi_\Delta f'') \\ &- \mathrm{K}^{\xi\pm}(\chi_\Delta f''))] - \frac{\lambda^2}{2}\,\mathrm{Tr}\,(\mathrm{K}^{\xi\pm}(\chi_\Delta f')\mathrm{K}^{\xi\pm}(\chi_\Delta f'')) \\ &+ \frac{\lambda^2}{2}\,\mathrm{Tr}\,[\mathrm{K}(s_\Gamma, 1)(\mathrm{K}^{\xi\pm}(s_\Gamma, \chi_\Delta f'f'') - \mathrm{K}^{\xi\pm}(\chi_\Delta f'f''))] \\ &+ \frac{\lambda^2}{2}\,\mathrm{Tr}\,[(\mathrm{K}(s_\Gamma, 1) - \mathrm{K}(1))\mathrm{K}^{\xi+}(\chi_\Delta f'f'')] \\ &+ \frac{\lambda^2}{2}\,\mathrm{Tr}\,(\mathrm{K}(1)\mathrm{K}^{\xi+}(\chi_\Delta f'f'')) + \frac{1}{2}m_c^2\int \chi_\Delta f'f''\,, \end{aligned} \quad (77)$$

where ξ_+ or ξ_- is chosen according to the value of $\Sigma(\Delta)$. Using the following two equalities:

$$-\frac{\lambda^2}{2}\,\mathrm{Tr}\,(\mathrm{K}^{\xi+}(1)\mathrm{K}^{\xi+}(\chi_\Delta f'f'')) + \frac{\lambda^2}{2}\,\mathrm{Tr}\,(\mathrm{K}(1)\mathrm{K}^{\xi+}(\chi_\Delta f'f'')) + \frac{1}{2}m_c^2\int \chi_\Delta f'f'' = 0\,, \quad (78)$$

$$\mathrm{Tr}\,(\mathrm{K}^{\xi+}(1)\mathrm{K}^{\xi+}(\chi_\Delta f'f'')) = \mathrm{Tr}\,(\mathrm{K}^{\xi\pm}(1)\mathrm{K}^{\xi\pm}(\chi_\Delta f'f''))\,, \quad (79)$$

we estimate the right hand side of (77) using Proposition A.I.4 and Proposition A.I.1. This gives

$$|(68)| \leqslant \left|\frac{\lambda^2}{2}\,\mathrm{Tr}\,(\mathrm{K}^{\xi\pm}(f'\chi_\Delta)\mathrm{K}^{\xi\pm}(f''\chi_\Delta - \mathrm{K}^{\xi\pm}(1)\mathrm{K}^{\xi\pm}(f'f''\chi_\Delta))\right| + 0(\lambda^2)m^{-\nu}(\mathrm{S}^\beta(f'\chi_\Delta)\mathrm{S}^\beta(f''\chi_\Delta) + \mathrm{S}^\beta(f'f''\chi_\Delta)) \quad (80)$$

for some positive ν and β. One term on the right hand side of the above inequality can be chosen by a combinatoric factor 3, thus $3^{|\Pi|}$ for the whole expression.

As far as the terms (69) are concerned we separate the product of them using the Hölder inequality and bound the resulting norm combining the methods of point 1 with Lemma IV.1. The result matches the right hand side of (15).

Finally we consider the expression (64). At first write

$$(64) = -\lambda^2 \sum_{\varepsilon=+,-} [\mathrm{Tr}\,(K(s_\Gamma, \chi_{\tilde{Z}^\varepsilon} h)K^h(s_\Gamma, \chi_{\tilde{Z}^\varepsilon} f) - K(s_\Gamma, \chi_{\tilde{Z}^\varepsilon})K^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^\varepsilon} hf))]$$

$$- \lambda^2 \sum_{\substack{\varepsilon_1,\varepsilon_2=+,0,- \\ (\varepsilon_1,\varepsilon_2)\neq(\pm,\pm)}} [\mathrm{Tr}\,(K(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_1}} h)K^h(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_2}} f)$$

$$- K(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_1}})K^{\xi+}(s_\Gamma, \chi_{\tilde{Z}^{\varepsilon_2}} hf))]$$

$$= -\lambda^2 \sum_{\varepsilon=+,-} \mathrm{Tr}\,[K(s_\Gamma, \xi_\varepsilon \chi_{\tilde{Z}^\varepsilon})(1 - \lambda K(s_\Gamma, \varepsilon h))^{-1}\lambda(K(s_\Gamma, \varepsilon h)$$

$$- K(s_\Gamma, \xi_+))(1 - \lambda K(s_\Gamma, \xi_+))^{-1}K(s_\Gamma, \chi_{\tilde{Z}^\varepsilon} f)]$$

$$- \lambda^2 \sum_{\substack{\Delta_1,\Delta_2\subset Z,\Delta_1\neq\Delta_2 \\ \Delta_1\subset\tilde{Z}^{\varepsilon_1},\Delta_2\subset\tilde{Z}^{\varepsilon_2},(\varepsilon_1,\varepsilon_2)\neq(\pm,\pm)}} [\mathrm{Tr}\,(K(s_\Gamma, \chi_{\Delta_1} h)K^h(s_\Gamma, \chi_{\Delta_2} f)$$

$$- K(s_\Gamma, \chi_{\Delta_1})K^{\xi+}(s_\Gamma, \chi_{\Delta_2} hf))] - \lambda^2 \sum_{\Delta\subset\tilde{Z}^0} [\lambda\,\mathrm{Tr}\,(K(s_\Gamma, \chi_\Delta h)K^h(s_\Gamma, h)K(s_\Gamma, \chi_\Delta f))$$

$$- \lambda\,\mathrm{Tr}\,(K(s_\Gamma, \chi_\Delta)K^{\xi+}(s_\Gamma, \xi_+)K(s_\Gamma, \chi_\Delta hf))]. \tag{81}$$

Now if differentiation and localization is performed we obtain expressions of the types considered before, in points 5.-7. There are either two squares Δ_1, Δ_2 with $d(\Delta_1, \Delta_2) \geqslant L$ or a square Δ and a bond b of a differentiation creating this expression with $d(b, \Delta) \geqslant \frac{1}{2} L$, so Lemma IV.1 can be applied and we may estimate the right hand side of (81) by the factors of (15) and $S^\beta(f\chi_\Delta)$.

9. After the estimations of point 8 we finally need a bound on the norm $\|\cdot\|^*_{L^q}$ of a product of the expressions:

$$S^\beta(f\chi_\Delta) \tag{82}$$

$$S^\beta(f'f''\chi_\Delta) \tag{83}$$

$$S^\beta(:\varphi^2:_{\tau_\Pi}\chi_\Delta) \tag{84}$$

$$\left| \frac{\lambda^2}{2} \mathrm{Tr}\,(K^{\xi\pm}(f'\chi_\Delta)K^{\xi\pm}(f''\chi_\Delta) - K^{\xi\pm}(1)K^{\xi\pm}(f'f''\chi_\Delta)) \right| \tag{85}$$

where f, f', f'' are equal $\varphi\Lambda$, $\varphi^{\pi_j}\Lambda$ or $(g - h)\Lambda + (\xi_+ - h)(1 - \Lambda)$ and at least one of the functions f', f'' in (85) is equal $\varphi^{\pi_j}\Lambda$, we can assume for example $f'' = \varphi^{\pi_j}\Lambda$. Using the Hölder inequality we can separate the four types of factors. The factors (82) can be further separated and we obtain the norms (31), (32), estimated in Lemmas IV.2, 3, and the factors with $f = (g - h)\Lambda + (\xi_+ - h)(1 - \Lambda)$ which are estimated by (28) and Lemma IV.1. We can separate also the factors (83) for which one of the functions f', f'' is equal φ, and the second is equal $(g-h)\Lambda+(\xi_+ -h)(1-\Lambda)$, and estimate the norm of a product of these factors by an obvious modification of Lemma IV.3, and Lemma IV.1.

A very similar situation occurs for factors (84). We have

LEMMA IV.4.

$$\left\| \prod_{k=1}^{K} S^{\beta}(:\varphi^2:_{\tau_\Pi} \chi_{\Delta_k}) \right\|_{L^q} \leqslant \prod_{\text{fermion propagators}}' e^{3d(\Delta,\Delta')} \cdot \prod_{\text{differentiated fermion propagators}}' e^{3\varepsilon d(\gamma,\tilde{\Delta},\tilde{\Delta}')} \prod_{\text{vertices}}' e^{0(l)} . \quad (86)$$

The products are over factors attached to elementary expressions giving rise to $S^{\beta}(:\varphi^2:_{\tau_\Pi}\chi_{\Delta_k})$, $k = 1, \ldots, K$.

Proof of this lemma is identical to the proof of Lemma IV.3. □

The situation with the factors (83), (85) is a little more complicated. We have to estimate two norms:

$$\left\| \prod_{k=1}^{K} S^{\beta}(f_k \varphi^{\pi_k} \Lambda \chi_{\Delta_k}) \right\|_{L^q}^{*}, \quad (87)$$

$$\left\| \prod_{k=1}^{K} \frac{\lambda^2}{2} \operatorname{Tr} \left(K^{\xi_\pm}(f'_k \chi_{\Delta_k}) K^{\xi_\pm}(\varphi^{\pi_k} \Lambda \chi_{\Delta_k}) - K^{\xi_\pm}(1) K^{\xi_\pm}(f'_k \varphi^{\pi_k} \Lambda \chi_{\Delta_k}) \right) \right\|_{L^q}^{*} \quad (88)$$

where $f_k = \varphi\Lambda$, $\varphi^{\pi}\Lambda$ or $(g - h)\Lambda + (\xi_+ - h)(1 - \Lambda)$.

We can separate further the factors in (87), (88) with

$$f_k = (g - h)\Lambda + (\xi_+ - h)(1 - \Lambda),$$

and the norms $\| S^{\beta}(f_k \varphi^{\pi_k} \Lambda \chi_{\Delta_k}) \|_{L^q}^{*}$ can be estimated by an obvious modification of Lemma IV.2, and Lemma IV.1. The norms

$$\left\| \frac{\lambda^2}{2} \operatorname{Tr} \left(K^{\xi_\pm}(((g - h)\Lambda + (\xi_+ - h)(1 - \Lambda))\chi_{\Delta_k}) K^{\xi_\pm}(\varphi^{\pi_k} \Lambda \chi_{\Delta_k}) - K^{\xi_\pm}(1) K^{\xi_\pm}(((g - h)\Lambda + (\xi_+ - h)(1 - \Lambda))\varphi^{\pi_k} \Lambda \chi_{\Delta_k}) \right) \right\|_{L^q}^{*} \quad (89)$$

will be considered later.

Thus we can assume that in (87), (88), $f_k = \varphi\Lambda$ or $\varphi^{\pi}\Lambda$. The needed estimations of (87), (88) are now contained in the following two lemmas.

LEMMA IV.5.

$$(87) \leqslant \prod_{\text{fields } \varphi^\pi \text{ in } (87)} e^{0(1)\varepsilon d(\pi,\tilde{\Delta})} 0(1)^{|\pi|} e^{-\delta m_c d(\pi,\tilde{\Delta})} \cdot \mathrm{G}_1(\pi, \delta_1) e^{-\delta_2 l|\pi|} \prod_{k=1}^{\mathrm{K}} e^{0(l)} m^{-\frac{1}{16}\nu_2}, \quad (90)$$

and the last product above can be omitted for m sufficiently large.

To prove the lemma we begin with a representation of the power q of (87), similar to the representation (36) for (31):

$$\int \prod_{j=1}^{\mathrm{K}q} dx_j \prod_{k=1}^{\mathrm{K}} \prod_{i=1}^{q/4} w_k(x_{4i-3+(k-1)q}, \ldots, x_{4i+(k-1)q}) \cdot \sum_{\text{double pairings}}{}' \prod_{\text{pairs } j=\{j+,j-\}} \mathrm{C}^{\#}_{m_c}(\tau\,; x_{j+}, x_{j-}), \quad (91)$$

where the summation over double pairings is understood as the summation over sets of pairs such, that every index $j = 1, \ldots, \mathrm{K}_q$ occurs exactly in two pairs (obviously not over all such sets) and $\mathrm{C}^{\#}_{m_c}(\tau)$ denotes the corresponding propagator $\mathrm{C}_{m_c}(\tau)$ or $\mathrm{C}^{\pi_k}_{m_c}(\tau)$ depending on a pair j. Applying the Hölder inequality with exponents p_1, q_1, and next Schwartz inequality to the product of propagators, we obtain

$$(91) \leqslant \prod_{k=1}^{\mathrm{K}} \| w_k \|^{q/4}_{\mathrm{L}^{p_1}(\mathbb{R}^8)} \cdot \sum_{\text{double pairings}}{}' \prod_{\text{pairs}} \| \chi_{\tilde{\Delta}_{k(j+)}} \mathrm{C}^{\#}_{m_c}(\tau) \chi_{\tilde{\Delta}_{k(j-)}} \|_{\mathrm{L}^{2q_1}(\mathbb{R}^4)} \quad (92)$$

where $k(j)$ denote this index k, for which the variable x_j occurs in w_k. We fix now one term in the sum over double pairings with the help of a proper combinatoric factor. Let $b_k \in \pi_k$. We have

$$\sum_{\text{double pairings}} \prod_{\text{pairs } j=\{j+,j-\}} e^{-\varepsilon d(b_{k(j+)}, b_{k(j-)})} = \sum_{\text{permutations } \pi \text{ of } \{1,\ldots,\mathrm{K}q\}} \prod_{i=1}^{\mathrm{K}q} e^{-\varepsilon d(b_{k(i)}, b_{k(\pi(i))})}$$

$$\leqslant \prod_{i=1}^{\mathrm{K}q} \sum_{i'=1}^{\mathrm{K}q} e^{-\varepsilon d(b_{k(i)}, b_{k(i')})} \leqslant \prod_{i=1}^{\mathrm{K}q} 0(1) \sum_{b'} e^{-\varepsilon d(b_{k(i)}, b')} \leqslant 0(1)^{\mathrm{K}q}, \quad (93)$$

hence the corresponding factor is

$$\prod_{\text{pairs } j=\{j+,j-\}} 0(q)e^{\varepsilon d(b_{k(j+)},b_{k(j-)})} \leqslant \prod_{i=1}^{\mathrm{K}q} e^{0(l)}e^{2\varepsilon d(b_{k(i)},\tilde{\Delta}_{k(i)})} \cdot \prod_{\text{pairs } j} e^{\varepsilon d(\tilde{\Delta}_{k(j+)},\tilde{\Delta}_{k(j-)})}. \quad (94)$$

Thus, using (92)-(94) and (34), the following inequality is obtained

$$(91) \leqslant \sup_{\text{double pairings}} \left(\prod_{k=1}^{\mathrm{K}} 0(q)e^{0(l)}m^{-\frac{1}{4}v_2}e^{2\varepsilon d(b_k,\tilde{\Delta}_k)} \right)^q \cdot \prod_{\text{pairs } j} e^{\varepsilon d(\tilde{\Delta}_{k(j+)},\tilde{\Delta}_{k(j-)})} \| \chi_{\tilde{\Delta}_{k(j+)}} \mathrm{C}^{\#}_{m_c}(\tau)\chi_{\tilde{\Delta}_{k(j-)}} \|_{\mathrm{L}^{2q_1}(\mathbb{R}^4)}. \quad (95)$$

Further, if for some j $\mathrm{C}^{\#}_{m_c}(\tau) = \mathrm{C}_{m_c}(\tau)$, then the corresponding factor in the product over pairs can be estimated by $0(l^{\frac{4}{2q_1}}) \leqslant e^{0(l)}$. If $\mathrm{C}^{\#}_{m_c}(\tau) = \mathrm{C}^{\pi_i}_{m_c}(\tau)$, then the geometric mean of the two choices of covariances for this factor can be estimated by

$$e^{0(l)}0(1)^{|\pi_i|}e^{-\delta m_c d(\pi_i,\tilde{\Delta}_{k(j+)},\tilde{\Delta}_{k(j-)})} \cdot \mathrm{G}_1(\pi_i,\,\delta_1)e^{-\delta_2 l|\pi_i|}e^{\varepsilon d(\pi_i,\tilde{\Delta}_{k(j+)},\tilde{\Delta}_{k(j-)})}.$$

Using these estimations we obtain (90). □

LEMMA IV.6.

$$(88) \leqslant \prod_{\text{fields } \varphi^\pi \text{ in } (88)} e^{0(1)\varepsilon d(\pi,\tilde{\Delta})}0(1)^{|\pi|}e^{-\delta m_c d(\pi,\tilde{\Delta})} \cdot \mathrm{G}_1(\pi,\,\delta_1)e^{-\delta_2 l|\pi|}\lambda^{-2\mathrm{K}}\prod_{k=1}^{\mathrm{K}} \lambda^2 e^{0(l)}m^{-2v} \quad (96)$$

and the last product can be omitted for m large.

We use a representation

$$\mathrm{Tr}\,(\mathrm{K}^{\xi\pm}(f'\chi_\Delta)\mathrm{K}^{\xi\pm}(f''\chi_\Delta) - \mathrm{K}^{\xi\pm}(1)\mathrm{K}^{\xi\pm}(f'f''\chi_\Delta)) = \frac{1}{8\pi^4}\int \mathrm{F}(k)\widehat{f'\chi_\Delta}(k)\widehat{f''\chi_\Delta}(k)dk = \frac{1}{2\pi^2}\int dxdy(f'\chi_\Delta)(x)\tilde{\mathrm{F}}(x-y)(f''\chi_\Delta)(y) \quad (97)$$

where $\mathrm{F}(k)$ is given in (V.280) in Chapter V and $\tilde{\mathrm{F}}$ is its inverse Fourier transform.

For (88) raised to the power q we have the formula

$$\int \prod_{j=1}^{2Kq} dx_j \prod_{i=1}^{Kq} \left(\frac{\lambda}{2\pi}\right)^2 \chi_{\Delta_{k(2i-1)}}(x_{2i-1}) \tilde{F}(x_{2i-1} - x_{2i}) \chi_{\Delta_{k(2i)}}(x_{2i}) \sum_{\text{pairings}}{}' \prod_{\text{pairs } j = \{j+, j-\}} C^{\#}_{m_c}(\tau; x_{j+}, x_{j-}) \quad (98)$$

where the same notation is used as in Proof of Lemma V.5. Let us notice that the indices in a given pair cannot occur in one function $\tilde{F}$, so each pair connects two such functions.

A pairing in the above sum can be fixed, identically as in Proof of Lemma IV.5, by the factor on the right hand side of (94), only i changes from 1 to $2Kq$. This factor will be estimated next the same way as there, we need only to extract the usual factors for the propagators $C^{\#}_{m_c}(\tau)$.

Now let us consider a term in (98) with fixed pairings. This pairing can be decomposed into disjoint chains of pairs $\{j_1, j_2\}, \{j_3, j_4\}, \ldots, \{j_{2r-1}, j_{2r}\}$ with the property that each pair $\{j_2, j_3\}, \{j_4, j_5\}, \ldots, \{j_{2r}, j_1\}$ occurs in one function $\tilde{F}$. Then the term in (98) factorizes into a product of terms corresponding to the chains. It suffices to estimate a term for one chain. It is easily seen that this may be interpreted as the trace of a product of operators

$$\mathrm{Tr}\left(\prod_{j=1}^{r} C^{\#}_{m_c}(\tau)\left(\frac{\lambda}{2\pi}\right)^2 \chi_{\Delta'_j} F(P) \chi_{\Delta'_j}\right). \quad (99)$$

This expression will be transformed now. If $C^{\#}_{m_c}(\tau)$ equals to one of the propagators $C^{\pi_j,\varepsilon_j}_{m_c}(\tau)$, then we insert on its both sides the operators $D^{\alpha}_c D^{-\alpha}_c$, $D_c = (P^2 + m_c^2)^{\frac{1}{2}}$. If $C^{\#}_{m_c}(\tau)$ is one of the propagators $C^{\pi_j,\rho_j,b_j}_{m_c}(\tau)$, $C_{m_c}(\tau)$, then we insert on its both sides the operators $\zeta_{\Delta'_j} D^{\alpha}_c D^{-\alpha}_c$, where

$$\zeta_{\Delta'_j} \in C_0^{\infty}(\mathbb{R}^2), \qquad \zeta_{\Delta'_j} = 1$$

on Δ'_j, supp $\zeta_{\Delta'_j}$ is contained in sufficiently small neighbourhood of Δ'_j, Δ'_j is the localization of the neighbouring operator $F(P)$. The expression obtained this way is estimated next by the corresponding product of norms

$$\| D^{\alpha}_c \zeta_{\Delta'_j} C^{\pi_j,\rho_j,b_j}_{m_c}(\tau) \zeta_{\Delta'_{j+1}} D^{\alpha}_c \|_{L^2}, \quad \text{including } \pi_j = \varnothing, \quad (100)$$

$$\| D^{\alpha}_c C^{\pi_j,\varepsilon_j}_{m_c}(\tau) D^{\alpha}_c \|_{L^2}, \quad (101)$$

$$\| D^{-\alpha}_c \chi_{\Delta'_j} F(P) \chi_{\Delta'_j} D^{-\alpha}_c \|_{\infty}. \quad (102)$$

In this place we use the estimates of the propagators contained in [3] Corollary VI.2, Lemma VI.11.

$$\| D^{\alpha}_c \zeta_{\Delta'_j} C^{\pi_j,\rho_j,b_j}_{m_c}(\tau) \zeta_{\Delta'_{j+1}} D^{\alpha}_c \|_{L^2} \leqslant e^{O(l)} e^{-\delta m_c d(\pi_j, \tilde{\Delta}'_j, \tilde{\Delta}'_{j+1})}, \quad (103)$$

$$\| D^{\alpha}_c C^{\pi_j,\varepsilon_j}_{m_c}(\tau) D^{\alpha}_c \|_{L^2} \leqslant e^{O(l)} O(1)^{|\pi_j|} G_1(\pi_j, \delta_1) e^{-\delta_2 l |\pi_j|}, \quad (104)$$

and holding for α properly chosen, for example for $\alpha = \frac{3}{4}$. These estimates provide all the necessary factors connected with the fields φ^{π} occuring in (88).

Finally, let us consider the norm (102). By the result of [*25*], Lemma 2.3, see also Lemma A.I.4.,

$$[D_c^{-\lambda}, \chi_\Delta]D_c^{\nu} \in C_4 \qquad \text{for} \qquad 0 \leqslant \nu < \frac{1}{4}, \qquad \lambda - 2\nu > \frac{1}{4}. \tag{105}$$

Hence taking for example $\alpha = \frac{3}{4}$, $\nu = \frac{1}{8}$ we obtain

$$\begin{aligned}(102) = \| \, [([D_c^{-\alpha}, \chi_{\Delta_j'}]D_c^{\nu})D_c^{-\nu} + \chi_{\Delta_j'}D_c^{-\alpha}]F(P) \\ \cdot [D_c^{-\alpha}\chi_{\Delta_j'} + D_c^{-\nu}(D_c^{\nu}[\chi_{\Delta_j'}, D_c^{-\alpha}])] \, \|_\infty \leqslant 0(1) \, \| \, D_c^{-\nu}F(P)D_c^{-\nu} \, \|_\infty .\end{aligned} \tag{106}$$

But

$$\begin{aligned}\| \, D_c^{-\nu}F(P)D_c^{-\nu} \, \|_\infty &= \sup_k \, (k^2 + m_c^2)^{-\nu} \, | \, F(k) \, | \\ &\leqslant \sup_k \, | \, k^2 \, |^{-\nu} \, | \, F(k) \, | \leqslant 0(1) \sup_k \, | \, k^2 \, |^{-\nu} \log\left(1 + \frac{k^2}{m^2}\right) \\ &= 0(m^{-2\nu}) \sup_{\rho > 0} \rho^{-\nu} \log(1 + \rho) = 0(m^{-2\nu}),\end{aligned} \tag{107}$$

where we have used the estimate (V.281) to be proven in Chapter V. This ends Proof of Lemma IV.6. □

Now it is easily seen, that the norms (89) can be estimated by the methods used in the proof of the above lemma, and by Lemma IV.1 of course, and we get the needed combinatoric factors for the fields φ^{π_k}.

Gathering together all estimations obtained up to this point we obtain all necessary factors on the right hand side of (15). This completes Proof of Proposition IV.1. □

CHAPTER V

LOWER LINEAR BOUND

The present chapter is devoted to the proof of Theorem III.1. To this end we shall estimate an $L^p(d\mu_{m_c}(\tau)) \equiv L^p(\tau)$ norm of $\tau_K\left(z^K \cdot \bigwedge_{j=1}^{K} Q_j\right)$

for random variables Q_j with values in trace-class operators on $L^2(Z)$.

$$
\begin{aligned}
z^K = \prod_{\Delta \subset Z} & \chi^{\#}_{\Sigma(\Delta)}((\varphi + g)^{\Lambda}_{\Delta})\Lambda^K(1 - \lambda K^h(s, \psi\!\!\!/))^{-1} \det{}_3 (1 - \lambda K^h(s, \psi\!\!\!/)) \\
& \cdot \exp\left[-\frac{\lambda^2}{2} : \mathrm{Tr}\, K^h(s, \psi\!\!\!/)^2 :_\tau + \frac{\lambda^2}{2} \int B^{\xi+}(s) : \psi\!\!\!/^2 :_\tau \right. \\
& \left. + \frac{\eta}{2} \int : \psi\!\!\!/^2 :_\tau + \frac{m_c^2 - \eta}{2} \int_Z : (\varphi\Lambda)^2 :_\tau \right] \\
& \cdot \exp\left[\frac{\lambda^2}{2} \int \mathrm{Tr}\, (K^{\xi+}(s, \varphi\Lambda\chi_Z)^2 - K^h(s, \varphi\Lambda\chi_Z)^2) d\mu_{m_c}(\tau) \right] \\
& \cdot \exp\left[- E_1(Z, \Sigma) - E_2(Z, \Sigma) - F(Z, \Sigma) \right], \qquad (1)
\end{aligned}
$$

s and τ vanish along ∂Z. z^K is a random variable with values in $B(\Lambda^K L^2(Z))$.

The first idea of the proof is to compare the partially decoupled action entering in z^K with the completely decoupled one based on operators K_Δ acting in $L^2(\Delta)$, Δ running through squares of the d-lattice. We shall define K_Δ using periodic boundary conditions. This choice leaves computation of traces we have to perform still manageable.

First some notation. By P^i_Δ we shall denote the operator $\frac{1}{i}\frac{\partial}{\partial x^j}$ on $L^2(\Delta)$ (or $\mathbb{C}^2 \otimes L^2(\Delta)$) with periodic boundary conditions. We shall also write $F(P)_\Delta \equiv F(P_\Delta)$ for a function F of two variables. $F(P)_\Delta$ will be also considered as an operator on $L^2(\mathbb{R}^2)$ giving zero on $L^2(\sim \Delta)$, or as an operator on $\mathbb{C}^2 \otimes L^2(\mathbb{R}^2)$. With $D_\Delta := (P^2 + m^2)^{1/2}_\Delta$ put

$$K_\Delta(f) := (P\!\!\!/ + m)_\Delta D_\Delta^{-3/2} f D_\Delta^{-1/2}, \qquad (2)$$

$$K^x_\Delta(f) := (1 - \lambda K_\Delta(x))^{-1} K_\Delta(f). \qquad (3)$$

Let

$$B'' := \lambda\left(K^h(s, \psi\!\!\!/) - \sum_{\Delta \subset Z^\pm} K^{\xi\pm}_\Delta(\psi\!\!\!/) \right), \qquad (4)$$

where $Z^\pm := \bigcup_{\Delta \subset Z : \Sigma(\Delta) = \pm} \Delta.$

Similarly as in [*13*] the proof of the linear lower bound will follow in steps. First we shall obtain a bound with $\exp\,[0(1)\lambda^3 \,|\, Z \,|]$ instead of $\exp\,[0(1) m^{-\nu} \,|\, Z \,|]$ but uniform in parameters $t = (t(\Delta))$ being superficial coupling constants introduced for d-squares Δ. The final result follows via expansion in parameters t.

Thus for each d-lattice square $\Delta \subset Z$ introduce $t(\Delta) \in [0, 1]$. Assume additionally that $t(\Delta) = 1$ for $\Delta \subset Z^0 \cup Z'$. Write

$$z^{K}(t) := \prod_{\Delta \subset Z^{\pm}} \chi_{\pm}^{\#}((\varphi + g)_{\Delta}^{\Lambda})\Lambda^{K}\Big(1 - \lambda \sum_{\Delta \subset Z^{\pm}} t(\Delta)K_{\Delta}^{\xi\pm}(\psi) - B''\Big)^{-1}$$
$$\cdot \det_3\Big(1 - \lambda \sum_{\Delta \subset Z^{\pm}} t(\Delta)K_{\Delta}^{\xi\pm}(\psi) - B''\Big)$$
$$\cdot \exp\Big[- \sum_{\Delta \subset Z^{\pm}} \frac{1}{2} t(\Delta)^2\Big(\lambda^2 : \mathrm{Tr}\, K_{\Delta}^{\xi\pm}(\psi)^2 :_\tau - \lambda^2 \int B_{\Delta}^{\xi+} : \psi^2 :_\tau$$
$$- \eta \int_{\Delta} : \psi^2 :_\tau - (m_c^2 - \eta)\int_{\Delta} : (\varphi\Lambda)^2 :_\tau\Big)\Big]$$
$$\cdot \exp\Big[-\frac{1}{2}\mathrm{Tr}\, B''^2 - \lambda \sum_{\Delta \subset Z^{\pm}} \mathrm{Tr}\big(K_{\Delta}^{\xi\pm}(\psi)B''\big) - D_1 - D_2 - E_1 - E_2 - F\Big], \quad (5)$$

where

$$D_1 := \frac{\lambda^2}{2}\int\Big(\sum_{\Delta \subset Z} B_{\Delta}^{\xi+} - B^{\xi+}(s)\Big) : \psi^2 :_\tau, \quad (6)$$

$$D_2 := \frac{\lambda^2}{2}\int \mathrm{Tr}\Big(\sum_{\Delta \subset Z^{\pm}} K_{\Delta}^{\xi\pm}(\varphi\Lambda)^2 - K^{\xi+}(s, \varphi\Lambda\chi_Z)^2\Big) d\mu_{m_c}(\tau), \quad (7)$$

$$B_{\Delta}^{\xi+}(x) := \frac{1}{2}\frac{\delta}{\delta\varphi(x)}\Big|_{\varphi=1} \mathrm{Tr}\,(K_{\Delta}(\varphi)K_{\Delta}^{\xi+}(\varphi)) \quad (8)$$

and we have written $E_2 = E_2' + E_2''$, see (III.7) and (III.8). Note that for $t(\Delta) \equiv 1$, $z^{K}(t) = z^{K}$.

Put

$$\mathscr{Z}^{K}(t) := \| z^{K}(t) \|, \quad (9)$$

where on the right hand side we have the operator norm in $\Lambda^{K}L^2(Z)$.

We shall prove

PROPOSITION V.1. — There exist η, $0 < \eta < \frac{1}{4}m_c^2$, $p > 1$, $d > 0$ and constants $0(1)$ such that for all Λ, K, λ, t, $\alpha > 0$ and $m \geqslant m_0(\lambda, \alpha)$

$$\| \mathscr{Z}^{K}(t) \|_{L^{p}(\tau)} \leqslant 0(1)^{K}\Big(\prod_{\Delta \subset Z'} \exp\,[0(1)n_\Delta](n_\Delta !)^{0(1)}\Big)$$
$$\cdot \exp\,[- 0(1)m^{2-\alpha} | Z^0 \cup Z' | + 0(1)\lambda^3 | Z_t | + 0(1)m^{-\frac{1}{4}+\alpha} | Z_v |], \quad (10)$$

where

$$Z_t = \bigcup_{t(\Delta) > 0} \Delta \quad (11)$$

and

$$Z_v = \bigcup_{t(\Delta)=0} \Delta \tag{12}$$

To obtain (10) we follow the general idea of G – J – S [13]. In each square of the d-lattice the field will be decomposed into its average and quantum fluctuations around the average. The terms involving a 1-loop effective potential of the average factor out in each square and are easily estimated with our knowledge of the shape of the potential. The terms with fluctuations are estimated basically as in the standard proof of the linear lower bound for $(Yu)_2$ [25]. The only complication is that the terms with fluctuations involve also the averages of the field.

Thus divide $K_\Delta^{\xi\pm}(\psi)$ into two parts:

$$K_\Delta^{\xi\pm}(\psi) = K_\Delta^{\xi\pm}(\psi_\Delta) + K_\Delta^{\xi\pm}(\delta\psi), \tag{13}$$

where $\psi_\Delta := d^{-2}\int_\Delta \psi$ and δ is the fluctuation operator on $L^2(\mathbb{R}^2)$ defined by

$$\delta\psi := \psi - \psi_\Delta \qquad \text{on } \Delta . \tag{14}$$

Again easy transformations give

$$\begin{aligned}
\mathscr{Z}^K(t) = {} & \prod_{\Delta\subset Z^\pm} \chi_\pm^\#((\varphi + g)_\Delta^\Lambda) \det_3\Big(1 - \lambda \sum_{\Delta\subset Z^\pm} t(\Delta) K_\Delta^{\xi\pm}(\psi_\Delta)\Big) \\
& \cdot \exp\Big[-\sum_{\Delta\subset Z^\pm} \frac{1}{2} t(\Delta)^2 \Big(\lambda^2 : \operatorname{Tr} K_\Delta^{\xi\pm}(\psi_\Delta)^2 :_\tau - \lambda^2 \int B_\Delta^{\xi+} : \psi_\Delta^2 :_\tau - \eta \int_\Delta : \psi_\Delta^2 :_\tau\Big)\Big] \\
& \cdot \Big\| \Lambda^K\Big(1 - \lambda \sum_{\Delta\subset Z^\pm} t(\Delta) K_\Delta^{\eta\pm}(\delta\psi) - B'\Big)^{-1} \cdot \Lambda^K\Big(1 - \lambda \sum_{\Delta\subset Z^+} t(\Delta) K_\Delta^{\xi\pm}(\psi_\Delta)\Big)^{-1} \Big\| \\
& \cdot \det_3\Big(1 - \lambda \sum_{\Delta\subset Z^\pm} t(\Delta) K_\Delta^{\eta\pm}(\delta\psi) - B'\Big) \\
& \cdot \exp\Big[-\sum_{\Delta\subset Z^\pm} \frac{1}{2} t(\Delta)^2 \Big(\operatorname{Tr} K_\Delta^{\eta\pm}(\delta\psi)^2 - \lambda^2 \int \operatorname{Tr} K_\Delta^{\xi\pm}(\delta(\varphi\Lambda))^2 d\mu_{m_c}(\tau) \\
& - \lambda^2 \int B_\Delta^{\xi+} : (\delta\psi)^2 :_\tau - \eta \int_\Delta : (\delta\psi)^2 :_\tau - (m_c^2 - \eta) \int_\Delta : (\varphi\Lambda)^2 :_\tau\Big)\Big] \\
& \cdot \exp\Big[-\lambda \sum_{\Delta\subset Z^\pm} (1 - t(\Delta)) \operatorname{Tr}(K_\Delta^{\xi\pm}(\psi) B'') - \lambda \sum_{\Delta\subset Z^\pm} t(\Delta) \operatorname{Tr}(K_\Delta^{\xi\pm}(\psi_\Delta) B') \\
& - \lambda \sum_{\Delta\subset Z^\pm} t(\Delta) \operatorname{Tr}(K_\Delta^{\eta\pm}(\delta\psi) B') - \frac{1}{2} \operatorname{Tr} B'^2\Big] \cdot \exp[-D_1 - D_2 - E_1 - E_2 - F],
\end{aligned} \tag{15}$$

where

$$\eta_\pm := \xi_\pm + t(\Delta)\psi_\Delta \tag{16}$$

and

$$B' := \Big(1 - \lambda \sum_{\Delta \subset Z^{\pm}} t(\Delta) K_{\Delta}^{\xi_{\pm}}(\psi_{\Delta})\Big)^{-1} B'' . \tag{17}$$

In some expressions above it will be more convenient to use operators

$$\overline{K}_{\Delta}(f) := W_{\Delta} K_{\Delta}(f) W_{\Delta}^{-1} \tag{18}$$

and

$$\overline{K}_{\Delta}^{x}(f) := (1 - \lambda \overline{K}_{\Delta}(x))^{-1} \overline{K}_{\Delta}(f) \tag{19}$$

instead of $K_{\Delta}(f)$ and $K_{\Delta}^{x}(f)$ with

$$W_{\Delta} := \overline{D}_{\Delta}^{1/2} D_{\Delta}^{-1/2} \tag{20}$$

where $\overline{D}_{\Delta} := (P^2 + \overline{m}^2)_{\Delta}^{1/2}$ and $\overline{m}^2 := m^2 + \lambda^2 \xi_{+}^2$.

Introduce the notation:

$$A := \lambda \sum_{\Delta \subset Z^{\pm}} t(\Delta) \overline{K}_{\Delta}^{\eta_{\pm}}(\delta\psi) \equiv \sum_{\Delta \subset Z^{\pm}} A_{\Delta} , \tag{21}$$

$$A' := \lambda \sum_{\Delta \subset Z^{\pm}} t(\Delta) \overline{K}_{\Delta}^{\xi_{\pm}}(\delta\psi) \equiv \sum_{\Delta \subset Z^{\pm}} A'_{\Delta} , \tag{22}$$

$$A'' := \lambda \sum_{\Delta \subset Z^{\pm}} t(\Delta) K_{\Delta}^{\xi_{\pm}}(\psi_{\Delta}) \equiv \sum_{\Delta \subset Z^{\pm}} A''_{\Delta} , \tag{23}$$

$$B := W B' W^{-1} , \tag{24}$$

where

$$W := \sum_{\Delta \subset Z^{\pm}} W_{\Delta} . \tag{25}$$

With some more transformations we obtain

$$\mathscr{Z}^{K}(t) = \prod_{b=1}^{14} \mathscr{Z}_{b}^{(K)}(t) , \tag{26}$$

where

$$\mathscr{Z}_1(t) := \prod_{\Delta \subset Z^{\pm}} \chi_{\pm}^{\#}((\varphi + g)_{\Delta}^{\Lambda}) , \tag{27}$$

$$\mathscr{Z}_2(t) := \det{}_3(1 - A'') \exp\Big[-\frac{1}{2} :\mathrm{Tr}\, A''^2 :_{\tau} + \sum_{\Delta \subset Z} \frac{1}{2} t(\Delta)^2 \Big(\lambda^2 \int B_{\Delta}^{\xi_{+}} :\psi_{\Delta}^2 :_{\tau} + \eta \int_{\Delta} :\psi_{\Delta}^2 :_{\tau} + \eta \int_{\Delta} \psi_{\Delta}^2 \Big) \Big] , \tag{28}$$

$$\mathscr{Z}_3^{\mathrm{K}}(t) := \| \Lambda^{\mathrm{K}}\mathrm{W}^{-1} \cdot \Lambda^{\mathrm{K}}(1 - \mathrm{A} - \mathrm{B})^{-1} \cdot \Lambda^{\mathrm{K}}\mathrm{W} \cdot \Lambda^{\mathrm{K}}(1 - \mathrm{A}'')^{-1} \|$$
$$\cdot \det_3 (1 - \mathrm{A} - \mathrm{B}) \exp\left[-\frac{1}{2}\mathrm{Tr}\,\mathrm{B}^2 - \mathrm{Tr}\,(\mathrm{AB}) - \frac{1}{2}\mathrm{Tr}\,(\mathrm{A}^2 + \mathrm{A}^*\mathrm{A} - \mathrm{A}'^2 - \mathrm{A}'^*\mathrm{A}') - \frac{1}{2} : \mathrm{Tr}\,(\mathrm{A}'^2 + \mathrm{A}'^*\mathrm{A}') :_\tau \right], \tag{29}$$

$$\mathscr{Z}_4(t) := \exp\left[-\lambda \sum_{\Delta \subset Z^\pm} (1 - t(\Delta))\,\mathrm{Tr}\left(\mathrm{K}_\Delta^{\xi\pm}(\psi)\mathrm{B}''\right) \right], \tag{30}$$

$$\mathscr{Z}_5(t) := \exp\,[-\mathrm{Tr}\,(\mathrm{A}''\mathrm{B}')], \tag{31}$$

$$\mathscr{Z}_6(t) := \exp\left[\frac{1}{2} : \mathrm{Tr}\,(\mathrm{A}'^*\mathrm{A}') :_\tau + \sum_{\Delta \subset \mathrm{Z}} \frac{1}{2} t(\Delta)^2 \left(\lambda^2 \int \mathrm{B}_\Delta^{\xi+} : (\delta\psi)^2 :_\tau + \eta \int_\Delta : (\delta\psi)^2 :_\tau \right) \right], \tag{32}$$

$$\mathscr{Z}_7(t) := \exp\left[\frac{1}{2}\mathrm{Tr}\,(\mathrm{A}^*\mathrm{A} - \mathrm{A}'^*\mathrm{A}') - \frac{\eta}{2} \sum_{\Delta \subset \mathrm{Z}} t(\Delta)^2 \int_\Delta \psi_\Delta^2 \right], \tag{33}$$

$$\mathscr{Z}_8(t) := \exp\,[-\mathrm{D}_1], \tag{34}$$

$$\mathscr{Z}_9(t) := \exp\,[-\mathrm{D}_2], \tag{35}$$

$$\mathscr{Z}_{10}(t) := \exp\,[-\mathrm{E}_1], \tag{36}$$

$$\mathscr{Z}_{11}(t) := \exp\,[-\mathrm{E}_2], \tag{37}$$

$$\mathscr{Z}_{12}(t) := \exp\,[-\mathrm{F}_3], \tag{38}$$

$$\mathscr{Z}_{13}(t) := \exp\left[-\mathrm{F}_2 - \mathrm{F}_4 + \frac{1}{2}(m_c^2 - \eta) \sum_{\Delta \subset \mathrm{Z}} t(\Delta)^2 \int_\Delta : (\varphi\Lambda)^2 :_\tau \right], \tag{39}$$

$$\mathscr{Z}_{14}(t) := \exp\,[-\mathrm{F}_1]. \tag{40}$$

By (26)

$$\| \mathscr{Z}^{\mathrm{K}}(t) \|_{\mathrm{L}^p(\tau)} \leqslant \prod_{b=1}^{14} \| \mathscr{Z}_b^{(\mathrm{K})}(t) \|_{\mathrm{L}^{q_b}(\tau)}^{\#} \tag{41}$$

with

$$\sum_b q_b^{-1} \leqslant p^{-1}. \tag{42}$$

$\| \mathrm{F} \|_{\mathrm{L}^q(\tau)}^{\#}$ denotes the $\mathrm{L}^q(\tau)$ norm of either F or of the function F′,

$$\mathrm{F}'(\mathrm{T}) := \begin{cases} \mathrm{F}(\mathrm{T}) & \text{if } \mathscr{Z}_1(t)(\mathrm{T}) \neq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{43}$$

We choose q_1, q_2, q_9, q_{11} and q_{14} to be equal ∞. Estimating $\| \mathscr{Z}_2(t)' \|_{\mathrm{L}^\infty(\tau)}$ we shall fix η. With η fixed $\| \mathscr{Z}_{13}(t) \|_{\mathrm{L}^{q_{13}}(\tau)}$ is finite for q_{13} close to 1. We choose q_{13} sufficiently close to 1, p between 1 and q_{13} and q_b for $b = 3, 4, 5,$

6, 7, 8, 10, 12 equal to q being sufficiently large. For given q we fix d estimating $\| \mathscr{Z}_6(t) \|_{L^q(\tau)}$ and $\| \mathscr{Z}_7(t)' \|_{L^q(\tau)}$.

Let us start with $\mathscr{Z}_1(t)$ which, as many other terms is in fact t-independent. First we specify the shape of $\chi_\pm$. Let

$$\chi(x) := \left(\int_{-1}^{1} \exp\left[-(1-y^2)^{-1} \right] dy \right)^{-1} \cdot \begin{cases} \exp\left[-(1-x^2)^{-1} \right] & \text{for } |x| \leqslant 1, \\ 0 & \text{for } |x| \geqslant 1. \end{cases} \tag{44}$$

χ is smooth, suppt $\chi \subset [-1, 1]$, $\int \chi = 1$. Put

$$\chi_+(x) := \int_0^\infty \chi(x-y)dy, \qquad \chi_-(x) := \int_{-\infty}^{0} \chi(x-y)dy. \tag{45}$$

$\chi_\pm$ are smooth, $0 \leqslant \chi_\pm \leqslant 1$, $\chi_+ + \chi_- = 1$, $\chi_+ = 0$ on $]-\infty, -1]$ and $\chi_- = 0$ on $[1, +\infty[$.

LEMMA V.1. — For $n = 0, 1, 2, \ldots$

$$\left\| \frac{d^n}{dx^n} \chi_\pm \right\|_{L^\infty} \leqslant \exp\left[0(1)n \right] (n!)^{0(1)}. \tag{46}$$

Proof. — For $|x| \leqslant 1$ $\frac{d^n}{dx^n}\chi(x)$ can be expressed as a sum of terms

$$ax^k(1-x^2)^{-l} \exp\left[-(1-x^2)^{-1} \right]$$

each of which is created by a differentiation of either x^k or $(1-x^2)^{-l}$ or $\exp\left[-(1-x^2)^{-1} \right]$ in a term corresponding to $\frac{d^{n-1}}{dx^{n-1}}\chi$. Thus the number of terms is smaller than 3^{n-1}, $k \leqslant n$, $l \leqslant 2n$. Notice that $|a| \leqslant 4^n n!$ which follows easily by induction. Moreover for $y \geqslant 0$

$$y^{-l} \exp\left[-y^{-1} \right] \leqslant l^l e^{-l}. \tag{47}$$

Hence

$$2 \left| \frac{d^n}{dx^n} \chi(x) \right| \leqslant 12^n e^{-2n} n! (2n)^{2n} \leqslant e^{0(1)n} (n!)^{0(1)}. \tag{48}$$

By definition of $\chi_\pm$ (48) yields (46). □

Now we pass to estimation of

$$\| \mathscr{Z}_2(t)' \|_{L^\infty(\tau)} = \left\| \prod_{\Delta \subset Z} \mathscr{Z}_{2,\Delta}(t)' \right\|_{L^\infty(\tau)} \leqslant \prod_{\Delta \subset Z} \| \mathscr{Z}_{2,\Delta}(t)' \|_{L^\infty(\tau)}, \tag{49}$$

where

$$\mathscr{Z}_{2,\Delta}(t) = \det{}_3(1 - A''_\Delta) \exp\left[-\frac{1}{2} : \operatorname{Tr} A''^2_\Delta :_\tau + \frac{1}{2} t(\Delta)^2 \left(\lambda^2 \int B^{\xi_+}_\Delta : \psi^2_\Delta :_\tau + \eta \int_\Delta : \psi^2_\Delta :_\tau + \eta \int_\Delta \psi^2_\Delta \right) \right]. \tag{50}$$

Boundedness of $\mathcal{L}_{2,\Delta}(t)'$ is guarantied by the shape of the effective potential, which can be bounded below by a quadratic term around each of its minima.

Define the 1-loop effective potential in Δ with periodic boundary conditions by

$$V_\Delta(x) := d^{-2}\left(- \ln \det_3 (1 - \lambda K_\Delta(x)) + \frac{\lambda^2}{2} \operatorname{Tr} K_\Delta(x)^2 - \frac{\lambda^2}{2} \int B_\Delta^{\xi_+} x^2 \right). \quad (51)$$

The same proof as for $K(f)$ and $K(s, f)$ shows that $\operatorname{Tr} K_\Delta(f)^{2n+1} = 0$ and, consequently, that $V_\Delta(x) = V_\Delta(-x)$.

$$V_{\Delta,0}(x) = V_\Delta(x) - V_\Delta(\xi_+) = d^{-2}\Big(- \ln \det_3 (1 - \lambda K_\Delta^{\xi_\pm}(x - \xi_\pm)) + \frac{\lambda^2}{2} \operatorname{Tr} K_\Delta^{\xi_\pm}(x - \xi_\pm)^2 - \frac{\lambda^2}{2} \int B_\Delta^{\xi_\pm}(x - \xi_\pm)^2 \Big). \quad (52)$$

With use of $V_{\Delta,0}$ we can rewrite (50) as

$$\mathcal{L}_{2,\Delta}(t) = \exp [- d^2(V_{\Delta,0}(x) - \eta(x - \xi_\pm)^2)] |_{x=\xi_\pm + t(\Delta)\psi_\Delta = \eta_\pm} \cdot \exp \left[\frac{1}{2} t(\Delta)^2\left(\lambda^2 \operatorname{Tr} K_\Delta^{\xi_\pm}(1)^2 - \lambda^2 \int_\Delta B_\Delta^{\xi_+} - \eta d^2\right)\int (\varphi\Lambda)^2_\Delta d\mu_{m_c}(\tau) \right]. \quad (53)$$

The information on the shape of V_Δ which we need is gathered in Lemmas V.2 and 3.

LEMMA V.2. — There exists η, $0 < \eta < \frac{1}{4} m_c^2$, such that for all d, λ, $m \geqslant m_0(d, \lambda)$ and $\begin{cases} -1 \leqslant x \\ x \leqslant 1 \end{cases}$

$$V_{\Delta,0}(x) \geqslant \eta(x - \xi_\pm)^2 . \quad (54)$$

Proof. — From (51) and (52) it follows that

$$V'_{\Delta,0}(0) = 0 , \quad (55)$$

$$V''_{\Delta,0}(0) = \lambda^2 d^{-2}\left(\operatorname{Tr} K_\Delta(1)^2 - \int B_\Delta^{\xi_+} \right) = \lambda^2 d^{-2} \operatorname{Tr}\left(K_\Delta(1)^2 - K_\Delta(1) K_\Delta^{\xi_+}(1)\right). \quad (56)$$

But since

$$K_\Delta(x) = (\not{P} + m)_\Delta D_\Delta^{-2} \Gamma x ,$$

$$V''_{\Delta,0}(0) = \sum_{p \in \frac{2\pi}{d}\mathbb{Z}^2} \operatorname{tr}\left[\left(\frac{\not{p} + m}{p^2 + m^2}\Gamma\right)^2 - \frac{\not{p} + m}{p^2 + m^2}\Gamma \frac{\not{p} + m + \lambda\xi_+\Gamma}{p^2 + \overline{m}^2}\Gamma\right]. \quad (57)$$

We sum over the spectrum of the periodic boundary conditions momentum operator and tr denotes the trace over the spinor indices. Hence

$$V''_{\Delta,0}(0) = - 2\lambda^4 \xi_+^2 d^{-2} \sum_p (p^2 + m^2)^{-1}(p^2 + \overline{m}^2)^{-1} < 0 . \quad (58)$$

Moreover

$$V_{\Delta,0}(\xi_\pm) = 0\,, \qquad V'_{\Delta,0}(\xi_\pm) = 0\,,$$

$$V''_{\Delta,0}(\xi_\pm) = \lambda^2 d^{-2}\left(\mathrm{Tr}\, K_\Delta^{\xi_\pm}(1)^2 - \int B_\Delta^{\xi_+}\right)$$

$$= \lambda^2 d^{-2}\,\mathrm{Tr}\,(K_\Delta^{\xi_\pm}(1) - K_\Delta(1)K_\Delta^{\xi_+}(1)) = 4\lambda^4\xi_\pm^2 d^{-2}\sum_p (p^2 + \overline{m}^2)^2 > 0\,. \quad (59)$$

Thus V_Δ ($V_{\Delta,0}$) has a local maximum at zero and two local minima at $\xi_\pm$. We shall prove that for $m \geqslant m_0(d)$ V'''_Δ is positive for $x > 0$. This shows that those are the only local extrema, minima being the global ones.

$$V'''_{\Delta,0}(x) = 2\lambda^3 d^{-2}\,\mathrm{Tr}\, K_\Delta^x(1)^3$$

$$= 4\lambda^4 d^{-2} x \sum_p (3(p^2 + m^2) - \lambda^2 x^2)(p^2 + m^2 + \lambda^2 x^2)^{-3}\,. \quad (60)$$

When $d \to \infty$ we obtain in the limit

$$4\lambda^4 d^{-2} x (2\pi)^{-2}\int dp (3(p^2 + m^2) - \lambda^2 x^2)(p^2 + m^2 + \lambda^2 x^2)^{-3}$$

$$= \pi^{-1}\lambda^4 x (3m^2 + 2\lambda^2 x^2)(m^2 + \lambda^2 x^2)^{-2} = V'''_0(x)\,. \quad (61)$$

But by the mean value theorem

$$\left|(2\pi)^{-2}\int dp F(p) - d^{-2}\sum_{p\in\frac{2\pi}{d}\mathbb{Z}^2} F(p)\right| \leqslant \sqrt{2}(8\pi d)^{-1}\int dp \sup_{|p-p'|\leqslant\sqrt{2}\pi d^{-1}} |\nabla F(p')|\,. \quad (62)$$

We put

$$F(p) := (3(p^2 + m^2) - \lambda^2 x^2)(p^2 + m^2 + \lambda^2 x^2)^{-3} \quad (63)$$

so that

$$|\nabla F(p)| \leqslant 0(1)(p^2 + m^2 + \lambda^2 x^2)^{-5/2}$$

$$\inf_{|p-p'|\leqslant\sqrt{2}\pi d^{-1}} (p'^2 + m^2 + \lambda^2 x^2) \geqslant \frac{1}{2}(p^2 + m^2 + \lambda^2 x^2) \quad (64)$$

for $m \geqslant m_0(d)$. Hence

$$V'''_{\Delta,0}(x) \geqslant V'''_0(x) - |V'''_\Delta(x) - V'''_{\Delta,0}(x)|$$

$$\geqslant \pi^{-1}\lambda^4 x[(3m^2 + 2\lambda^2 x^2)(m^2 + \lambda^2 x^2)^{-2} - 0(1)d^{-1}(m^2 + \lambda^2 x^2)^{-3/2}] > 0 \quad (65)$$

for $x > 0$ and $m \geqslant m_0(d)$. In the sequel we shall consider only those m for which (65) holds. Since $V''_{\Delta,0}(x)$ grows for $x > 0$ it follows that

$$V_{\Delta,0}(x) - \frac{1}{2}V''_{\Delta,0}(\xi_\pm)(x - \xi_\pm)^2 \geqslant 0 \quad (66)$$

for $|x| \geqslant \xi_+$. Moreover

$$V_{\Delta,0}(x) \geqslant \eta(x - \xi_\pm)^2 \qquad \text{for} \quad \begin{cases} -1 \leqslant x \leqslant \xi_+ \\ \xi_- \leqslant x \leqslant 1 \end{cases} \quad (67)$$

if and only if

$$V_{\Delta,0}(-1) \geqslant \eta(1+\xi_+)^2 .$$

From (I.4) it is easy to see that there exists η, $0 < \eta < \frac{1}{4} m_c^2$, such that for $m \geqslant m_0(\lambda)$

$$V_0(-1) \geqslant \eta(1+\xi_+)^2 .$$

Taking slightly smaller η we also see that there exist η, $\alpha > 0$ such that

$$V_0(-1) \geqslant \eta(1+\xi_+)^2 + \alpha\xi_+^2 . \tag{68}$$

Now we have to compare V_0 and $V_{\Delta,0}$. But for $m \geqslant m_0(d)$

$$|V(x) - V_\Delta(x)| \leqslant 0(1)\lambda^2 x^2 m^{-1} . \tag{69}$$

Indeed,

$$V(x) - V_\Delta(x) = \left((2\pi)^{-2}\int dp - d^{-2}\sum_p\right)(-\ln(1+\lambda^2x^2(p^2+m^2)^{-1}) + \lambda^2x^2(p^2+m^2)^{-1} - 2\lambda^4\xi_+^2x^2(p^2+m^2)^{-1}(p^2+\bar{m}^2)^{-1}) \tag{70}$$

and (69) follows by (62).

(68) and (69) give

$$\begin{aligned} V_{\Delta,0}(-1) &\geqslant V_0(-1) - |V(-1) - V_\Delta(-1)| - |V(\xi_+) - V_\Delta(\xi_+)| \\ &\geqslant \eta(1+\xi_+)^2 + \alpha\xi_+^2 - 0(1)\lambda^2m^{-1} - 0(1)\lambda^2\xi_+^2m^{-2} \\ &\geqslant \eta(1+\xi_+)^2 \quad \text{for} \quad m \geqslant m_0(\lambda, d) . \end{aligned}$$

Thus (67) holds. But since

$$V''_\Delta(\xi_\pm) \geqslant V''(\xi_\pm) - |V''(\xi_\pm) - V''_\Delta(\xi_\pm)| \geqslant m_c^2 - 0(1)\lambda^2m^{-1} \geqslant \frac{1}{2}m_c^2 \tag{71}$$

for $m \geqslant m_0(\lambda, d)$, (54) holds too, see (66). □

We shall also need

LEMMA V.3. — There exist η, $0 < \eta < \frac{1}{4}m_c^2$, and $\beta > 0$ such that for all d, λ, $m \geqslant m_0(\lambda, d)$ and $-1 \leqslant x \leqslant 1$

$$V_{\Delta,0}(x) \geqslant \eta(x-\xi_\pm)^2 + \beta\xi_+^2 . \tag{72}$$

Proof. — By (54)

$$V_{\Delta,0}(x) \geqslant \frac{\eta}{2}(x-\xi_\pm)^2 + \frac{\eta}{2}(\xi_+ - 1)^2 \geqslant \frac{\eta}{2}(x-\xi_\pm)^2 + \frac{\eta}{4}\xi_+^2$$

for $m \geqslant m_0(\lambda, d)$. □

Now we are ready to estimate $\| \mathscr{Z}_2(t)' \|_{L^\infty(\tau)}$.

LEMMA V.4. — There exists η, $0 < \eta < \frac{1}{4}m_c^2$, such that for all d, λ, $m \geqslant m_0(\lambda, d)$

$$\| \mathscr{Z}_2(t)' \|_{L^\infty(\tau)} \leqslant \exp\left[-0(1)\xi_+^2 |Z'| + 0(1)|Z_t|\right] . \tag{73}$$

Proof. — In virtue of (49) it is enough to estimate $\mathscr{Z}_{2,\Delta}(t)$ given by (53) at points for which $\mathscr{Z}_1(t) \neq 0$. If $\Lambda \restriction_\Delta \equiv 0$ or $\Delta \not\subset Z_v$ then $\mathscr{Z}_{2,\Delta}(t) \equiv 1$. Suppose that $\Lambda \restriction_\Delta \not\equiv 0$ and $\Delta \not\subset Z_v$. For $\mathscr{Z}_1(t)$ to be non-zero it is necessary that $(\varphi + g)_\Delta^\Lambda \geqslant -1$, $(\varphi + g)_\Delta^\Lambda \leqslant 1$ or both if $\Delta \subset Z^-$ or $\Delta \subset Z'$ respectively. But comparing (II.3) and (III.2) we obtain

$$(\varphi + g)_\Delta^\Lambda = \psi_\Delta + h_\Delta = \psi_\Delta + \xi_\pm .$$

Hence if $\mathscr{Z}_1(t) \neq 0$ then

$$\xi_\pm + t(\Delta)\psi_\Delta \begin{Bmatrix} \geqslant -1 \\ \leqslant 1 \end{Bmatrix} \quad \text{for} \quad \Delta \subset Z^\pm$$

and

$$-1 \leqslant \xi_\pm + \psi_\Delta \leqslant 1 \quad \text{for} \quad \Delta \subset Z' \quad (t(\Delta) = 1 \quad \text{for} \quad \Delta \subset Z').$$

Thus (53) and Lemmas V.2, 3 yield

$$\| \mathscr{Z}_{2,\Delta}(t)' \|_{L^\infty(\tau)} \leqslant \exp\left[-\beta\xi_+^2 \,|\, Z' \cap \Delta \,| \right] \cdot \exp\left[\frac{1}{2} t(\Delta)^2 \left(\lambda^2 \operatorname{Tr} K_\Delta^{\xi_\pm}(1)^2 - \lambda^2 \int B_\Delta^{\xi_+} - \eta d^2 \right) \int (\varphi\Lambda)_\Delta^2 d\mu_{m_c}(\tau) \right]. \quad (74)$$

But

$$\lambda^2 \operatorname{Tr} K_\Delta^{\xi_\pm}(1)^2 - \lambda^2 \int B_\Delta^{\xi_+} - \eta d^2 \leqslant d^2(V_\Delta''(\xi_\pm) - \eta) \leqslant 0(1), \quad (75)$$

compare (71).

Moreover

$$\int (\varphi\Lambda)_\Delta^2 d\mu_{m_c}(\tau) = d^{-4}(\Lambda\chi_\Delta \,|\, C_{m_c}(\tau)\Lambda\chi_\Delta)_{L^2} \leqslant d^{-4}(\chi_\Delta \,|\, C_{m_c}(\tau)\chi_\Delta)_{L^2} \leqslant 0(1). \quad (76)$$

(49), (74) and (76) give (73) (constants 0(1) depend on d but not on λ nor m). This ends Proof of Lemma V.4. □

Next term which we shall estimate is $\mathscr{Z}_3(t)$, see (29). The general idea is taken from [*23*] [*24*] [*25*]. Let us start with a determinant inequality.

LEMMA V.5. — For sufficiently regular operators A and B

$$\left\| \Lambda^K(1 - A - B)^{-1} \det_3(1 - A - B) \exp\left[-\frac{1}{2}\operatorname{Tr} B^2 - \operatorname{Tr}(AB) - \frac{1}{2}\operatorname{Tr} A^2 - \frac{1}{2}\operatorname{Tr}(A^*A) \right] \right\|$$
$$\leqslant (\det_3(1 + \mathcal{O}_A^+))^{1/2} \exp\left[-\frac{1}{2}\operatorname{Re}\operatorname{Tr} A^2 - \frac{1}{2}\operatorname{Tr}(A^*A) + \operatorname{Re}\operatorname{Tr}(A^*A^2) - \frac{1}{4}\operatorname{Tr}(A^*A)^2 \right] \cdot \exp\left[a \| AB \|_1 + \frac{1}{2} \| B^*B \|_1 + cK \right], \quad (77)$$

where

$$\mathcal{O}_A := -A - A^* + A^*A, \quad (78)$$

$\mathcal{O}_A^\pm$ are its positive and negative parts ($\mathcal{O}_A^\pm \geqslant 0$), $a, c > 0$.

Proof. — (77) can be easily obtained from Theorem 4.1 of [*25*]:

$$|\det(1-A-B-C)\exp[\mathrm{Tr}\,A+\mathrm{Tr}\,B]| \leqslant (\det(1+\mathcal{O}_A^+))^{1/2}$$
$$\cdot\exp\left[-\frac{1}{2}\mathrm{Tr}\,\mathcal{O}_A^- - \frac{1}{4}\mathrm{Tr}\,(\mathcal{O}_A^-)^2 + \mathrm{Re}\,\mathrm{Tr}\,A + a\|AB\|_1 + \frac{1}{2}\|B^*B\|_1 + b\|C\|_1\right]. \tag{79}$$

Indeed, putting $C = \sum_{j=1}^{K}\lambda_j C_j$ and using the Cauchy integral formula to estimate

$$\frac{\partial^K}{\partial\lambda_1\ldots\partial\lambda_K}\bigg|_{\lambda=0}\det\left(1-A-B-\sum_j\lambda_jC_j\right)\exp[\mathrm{Tr}\,A+\mathrm{Tr}\,B]$$

we get (compare Proof of Proposition 1 of Appendix of [*24*])

$$\|\Lambda^K(1-A-B)^{-1}\det(1-A-B)\exp[\mathrm{Tr}\,A+\mathrm{Tr}\,B]\|$$
$$\leqslant (\det(1+\mathcal{O}_A^+))^{1/2}\exp\left[-\frac{1}{2}\mathrm{Tr}\,\mathcal{O}_A^- - \frac{1}{4}\mathrm{Tr}\,(\mathcal{O}_A^-)^2 + \mathrm{Re}\,\mathrm{Tr}\,A\right]$$
$$\cdot\exp\left[a\|AB\|_1 + \frac{1}{2}\|B^*B\|_1 + cK\right]. \tag{80}$$

(80) gives

$$\left\|\Lambda^K(1-A-B)^{-1}\det{}_3(1-A-B)\exp\left[-\frac{1}{2}\mathrm{Tr}\,B^2 - \mathrm{Tr}\,(AB)\right.\right.$$
$$\left.\left.-\frac{1}{2}\mathrm{Tr}\,A^2 - \frac{1}{2}\mathrm{Tr}\,(A^*A)\right]\right\|$$
$$\leqslant (\det{}_2(1+\mathcal{O}_A^+))^{1/2}\exp\left[\frac{1}{2}\mathrm{Tr}\,\mathcal{O}_A - \frac{1}{4}\mathrm{Tr}\,(\mathcal{O}_A^-)^2 + \mathrm{Re}\,\mathrm{Tr}\,A - \frac{1}{2}\mathrm{Tr}\,(A^*A)\right]$$
$$\cdot\exp\left[a\|AB\|_1 + \frac{1}{2}\|B^*B\|_1 + cK\right] \tag{81}$$

which is easily seen to coincide with (77). □

In virtue of (29) and Lemma V.5

$$|\mathcal{Z}_3^K(t)| \leqslant (e^c\|W^{-1}\|\,\|W\|\,\|(1-A'')^{-1}\|)^K\mathcal{Z}_3'(t)\mathcal{Z}_3''(t), \tag{82}$$

where

$$\mathcal{Z}_3'(t) := (\det{}_3(1+\mathcal{O}_A^+))^{1/2}\exp\left[-\frac{1}{2}\mathrm{Re}\,\mathrm{Tr}\,A^2 + A^*A) + \mathrm{Re}\,\mathrm{Tr}\,(A^*A^2)\right.$$
$$\left.-\frac{1}{4}\mathrm{Tr}\,(A^*A)^2 + \frac{1}{2}\mathrm{Re}\int\mathrm{Tr}\,(A'^2 + A'^*A')d\mu_{m_c}(\tau)\right], \tag{83}$$

$$\mathcal{Z}_3''(t) := \exp\left[a\|AB\|_1 + \frac{1}{2}\|B^*B\|_1\right]. \tag{84}$$

From (20), (25) and (23) it is easy to see that

$$\| W^{-1} \| \| W \| \| (1 - A'')^{-1} \| \leqslant 0(1) . \tag{85}$$

Also

$$\mathscr{Z}'_3(t) = \prod_{\Delta \subset Z^{\pm}} \mathscr{Z}'_{3,\Delta}(t) , \tag{86}$$

where $\mathscr{Z}'_{3,\Delta}(t)$ are given by (83) but with A and A′ replaced by A_Δ and A'_Δ. By the checkerboard estimate for the measures $d\mu_{m_c}(\tau)$ (uniform in τ)

$$\| \mathscr{Z}'_3(t) \|_{L^{2q}(\tau)} \leqslant \prod_{\Delta \subset Z} \| \mathscr{Z}'_{3,\Delta}(t) \|_{L^r(\tau)} \tag{87}$$

for $r = r(q, d)$.

$\| \mathscr{Z}'_3(t) \|_{L^r(\tau)}$ will be bounded by using the standard Nelson argument. First we consider the expressions $\mathscr{Z}'_{3,\Delta,\varkappa}(t)$ obtained from $\mathscr{Z}'_{3,\Delta}(t)$ by replacing $\psi = ((\varphi + g - h)\Lambda + (\xi_+ - h)(1 - \Lambda))\chi_Z$ with its ultraviolet cut-off version $\psi^\varkappa := ((\varphi\Lambda)_\varkappa + (g - h)\Lambda + (\xi_+ - h)(1 - \Lambda))\chi_Z$. The detailed description and the properties of the cut-off which we use are inclosed in Appendix II. Here let us remind that $\psi^\varkappa \chi_\Delta$ depends on $\varphi \restriction_\Delta$ only and that $(\psi^\varkappa)_\Delta = \psi_\Delta$. The bound on $\| \mathscr{Z}'_{3,\Delta}(t) \|_{L^r(\tau)}$ follows from two estimates which hold for all λ and sufficiently large m:

i) $\mathscr{Z}'_{3,\Delta,\varkappa}(t) \leqslant \exp\,[0(1)\lambda^2 \ln \varkappa]$, (88)

ii) $\| \ln \mathscr{Z}'_{3,\Delta,\varkappa}(t) - \ln \mathscr{Z}_{3,\Delta,\varkappa'}(t) \|_{L^s(\tau)}$

$$\leqslant \begin{cases} 0(1)\lambda^6 s^3 \min(\varkappa, \varkappa')^{-\varepsilon} & \text{for } \Delta \not\subset Z^0 , \\ 0(1)\lambda^6 m^5 s^3 \min(\varkappa, \varkappa')^{-\varepsilon} & \text{for } \Delta \subset Z^0 , \end{cases} \tag{89}$$

where $\varepsilon > 0$ and $1 \leqslant s \leqslant \infty$.

Now

$$\begin{aligned} \mathscr{Z}'_{3,\Delta,\varkappa}(t) &= (\det_3 (1 + \mathcal{O}^+_{A_{\Delta,\varkappa}}))^{1/2} \exp\Big[-\frac{1}{2} \operatorname{Re} \operatorname{Tr} (A^2_{\Delta,\varkappa} + A^*_{\Delta,\varkappa} A_{\Delta,\varkappa}) \\ &+ \operatorname{Re} \operatorname{Tr} (A^*_{\Delta,\varkappa} A^2_{\Delta,\varkappa}) - \frac{1}{4} \operatorname{Tr} (A^*_{\Delta,\varkappa} A_{\Delta,\varkappa})^2 + \frac{1}{2} \operatorname{Re} \int \operatorname{Tr} (A'^2_{\Delta,\varkappa} + A'^*_{\Delta,\varkappa} A'_{\Delta,\varkappa}) d\mu_{m_c}(\tau) \Big] \\ &\leqslant \exp\left[\frac{1}{2} \operatorname{Re} \int \operatorname{Tr} (A'^2_{\Delta,\varkappa} + A'^*_{\Delta,\varkappa} A'_{\Delta,\varkappa}) d\mu_{m_c}(\tau) \right] \end{aligned} \tag{90}$$

since

$$\begin{aligned} (\det_3 (1 + \mathcal{O}^+_{A_{\Delta,\varkappa}}))^{1/2} &\leqslant \exp\left[\frac{1}{4} \operatorname{Tr} (\mathcal{O}^+_{A_{\Delta,\varkappa}})^2 \right] \leqslant \exp\left[\frac{1}{4} \operatorname{Tr} \mathcal{O}^2_{A_{\Delta,\varkappa}} \right] \\ &= \exp\left[\frac{1}{2} \operatorname{Re} \operatorname{Tr} (A^2_{\Delta,\varkappa} + A^*_{\Delta,\varkappa} A_{\Delta,\varkappa}) - \operatorname{Re} \operatorname{Tr} (A^*_{\Delta,\varkappa} A^2_{\Delta,\varkappa}) + \frac{1}{4} \operatorname{Tr} (A^*_{\Delta,\varkappa} A_{\Delta,\varkappa})^2 \right]. \end{aligned} \tag{91}$$

Hence (88) follows from

LEMMA V.6.

$$\operatorname{Re} \int \operatorname{Tr}\,(A'^{2}_{\Delta,\varkappa} + A'^{*}_{\Delta,\varkappa}A_{\Delta,\varkappa})d\mu_{m_c}(\tau) \leqslant 0(1)\lambda^2 \ln \varkappa \tag{92}$$

for $m \geqslant m_0(d)$.

Proof. — Using (22) and computing the traces we obtain

$$\begin{aligned}\int \operatorname{Tr}\,(A'^{2}_{\Delta,\varkappa} + A'^{*}_{\Delta,\varkappa}A_{\Delta,\varkappa})d\mu_{m_c}(\tau) &= t(\Delta)^2\lambda^2 \int \operatorname{Tr}\,(\overline{K}^{\xi_\pm}_{\Delta}(\delta(\varphi\Lambda)_\varkappa)^2 \\ &\quad + \overline{K}^{\xi_\pm}_{\Delta}(\delta(\varphi\Lambda)_\varkappa)^*\overline{K}^{\xi_\pm}_{\Delta}(\delta(\varphi\Lambda)_\varkappa))d\mu_{m_c}(\tau) \\ &= (2\pi^2 d^2)^{-1} t(\Delta)^2\lambda^2 \int \sum_{k\in\frac{2\pi}{d}\mathbb{Z}^2} I_\Delta(k)\,|\,\widehat{\delta(\varphi\Lambda\chi_\Delta)_\varkappa}(k)\,|^2\, d\mu_{m_c}(\tau)\,, \end{aligned} \tag{93}$$

where

$$\begin{aligned} I_\Delta(k) := \sum_{q\in\frac{2\pi}{d}\mathbb{Z}^2} (2\pi)^2 d^{-2}\big[(-(q+k\,|\,q) - m^2 + \lambda^2\xi_+^2)((q+k)^2 + \overline{m}^2)^{-1} \\ \cdot (q^2 + \overline{m}^2)^{-1} + ((q+k)^2 + \overline{m}^2)^{-1/2}(q^2 + \overline{m}^2)^{-1/2}\big]. \end{aligned} \tag{94}$$

In the limit $d \to \infty$ we obtain the function $I(k)$ given by (94) with $\sum_q (2\pi)^2 d^{-2}$ replaced by $\int dq$. This can be explicitely computed.

$$\begin{aligned} I(k) = \pi(1 + 4\overline{\alpha}\rho^{-1})(1 + 4(1+\overline{\alpha})\rho^{-1})^{-1/2} \ln \frac{(1 + 4(1+\overline{\alpha})\rho^{-1})^{1/2} + 1}{(1 + 4(1+\overline{\alpha})\rho^{-1})^{1/2} - 1} \\ - 2\pi \ln (1 + (1 + 4^{-1}(1+\overline{\alpha})^{-1}\rho)^{1/2}) + 2\pi \ln 2 \equiv \tilde{I}(\rho, \overline{\alpha})\,, \end{aligned} \tag{95}$$

where

$$\rho := \frac{k^2}{m^2} \quad \text{and} \quad \overline{\alpha} := \frac{\lambda^2\xi_\pm^2}{m^2}\,. \tag{96}$$

Since $\overline{\alpha}$ is uniformly bounded (λ is bounded away from zero)

$$\tilde{I}(\rho, \overline{\alpha}) = 4\pi \ln 2 + \cdot\, 0(\rho^{-1/2}) \quad \text{when} \quad \rho \to \infty\,, \tag{97}$$

$$\tilde{I}(0, \overline{\alpha}) = 2\pi\overline{\alpha}(1 + \overline{\alpha})^{-1}\,. \tag{98}$$

Thus $I(k)$ is bounded uniformly in λ and m. Using (62) we also get for large m

$$\begin{aligned} |\, I(k) - I_\Delta(k)\,| &\leqslant 0(1) \int dq((q+k)^2 + \overline{m}^2)^{-1/2}(q^2 + \overline{m}^2)^{-1} \\ &\leqslant 0(1)\,\|\,(q^2 + \overline{m}^2)^{-1/2}\,\|_{L^3}\,\|\,(q^2 + \overline{m}^2)^{-1}\,\|_{L^{3/2}} \leqslant 0(1)m^{-1}\,. \end{aligned} \tag{99}$$

Hence also $I_\Delta(k)$ is bounded uniformly in λ and $m \geqslant m_0(d)$. Now using (A.II.1) we obtain

$$d^{-2}\int \sum_{k\in\frac{2\pi}{d}\mathbb{Z}^2} I_\Delta(k)\,|\,\widehat{\delta(\varphi\Lambda\chi_\Delta)_\varkappa}(k)\,|^2\, d\mu_{m_c}(\tau) = \mathrm{Tr}\,(I_\Delta(P)_\Delta\delta R_\varkappa\Lambda\chi_\Delta C_{m_c}(\tau)\chi_\Delta\Lambda R_\varkappa)$$

$$\leqslant 0(1)\,\|\,\delta R_\varkappa\Lambda\chi_\Delta C_{m_c}(\tau)\chi_\Delta\Lambda R_\varkappa\delta\,\|_1 \leqslant 0(1)\,\|\,R_\varkappa\Lambda\chi_\Delta C_{m_c}(\tau)\chi_\Delta\Lambda R_\varkappa\,\|_1$$

$$= 0(1)\int_{\Delta x\Delta x\Delta} dxdydz \sum_{i,j=1}^{4} \rho_\varkappa(x^i - y)\Lambda(y)C_{m_c}(\tau)(y, z)\Lambda(z)\rho_\varkappa(x^j - z)$$

$$\leqslant 0(1)\int_{\Delta x\Delta x\Delta} dxdydz \sum_{i,j=1}^{4} \rho_\varkappa(x^i - y)C_{m_c}(y - z)\rho_\varkappa(x^j - z)$$

$$\leqslant 0(1)\,\|\,\rho_\varkappa * C_{m_c} * \rho_\varkappa\,\|_{L^\infty} \leqslant 0(1)\int (p^2 + m_c^2)^{-1}\,|\,\hat\rho_1(\varkappa^{-1}p)\,|^2\, dp$$

$$\leqslant 0(1)\int_{|p|\leqslant\varkappa} dp(p^2 + m_c^2)^{-1} + 0(1)\int_{|p|\geqslant\varkappa} p^{-2}\,|\,\hat\rho_1(\varkappa^{-1}p)\,|^2\, dp$$

$$\leqslant 0(1)\ln\varkappa + 0(1) \leqslant 0(1)\ln\varkappa\,. \tag{100}$$

(93) and (100) give (92). □

The proof of (89) is slightly more complicated. We write

$$\ln \mathscr{Z}'_{3,\Delta,\varkappa}(t) - \ln \mathscr{Z}'_{3,\Delta,\varkappa'}(t) = M_1 + M_2 + M_3 + M_4\,, \tag{101}$$

where

$$\begin{aligned} M_1 := \frac{1}{2}\ln\det{}_3\,(1 + \mathcal{O}^+_{A_{\Delta,\varkappa}}) + \mathrm{Re}\,\mathrm{Tr}\,(A^*_{\Delta,\varkappa}A^2_{\Delta,\varkappa}) \\ - \frac{1}{4}\mathrm{Tr}\,(A^*_{\Delta,\varkappa}A_{\Delta,\varkappa})^2 - \frac{1}{2}\ln\det{}_3\,(1 + \mathcal{O}^+_{A_{\Delta,\varkappa'}}) \\ - \mathrm{Re}\,\mathrm{Tr}\,(A^*_{\Delta,\varkappa'}A^2_{\Delta,\varkappa'}) + \frac{1}{4}\mathrm{Tr}\,(A^*_{\Delta,\varkappa'}A_{\Delta,\varkappa'})^2\,, \end{aligned} \tag{102}$$

$$M_2 := -\frac{1}{2}\mathrm{Re}\,\mathrm{Tr}\,(A^2_{\Delta,\varkappa} - A'^2_{\Delta,\varkappa} - A^2_{\Delta,\varkappa'} + A'^2_{\Delta,\varkappa'})\,, \tag{103}$$

$$M_3 := -\frac{1}{2}\mathrm{Re}\,\mathrm{Tr}\,(A^*_{\Delta,\varkappa}A_{\Delta,\varkappa} - A'^*_{\Delta,\varkappa}A'_{\Delta,\varkappa} - A^*_{\Delta,\varkappa'}A_{\Delta,\varkappa'} + A'^*_{\Delta,\varkappa'}A'_{\Delta,\varkappa'})\,, \tag{104}$$

$$M_4 := -\frac{1}{2}\mathrm{Re}\,:\mathrm{Tr}\,(A'^2_{\Delta,\varkappa} + A'^*_{\Delta,\varkappa}A'_{\Delta,\varkappa} - A'^2_{\Delta,\varkappa'} - A'^*_{\Delta,\varkappa'}A'_{\Delta,\varkappa'}):_\tau\,. \tag{105}$$

As follows from formulae (2.23) and (2.25) of [*24*]

$$|\,M_1\,| \leqslant 0(1)\sum_{\substack{2\leqslant k+l\leqslant 5\\ k,l\geqslant 0}} \|\,A_{\Delta,\varkappa} - A_{\Delta,\varkappa'}\,\|_4\,\|\,A_{\Delta,\varkappa}\,\|^k_{8/3}\,\|\,A_{\Delta,\varkappa'}\,\|^l_{8/3}\,. \tag{106}$$

Notice that

$$
\begin{aligned}
A^*_{\Delta,\varkappa}A_{\Delta,\varkappa} &= t(\Delta)^2\lambda^2\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa})^*\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa}) \\
&= t(\Delta)^2\lambda^2\overline{D}_{\Delta}^{-1/2}(\delta\psi^{\varkappa})\overline{D}_{\Delta}(P^2 + m^2 + \lambda^2\eta^2_{\pm})^{-1}_{\Delta}(\delta\psi^{\varkappa})\overline{D}_{\Delta}^{-1/2} \\
&\leqslant \lambda^2(1+\overline{\alpha})\overline{D}_{\Delta}^{-1/2}(\delta\psi^{\varkappa})\overline{D}_{\Delta}^{-1}(\delta\psi^{\varkappa})\overline{D}_{\Delta}^{-1/2}, \qquad (107)
\end{aligned}
$$

where $\overline{\alpha}$ is given by (96). Hence by the min-max principle the trace q-norms of $A_{\Delta,\varkappa}$ are bounded by the trace q-norms of $\lambda(1+\overline{\alpha})^{1/2}D_{\Delta}^{-1/2}(\delta\psi^{\varkappa})D_{\Delta}^{-1/2}$ and analogically for $A_{\Delta,\varkappa} - A_{\Delta,\varkappa'}$. Hence, denoting $g^{\Lambda} := g\Lambda + \xi_{+}(1-\Lambda)$, we have

$$
\begin{aligned}
|M_1| &\leqslant 0(1)\lambda^6 \sum_{\substack{2\leqslant k+l+m\leqslant 5\\ k,l,m\geqslant 0}} \|\overline{D}_{\Delta}^{-1/2}[\delta((\varphi\Lambda)_{\varkappa} - (\varphi\Lambda)_{\varkappa'})]\overline{D}_{\Delta}^{-1/2}\|_4 \\
&\cdot \|\overline{D}_{\Delta}^{-1/2}(\delta(\varphi\Lambda)_{\varkappa})\overline{D}_{\Delta}^{-1/2}\|^k_{8/3}\|\overline{D}_{\Delta}^{-1/2}(\delta(\varphi\Lambda)_{\varkappa'})\overline{D}_{\Delta}^{-1/2}\|^l_{8/3}\|\overline{D}_{\Delta}^{-1/2}(\delta g^{\Lambda})\overline{D}_{\Delta}^{-1/2}\|^m_{8/3} \\
&\leqslant 0(1)\lambda^6 m^5 \sum_{\substack{0\leqslant k+l\leqslant 5\\ k,l\geqslant 0}} \|\overline{D}_{\Delta}^{-1/2}[\delta((\varphi\Lambda)_{\varkappa} - (\varphi\Lambda)_{\varkappa'})]\overline{D}_{\Delta}^{-1/2}\|_4 \\
&\cdot \|\overline{D}_{\Delta}^{-1/2}(\delta(\varphi\Lambda)_{\varkappa})\overline{D}_{\Delta}^{-1/2}\|^k_{8/3}\|\overline{D}_{\Delta}^{-1/2}(\delta(\varphi\Lambda)_{\varkappa'})\overline{D}_{\Delta}^{-1/2}\|^l_{8/3} \qquad (108)
\end{aligned}
$$

since for $\Delta \subset Z^0$

$$
\|\overline{D}_{\Delta}^{-1/2}(\delta g^{\Lambda})\overline{D}_{\Delta}^{-1/2}\|_{8/3} \leqslant \|g^{\Lambda}\|_{L^\infty}\|\overline{D}_{\Delta}^{-1/2}\|^2_{16/3} \leqslant 0(1)\xi_{+} \leqslant 0(1)m. \qquad (109)
$$

For $\Delta \not\subset Z^0$, m^5 on the right hand side of (108) may be omitted. In virtue of Lemma A.II.2 (108) gives

$$
\begin{aligned}
\|M_1\|_{L^s(\tau)} &\leqslant 0(1)\lambda^6 m^5 \sum_{\substack{0\leqslant k+l\leqslant 5\\ k,l\geqslant 0}} \|\overline{D}_{\Delta}^{-1/2}[\delta((\varphi\Lambda)_{\varkappa} - (\varphi\Lambda)_{\varkappa'})]\overline{D}_{\Delta}^{-1/2}\|_{4,6s,\tau} \\
&\quad\cdot \|\overline{D}_{\Delta}^{-1/2}(\delta(\varphi\Lambda)_{\varkappa})\overline{D}_{\Delta}^{-1/2}\|^k_{8/3,6s,\tau}\|\overline{D}_{\Delta}^{-1/2}(\delta(\varphi\Lambda)_{\varkappa'})\overline{D}_{\Delta}^{-1/2}\|_{8/3,6s,\tau} \\
&\quad\leqslant 0(1)\lambda^6 m^5 s^3 \min(\varkappa,\varkappa')^{-\varepsilon} \qquad (110)
\end{aligned}
$$

and again m^5 may be omitted for $\Delta \not\subset Z^0$.

Next we estimate the M_2 term. Since (see(16))

$$
\begin{aligned}
A_{\Delta,\varkappa} - A'_{\Delta,\varkappa} &= t(\Delta)\lambda(\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa}) - \overline{K}^{\xi\pm}_{\Delta}(\delta\psi^{\varkappa})) \\
&= t(\Delta)\lambda^2\overline{K}^{\xi\pm}_{\Delta}(\eta_{\pm} - \xi_{\pm})\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa}) = t(\Delta)^2\lambda^2\overline{K}^{\xi\pm}_{\Delta}(\psi_{\Delta})\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa}), \qquad (111)
\end{aligned}
$$

$$
\begin{aligned}
|M_2| &\leqslant \frac{1}{2}t(\Delta)^3\lambda^3[|\operatorname{Tr}(\overline{K}^{\xi\pm}_{\Delta}(\psi_{\Delta})(\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa})^2 - \overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa'})^2))| \\
&+ |\operatorname{Tr}(\overline{K}^{\xi\pm}_{\Delta}(\delta\psi^{\varkappa})\overline{K}^{\xi\pm}_{\Delta}(\psi_{\Delta})\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa}) - \overline{K}^{\xi\pm}_{\Delta}(\delta\psi^{\varkappa'})\overline{K}^{\xi\pm}_{\Delta}(\psi_{\Delta})\overline{K}^{\eta\pm}_{\Delta}(\delta\psi^{\varkappa'}))|] \\
&\leqslant 0(1)\lambda^3\|\overline{D}_{\Delta}^{-1/2}[\delta((\varphi\Lambda)_{\varkappa} - (\varphi\Lambda)_{\varkappa'})]\overline{D}_{\Delta}^{-1/2}\|_3\|\overline{D}_{\Delta}^{-1/2}\psi_{\Delta}\overline{D}_{\Delta}^{-1/2}\|_3 \\
&\cdot (\|\overline{D}_{\Delta}^{-1/2}(\delta\psi^{\varkappa})\overline{D}_{\Delta}^{-1/2}\|_3 + \|\overline{D}_{\Delta}^{-1/2}(\delta\psi^{\varkappa'})\overline{D}_{\Delta}^{-1/2}\|_3) \\
&\leqslant 0(1)\lambda^3 m^2\|\overline{D}_{\Delta}^{-1/2}[\delta((\varphi\Lambda)_{\varkappa} - (\varphi\Lambda)_{\varkappa'})]\overline{D}_{\Delta}^{-1/2}\|_3 \sum_{\substack{k+l+m\leqslant 2\\ k,l,m\geqslant 0}} \|\overline{D}_{\Delta}^{-1/2}\varphi_{\Delta}\overline{D}_{\Delta}^{-1/2}\|^k_3
\end{aligned}
$$

$$\cdot \| \overline{D}_\Delta^{-1/2}(\delta(\varphi\Lambda)_\varkappa)\overline{D}_\Delta^{-1/2} \|_3^l \, \| \overline{D}_\Delta^{-1/2}(\delta(\varphi\Lambda)_{\varkappa'})\overline{D}_\Delta^{-1/2} \|_3^m . \tag{112}$$

$$\| M_2 \|_{L^s(\tau)} \leqslant 0(1)\lambda^3 m^2 \| \overline{D}_\Delta^{-1/2} [\delta((\varphi\Lambda)_\varkappa - (\varphi\Lambda)_{\varkappa'})]\overline{D}_\Delta^{-1/2} \|_{3,3s,\tau}$$
$$\cdot \sum_{\substack{k+l+m\leqslant 2 \\ k,l,m\geqslant 0}} \| \overline{D}_\Delta^{-1/2}\varphi_\Delta \overline{D}_\Delta^{-1/2} \|_{3,3s,\tau}^k \, \| \overline{D}_\Delta^{-1/2}(\delta(\varphi\Lambda)_\varkappa)\overline{D}_\Delta^{-1/2} \|_{3,3s,\tau}^l$$
$$\| \overline{D}_\Delta^{-1/2}(\delta(\varphi\Lambda)_{\varkappa'})\overline{D}_\Delta^{-1/2} \|_{3,3s,\tau}^m \leqslant 0(1)\lambda^3 m^2 s^{3/2} \min(\varkappa, \varkappa')^{-\varepsilon} . \tag{113}$$

M_3 is estimated the same way yielding

$$\| M_3 \|_{L^s(\tau)} \leqslant 0(1)\lambda^3 m^2 s^{3/2} \min(\varkappa, \varkappa')^{-\varepsilon} . \tag{114}$$

In both cases m^2 can be omitted for $\Delta \not\subset Z^0$.

For M_4 we obtain

$$M_4 \leqslant \frac{1}{2} t(\Delta)^2 \lambda^2 [| : \mathrm{Tr}\,(\overline{K}_\Delta^{\xi\pm}(\delta(\varphi\Lambda)_\varkappa)^2 + \overline{K}_\Delta^{\xi\pm}(\delta(\varphi\Lambda)_\varkappa)^* \overline{K}_\Delta^{\xi\pm}(\delta(\varphi\Lambda)_\varkappa)$$
$$- \overline{K}_\Delta^{\xi\pm}(\delta(\varphi\Lambda)_{\varkappa'})^2 - \overline{K}_\Delta^{\xi\pm}(\delta(\varphi\Lambda)_{\varkappa'})^* \overline{K}_\Delta^{\xi\pm}(\delta(\varphi\Lambda)_{\varkappa'})) :_\tau |$$
$$+ | \mathrm{Tr}\,(\overline{K}_\Delta^{\xi\pm}(\delta((\varphi\Lambda)_\varkappa - (\varphi\Lambda)_{\varkappa'}))\overline{K}_\Delta^{\xi\pm}(\delta g^\Lambda)$$
$$+ \overline{K}_\Delta^{\xi\pm}(\delta((\varphi\Lambda)_\varkappa - (\varphi\Lambda)_{\varkappa'}))^* \overline{K}_\Delta^{\xi\pm}(\delta g^\Lambda)$$
$$+ \overline{K}_\Delta^{\xi\pm}(\delta g^\Lambda)\overline{K}_\Delta^{\xi\pm}(\delta((\varphi\Lambda)_\varkappa - (\varphi\Lambda)_{\varkappa'}))$$
$$+ \overline{K}_\Delta^{\xi\pm}(\delta g^\Lambda)^* \overline{K}_\Delta^{\xi\pm}(\delta((\varphi\Lambda)_\varkappa - (\varphi\Lambda)_{\varkappa'}))) |] := M_4' + M_4'' \tag{115}$$

with

$$M_4' = \frac{1}{2} t(\Delta)^2 \lambda^2 | : (\varphi \mid L_{\varkappa,\varkappa'}\varphi)_{L^2} :_\tau | . \tag{116}$$

$L_{\varkappa,\varkappa'}$ is the operator on $L^2(\mathbb{R}^2)$ given by (see (93) and (94))

$$L_{\varkappa,\varkappa'} := (2\pi^2)^{-1} \chi_\Delta \Lambda (R_\varkappa \delta I_\Delta(P)_\Delta R_\varkappa - R_{\varkappa'} \delta I_\Delta(P)_\Delta R_{\varkappa'}) \Lambda \chi_\Delta . \tag{117}$$

$$\| M_4' \|_{L^2(\tau)} = 2^{-1/2} t(\Delta)^2 \lambda^2 \| C_{m_c}(\tau)^{1/2} L_{\varkappa,\varkappa'} C_{m_c}(\tau)^{1/2} \|_2$$
$$\leqslant 0(1)\lambda^2 \| C_{m_c}(\tau)^{1/2}\chi_\Delta\Lambda(R_\varkappa - R_{\varkappa'}) \|_4 \, (\| C_{m_c}(\tau)^{1/2}\chi_\Delta\Lambda R_\varkappa \|_4$$
$$+ \| C_{m_c}(\tau)^{1/2}\chi_\Delta\Lambda R_{\varkappa'} \|_4) = 0(1)\lambda^2 \| (R_\varkappa - R_{\varkappa'})\Lambda\chi_\Delta C_{m_c}(\tau)\chi_\Delta\Lambda(R_\varkappa - R_{\varkappa'}) \|_2^{1/2}$$
$$(\| R_\varkappa\Lambda\chi_\Delta C_{m_c}(\tau)\chi_\Delta\Lambda R_\varkappa \|_2^{1/2} + \| R_{\varkappa'}\Lambda\chi_\Delta C_{m_c}(\tau)\chi_\Delta\Lambda R_{\varkappa'} \|_2^{1/2})$$
$$\leqslant 0(1)\lambda^2 \min(\varkappa, \varkappa')^{-\varepsilon} , \tag{118}$$

where we have used the uniform boundedness of $I_\Delta(k)$, see Proof of Lemma V.6, and Lemma A.II.1. The hypercontractivity allows to extend (118) to

$$\| M_4' \|_{L^s(\tau)} \leqslant 0(1)\lambda^2 s \min(\varkappa, \varkappa')^{-\varepsilon} . \tag{119}$$

For M_4'' we have the representation

$$M_4'' = t(\Delta)^2 \lambda^2 | (\varphi \mid \tilde{L}_{\varkappa,\varkappa'} g^\Lambda)_{L^2} | \tag{120}$$

with

$$\tilde{L}_{\varkappa,\varkappa'} := (2\pi^2)^{-1} \chi_\Delta \Lambda (R_\varkappa - R_{\varkappa'}) \delta(I_\Delta(P))_\Delta . \tag{121}$$

Hence

$$\| M''_4 \|_{L^2(\tau)} = t(\Delta)^2 \lambda^2 (\tilde{L}_{\varkappa,\varkappa'} g^\Lambda \mid C_{m_c}(\tau) \tilde{L}_{\varkappa,\varkappa'} g^\Lambda)^{1/2}_{L^2}$$
$$= (2\pi^2)^{-1} t(\Delta)^2 \lambda^2 (\delta I_\Delta(P)_\Delta g^\Lambda \mid (R_\varkappa - R_{\varkappa'}) \Lambda \chi_\Delta C_{m_c}(\tau) \chi_\Delta \Lambda (R_\varkappa - R_{\varkappa'}) \delta I_\Delta(P)_\Delta g^\Lambda)^{1/2}_{L^2}$$
$$\leqslant 0(1)\lambda^2 \| (R_\varkappa - R_{\varkappa'}) \Lambda \chi_\Delta C_{m_c}(\tau) \chi_\Delta \Lambda (R_\varkappa - R_{\varkappa'}) \|_2^{1/2}$$
$$\cdot \| \delta I_\Delta(P)_\Delta g^\Lambda \|_{L^2} \leqslant 0(1)\lambda^2 m \min(\varkappa, \varkappa')^{-\varepsilon} \tag{122}$$

for $\Delta \subset Z^0$ ($M''_4 = 0$ for $\Delta \not\subset Z^0$).

Again the hypercontractivity yields

$$\| M''_4 \|_{L^s(\tau)} \leqslant 0(1)\lambda^2 m s^{1/2} \min(\varkappa, \varkappa')^{-\varepsilon}. \tag{123}$$

(101), (110), (113)-(115), (119) and (123) give (89).

With (88) and (89) given, the estimation of $\| \mathscr{Z}'_{3,\Delta}(t) \|_{L^r(\tau)}$ follows in a standard way, see Proof of Lemma II.3.2.6 in [*13*]:

LEMMA V.7. — For any $r < \infty$ and m sufficiently large

$$\| \mathscr{Z}'_{3,\Delta}(t) \|_{L^r(\tau)} \leqslant \begin{cases} \exp[0(1)\lambda^3] & \text{for } \Delta \subset Z_t, \Delta \not\subset Z^0, \\ \exp[0(1)\lambda^3 \ln m] & \text{for } \Delta \subset Z_t, \Delta \subset Z^0, \\ 1 & \text{otherwise}. \end{cases} \tag{124}$$

Proof. — The case $\Delta \subset Z_v$ is trivial. So take the case $\Delta \subset Z_t$. By (88) and (89)

$$\| \mathscr{Z}'_{3,\Delta}(t) \|^r_{L^r(\tau)} \leqslant \exp[1 + r \sup \ln \mathscr{Z}_{3,\Delta,\varkappa_{j_0}}(t)]$$
$$+ \sum_{j=j_0}^{\infty} \| r(\ln \mathscr{Z}'_{3,\Delta}(t) - \ln \mathscr{Z}'_{3,\Delta,\varkappa_j}(t)) \|^{q_j}_{q_j} \exp[1 + r \sup \ln \mathscr{Z}_{3,\Delta,\varkappa_{j+1}}(t)]$$
$$\leqslant \exp[1 + 0(1)\lambda^2 r \ln \varkappa_{j_0}] + \sum_{j=j_0}^{\infty} (0(1) r \lambda^6)^{q_j} q_j^{3q_j} \varkappa_j^{-\frac{1}{2}\varepsilon q_j}$$
$$\cdot \exp[1 + 0(1)\lambda^2 r \ln \varkappa_{j+1}] \tag{125}$$

provided $\varkappa_j \geqslant m^{10/\varepsilon}$ for $\Delta \subset Z^0$. Choosing

$$\varkappa_j = 0(1)^{\lambda j}, \qquad q_j = \lambda^2 j,$$
$$j_0 = \begin{cases} 0(1) & \text{if } \Delta \not\subset Z^0, \\ 0(1) \ln m & \text{if } \Delta \subset Z^0, \end{cases} \tag{126}$$

we obtain

$$\| \mathscr{Z}'_{3,\Delta}(t) \|_{L^r(\tau)} \leqslant \exp[0(1)\lambda^2 \ln \varkappa_{j_0}],$$

which gives (124). □

To end the estimation of $\| \mathscr{Z}_3(t) \|_{L^q(\tau)}$ we have to bound $\| \mathscr{Z}''_3(t) \|_{L^{2q}(\tau)}$, see (84). This is done again by following [*25*] and bounding $\mathscr{Z}''_3(t)$ by product of exponentials of the square roots of the fourth order non-linear monomials in φ. The integrability of such terms follows as in [*25*]. We start with technical

LEMMA V.8. — For any r, $\frac{16}{9} < r \leqslant 2$, and $\alpha > 0$ there exist β, $\varepsilon > 0$ such that for all d-squares Δ, Δ', Δ''

$$\| \chi_{\Delta'}(\mathrm{K}^h(s, f\chi_\Delta) - \mathrm{K}_\Delta^{\xi\pm}(f))\chi_{\Delta''} \|_r \leqslant 0(1)m^{-9/4+4/r+\alpha} \cdot \exp[-\varepsilon m(d(\Delta', \Delta) + d(\Delta, \Delta''))]\mathrm{S}^\beta(f\chi_\Delta). \quad (127)$$

Proof. — By Proposition A.I.4 and Corollary A.I.4

$$\| \chi_{\Delta'}(\mathrm{K}^h(s, f\chi_\Delta) - \mathrm{K}^{\xi\pm}(f\chi_\Delta))\chi_{\Delta''} \|_r \leqslant 0(1)m^{-9/4+4/r+\alpha} \exp[-\varepsilon m(d(\Delta', \Delta) + d(\Delta, \Delta''))]\mathrm{S}^\beta(f\chi_\Delta) \quad (128)$$

for $\frac{16}{9} < r \leqslant 2$.

Now notice that

$$\| \chi_{\Delta'}(\mathrm{K}^{\xi\pm}(f\chi_\Delta) - \chi_\Delta \mathrm{K}^{\xi\pm}(f\chi_\Delta)\chi_\Delta)\chi_{\Delta''} \|_r = \begin{cases} 0 & \text{if } \Delta' = \Delta = \Delta'', \\ \| \chi_{\Delta'}\mathrm{K}^{\xi\pm}(f\chi_\Delta)\chi_{\Delta''} \|_r & \text{otherwise} \end{cases}$$
$$\leqslant 0(1)m^{-9/4+4/r+\alpha} \exp[-\varepsilon m(d(\Delta', \Delta) + d(\Delta, \Delta''))]\mathrm{S}^\beta(f\chi_\Delta) \quad (129)$$

for $\frac{16}{9} < r \leqslant 2$ by Corollary A.I.4.

The last step in proving Lemma V.8 consist in estimating

$$\| \chi_\Delta \mathrm{K}^{\xi\pm}(f\chi_\Delta)\chi_\Delta - \mathrm{K}_\Delta^{\xi\pm}(f) \|_r = \| \chi_\Delta \mathrm{D}^{1/2}(\not{\mathrm{P}} + m + \lambda\xi_\pm\Gamma)\overline{\mathrm{D}}^{-2}\Gamma f\chi_\Delta \mathrm{D}^{-1/2}\chi_\Delta - \mathrm{D}_\Delta^{1/2}(\not{\mathrm{P}} + m + \lambda\xi_\pm\Gamma)_\Delta \overline{\mathrm{D}}_\Delta^{-2}\Gamma f \mathrm{D}_\Delta^{-1/2} \|_r .$$

It is easy to see that if $\mathrm{H}(x - y)$ is the kernel of $f(\mathrm{P})$ then $\chi_\Delta(x)\mathrm{H}_\Delta(x-y)\chi_\Delta(y)$ is the kernel of $f(\mathrm{P})_\Delta$, where

$$\mathrm{H}_\Delta(x) := \sum_{z\in d\mathbb{Z}^2} \mathrm{H}(x + z).$$

Thus

$$f(\mathrm{P})_\Delta = \sum_{z\in d\mathbb{Z}^2} \chi_\Delta f(\mathrm{P})\chi_{\Delta+z}\mathrm{T}_z = \sum_z \mathrm{T}_z\chi_{\Delta-z} f(\mathrm{P})\chi_\Delta , \quad (130)$$

where T_z is the translation by z in $\mathrm{L}^2(\mathbb{R}^2)$.

Hence

$$\| \chi_\Delta \mathrm{K}^{\xi\pm}(f\chi_\Delta)\chi_\Delta - \mathrm{K}_\Delta^{\xi\pm}(f) \|_r \leqslant \sum_{z,z'} \| \chi_\Delta \mathrm{K}^{\xi\pm}(f\chi_\Delta)\chi_\Delta - \mathrm{T}_z\chi_{\Delta-z}\mathrm{K}^{\xi\pm}(f\chi_\Delta)\chi_{\Delta+z'}\mathrm{T}_{z'} \|_r \leqslant \sum_{\Delta' \text{ or } \Delta'' \neq \Delta} \| \chi_{\Delta'}\mathrm{K}^{\xi\pm}(f\chi_\Delta)\chi_{\Delta''} \|_r$$
$$\leqslant 0(1)m^{-9/4+4/r+\alpha}\mathrm{S}^\beta(f\chi_\Delta) \quad (131)$$

as in (129), (128), (129) and (131) give (127). □

Lemma V.9. — For $q < \infty$, $\alpha > 0$ and m sufficiently large

$$\| \exp [0(1) \| AB \|_1] \|_{L^q(\tau)} \leqslant \exp \left[0(1) m^{-1/4+\alpha}(\xi_+^2 \mid Z^0 \mid + \mid Z \mid)\right]. \quad (132)$$

Proof. — Denote

$$B''_{\Delta',\Delta,\Delta''} := \lambda \chi_{\Delta'}(K^h(s, \psi\chi_\Delta) - K^{\xi\pm}_{\Delta}(\psi))\chi_{\Delta''}. \quad (133)$$

We have (see (4))

$$B'' = \sum_{\Delta',\Delta,\Delta''} B''_{\Delta',\Delta,\Delta''} \quad (134)$$

Moreover by (17)-(21), (24) and (25)

$$\| AB \|_1 \leqslant \sum_{\Delta',\Delta,\Delta''} t(\Delta')\lambda \| W_{\Delta'} K^{\eta\mp}_{\Delta'}(\delta\psi)(1 - \lambda t(\Delta')K^{\xi\mp}_{\Delta'}(\psi_\Delta))^{-1} B''_{\Delta',\Delta,\Delta''} W^{-1}_{\Delta''} \|_1$$

$$\leqslant 0(1)\lambda \sum_{\Delta',\Delta,\Delta''} \| W_{\Delta'} \| \, \| W^{-1}_{\Delta''} \| \, \| (1 - \lambda t(\Delta')K^{\xi\pm}_{\Delta'}(\psi_\Delta))^{-1} \|^2 \, \| K^{\xi\mp}_{\Delta'}(\delta\psi) \|_{s'}$$

$$\cdot \; \| B''_{\Delta',\Delta,\Delta''} \|_s ,$$

where we have used the relation

$$K^{\eta\mp}_{\Delta'}(\delta\psi) = (1 - \lambda t(\Delta')K^{\xi\mp}_{\Delta'}(\psi_\Delta))^{-1} K^{\xi\pm}_{\Delta'}(\delta\psi).$$

Straightforward estimates

$$\| W_\Delta \| \leqslant 0(1), \qquad \| W_\Delta^{-1} \| \leqslant 0(1), \qquad \| (1 - \lambda K^{\xi\pm}_{\Delta}(x))^{-1} \| \leqslant 0(1) \quad (135)$$

give

$$\| AB \|_1 \leqslant 0(1)\lambda \sum_{\Delta',\Delta,\Delta''} \| K^{\xi\mp}_{\Delta'}(\delta\psi) \|_{s'} \, \| B''_{\Delta',\Delta,\Delta''} \|_s$$

$$\leqslant 0(1)\lambda \sum_{\Delta',\Delta,\Delta''} \| D^{-1/2+\beta}_{\Delta'} \|_{s_1} S^\beta_{\Delta'}(\delta\psi) \, \| B''_{\Delta',\Delta,\Delta''} \|_s \quad (136)$$

with $s_1^{-1} + 4^{-1} = s'^{-1} = 1 - s^{-1}$.

We choose $s < 2$ but close to 2. Then $s_1 > 4$ but is close to 4 and for small β

$$\| D^{-1/2+\beta}_{\Delta'} \|_{s_1} \leqslant \sum_{\Delta'''} \| \chi_{\Delta'} D^{-1/2+\beta} \chi_{\Delta'''} \|_{s_1}$$

$$\leqslant 0(1) m^{-1/2+\beta+2/s_1} \sum_{\Delta'''} \exp [- \varepsilon m d(\Delta', \Delta''')] \leqslant 0(1) m^{-1/2+\beta+2/s_1}, \quad (137)$$

where we have used (130), Lemma 2.1 of [*25*] and Lemma A.I.5. From Lemma V.8 we get

$$\| B''_{\Delta',\Delta,\Delta''} \|_s \leqslant 0(1)\lambda m^{-9/4+4/s+\alpha} \exp [- \varepsilon m(d(\Delta', \Delta) + d(\Delta, \Delta''))] S^\beta(\psi\chi_\Delta). \quad (138)$$

(136)-(138) give (compare formula (5.18) of [25])

$$\| AB \|_1 \leqslant 0(1)\lambda^2 m^{-1/4+\alpha} \sum_{\Delta,\Delta',\Delta'' \subset Z} \exp [-\varepsilon m(d(\Delta',\Delta) + d(\Delta,\Delta''))]$$

$$\cdot S^\beta_{\Delta'}(\delta\psi) S^\beta(\psi\chi_\Delta) \leqslant 0(1)\lambda^2 m^{-1/4+\alpha} \sum_{\Delta\subset Z} (S^\beta_\Delta(\delta\psi)^2 + S^\beta(\psi\chi_\Delta)^2) \quad (139)$$

for any $\alpha > 0$ provided β is sufficiently small and m sufficiently large. Since

$$\psi = ((\varphi + g - h)\Lambda + (\xi_+ - h)(1 - \Lambda))\chi_Z$$

and $S^\beta(f\chi_\Delta) \leqslant 0(1) \| f\chi_\Delta \|_{L^\infty}$, $S^\beta_\Delta(f) \leqslant 0(1) \| f\chi_\Delta \|_{L^\infty}$, we have

$$\| AB \|_1 \leqslant 0(1)\lambda^2 m^{-1/4+\alpha} \left[\sum_{\Delta\subset Z} (S^\beta_\Delta(\delta(\varphi\Lambda))^2 + S^\beta(\varphi\Lambda\chi_\Delta)^2) + \xi_+^2 \, | Z^0 | \right]. \quad (140)$$

Now in virtue of (140), the checkerboard estimate, the Hölder inequality and Lemma A.II.5

$$\| \exp [0(1) \| AB \|_1] \|_{L^q(\tau)} \leqslant \prod_{\Delta\subset Z} (\| \exp [0(1)m^{-1/4+\alpha} S^\beta_\Delta(\delta(\varphi\Lambda))^2] \|_{L^{q_1}(\tau)}$$
$$\cdot \| \exp [0(1)m^{-1/4+\alpha} S^\beta(\varphi\Lambda\chi_\Delta)^2] \|_{L^{q_1}(\tau)}) \exp [0(1)m^{-1/4+\alpha}\xi_+^2 \, | Z^0 |]$$
$$\leqslant \exp [0(1)m^{-1/4+\alpha}(\xi_+^2 \, | Z^0 | + | Z |)]$$

for β sufficiently small and m sufficiently large (λ^2 has been swallowed by a small power of m). □

LEMMA V.10. — For $q < \infty$, $\alpha > 0$ and m sufficiently large

$$\| \exp [0(1) \| B^*B \|_1] \|_{L^q(\tau)} \leqslant \exp [0(1)m^{-1/2+\alpha}(\xi_+^2 \, | Z^0 | + | Z |)]. \quad (141)$$

Proof. — Similarly as in Proof of Lemma V.9 one shows that

$$\| B^*B \|_1 \leqslant 0(1) \sum_{\substack{\Delta,\Delta',\Delta'',\\ \Delta''',\Delta^{IV}}} \| B''^*_{\Delta',\Delta,\Delta''} B''_{\Delta',\Delta''',\Delta^{IV}} \|_1$$

$$\leqslant 0(1) \sum_{\Delta,\dots,\Delta^{IV}} \| B''_{\Delta',\Delta,\Delta''} \|_2 \, \| B''_{\Delta',\Delta''',\Delta^{IV}} \|_2$$

$$\leqslant 0(1)m^{-1/2+\alpha} \sum_{\Delta,\dots,\Delta^{IV}} \exp [-\varepsilon m(d(\Delta',\Delta) + d(\Delta,\Delta'') + d(\Delta',\Delta''') + d(\Delta''',\Delta^{IV}))]$$

$$\cdot \; S^\beta(\psi\chi_\Delta)S^\beta(\psi\chi_{\Delta'''}) \leqslant 0(1)m^{-1/2+\alpha} \sum_{\Delta\subset Z} S^\beta(\varphi\Lambda\chi_\Delta)^2 + 0(1)m^{-1/2+\alpha}\xi_+^2 \, | Z^0 | . \quad (142)$$

(142) yields (141) as (140) did (132). □

(82)-(85), (87) and Lemmas V.7, 9, 10 give

LEMMA V.11. — For each $q < \infty$, $\alpha > 0$ and λ

$$\|\mathscr{Z}_3^{\mathrm{K}}(t)\|_{\mathrm{L}^q(\tau)} \leqslant \exp\big[0(1)\mathrm{K} + 0(1)m^{-1/4+\alpha}\xi_+^2 \,|\,\mathrm{Z}^0\,| + 0(1)m^{-1/4+\alpha}\,|\,\mathrm{Z}_v\,| + 0(1)\lambda^3\,|\,\mathrm{Z}_t\,|\big] \quad (143)$$

provided m is large enough. □

Similarly as in Proof Lemma V.9 we also show

LEMMA V.12. — For each $q < \infty$, $\alpha > 0$ and λ

$$\|\mathscr{Z}_b(t)\|_{\mathrm{L}^q(\tau)} \leqslant \exp\big[0(1)m^{-1/4+\alpha}(\xi_+^2\,|\,\mathrm{Z}^0\,| + |\,\mathrm{Z}\,|)\big] \quad (144)$$

for $b = 4,5$ and for m sufficiently large. □

Estimation of $\|\mathscr{Z}_6(t)\|_{\mathrm{L}^q(\tau)}$ and $\|\mathscr{Z}_7(t)\|_{\mathrm{L}^q(\tau)}$ are the steps which introduce restrictions on the lattice diameter d. Because of that we shall trace more carefully the d-dependence of the appearing constants. The reason for these restrictions is that $q \ln \mathscr{Z}_6(t)$ and $q \ln \mathscr{Z}_7(t)$ are essentially quadratic forms in fluctuation field $\delta\varphi$. For small d the $\delta\varphi$ field has a high effective mass and the $-\frac{1}{2}(\delta\varphi \mid \mathrm{C}_{m_c}(\tau)^{-1}\delta\varphi)$ term of the Gaussian measure dominates the other quadratic terms rendering the integral finite.

Let us start with the purely Gaussian term $\mathrm{Z}_6(t)$. Conditioning with respect to Neumann boundary conditions on the boundaries $\partial\Delta$, see [*14*] [*13*], we obtain (in self-explanatory notation)

$$\|\mathscr{Z}_6(t)\|_{\mathrm{L}^q(\tau)} \leqslant \prod_{\Delta\subset \mathrm{Z}_t} \|\mathscr{Z}_{6,\Delta,\mathrm{N}}(t)\|_{\mathrm{L}^q(\mathrm{N})}, \quad (145)$$

$$\begin{aligned}\mathscr{Z}_{6,\Delta,\mathrm{N}}(t) &= \exp\Big[\frac{1}{2}t(\Delta)^2 : \Big(\lambda^2 \operatorname{Tr} \overline{\mathrm{K}}_\Delta^{\xi\pm}(\delta\psi)^*\overline{\mathrm{K}}_\Delta^{\xi\pm}(\delta\psi) \\ &\quad + \lambda^2\int \mathrm{B}_\Delta^{\xi+}(\delta\psi)^2 + \eta\int_\Delta(\delta\psi)^2 :_{\mathrm{N}}\Big] \\ &= \exp\Big[(2\pi d)^{-2}t(\Delta)^2 \sum_{k\in\frac{2\pi}{d}\mathbb{Z}^2}(\lambda^2\mathrm{G}_\Delta(k)+\eta) : |\,\widehat{(\delta\psi\chi_\Delta)}(k)\,|^2 :_{\mathrm{N}}\Big]\end{aligned}$$

by explicite computation of traces.

$$\mathrm{G}_\Delta(k) := \sum_{q\in\frac{2\pi}{d}\mathbb{Z}^2}(2\pi)^2 d^{-2}\,[((q+k)^2+\overline{m}^2)^{-1/2}(q^2+\overline{m}^2)^{-1/2} - (q^2+\overline{m}^2)^{-1}]. \quad (146)$$

In the limit $d \to \infty$ we obtain

$$G(k) = -\pi \ln\left(1 + \frac{1}{4}(1+\bar{\alpha})^{-1}\rho\right) - 2\pi \ln\left(\frac{1}{2} + (4 + (1+\bar{\alpha})^{-1}\rho)^{-1/2}\right) \equiv \tilde{G}((1+\bar{\alpha})^{-1}\rho), \quad (147)$$

see formulae (A.6) and (A.12) of [23] (ρ and $\bar{\alpha}$ are given by (96)).

$$\tilde{G}((1+\bar{\alpha})^{-1}\rho) = -\pi \ln((1+\bar{\alpha})^{-1}\rho) + 4\pi \ln 2 + 0((1+\bar{\alpha})^{1/2}\rho^{-1/2}) \quad (148)$$

when $\rho \to \infty$, see formula (A.14) of [23],

$$\tilde{G}(0) = 0, \qquad \tilde{G} \leqslant 0. \quad (149)$$

By standard estimating procedure using (62) and the Hausdorff-Yang theorem

$$| G(k) - G_\Delta(k) | \leqslant 0(1)m^{-1} \quad (150)$$

with d-dependent 0(1), compare (99).

In virtue of (147)-(150) there exists $\beta > 0$ such that for each d and $m \geqslant m_0(\lambda, d)$

$$\lambda^2 G_\Delta(k) + \eta \leqslant \beta \quad (151)$$

and

$$| \lambda^2 G_\Delta(k) + \eta | \leqslant \lambda^2 \beta \ln\left(1 + \frac{k^2}{m^2}\right) + \eta + \beta. \quad (152)$$

By (145)

$$\mathscr{Z}_{6,\Delta,N}(t) = \exp\left[\frac{1}{2} : (\psi \mid L_\Delta \psi)_{L^2} :_N\right], \quad (153)$$

where

$$L_\Delta := (2\pi^2)^{-1} t(\Delta)^2 \chi_\Delta \delta(\lambda^2(G_\Delta(P) + \eta)_\Delta \chi_\Delta. \quad (154)$$

By explicite Gaussian integration

$$\| \mathscr{Z}_{6,\Delta,N}(t) \|_{L^q(N)} = \exp\left[\frac{1}{2}(g^\Lambda \mid L_\Delta g^\Lambda)_{L^2}\right] \left(\int \exp\left[\frac{1}{2} q : (\varphi \mid \Lambda L_\Delta \Lambda \varphi)_{L^2} :_N + q(\varphi \mid \Lambda L_\Delta g^\Lambda)_{L^2}\right] d\mu_{m_c}(N)\right)^{1/2}$$
$$= \exp\left[\frac{1}{2}(g^\Lambda \mid L_\Delta g^\Lambda)_{L^2} + \frac{1}{2} q (g^\Lambda \mid L_\Delta \Lambda C_{m_c}(N)^{1/2}(1 - qL'_\Delta)^{-1} C_{m_c}(N)^{1/2} \Lambda L_\Delta g^\Lambda)_{L^2}\right] \cdot (\det_2(1 - qL'_\Delta))^{-1/2q} \quad (155)$$

provided

$$L'_\Delta := C_{m_c}(N)^{1/2} \Lambda L_\Delta \Lambda C_{m_c}(N)^{1/2} < \frac{1}{q}. \quad (156)$$

$C_{m_c}(N)$ is the covariance corresponding to mass m_c and Neumann boundary conditions on $\partial\Delta$, $g^\Lambda = g\Lambda + \xi_+(1 - \Lambda)$.

$$L'_\Delta \leqslant (2\pi^2)^{-1} \beta C_{m_c}(N)^{1/2} \chi_\Delta \Lambda \delta \Lambda \chi_\Delta C_{m_c}(N)^{1/2}. \quad (157)$$

Hence to prove that

$$L'_\Delta \leqslant \frac{1}{2q} \tag{158}$$

for sufficiently small d, it is sufficient to show that

$$\| C_{m_c}(N)^{1/2}\chi_\Delta \Lambda\delta \| \leqslant 0(1)d\,. \tag{159}$$

But

$$\| C_{m_c}(N)^{1/2}\chi_\Delta \Lambda\delta \| \leqslant \| C_{m_c}(N)^{1/2}\chi_\Delta \| \, \| [\Lambda, \delta] \| + \| C_{m_c}(N)^{1/2}\chi_\Delta\delta \| \, \| \Lambda \|_{L^\infty}\,.$$

We have

$$\| [\Lambda, \delta] \| \leqslant 0(1)d \qquad \text{since} \qquad \| \nabla\Lambda \|_{L^\infty} \leqslant 0(1)\,.$$

Moreover

$$\| C_{m_c}(N)^{1/2}\chi_\Delta \| = m_c^{-1}\,, \qquad \| \Lambda \|_{L^\infty} \leqslant 1\,,$$
$$\| C_{m_c}(N)^{1/2}\chi_\Delta\delta \| = (\pi^2 d^{-2} + m_c^2)^{-1/2}\,.$$

Hence (159) and consequently (158) hold if d is small enough. This is one of the restrictions on d we shall encounter.

When (158) holds then

$$(\det_2(1 - qL'_\Delta))^{-1/2q} = \det_2(1 + q(1 - qL'_\Delta)^{-1}L'_\Delta)^{1/2q} \cdot \exp\left[\frac{1}{2q}\operatorname{Tr}((1 - qL'_\Delta)^{-1}q^2 L'^2_\Delta)\right] \leqslant \exp\left[q \| L'_\Delta \|_2^2\right]. \tag{160}$$

$\| L'_\Delta \|_2$ will be estimated also with taking care of d-dependence. We shall consider $0 < d \leqslant \frac{1}{2}$ and, until otherwise specified, 0(1) will usually denote constants independent of Λ, λ, m and d.

In virtue of (156), (154) and (152)

$$\| L'_\Delta \|_2 \leqslant 0(1) \| C_{m_c}(N)^{1/2}\Lambda\delta(P^2 + m_c^2)_\Delta^{1/30}\Lambda C_{m_c}(N)^{1/2} \|_2 \leqslant 0(1) \| \delta(P^2 + m_c^2)_\Delta^{1/60}\Lambda C_{m_c}(N)^{1/2} \|_4^2\,. \tag{161}$$

$$\| \delta(P^2 + m_c^2)_\Delta^{1/60}\Lambda C_{m_c}(N)^{1/2} \|_4 \leqslant \| \delta(P^2 + m_c^1)_\Delta^{1/60}\Lambda(1 - \delta)C_{m_c}(N)^{1/2} \|_4 + \| \delta(P^2 + m_c^2)_\Delta^{1/60}\Lambda\delta \cdot C_{m_c}(N)^{1/2} \|_4\,. \tag{162}$$

But

$$(1 - \delta)C_{m_c}(N)^{1/2} = m_c^{-1}(1 - \delta)\,. \tag{163}$$

Hence

$$\| \delta(P^2 + m_c^2)_\Delta^{1/60}\Lambda(1 - \delta)C_{m_c}(N)^{1/2} \|_4 = (m_c d)^{-1} \| \delta(P^2 + m_c^2)_\Delta^{1/60}\Lambda\chi_\Delta \|_{L^2} = (m_c d^2)^{-1}\left(\sum_{\substack{p\in\frac{2\pi}{d}\mathbb{Z}^2 \\ p\neq 0}} (p^2 + m_c^2)^{1/30} \, |\widehat{\Lambda\chi_\Delta}(p)|^2\right)^{1/2}. \tag{164}$$

LEMMA V.13. — For any $\alpha > 0$

$$|\widehat{\Lambda\chi_\Delta}(p)| \leqslant 0(1)d^{2-\alpha}(1 + |p_0|d)^{-1+\alpha}(1 + |p_1|d)^{-1+\alpha} \tag{165}$$

uniformly in Λ.

Proof.

$$(2\pi)^2\widehat{\Lambda\chi_\Delta}(p) = \int \widehat{\chi_\Delta}(p-q)\hat{\Lambda}(q)dq \tag{166}$$

and since only $\Lambda \restriction_\Delta$ counts we may assume that we take Λ with bounded diameter of the support. Then

$$|\hat{\Lambda}(q)| \leqslant 0(1)(1+|q_0|)^{-1}(1+|q_1|)^{-1} \tag{167}$$

uniformly in Λ. Hence

$$|\widehat{\Lambda\chi_\Delta}(p)| \leqslant 0(1)d^2\int(1+|p_0-q_0|d)^{-1}(1+|p_1-q_1|d)^{-1} \cdot(1+|q_0|)^{-1}(1+|q_1|)^{-1}dq. \tag{168}$$

But

$$\begin{aligned}\int(1+|p_i-q_i|d)^{-1}(1+|q_i|)^{-1}dq_i &\leqslant 0(1)d\int_{|q_i|d\geqslant\frac{1}{2}}(1+|p_i-q_i|d)^{-1}\\ &\cdot(1+|q_i|d)^{-1}dq_i + 0(1)\int_{|q_i|d\leqslant\frac{1}{2}}(1+|p_i|d)^{-1}(1+|q_i|)^{-1}dq_i\\ &\leqslant 0(1)(1+|p_i|d)^{-1+\alpha} + 0(1)\ln(1+d^{-1})(1+|p_i|d)^{-1}\\ &\leqslant 0(1)d^{-1/2\alpha}(1+|p_i|d)^{-1+\alpha}.\end{aligned} \tag{168}$$

(167) and (168) give (165). □

With (165), (164) yields

$$\begin{aligned}&\|\delta(P^2+m_c^2)_\Delta^{1/60}\Lambda(1-\delta)C_{m_c}(N)^{1/2}\|_4\\ &\leqslant 0(1)d^{-\alpha}\sum_{p\in\frac{2\pi}{d}\mathbb{Z}^2}(p^2+m_c^2)^{1/30}(1+|p_0|d)^{-2+2\alpha}(1+|p_1|d)^{-2+2\alpha}\\ &\leqslant 0(1)d^{-\frac{1}{15}-\alpha}\sum_{p\in\frac{2\pi}{d}\mathbb{Z}^2}((pd)^2+m_c^2)^{1/30}(1+|p_0|d)^{-2+2\alpha}(1+|p_1|d)^{-2+2\alpha}\\ &= 0(1)d^{-\frac{1}{15}-\alpha}.\end{aligned} \tag{169}$$

Now we estimate the second term on the right hand side of (162).

$$\begin{aligned}&\|\delta(P^2+m_c^2)_\Delta^{1/60}\Lambda\delta C_{m_c}(N)^{1/2}\|_4 \leqslant \|\delta[(P^2+m_c^2)_\Delta^{1/60},\Lambda]\delta C_{m_c}(N)^{1/2}\|_4\\ &+\|\delta\Lambda(P^2+m_c^2)_\Delta^{1/60}\delta C_{m_c}(N)^{1/2}\|_4 \leqslant \|\delta[(P^2+m_c^2)_\Delta^{1/60},\Lambda]\delta(P^2+m_c^2)_\Delta^{-1/40}\|\\ &\cdot\|(P^2+m_c^2)_\Delta^{1/10}\delta C_{m_c}(N)^{1/2}\|_4+\|(P^2+m_c^2)_\Delta^{1/60}\delta C_{m_c}(N)^{1/2}\|_4.\end{aligned} \tag{170}$$

LEMMA V.14. — For $0 \leqslant \nu < \frac{1}{16}$, $\lambda - 2\nu > \frac{1}{16}$ and $\alpha > 0$

$$\|\delta(P^2+m_c^2)_\Delta^\nu[\Lambda,(P^2+m_c^2)_\Delta^{-2}]\delta\|_8 \leqslant 0(1)d^{2\lambda-2\nu-\alpha} \tag{171}$$

uniformly in Λ.

Proof. — Notice that in the basis composed of periodic momentum eigenvectors

$$\delta(P^2 + m_c^2)_\Delta^\nu[\Lambda, (P^2 + m_c^2)_\Delta^{-\lambda}]\delta$$

has matrix elements equal

$$d^{-2}(p^2 + m_c^2)^\nu \widehat{\Lambda\chi_\Delta}(p - q)((q^2 + m_c^2)^{-\lambda} - (p^2 + m_c^2)^{-\lambda}) \tag{172}$$

for $p, q \in \frac{2\pi}{d}\mathbb{Z}^2$, $p \neq 0$, $q \neq 0$ and vanishing for $p = 0$ or $q = 0$. We repeat the arguments used in Proof of Lemma 2.3 of [*25*] by adapting them to the discrete case. To this end notice that since

$$(p^2 + m_c^2)^{-a} \leqslant 0(1)d^{2a}((pd)^2 + m_c^2)^{-a} \tag{173}$$

for $p \neq 0$, $p \in \frac{2\pi}{d}\mathbb{Z}^2$, we may account for the d-dependence by bounding the matrix elements by functions of pd and extracting an overall d power. When we sum over the discrete momenta using (165), only the overall d power survives yielding (171). □

Using Lemma V.14 we estimate

$$\begin{aligned} &\| \delta[(P^2 + m_c^2)_\Delta^{1/60}, \Lambda]\delta(P^2 + m_c^2)_\Delta^{-1/10} \|_8 \\ &\qquad \leqslant \|\delta(P^2 + m_c^2)_\Delta^{1/60}[\Lambda, (P^2 + m_c^2)_\Delta^{-1/10}]\delta \|_8 \\ &\qquad + \|\delta[(P^2 + m_c^2)_\Delta^{-5/60}, \Lambda]\delta \|_8 \leqslant 0(1)d^{1/10}. \end{aligned} \tag{174}$$

We shall also need

Lemma V.15. — For $\nu < \frac{1}{8}$

$$\| (P^2 + m_c^2)_\Delta^\nu \delta C_{m_c}(N)^{1/2} \|_4 \leqslant 0(1)d^{3/4}. \tag{175}$$

Proof. — Choose $\Delta = \left[-\frac{1}{2}d, \frac{1}{2}d \right]^2$. The spectrum of Δ_N is $\pi d^{-1}\mathbb{Z}_+^2$. The eigenfunctions e_β corresponding to p are given by

$$e_p(x_1, x_2) = 2d^{-1}{\textstyle\frac{\sin}{\cos}}(p_0x^0)\,{\textstyle\frac{\sin}{\cos}}(p_1x^1)\chi_\Delta(x), \tag{176}$$

where sin or cos is choosen for p_i being and odd or even multiplicity of πd^{-1} respectively. Explicite computation gives for

$$H := C_{m_c}(N)^{1/2}\delta(P^2 + m_c^2)_\Delta^{2\nu}C_{m_c}(N)^{1/2}$$

$$\begin{aligned} (e_p \mid He_q) = {} & 4d^{-4}(p^2 + m_c^2)^{-1/2} \sum_{0 \neq r \in \frac{2\pi}{d}\mathbb{Z}^2} (r^2 + m_c^2)^{2\nu} \sum_{\bar{p},\bar{q}} i^{\sigma(\bar{p},\bar{q})} \\ & \cdot \prod_{i=0}^{1} \left((\bar{p}_i - r_i)^{-1} \sin\left(\frac{1}{2}(\bar{p}_i - r_i)d\right)(r_i - \bar{q}_i)^{-1} \sin\left(\frac{1}{2}(r_i - \bar{q}_i)d\right)\right) \\ & \cdot (q^2 + m_c^2)^{-1/2} \qquad \text{if} \qquad p \neq 0, \qquad q \neq 0 \end{aligned} \tag{177}$$

and

$$(e_p \mid He_q) = 0 \qquad \text{if either} \qquad p = 0 \quad \text{or} \quad q = 0\,.$$

Here for given p, $\bar{p}$ runs through points of $\frac{\pi}{d}\mathbb{Z}^2$ obtained from p via reflections in coordinate axes. Similarly for $\bar{q}$.

$\sigma(p, q)$ is an integer. Using the Schwartz inequality for $\sum_{\bar{p},\bar{q}}$ we obtain

$$\| H \|_2^2 = \sum_{\substack{p,q\in\frac{\pi}{d}\mathbb{Z}_+^2 \\ p,q\neq 0}} | (e_p \mid He_q) |^2 \leqslant 0(1)d^{-8} \sum_{\substack{p,q\in\frac{\pi}{d}\mathbb{Z}^2 \\ p,q\neq 0}} (p^2 + m_c^2)^{-1}$$

$$\cdot (q^2+m_c^2)^{-1}\Bigg(\sum_{\substack{r\in\frac{2\pi}{d}\mathbb{Z}^2 \\ r\neq 0}} (r^2+m_c^2)^{2\nu}\Bigg(\prod_i (p_i-r_i)^{-1} \sin\left(\frac{1}{2}(p_i-r_i)d\right)$$

$$\cdot (r_i-q_i)^{-1} \sin\left(\frac{1}{2}(r_i-q_i)d\right)\Bigg)\Bigg)^2 \leqslant 0(1)d^{4-8\nu} \sum_{p,q\in\frac{\pi}{d}\mathbb{Z}^2} ((pd)^2 + m_c^2)^{-1}$$

$$\cdot ((qd)^2 + m_c^2)^{-1}\Bigg(\sum_{r\in\frac{2\pi}{d}\mathbb{Z}^2} ((rd)^2 + m_c^2)^{2\nu} \prod_i (1 + | p_i - r_i | d)^{-1}$$

$$\cdot (1 +| r_i-q_i | d)^{-1}\Bigg)^2 \leqslant 0(1)d^{4-8\nu} \sum_{p,q\in\frac{\pi}{d}\mathbb{Z}^2} ((pd)^2+m_c^2)^{-1}$$

$$\cdot ((qd)^2 + m_c^2)^{-1}\Bigg(\sum_{r\in\frac{2\pi}{d}\mathbb{Z}^2} ((pd)^2 + m_c^2)^\nu(((p-r)d)^2 + m_c^2)^\nu((qd)^2 + m_c^2)^\nu$$

$$\cdot (((q-r)d)^2 + m_c^2)^\nu \prod_i (1 + | p_i - r_i | d)^{-1}(1 + | r_i - q_i | d)^{-1}\Bigg)^2$$

$$\leqslant 0(1)d^{4-8\nu} \sum_{p,q\in\frac{\pi}{d}\mathbb{Z}^2} ((pd)^2 + m_c^2)^{-1+2\nu}((qd)^2 + m_c^2)^{-1+2\nu}$$

$$\cdot \prod_i \Bigg(\sum_{r_i\in\frac{2\pi}{d}\mathbb{Z}} (1 + | p_i - r_i | d)^{-1+2\nu}(1 + | r_i - q_i | d)^{-1+2\nu}\Bigg)^2 . \tag{178}$$

Now it is easy to see that

$$\sum_{r_i\in\frac{2\pi}{d}\mathbb{Z}} (1 + | p_i - r_i | d)^{-1+2\nu}(1 + | r_i - q_i | d)^{-1+2\nu} \leqslant 0(1)(1 + | p_i - q_i | d)^{-\gamma} \qquad \text{if} \qquad \gamma < 1 - 4\nu\,. \tag{179}$$

Hence

$$\| \mathrm{H} \|_2^2 \leqslant 0(1) d^{4-8\nu} \sum_{p,q \in \frac{\pi}{d}\mathbb{Z}^2} ((pd)^2 + m_c^2)^{-1+2\nu} ((qd)^2 + m_c^2)^{-1+2\nu}$$

$$\cdot \prod_i (1 + | p_i - q_i | d)^{-2r}$$

$$\leqslant 0(1) d^{4-8\nu} \Big(\sum_{p \in \frac{\pi}{d}\mathbb{Z}^2} ((pd)^2 + m_c^2)^{-2+4\nu} \Big) \Big(\sum_{p \in \frac{\pi}{d}\mathbb{Z}} (1 + | p | d)^{-2\gamma} \Big)^2, \quad (180)$$

where we have used the discrete version of the Hausdorff-Yang theorem.

We conclude that

$$\| \mathrm{H} \|_2 \leqslant 0(1) d^{3/2} \quad (181)$$

which gives (175). □

Gathering (161), (162), (169), (170), (174) and (175) gives

$$\| \delta(\mathrm{P}^2 + m_c^2)_\Delta^{1/60} \Lambda \mathrm{C}_{m_c}(\mathrm{N})^{1/2} \|_4 \leqslant 0(1) d^{-1/10} \quad (182)$$

and

$$\| \mathrm{L}'_\Delta \|_2 \leqslant 0(1) d^{-1/5}. \quad (183)$$

(160) and (183) bound the second factor on the right hand side of (155). To estimate the first factor notice that in virtue of (151) and (154)

$$\exp\left[\frac{1}{2} (g^\Lambda \mid \mathrm{L}_\Delta g^\Lambda)_{\mathrm{L}^2} \right] \leqslant \exp\left[(2\pi)^{-2} \beta \int_\Delta (\delta g^\Lambda)^2 \right]. \quad (184)$$

By (158)

$$\exp\left[\frac{1}{2} q (g^\Lambda \mid \mathrm{L}_\Delta \Lambda \mathrm{C}_{m_c}(\mathrm{N})^{1/2} (1 - q\mathrm{L}'_\Delta)^{-1} \mathrm{C}_{m_c}(\mathrm{N})^{1/2} \Lambda \mathrm{L}_\Delta g^\Lambda)_{\mathrm{L}^2} \right]$$

$$\leqslant \exp\left[q \, \| \mathrm{L}_\Delta \Lambda \mathrm{C}_{m_c}(\mathrm{N})^{1/2} \|^2 \int_\Delta (\delta g^\Lambda)^2 \right]$$

$$\leqslant \exp\left[0(1) d^{-1/5} \int_\Delta (\delta g^\Lambda)^2 \right] \quad (185)$$

since

$$\| \mathrm{L}_\Delta \Lambda \mathrm{C}_{m_c}(\mathrm{N})^{1/2} \|_4 \leqslant 0(1) d^{-1/10}$$

in virtue of (154), (152) and (182).

Inserting (160), (183), (184) and (185) to (155) we obtain

$$\| \mathscr{L}_{6,\Delta,\mathrm{N}}(t) \|_{\mathrm{L}^q(\mathrm{N})} \leqslant \exp\left[0(1) d^{-1/5} \int_\Delta (\delta g^\Lambda)^2 + 0(1) d^{-2/5} \right] \quad (186)$$

for d sufficiently small and $m \geqslant m_0(\lambda, d)$.

Now on Δ

$$| \delta g^\Lambda | = | \delta(g\Lambda + \xi_+(1-\Lambda)) | \leqslant | \delta((g-h)\Lambda) | + | (\xi_+ - h)\delta\Lambda |$$

$$\leqslant | \delta(g-h) | + 0(1) d (| g-h |_\Delta + | \xi_+ - h |), \quad (187)$$

where we have used the mean value theorem and uniform boundedness of Λ and its first derivatives. The $|\xi_+ - h|$ term appears only if Λ is not constant on Δ. Hence

$$\int_\Delta (\delta g^\Lambda)^2 \leqslant 0(1) \int_\Delta (\delta g)^2 + 0(1)d^2 \int_\Delta (g-h)^2 + 0(1)d^2 \int_\Delta (\xi_+ - h)^2 . \quad (188)$$

But

$$\int_\Delta (\delta g)^2 \leqslant 2d^2 \int_\Delta (\nabla g)^2 . \quad (189)$$

Indeed, for $x \in \Delta$

$$| \delta g(x) | = d^{-2} \left| \int_\Delta dy(g(x) - g(y)) \right| \leqslant d^{-2} \int_\Delta dy \left(\left| \int_{y^0}^{x^0} ds \frac{\partial g}{\partial s}(s, x^1) \right| \right.$$
$$\left. + \left| \int_{y^1}^{x^1} dt \frac{\partial g}{\partial s}(y^0, t) \right| \right) \leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}d} ds \left| \frac{\partial g}{\partial s}(s, x^1) \right| + d^{-1} \int_\Delta dy^0 dt \left| \frac{\partial g}{\partial t}(y^0, t) \right|$$
$$\leqslant d^{1/2} \left(\int_{-\frac{1}{2}d}^{\frac{1}{2}d} ds \left| \frac{\partial g}{\partial s}(s, x^1) \right|^2 \right)^{1/2} + \left(\int_\Delta dy^0 dt \left| \frac{\partial g}{\partial t}(y^0, t) \right|^2 \right)^{1/2} \quad (190)$$

and (190) yields (189).

(145), (186), (188) and (189) give

$$\| \mathscr{Z}_6(t) \|_{L^2(\tau)} \leqslant \exp \left[0(1)d^{-12/5} | Z_t | + 0(1)d^{9/5} \int_Z (\nabla g)^2 \right.$$
$$\left. + 0(1)d^{9/5} \int_Z (g - h)^2 + 0(1)d^{9/5} \int_{Z^b} (\xi_+ - h)^2 \right], \quad (191)$$

where

$$Z^b := \bigcup_{\substack{\Delta \subset Z \\ \Lambda \restriction_\Delta \text{ is not constant}}} \Delta .$$

Now

$$d(\xi_+ - \xi_-)^2 | \Sigma \cap Z | \leqslant 0(1) F_1(Z, \Sigma) , \quad (192)$$

where $| \Sigma \cap Z |$ is the length of countours in Z separating the + and − regions. This is the second inequality of Proposition 2.4.1 of [*13*] adapted to the d-scale. Since the number of d-squares in Z^- on which Λ is neither identically zero nor 1 is bounded by $0(1)d^{-2} | \Sigma \cap Z |$,

$$d^{9/5} \int_{Z^b} (\xi_+ - h)^2 \leqslant 0(1)d^{9/5}\xi_+^2 | \Sigma \cap Z | \leqslant 0(1)d^{4/5} F_1(Z, \Sigma) , \quad (193)$$

where we have also used (192).

With (193), (191) reads

$$\| \mathscr{Z}_6(t) \|_{L^{q(\tau)}} \leqslant \exp\, [0(1)d^{4/5}F_1(Z, \Sigma) + 0(1)d^{-12/5} \mid Z_t \mid]. \quad (194)$$

Summarizing, we have proven.

LEMMA V.16. — Let $1 \leqslant q < \infty$ be fixed. Choose d sufficiently small. Then for all λ, $m \geqslant m_0(\lambda, d)$

$$\| \mathscr{Z}_6(t) \|_{L^{q(\tau)}} \leqslant \exp\left[\frac{1}{30} F_1(Z, \Sigma) + 0(1) \mid Z_t \mid\right]. \quad (195)$$

□

(188), (189) and (193) also prove

LEMMA V.17.

$$\int_Z (\delta g^\Lambda)^2 \leqslant 0(1)dF_1(Z, \Sigma). \quad (196)$$

□

We pass to estimation of the $\mathscr{Z}_7(t)$ term, see (33).This term is not pure Gaussian but we shall reduce estimation of it to estimation of a Gaussian term.

Suppose that $\xi_+ \geqslant 1$ (which holds for $m \geqslant m_0(\lambda)$) and denote by $\chi_1^\pm, \chi_2^\pm, \chi_3^\pm$ the characteristic functions of the sets $[\mp \xi_+^{1/2}, 0]$, $[\mp 2\xi_+, \mp \xi_+^{1/2}[$, $]\mp \infty, \mp 2\xi_+[\cup]0, \pm \infty[$ respectively. Write

$$\begin{aligned} \mathscr{Z}_7(t) &= \prod_{\Delta \subset Z_t \cap Z^\pm} (\chi_1^\pm(t(\Delta)\psi_\Delta) + \chi_2^\pm(t(\Delta)\psi_\Delta) + \chi_3^\pm(t(\Delta)\psi_\Delta))\mathscr{Z}_{7,\Delta}(t) \\ &= \sum_{\substack{Z_i^\pm \subset Z_t \cap Z^\pm \\ i=1,2,3}} \left(\prod_{\Delta \subset Z_1^\pm} \chi_1^\pm(t(\Delta)\psi_\Delta)\mathscr{Z}_{7,\Delta}(t)\right)\left(\prod_{\Delta \subset Z_2^\pm} \chi_2^\pm(t(\Delta)\psi_\Delta)\mathscr{Z}_{7,\Delta}(t)\right) \\ &\cdot \left(\prod_{\Delta \subset Z_3^\pm} \chi_3^\pm(t(\Delta)\psi_\Delta)\mathscr{Z}_{7,\Delta}(t)\right), \end{aligned} \quad (197)$$

where

$$0 \leqslant \mathscr{Z}_{7,\Delta}(t) = \exp\left[\frac{1}{2} \operatorname{Tr}\left(A_\Delta^* A_\Delta - A_\Delta'^* A_\Delta'\right) - \frac{1}{2} t(\Delta)^2 \eta d^2 \psi_\Delta^2\right]. \quad (198)$$

$$\begin{aligned} \operatorname{Tr}\left(A_\Delta^* A_\Delta - A_\Delta'^* A_\Delta'\right) &= t(\Delta)^2\lambda^2 \operatorname{Tr}\left(\overline{K}_\Delta^{\eta_\pm}(\delta\psi)^*\overline{K}_\Delta^{\eta_\pm}(\delta\psi)\right. \\ &\quad \left. - \overline{K}_\Delta^{\xi_\pm}(\delta\psi)^*\overline{K}_\Delta^{\xi_\pm}(\delta\psi)\right) = 2t(\Delta)^2\lambda^4(\xi_\pm^2 - \eta_\pm^2) \operatorname{Tr}\left(\overline{D}_\Delta^{-1/2}(\delta\psi)\right. \\ &\quad \left. \cdot (P^2 + m^2 + \lambda^2\eta_\pm^2)_\Delta^{-1}\overline{D}_\Delta^{-1}(\delta\psi)\overline{D}_\Delta^{-1/2}\right). \end{aligned} \quad (199)$$

Since on the support of $\chi_3^{\mp}(t(\Delta)\psi_\Delta)\xi_\pm^2 - \eta_\pm^2 < 0$,

$$0 \leqslant \chi_3^{\mp}(t(\Delta)\psi_\Delta)\mathscr{X}_{7,\Delta}(t) \leqslant 1 . \tag{200}$$

On the support of $\chi_1^{\mp}(t(\Delta)\psi_\Delta)$

$$0 \leqslant \xi_\pm^2 - \eta_\pm^2 = -2t(\Delta)\psi_\Delta\xi_\pm - t(\Delta)^2\psi_\Delta^2 \leqslant 2\xi_+^{3/2} , \tag{201}$$

see (16).

On the support of $\chi_2^{\mp}(t(\Delta)\psi_\Delta)$

$$0 \leqslant \xi_\pm^2 - \eta_\pm^2 \leqslant \xi_\pm^2 \tag{202}$$

and

$$t(\Delta)^2\psi_\Delta^2 \geqslant \xi_+ . \tag{203}$$

Moreover

$$\begin{aligned} &\operatorname{Tr}\left(\overline{\mathrm{D}}_\Delta^{-1/2}(\delta\psi)(\mathrm{P}^2 + m^2 + \lambda^2\eta_\pm^2)_\Delta^{-1}\overline{\mathrm{D}}_\Delta^{-1}(\delta\psi)\overline{\mathrm{D}}_\Delta^{-1/2}\right) \\ &\qquad \leqslant \operatorname{Tr}\left(\mathrm{D}_\Delta^{-1/2}(\delta\psi)\mathrm{D}_\Delta^{-3}(\delta\psi)\mathrm{D}_\Delta^{-1/2}\right) =: (2\pi)^{-2}(\delta\psi \mid \mathrm{H}_\Delta(\mathrm{P})_\Delta\delta\psi)_{\mathrm{L}^2} , \end{aligned} \tag{204}$$

where

$$\mathrm{H}_\Delta(k) := (2\pi)^2 d^{-2} \sum_{q \in \frac{2\pi}{d}\mathbb{Z}^2} (q^2 + m^2)^{-3/2}((q + k)^2 + m^2)^{-1/2} . \tag{205}$$

LEMMA V.18. — There exists $\beta > 0$ such that for each d and $m \geqslant m_0(d)$

$$0 \leqslant \mathrm{H}_\Delta(k) \leqslant \beta m^{-3/2}(k^2 + m^2)^{-1/4} . \tag{206}$$

Proof. — $\frac{1}{2}m^2(k^2+m^2) \leqslant m^2q^2 + m^2(q+k)^2 + m^4 \leqslant (q^2+m^2)((q+k)^2+m^2)$.

Hence

$$\begin{aligned} &0 \leqslant \mathrm{H}_\Delta(k) \\ &\leqslant 2^{1/4}m^{-1/2}(k^2+m^2)^{-1/4} \sum_{q \in \frac{2\pi}{d}\mathbb{Z}^2} (2\pi)^2 d^{-2}(q^2+m^2)^{-5/4}((q+k)^2+m^2)^{-1/4} \\ &\qquad \leqslant 2^{1/4}m^{-1}(k^2+m^2)^{-1/4} \sum_q (2\pi)^2 d^{-2}(q^2+m^2)^{-5/4} . \end{aligned} \tag{207}$$

Now

$$\int dq(q^2 + m^2)^{-5/4} = 4\pi m^{-1/2}$$

and by the mean value theorem (see (62))

$$\left| \int dq(q^2 + m^2)^{-5/4} - \sum_q \left(\frac{2\pi}{d}\right)^2 (q^2 + m^2)^{-5/4} \right| \leqslant 0(1)d^{-1}m^{-3/2} . \tag{208}$$

Hence

$$H_\Delta(k) \leqslant 8\pi m^{-3/2}(k^2 + m^2)^{-1/4}$$

for $m \geqslant m_0(d)$. □

With use of (206) it follows that

$$\begin{aligned}(2\pi)^{-2}(\delta\psi \mid H_\Delta(P)_\Delta \delta\psi)_{L^2} &\leqslant (2\pi^2)^{-1}(\delta(\varphi\Lambda) \mid H_\Delta(P)_\Delta \delta(\varphi\Lambda))_{L^2} \\ &+ (2\pi^2)^{-1}(\delta g^\Lambda \mid H_\Delta(P)_\Delta \delta g^\Lambda)_{L^2} \leqslant (2\pi^2)^{-1}(\delta(\varphi\Lambda) \mid H_\Delta(P)_\Delta \delta(\varphi\Lambda))_{L^2} \\ &+ (2\pi^2)^{-1}\beta m^{-2}\int_\Delta (\delta g^\Lambda)^2 . \end{aligned} \tag{209}$$

(198), (199), (201), (204) and (209) yield

$$0 \leqslant \chi_1^\pm(t(\Delta)\psi_\Delta \mathscr{Z}_{7,\Delta}(t) \leqslant \exp\left[\pi^{-2}t(\Delta)^2\lambda^4\xi_+^{3/2} \cdot (\delta(\varphi\Lambda) \mid H_\Delta(P)_\Delta\delta(\varphi\Lambda))_{L^2} + 0(1)\xi_+^{-1/2}\int_\Delta (\delta g^\Lambda)^2\right]. \tag{210}$$

In obtaining (210) we also used the fact that $\lambda^4\xi_+^2 m^{-2}$ is bounded uniformly in λ and m. In turn (198), (199), (202)-(204), (209) give

$$\begin{aligned}0 \leqslant \chi_2^\pm(t(\Delta)\psi_\Delta)\mathscr{Z}_{7,\Delta}(t) &\leqslant \exp\Big[(2\pi^2)^{-1}t(\Delta)^2\lambda^4\xi_+^2(\delta(\varphi\Lambda) \mid H_\Delta(P)_\Delta\delta(\varphi\Lambda))_{L^2} \\ &- \frac{1}{2}\eta d^2\xi_+ + 0(1)\int_\Delta (\delta g^\Lambda)^2\Big]. \end{aligned} \tag{211}$$

In virtue of (197) we obtain with use of (200), (210) and (211)

$$\begin{aligned}\mathscr{Z}_7(t) \leqslant \sum_{Z_i^\pm \subset Z_t \cap Z^\pm} &\left(\prod_{\Delta \subset Z_1^\pm} \exp\left[\pi^{-2}t(\Delta)^2\lambda^4\xi_+^{3/2}(\delta(\varphi\Lambda) \mid H_\Delta(P)_\Delta\delta(\varphi\Lambda))_{L^2}\right]\right) \\ &\cdot\left(\prod_{\Delta \subset Z_2^\pm} \exp\left[(2\pi^2)^{-1}t(\Delta)^2\lambda^4\xi_+^2(\delta(\varphi\Lambda) \mid H_\Delta(P)_\Delta\delta(\varphi\Lambda))_{L^2}\right]\right) \\ &\cdot \exp\left[-\frac{1}{2}\eta\xi_+ \mid Z_2^+ \cup Z_2^- \mid + 0(1)\int_{Z_t}(\delta g^\Lambda)^2\right] \end{aligned} \tag{212}$$

for $m \geqslant m_0(\lambda, d)$.

This bound reduces estimation of $\| \mathscr{Z}_7(t) \|_{L^{q(\tau)}}$ to estimation of Gaussian integrals. To separate different squares we Wick order the expressions in the exponent and bound them by conditioning with respect to the Gaussian measure with Neumann boundary conditions on $\partial\Delta$. With the Wick ordering redone and the Gaussian integrals obtained this way explicitely computed this yields

$$\| \mathscr{L}_7(t) \|_{L^q(\tau)} \leqslant \sum_{Z_t^{\pm}} \Big[\prod_{\Delta \subset Z_1^{\pm}} (\det (1 - 2\pi^{-2} t(\Delta)^2 q \lambda^4 \xi_+^{3/2} C_{m_c}(N)^{1/2} \Lambda \cdot \delta H_\Delta(P)_\Delta \Lambda C_{m_c}(N)^{1/2}))^{-1/2q} \Big]$$
$$\cdot \Big[\prod_{\Delta \subset Z_2^{\pm}} (\det (1 - \pi^{-2} t(\Delta)^2 q \lambda^4 \xi_+^2 C_{m_c}(N)^{1/2} \Lambda \delta H_\Delta(P)_\Delta \Lambda C_{m_c}(N)^{1/2}))^{-1/2q} \Big]$$
$$\cdot \Big[\prod_{\Delta \subset Z_1^{\pm}} \exp \big[\pi^{-2} t(\Delta)^2 \lambda^4 \xi_+^{3/2} \operatorname{Tr} (\Lambda \delta H_\Delta(P)_\Delta \Lambda (C_{m_c}(\tau) - C_{m_c}(N))) \big] \Big]$$
$$\cdot \Big[\prod_{\Delta \subset Z_2^{\pm}} \exp \big[(2\pi^2)^{-1} t(\Delta)^2 \lambda^4 \xi_+^2 \operatorname{Tr} (\Lambda \delta H_\Delta(P)_\Delta \Lambda (C_{m_c}(\tau) - C_{m_c}(N))) \big] \Big]$$
$$\cdot \exp \Big[- 0(1) \xi_+ \, | Z_2^+ \cup Z_2^- | + 0(1) \int_{Z_t} (\delta g^\Lambda)^2 \tag{213}$$

provided the operators under the determinants are positive. This holds if for example

$$\pi^{-2} t(\Delta)^2 \lambda^4 \xi_+^2 C_{m_c}(N)^{1/2} \Lambda \delta H_\Delta(P)_\Delta \Lambda C_{m_c}(N)^{1/2} \leqslant \frac{1}{2q}. \tag{214}$$

But (see (206) and (159))

$$C_{m_c}(N)^{1/2} \Lambda \delta H_\Delta(P)_\Delta \Lambda C_{m_c}(N)^{1/2} \leqslant \beta m^{-2} C_{m_c}(N)^{1/2} \chi_\Delta \Lambda \delta \Lambda \chi_\Delta C_{m_c}(N)^{1/2} \leqslant 0(1) m^{-2} d^2 . \tag{215}$$

Using the fact that $\lambda^4 \xi_+^2 m^{-2}$ is uniformly bounded we conclude that (214) holds for sufficiently small d. Similarly as in (160) we can bound the determinants in negative powers by exponential of the trace norms, provided (214) holds. Thus

$$\| \mathscr{L}_7(t) \|_{L^q(\tau)} \leqslant \sum_{Z_t^{\pm}} \Big(\prod_{\Delta \subset Z_1^{\pm}} \exp \big[0(1) \lambda^4 \xi_+^{3/2} \| C_{m_c}(N)^{1/2} \Lambda \delta H_\Delta(P)_\Delta \Lambda C_{m_c}(N)^{1/2} \|_1 \big] \Big)$$
$$\cdot \Big(\prod_{\Delta \subset Z_2^{\pm}} \exp \big[0(1) \lambda^4 \xi_+^2 \| C_{m_c}(N)^{1/2} \Lambda \delta H_\Delta(P)_\Delta \Lambda C_{m_c}(N)^{1/2} \|_1 \big] \Big)$$
$$\cdot \Big(\prod_{\Delta \subset Z_1^{\pm}} \exp \big[0(1) \lambda^4 \xi_+^{3/2} \operatorname{Tr} (\Lambda \delta H_\Delta(P)_\Delta \Lambda (C_{m_c}(\tau) - C_{m_c}(N))) \big] \Big)$$
$$\cdot \Big(\prod_{\Delta \subset Z_2^{\pm}} \exp \big[0(1) \lambda^4 \xi_+^2 \operatorname{Tr} (\Lambda \delta H_\Delta(P)_\Delta \Lambda (C_{m_c}(\tau) - C_{m_c}(N))) \big] \Big)$$
$$\cdot \exp \Big[- 0(1) \xi_+ \, | Z_2^+ \cup Z_2^- | + 0(1) \int_{Z_t} (\delta g^\Lambda)^2 . \tag{216}$$

Using Lemma V.18 it is easy to bound the trace norms and the traces. Together with Lemma V.17 this gives

LEMMA V.19. — Fix $1 \leqslant q < \infty$. If d is sufficiently small then for $m \geqslant m_0(\lambda, d)$

$$\| \mathscr{Z}_7(t) \|_{L^{q(\tau)}} \leqslant \exp\left[\frac{1}{30} F_1(Z, \Sigma) + 0(1) | Z_t |\right]. \tag{217}$$

□

In other terms which we shall estimate now, d will be fixed to meet the requirements stated above and we shall not trace the d-dependence of the constants.

The next term is $Z_8(t)$, see (6) and (34).

$$\begin{aligned} D_1 &= \frac{1}{2}\lambda^2 \int \Big(\sum_{\Delta \subset Z} B_\Delta^{\xi+} - B^{\xi+}(s) \Big) : \psi^2 :_\tau \\ &= \frac{1}{2}\lambda^2 \sum_{\Delta \subset Z} \operatorname{Tr}\big(K_\Delta(1) K_\Delta^{\xi+}(: \psi^2 :_\tau) - K(s, 1) K^{\xi+}(s, : \psi^2 :_\tau \chi_\Delta) \big). \end{aligned} \tag{218}$$

But

$$\begin{aligned} & | \operatorname{Tr}(K_\Delta(1) K_\Delta^{\xi+}(f) - K(s, 1) K^{\xi+}(s, f\chi_\Delta)) | \\ & \quad \leqslant | \operatorname{Tr}((K_\Delta(1) - K(s, 1)) K_\Delta^{\xi+}(f) | + | \operatorname{Tr}(K(s, 1)(K_\Delta^{\xi+}(f) - K^{\xi+}(s, f\chi_\Delta))) | \\ & \quad \leqslant \sum_{\Delta', \Delta''} \| \chi_{\Delta'}(K_{\Delta''}(1) - K(\chi_{\Delta''}))\chi_\Delta \|_p \, \| K_\Delta^{\xi+}(f) \|_{p'} \\ & \quad + \sum_{\Delta', \Delta''} \| \chi_{\Delta'}(K(\chi_{\Delta''}) - K(s, \chi_{\Delta''}))\chi_\Delta \|_p \, \| K_\Delta^{\xi+}(f) \|_{p'} \\ & \quad + \sum_{\Delta', \Delta'', \Delta''', \Delta^{IV}} \| \chi_{\Delta'} K(s, \chi_{\Delta''})\chi_{\Delta'''} \|_{p'} \, \| \chi_{\Delta'''}(K_\Delta^{\xi+}(f) - K^{\xi+}(s, f\chi_\Delta))\chi_{\Delta^{IV}} \|_p \\ & \quad \leqslant 0(1) m^{-1/4+\alpha} S^\beta(f\chi_\Delta), \end{aligned}$$

where we have taken $p < 2$ but close to 2 and used Proposition A.I.3, Corollaries A.I.3,4, Lemma V.8 and the relations

$$\begin{aligned} & \sum_{\Delta', \Delta''} \| \chi_{\Delta'}(K_{\Delta''}(1) - K(\chi_{\Delta''}))\chi_\Delta \|_p \leqslant \| K_\Delta(1) - \chi_\Delta K(\chi_\Delta)\chi_\Delta \|_p \\ & \qquad + \sum_{\substack{\Delta', \Delta'' \\ \Delta' \text{ or } \Delta'' \neq \Delta}} \| \chi_{\Delta'} K(\chi_{\Delta''})\chi_\Delta \|_p \leqslant 2 \sum_{\substack{\Delta', \Delta'' \\ \Delta' \text{ or } \Delta'' \neq \Delta}} \| \chi_{\Delta'} K(\chi_{\Delta''})\chi_\Delta \|_p , \\ & \qquad \| K_\Delta^{\xi+}(f) \|_{p'} \leqslant \sum_{\Delta', \Delta''} \| \chi_{\Delta'} K^{\xi+}(f\chi_\Delta)\chi_{\Delta''} \|_{p'} , \end{aligned} \tag{219}$$

which easily follow in virtue of (130).

Hence

$$\exp\,[-\mathrm{D}_1] \leqslant \exp\left[0(1)\lambda^2 m^{-1/4+\alpha}\sum_{\Delta\subset Z} S^\beta(:\psi^2:_\tau\chi_\Delta)\right]. \tag{220}$$

But

$$S^\beta(:\psi^2:_\tau\chi_\Delta)\leqslant S^\beta(:(\varphi\Lambda)^2:_\tau\chi_\Delta)+2S^\beta(\varphi\Lambda(g^\Lambda-h)\chi_\Delta)+0(1)\xi_+^2\,. \tag{221}$$

For $\Delta \subset Z^0$

$$S^\beta(\varphi\Lambda(g^\Lambda - h)\chi_\Delta) \leqslant \frac{1}{2}\xi_+^2 + \frac{1}{2}S^\beta(\varphi\Lambda\xi_+^{-1}(g^\Lambda - h)\chi_\Delta)^2\,. \tag{222}$$

Using the checkerboard estimate to separate fields in different squares Δ and Lemma A.II.5 we obtain from (220)-(222).

LEMMA V.20. — Given $q < \infty$ and $\alpha > 0$,

$$\|\mathscr{Z}_8(t)\|_{L^q(\tau)} \leqslant \exp\,[0(1)m^{7/4+\alpha}\,|Z^0| + 0(1)m^{-1/4+\alpha}\,|Z|\,] \tag{223}$$

provided $m \geqslant m_0(\lambda)$. □

$Z_9(t)$ term, see (7) and (35), is field independent

$$|\mathrm{D}_2| = \frac{1}{2}\lambda^2\left|\int \mathrm{Tr}\left(\sum_{\Delta\subset Z} K_\Delta^{\xi_+}(\varphi\Lambda)^2 - K^{\xi_+}(s,\varphi\Lambda\chi_Z)^2\right)d\mu_{m_c}(\tau)\right|$$

$$\leqslant \frac{1}{2}\lambda^2 \sum_{\Delta,\Delta',\Delta''\subset Z}\int\Big(\|\chi_{\Delta'}(K_\Delta^{\xi_+}(\varphi\Lambda) - K^{\xi_+}(s,\varphi\Lambda\chi^\Delta))\chi_{\Delta''}\|_p\;\|K_{\Delta'}^{\xi_+}(\varphi\Lambda)\|_{p'}\Big)d\mu_{m_c}(\tau)$$

$$+\frac{1}{2}\lambda^2 \sum_{\Delta,\Delta',\Delta'',\Delta''',\Delta^{\mathrm{IV}}\subset Z}\int\Big(\|\chi_{\Delta'}K^{\xi_+}(s,\varphi\Lambda\chi_\Delta)\chi_{\Delta''}\|_{p'}\;\|\chi_{\Delta''}(K_{\Delta'''}^{\xi_+}(\varphi\Lambda) - K^{\xi_+}(s,\varphi\Lambda\chi_{\Delta'''}))\chi_{\Delta^{\mathrm{IV}}}\|_p\Big)d\mu_{m_c}(\tau)$$

$$\leqslant 0(1)\lambda^2 m^{-1/4+\alpha}\sum_{\Delta,\Delta'''\subset Z}\exp\,[-d(\Delta,\Delta''')]\int S^\beta(\varphi\Lambda\chi_\Delta)S^\beta(\varphi\Lambda\chi_{\Delta'''})d\mu_{m_c}(\tau), \tag{224}$$

where we have used (219), Corollary A.I.4 and Lemma V.8. In virtue of Lemma A.II.3, (224) yields

$$|\mathrm{D}_2| \leqslant 0(1)\lambda^2 m^{-1/4+\alpha}\sum_{\Delta\subset Z}\int S^\beta(\varphi\Lambda\chi_\Delta)^2 d\mu_{m_c}(\tau)$$

$$\leqslant 0(1)\lambda^2 m^{-1/4+\alpha}\,|Z|\,\|S^\beta(\varphi\Lambda\chi_\Delta)\|^2_{L^8(\tau)} \leqslant 0(1)\lambda^2 m^{-1/4+\alpha}\,|Z|\,. \tag{225}$$

We have proven

LEMMA V.21. — Given $\alpha > 0$

$$\| \mathscr{L}_9(t) \|_{L^\infty(\tau)} \leqslant \exp\, [0(1)m^{-1/4+\alpha} | Z |], \tag{226}$$

provided $m \geqslant m_0(\lambda')$. □

The terms with E_1 and E_2, see (III.6-8), (36) and (37) are estimated essentially the same way as their s-derivatives, see Chapter 4, except that we must be more careful in choosing the trace norms when applying the Hölder inequality in order to obtain the right powers of m, which were not traced for in the estimation of the s-derivatives.

We start with the E_1 term

$$\begin{aligned} | E_1 | &= \lambda^2 \left| \mathrm{Tr}\, (K(s, h)K^h(s, \psi)) - \int B^{\xi_+}(h)h\psi \right| \\ &= \lambda^2 \, | \mathrm{Tr}\, (K(s, h)K^h(s, \psi) - K(s, 1)K^{\xi_+}(s, h\psi)) | \\ &= \lambda^2 \left| \sum_{+,-} \mathrm{Tr}\, (K(s, h)K^h(s, \psi\chi_{Z^\pm}) - K(s, 1)K^{\xi_\pm}(s, h\psi\chi_{Z^\pm})) \right|, \end{aligned} \tag{227}$$

where we have used the $\varphi \to -\varphi$ symmetry.

Thus

$$\begin{aligned} | E_1 | \leqslant{} & \lambda^2\xi_+ \sum_{+,-} \| K(s, \chi_{Z^\pm})K^h(s, \psi\chi_{Z^\pm}) - K(s, \chi_{Z^\pm})K^{\xi_\pm}(s, \psi\chi_{Z^\pm}) \|_1 \\ & + \lambda^2\xi_+ \sum_{\substack{i,j=\pm \\ i\neq j}} (\| K(s, \chi_{Z^j})K(s, \psi\chi_{Z^i}) \|_1 + \| K(s, \chi_{Z^j})K^{\xi_i}(s, \psi\chi_{Z^i}) \|_1). \end{aligned} \tag{228}$$

The first term on the right hand side of (228) is equal to

$$\lambda^2\xi_+^2 \sum_{+,-} \| K(s, \chi_{Z^\pm})K^h(s, \chi_{Z^\pm})K^{\xi_\pm}(s, \psi\chi_{Z^\pm}) \|_1 . \tag{229}$$

Inserting localizations, using the Hölder inequality with $p_1 = 2$, $p_2 = 4$, $p_3 = 4$ where p_1 trace-norm bounds the operator K with non-coinciding localizations we obtain for the term in question the bound

$$0(1)m^{3/4+\alpha} \sum_{\substack{\Delta\subset Z^+ \\ \Delta'\subset Z^-}} \exp\, [-d(\Delta, \Delta')](S^\beta(\psi\chi_\Delta) + S^\beta(\psi\chi_{\Delta'})) . \tag{230}$$

The same way we show that the second and the third term on the right hand side of (228) are also bounded by (230). Hence

$$| E_1 | \leqslant 0(1)m^{7/4+\alpha} | Z^0 | + 0(1)m^{3/4+\alpha} \sum_{\substack{\Delta \subset Z^+ \\ \Delta' \subset Z^-}} \exp [- d(\Delta, \Delta')]$$

$$\cdot (S^\beta(\varphi\Lambda\chi_\Delta) + S^\beta(\varphi\Lambda\chi_{\Delta'})) \leqslant 0(1)m^{7/4+\alpha} | Z^0 |$$

$$+ 0(1) \sum_{\substack{\Delta \subset Z^+ \\ \Delta' \subset Z^-}} \exp [- d(\Delta, \Delta')](m^{7/4+\alpha} + m^{-1/4+\alpha}(S^\beta(\varphi\Lambda\chi_\Delta) + S^\beta(\varphi\Lambda\chi_{\Delta'}))^2)$$

$$\leqslant 0(1)m^{7/4+\alpha} | Z^0 | + 0(1)m^{7/4+\alpha} \sum_{\substack{\Delta \subset Z^+ \\ \Delta' \subset Z^- \sim Z^0}} \exp [- d(\Delta, \Delta')]$$

$$+ 0(1)m^{-1/4+\alpha} \sum_{\Delta \subset Z} S^\beta(\varphi\Lambda\chi_\Delta)^2 \leqslant 0(1)m^{7/4+\alpha} | Z^0 |$$

$$+ 0(1)m^{-1/4+\alpha} | Z | + 0(1)m^{-1/4+\alpha} \sum_{\Delta \subset Z} S^\beta(\varphi\Lambda\chi_\Delta)^2 . \tag{231}$$

In obtaining the fourth inequality we have used the fact that for $\Delta \subset Z^+$ and $\Delta \subset Z^- \sim Z^0$ $d(\Delta, \Delta') > L = (\ln m)^2$. (231) given, Lemma A.II.5 yields immediately

LEMMA V.22. — Fix $q < \infty$ and $\alpha > 0$. Then for $m \geqslant m_0(\lambda)$

$$\| \mathscr{Z}_{10}(t) \|_{L^q(\tau)} \leqslant \exp [0(1)m^{7/4+\alpha} | Z^0 | + 0(1)m^{-1/4+\alpha} | Z |] . \tag{232}$$

□

$\mathscr{Z}_{11}(t)$ is estimated in

LEMMA V.23. — Given

$$\| \mathscr{Z}_{11}(t) \|_{L^\infty(\tau)} \leqslant \exp [0(1)m^{7/4+\alpha} | Z^0 | + 0(1)m^{-1/4+\alpha} | Z |] \tag{233}$$

for $m \geqslant m_0(\lambda)$.

Proof. — We proceed as when estimating $Z_{10}(t)$.

First write for $\tilde{\lambda} := \lambda'\lambda^{-1}$

$$\lambda^2 \,|\, \mathrm{Tr}\,(\mathrm{K}^{\tilde{\lambda}h}(s, h\chi_Z)\mathrm{K}(s, h\chi_Z)^2 - \mathrm{K}^{\tilde{\lambda}\xi_+}(s, \xi_+\chi_Z)\mathrm{K}(s, \xi_+\chi_Z)^2) \,|$$
$$= \lambda^2 \left| \sum_{+,-} \mathrm{Tr}\,(\mathrm{K}^{\tilde{\lambda}h}(s, h\chi_{Z^+})\mathrm{K}(s, h\chi_Z)^2 - \mathrm{K}^{\tilde{\lambda}\xi_+}(s, \xi_\pm\chi_{Z^+})\mathrm{K}(s, \xi_\pm\chi_Z)^2) \right|$$
$$\leqslant \sum_{+,-} \lambda^2\xi_+^3 \,\|\, \mathrm{K}^{\tilde{\lambda}h}(s, \chi_{Z^\pm})\mathrm{K}(s, \chi_{Z^\pm}) - \mathrm{K}^{\tilde{\lambda}\xi_\pm}(s, \chi_{Z^\pm})\mathrm{K}(s, \chi_{Z^\pm}) \,\|_1 \,\|\, \mathrm{K}(s, \chi_{Z^\pm}) \,\|$$
$$+ \sum_{\substack{i,j,k=\pm \\ i\neq j \text{ or } j\neq k}} \lambda^2\xi_+^3 (\,\|\, \mathrm{K}^{\tilde{\lambda}h}(s, \chi_{Z^i})\mathrm{K}(s, \chi_{Z^j})\mathrm{K}(s, \chi_{Z^k}) \,\|_1 + \|\, \mathrm{K}^{\tilde{\lambda}\xi_i}(s, \chi_{Z^i})\mathrm{K}(s, \chi_{Z^j})\mathrm{K}(s, \chi_{Z^k}) \,\|_1 \,)$$
$$\leqslant 0(1) m^{7/4+\alpha} \sum_{\substack{\Delta\subset Z^+ \\ \Delta'\subset Z^-}} \exp\,[-\,d(\Delta, \Delta')]\,, \tag{234}$$

where in the first step we have used the $\varphi \to -\varphi$ symmetry and to obtain the last bound we proceeded as when bounding the right hand side of (228) by (230).

Now

$$\sum_{\substack{\Delta\subset Z^+ \\ \Delta'\subset Z^-}} \exp\,[-\,d(\Delta, \Delta')] = \sum_{\substack{\Delta\subset Z^+ \\ \Delta'\subset Z^-\cap Z^0}} \exp\,[-\,d(\Delta, \Delta')]$$
$$+ \sum_{\substack{\Delta\subset Z^+ \\ \Delta'\subset Z^-\sim Z^0}} \exp\,[-\,d(\Delta, \Delta')] \leqslant 0(1)\,|\,Z^0\,| + 0(1) m^{-2}\,|\,Z\,|\,, \tag{235}$$

which together with (234) gives the right bound for the first factor of $\exp\,[-\,E_2]$.

The second term is $\exp\left[-\frac{1}{2}\lambda^2\,\mathrm{Tr}\,(\mathrm{K}(s, h\chi_Z)^2 - \mathrm{K}(s, \xi_+\chi_Z)^2) \right]$ and is still simpler to estimate along the same lines. □

To end Proof of Proposition V.1 we still have to estimate the F terms, see (III.9-13). As far as $\mathscr{X}_{12}(t)$ is concerned, see (38), we repeat the arguments of Proof of Proposition 2.5.1 of [*13*] and obtain

LEMMA V.24. — Given $q < \infty$,

$$\|\, \mathscr{X}_{12}(t) \,\|_{L^q(\tau)} \leqslant \exp\,[0(1) m^{-1}\,|\,Z^0\,|\,] \tag{236}$$

for $m \geqslant m_0$. □

For $\mathscr{X}_{13}(t)$, $\|\, \mathscr{X}_{13}(t) \,\|_{L^{q_{13}(\tau)}}$ is finite provided $q_{13} < q_{13}^0(\eta)$ (q_{13}^0 is independent of d). Indeed, conditioning with respect to the free Gaussian measure and performing explicitely the integration we obtain

$$\| \mathscr{Z}_{13}(t) \|_{L^{q_{13}(\tau)}} \leqslant \exp\left[\frac{1}{2} q_{13}(f \,|\, (-\Delta + m_c^2 - q_{13}L)^{-1} f)_{L^2} - F_2\right] \cdot (\det_2 (1 - q_{13} C_{m_c}^{1/2} L C_{m_c}^{1/2}))^{-1/2q_{13}}, \quad (237)$$

where

$$L := (m_c^2 - \eta) \sum_{\Delta \subset Z} t(\Delta)^2 \Lambda^2 \chi_\Delta \quad (238)$$

and

$$f := [\eta(1 - \Lambda)(h - \xi_+) + (m_c^2 - \eta)(1 - \Lambda^2)(g - \xi_+)]\chi_Z . \quad (239)$$

(237) holds if $C_{m_c}^{1/2} L C_{m_c}^{1/2} < \frac{1}{q_{13}}$. We choose $q_{13} = \left(1 - \frac{1}{3}\eta m_c^{-2}\right)^{-1}$ which gives

$$C_{m_c}^{1/2} L C_{m_c}^{1/2} \leqslant (1 - \eta m_c^{-2}) < \frac{1}{q_{13}} .$$

The argument used in (160) gives

$$(\det_2 (1 - q_{13} C_{m_c}^{1/2} L C_{m_c}^{1/2}))^{-1/(2q_{13})} \leqslant \exp [0(1) \| C_{m_c}^{1/2} \chi_{Z_t} C_{m_c}^{1/2} \|_2^2] \leqslant \exp [0(1) \,|\, Z_t \,|\,] . \quad (240)$$

Moreover

$$\begin{aligned}
&(-\Delta + m_c^2 - q_{13}L)^{-1} \\
&= (m_c^2 - q_{13}L)^{-1/2}[(m_c^2 - q_{13}L)^{-1/2}(-\Delta)(m_c^2 - q_{13}L)^{-1/2} + 1]^{-1} \cdot \\
&\cdot (m_c^2 - q_{13}L)^{-1/2} \leqslant (m_c^2 - q_{13}L)^{-1} \leqslant (m_c^2 - q_{13}(m_c^2 - \eta)\Lambda^2)^{-1} . \quad (241)
\end{aligned}$$

Hence

$$\begin{aligned}
&\exp\left[\frac{1}{2} q_{13}(f \,|\, (-\Delta + m_c^2 - q_{13}L)^{-1} f)_{L^2} - F_2\right] \\
&\leqslant \exp\left[\frac{1}{2} q_{13} m_c^{-2}(1 - q_{13}(1 - \eta m_c^{-2})\Lambda^2)^{-1} \int_Z (\eta(1 - \Lambda)(h - \xi_+)\right. \\
&+ (m_c^2 - \eta)(1 - \Lambda^2)(g - \xi_+))^2 - F_2] \\
&= \prod_{\Delta \subset Z} \exp\left[\int_\Delta \left(\frac{1}{2} q_{13} m_c^{-2}(1 - q_{13}(1 - \eta m_c^{-2})\Lambda^2)^{-1} \eta^2 (1 - \Lambda)^2 (h - \xi_+)^2\right.\right. \\
&+ \frac{1}{2} q_{13} m_c^{-2}(1 - q_{13}(1 - \eta m_c^{-2})\Lambda^2)^{-1}(m_c^2 - \eta)^2(1 - \Lambda^2)^2(g - \xi_+)^2 \\
&+ q_{13} m_c^{-2}(1 - q_{13}(1 - \eta m_c^{-2})\Lambda^2)^{-1}\eta(m_c^2 - \eta)(1 - \Lambda)(1 - \Lambda^2)(h - \xi_+)(g - \xi_+) \\
&- \frac{1}{2}\eta(1 - \Lambda)(g - \xi_+)^2 + \frac{1}{2}\eta(1 - \Lambda)(g - h)^2 - \frac{1}{2}\eta(1 - \Lambda)(h - \xi_+)^2 \\
&\left.- \frac{1}{2}(m_c^2 - \eta)(1 - \Lambda^2)(g - \xi_+)^2\right] . \quad (242)
\end{aligned}$$

The factor on the right hand side of (242) corresponding to $\Delta \subset Z^+$ is

$$\exp\left[\int_\Delta \left(\frac{1}{2} q_{13} m_c^{-2}(1-q_{13}(1-\eta m_c^{-2})\Lambda^2)^{-1}(m_c^2-\eta)^2(1-\Lambda^2)^2(g-\xi_+)^2 - \frac{1}{2}(m_c^2-\eta)(1-\Lambda^2)(g-\xi_+)^2\right)\right] \leqslant 1 \quad (243)$$

since $q_{13}(1 - \eta m_c^{-2}) < 1$.

For $\Delta \subset Z^-$ the estimation is slightly more complicated. The Δ factor is

$$\begin{aligned}
&\exp\left[\int_\Delta \left(2q_{13}m_c^{-2}(1-q_{13}(1-\eta m_c^{-2})\Lambda^2)^{-1}\eta^2(1-\Lambda)^2\xi_+^2\right.\right.\\
&+ \frac{1}{2} q_{13}m_c^{-2}(1-q_{13}(1-\eta m_c^{-2})\Lambda^2)^{-1}(m_c^2-\eta)^2(1-\Lambda^2)^2(g-\xi_+)^2\\
&- 2q_{13}m_c^{-2}(1-q_{13}(1-\eta m_c^{-2})\Lambda^2)^{-1}\eta(m_c^2-\eta)(1-\Lambda)(1-\Lambda^2)\xi_+(g-\xi_+)\\
&-\frac{1}{2}\eta(1-\Lambda)(g-\xi_+)^2 + \frac{1}{2}\eta(1-\Lambda)(g+\xi_+)^2 - 2\eta(1-\Lambda)\xi_+^2\\
&\left.\left.-\frac{1}{2}(m_c^2-\eta)(1-\Lambda^2)(g-\xi_+)^2\right)\right]\\
&\leqslant \exp\left[\int_\Delta \left(2q_{13}m_c^{-2}\eta^2\xi_+^2 - 2q_{13}\eta(1-\eta m_c^{-2})\xi_+(g-\xi_+)\right.\right.\\
&\left.\left.-\frac{1}{2}\eta(g-\xi_+)^2 + \frac{1}{2}\eta(g+\xi_+)^2 - 2\eta\xi_+^2\right)(1-\Lambda)\right]\\
&\leqslant \exp\left[\int_\Delta \frac{1}{2}\varepsilon(g+\xi_+)^2(1-\Lambda)\right]
\end{aligned} \quad (244)$$

for

$$q_{13} = \left(1 - \frac{1}{3}\eta m_c^{-2}\right)^{-1}, \qquad \varepsilon = \frac{4}{3} q_{13}\eta^2 m_c^{-2}. \quad (245)$$

Indeed, for this choice of q_{13} and ε

$$\begin{aligned}
&2q_{13}m_c^{-2}\eta^2\xi_+^2 - 2q_{13}\eta(1-\eta m_c^{-2})\xi_+(g-\xi_+) - \frac{1}{2}\eta(g-\xi_+)^2\\
&+ \frac{1}{2}\eta(g + \xi_+)^2 - 2\eta\xi_+^2 - \frac{1}{2}\varepsilon(g + \xi_+)^2\\
&= \left[2q_{13}m_c^{-2}\eta^2 + 2q_{13}\eta(1 - \eta m_c^{-2}) - 2\eta - \frac{1}{2}\varepsilon\right]\xi_+^2\\
&+ \left[- 2q_{13}\eta(1 - \eta m_c^{-2}) + 2\eta - \varepsilon\right]\xi_+ g - \frac{1}{2}\varepsilon g^2 = -\frac{1}{2}\varepsilon g^2 \leqslant 0,
\end{aligned} \quad (246)$$

which substantiates (244).

Gathering (237), (240), (242)-(244) and noting that $\varepsilon < \frac{4}{11}\eta$ since $\eta < \frac{1}{4} m_c^2$ we obtain

LEMMA V.25. — Given η, $0 < \eta < \frac{1}{4} m_c^2$, there exists $q_{13} > 1$ such that for all d, λ, m

$$\| \mathscr{X}_{13}(t) \|_{L^{q_{13}(\tau)}} \leqslant \exp\left[\frac{4}{11} F_1 + 0(1) | Z_t |\right]. \tag{247}$$

□

This way we have estimated the appropriate norms of all $\mathscr{X}_b^{(K)}(t)$ terms except $\mathscr{X}_{14}(t)$. Gathering Lemmas V.1, 4, 11, 16, 19-22, 24, 25 we obtain

$$\begin{aligned} \| \mathscr{X}^K(t) \|_{L^p(\tau)} \leqslant 0(1)^K \prod_{\Delta \subset Z} 0(1)^{n_\Delta}(n_\Delta\, !)^{0(1)} \\ \cdot \exp\Big[-\frac{1}{2} F_1 + 0(1) m^{7/4+\alpha} | Z^0 | - 0(1)\xi_+^2 | Z' | \\ + 0(1) m^{-1/4+\alpha} | Z_v | + 0(1)\lambda^3 | Z_t | \Big]. \end{aligned} \tag{248}$$

But since $0(1)F_1 \geqslant (\ln m)^{-4}\xi_+^2 | Z^0 |$, see Proposition 2.4.1 of [*13*], (10) follows immediately from (248). This completes Proof of Proposition V.1.

□

With Proposition V.1 proven, Theorem III.1 follows by a cluster expansion argument. Namely we write for z^K (see (1) and (5))

$$z^K = \sum_{Z_1 \subset Z \sim (Z^0 \cup Z')} \int dt_{Z_1} \partial_t^{Z_1} z^K(t_{Z_1}), \tag{249}$$

where Z_1 is built of d-lattice squares,

$$\int dt_{Z_1} = \prod_{\Delta \subset Z_1} \int_0^1 dt(\Delta), \tag{250}$$

$$\partial_t^{Z_1} = \prod_{\Delta \subset Z_1} \frac{\partial}{\partial t(\Delta)}, \tag{251}$$

$$t_{Z_1}(\Delta) = \begin{cases} t(\Delta) & \text{for} \quad \Delta \subset Z_1, \\ 1 & \text{for} \quad \Delta \subset Z^0 \cup Z', \\ 0 & \text{for} \quad \Delta \subset Z \sim (Z^0 \cup Z' \cup Z_1). \end{cases} \tag{252}$$

The t-dependent part of $z^K(t)$ is (see (5))

$$\Lambda^K\Big(1 - \lambda \sum_{\Delta \subset Z^\pm} t(\Delta) K_\Delta^{\xi\pm}(\psi) - B''\Big)^{-1} \det{}_3 \Big(1 - \sum_{\Delta \subset Z^\pm} t(\Delta) K_\Delta^{\xi\pm}(\psi) - B''\Big) \cdot \exp\,[-\mathscr{F}(t)], \tag{253}$$

where

$$\mathcal{F}(t) = \sum_{\Delta \subset Z^\pm \sim (Z^0 \cup Z')} \frac{1}{2} t(\Delta)^2 \Big[\lambda^2 : \mathrm{Tr}\, K_\Delta^{\xi\pm}(\varphi\Lambda)^2 :_\tau - \lambda^2 \int_\Delta B_\Delta^{\xi+} : (\varphi\Lambda)^2 :_\tau - m_c^2 \int_\Delta : (\varphi\Lambda)^2 :_\tau \Big] =: \sum_\Delta \frac{1}{2} t(\Delta)^2 \mathcal{F}_\Delta . \quad (254)$$

In virtue of (249) for $Q^K := \bigwedge_{j=1}^{K} Q_j$

$$\tau_K(z^K Q^K) = \sum_{Z_1} \int dt_{Z_1} \partial_t^{Z_1} \tau_K(z^K(t_{Z_1})Q^K) . \quad (255)$$

We apply the derivatives $\frac{\partial}{\partial t(\Delta)}$ in a fixed order. Each derivative produces either

$$\mathcal{A}_\Delta(t) := \lambda\Big(\lambda \sum_{\Delta' \subset Z^\pm} t(\Delta') K_{\Delta'}^{\xi\mp}(\psi) + B''\Big)^2 K_\Delta^{\xi\pm}(\varphi\Lambda) \quad (256)$$

or

$$\mathcal{E}_\Delta(t) := \lambda\Big(1 + \lambda \sum_{\Delta' \subset Z^\pm} t(\Delta') K_{\Delta'}^{\xi\mp}(\psi) + B''\Big) K_\Delta^{\xi\pm}(\varphi\Lambda) \quad (257)$$

or

$$t(\Delta)\mathcal{F}_\Delta$$

or

differentiates an $\mathcal{A}_{\tilde{\Delta}}(t)$ term produced earlier, compare (III.15).

Denote by $Z_{1,\mathcal{A}}$, $Z_{1,\mathcal{E}}$, $Z_{1,\mathcal{F}}$ and $Z_{1,\mathcal{R}}$ the subsets giving the decomposition of Z_1 corresponding to the choice of one of the described above types of action for each derivative. For given decomposition put

$$a := d^{-2} | Z_{1,\mathcal{A}} |, \quad e := d^{-2} | Z_{1,\mathcal{E}} |, \quad f := d^{-2} | Z_{1,\mathcal{F}} |, \quad r := d^{-2} | Z_{1,\mathcal{R}} | .$$

We have

$$\tau_K(z^K Q^K) = \sum_{Z_1} \int dt_{Z_1} \sum_{\text{decompositions of } Z_1} (-1)^{a+f} \cdot \partial_t^{Z_{1,\mathcal{R}}} \tau_{K+a}(z^{K+a}(t_{Z_1})(Q^K \Lambda \mathcal{A}(t_{Z_1}, Z_{1,\mathcal{A}})) \cdot d\Lambda^{K+a} \mathcal{E}(t_{Z_1}, Z_{1,\mathcal{E}})) \mathcal{F}(t_{Z_1}, Z_{1,\mathcal{F}}) , \quad (258)$$

where

$$\mathcal{A}(t_{Z_1}, Z_{1,\mathcal{A}}) := \bigwedge_{\Delta \subset Z_{1,\mathcal{A}}} \mathcal{A}_\Delta(t_{Z_1}) , \quad (259)$$

$$d\Lambda^{K+a} \mathcal{E}(t_{Z_1}, Z_{1,\mathcal{E}}) := \prod_{\Delta \subset Z_{1,\mathcal{E}}} d\Lambda \mathcal{E}_\Delta(t_{Z_1}) , \quad (260)$$

$$\mathcal{F}(t_{Z_1}, Z_{1,\mathcal{F}}) := \prod_{\Delta \subset Z_{1,\mathcal{F}}} t(\Delta)\mathcal{F}_\Delta \quad (261)$$

and (258) holds but the restrictions on the right hand side imposed by the order convention which are of no importance for us.

In the next step we introduce the d-lattice localizations into $\mathscr{A}_\Delta$ and $\mathscr{E}_\Delta$ terms writing (see (134))

$$\mathscr{A}_{\Delta,\mathrm{loc}}(t) := \lambda(\chi_{\Delta_1}\lambda t(\Delta_2)\mathrm{K}^{\xi\pm}_{\tilde\Delta_2}(\psi)\chi_{\Delta_3} + \mathrm{B}''_{\Delta_1,\Delta_2,\Delta_3}) \cdot (\chi_{\Delta_3}\lambda\, t(\Delta_4)\mathrm{K}^{\xi\pm}_{\tilde\Delta_4}(\psi)\chi_\Delta + \mathrm{B}''_{\Delta_3,\Delta_4,\Delta})\mathrm{K}^{\xi\pm}_{\tilde\Delta}(\varphi\Lambda), \quad (262)$$

$$\mathscr{E}_{\Delta,\mathrm{loc}}(t) := \lambda(\chi_{\Delta_1}\chi_{\Delta_2}\chi_\Delta + \chi_{\Delta_1}\lambda t(\Delta_2)\mathrm{K}^{\xi\pm}_{\tilde\Delta_2}(\psi)\chi_\Delta + \mathrm{B}''_{\Delta_1,\Delta_2,\Delta})\mathrm{K}^{\xi\pm}_{\tilde\Delta}(\varphi\Lambda). \quad (263)$$

After localization we distribute the derivatives of $\partial_t^{\mathrm{Z}_1,\mathscr{R}}$ among the $\mathscr{A}_{\Delta,\mathrm{loc}}$ terms (these are the only ones that may receive them). Next we distribute the $\mathscr{E}_{\Delta,\mathrm{loc}}$ terms according to the definition of $d\Lambda^{\mathrm{K}+a}$. For given $\tilde\Delta \subset \mathrm{Z}$ denote by $\mathrm{K}(\tilde\Delta)$ the number of Q_j terms right-localized in $\tilde\Delta$ and by $e(\tilde\Delta)$ the number of $\mathscr{E}_{\Delta,\mathrm{loc}}$ terms left-localized in $\tilde\Delta$. Distributing the latter among the $\mathrm{Q}_j - s$ and (at most one) $\mathscr{A}_{\Delta,\mathrm{loc}}$ or $\mathscr{E}_{\Delta,\mathrm{loc}}$ factors we generate at most $\binom{\mathrm{K}(\tilde\Delta)+1}{e(\tilde\Delta)} e(\tilde\Delta)\,! \leqslant 2^{\mathrm{K}(\tilde\Delta)+1}e(\tilde\Delta)\,!$ terms. Thus the distributions of the terms produces at most

$$\prod_{\substack{\tilde\Delta\subset \mathrm{Z}\\ e(\tilde\Delta)>0}} 2^{\mathrm{K}(\tilde\Delta)+1}e(\tilde\Delta)\,! \leqslant 2^{\mathrm{K}+e}\prod_{\tilde\Delta\subset\mathrm{Z}} e(\tilde\Delta)\,! \quad (264)$$

expressions of the form

$$\tau_{\mathrm{K}+a}(z^{\mathrm{K}+a}(t)\cdot \text{localized exterior monomial}),$$

where the localized exterior monomials are exterior products of Q_j, $\mathscr{A}_{\Delta,\mathrm{loc}}$ and $\dfrac{\partial}{\partial t(\Delta')}\mathscr{A}_{\Delta,\mathrm{loc}}$ factors, each possibly multiplied from the left by a chain of $\mathscr{E}_{\Delta,\mathrm{loc}}$ operators. After all these steps (258) becomes

$$| \tau_{\mathrm{K}}(z^{\mathrm{K}}\mathrm{Q}^{\mathrm{K}}) | \leqslant \sum_{\mathrm{Z}_1}\ \sum_{\substack{\text{decompositions}\\ \text{of } \mathrm{Z}_1}}\ \sum_{\text{localizations}}\ \sum_{\substack{\text{distributions}\\ \text{of } \mathrm{Z}_{1,\mathscr{R}} \text{ derivatives}}} 2^{\mathrm{K}+e}\prod_{\tilde\Delta\subset\mathrm{Z}} e(\tilde\Delta)\,!$$
$$\cdot \sup_{\substack{\text{localized}\\ \text{monomials}}}\ \sup_{t_{\mathrm{Z}_1}} | \tau_{\mathrm{K}+a}(z^{\mathrm{K}+a}(t_{\mathrm{Z}_1})\cdot \text{localized exterior monomial})$$
$$\cdot \mathscr{F}(t_{\mathrm{Z}_1}, \mathrm{Z}_{1,\mathscr{F}}) | . \quad (265)$$

In estimating the right hand side of (265) we use the combinatoric factors technic. The factor $4^{d^{-2}|\mathrm{Z}_1|}$ will take care of the sum over decompositions of Z_1. The sum over localizations is controlled by the $0(1)\exp[d(\Delta_i, \Delta)]$ factor introduced for each localization square Δ_i in each $\mathscr{A}_{\Delta,\mathrm{loc}}$ or $\mathscr{E}_{\Delta,\mathrm{loc}}$ and the sum over distributions of derivatives of $\partial_t^{\mathrm{Z}_1,\mathscr{R}}$ by the factor 2^a corresponding to the choice of the differentiated $\mathscr{A}_{\Delta,\mathrm{loc}}$ terms. Moreover, as in Proof of Lemma 10.2 of [11] one shows that

$$\prod_{\tilde{\Delta}\subset Z} e(\tilde{\Delta})\,! \leqslant 0(1)^{e} \prod_{\Delta\subset Z_{1,\mathscr{E}}} \exp\left[0(1)\sum_{\Delta_i} d(\Delta_i, \Delta)\right]. \tag{266}$$

Hence

$$| \tau_{K}(z^{K}Q^{K}) | \leqslant \sum_{Z_1} 0(1)^{K+|Z_1|} \operatorname{Sup} \prod_{\Delta\subset Z_{1,\mathscr{A}}\cup Z_{1,\mathscr{E}}} \prod_{i} \exp\left[0(1)d(\Delta_i, \Delta)\right]$$
$$\cdot | \tau_{K+a}(z^{K+a}(t_{Z_1}) \cdot \text{localized exterior monomial}) \cdot \mathscr{F}(t_{Z_1}, Z_{1,\mathscr{F}}) | , \tag{267}$$

where Sup denotes the supremum over decompositions of Z_1, localizations, distributions of $Z_{1,\mathscr{R}}$ derivatives, localized exterior monomials and parameters t. But

$$| \tau_{K+a}(z^{K+a}(t_{Z_1}) \cdot \text{localized exterior monomial}) |$$
$$\leqslant \| z^{K+a}(t_{Z_1}) \| \prod_{j=1}^{K} \| Q_j \|_1 \prod_{\Delta\subset Z_{1,\mathscr{A}}} \| \mathscr{A}^{\#}_{\Delta,\mathrm{loc}}(t_{Z_1}) \|_1 \prod_{\Delta\subset Z_{1,\mathscr{E}}} \| \mathscr{E}_{\Delta,\mathrm{loc}}(t_{Z_1}) \| , \tag{268}$$

where $\#$ in $\mathscr{A}^{\#}_{\Delta,\mathrm{loc}}$ indicates that the $\mathscr{A}_{\Delta,\mathrm{loc}}$ term may be differentiated.

LEMMA V.26. — For any $\alpha > 0$ and $C > 0$ there exists $\beta > 0$ and constant 0(1) such that for all t, λ and $m \geqslant m_0(\lambda, \alpha, C)$

$$\| \mathscr{A}^{\#}_{\Delta,\mathrm{loc}}(t_{Z_1}) \|_1 \leqslant 0(1)m^{-1+\alpha} \exp\left[-C\sum_{i=1}^{4} d(\Delta_i, \Delta)\right]$$
$$\cdot\left(\prod_{i=1}^{4}(1 + S^{\beta}(\varphi\Lambda\chi_{\Delta_i}))\right)S^{\beta}(\varphi\Lambda\chi_{\Delta}), \tag{269}$$

$$\| \mathscr{E}_{\Delta,\mathrm{loc}}(t_{Z_1}) \| \leqslant 0(1)m^{-1/2+\alpha} \exp\left[-C\sum_{i=1}^{2} d(\Delta_i, \Delta)\right]$$
$$\cdot\left(\prod_{i=1}^{2}(1 + S^{\beta}(\varphi\Lambda\chi_{\Delta_i}))\right)S^{\beta}(\varphi\Lambda\chi_{\Delta}). \tag{270}$$

Proof. — By (262)

$$\begin{aligned}\| \mathscr{A}_{\Delta,\mathrm{loc}}(t_{Z_1}) \|_1 &\leqslant \lambda \| \chi_{\Delta_1}\lambda t(\Delta_2)K^{\xi\pm}_{\Delta_2}(\psi)\chi_{\Delta_3} + B''_{\Delta_1,\Delta_2,\Delta_3} \|_{8/3} \\ &\quad\cdot \| \chi_{\Delta_3}\lambda t(\Delta_4)K^{\xi\pm}_{\Delta_4}(\varphi\Lambda)\chi_{\Delta} + B''_{\Delta_3,\Delta_4,\Delta} \|_{8/3} \; \| K^{\xi\pm}_{\Delta}(\varphi\Lambda) \|_4 \\ &\leqslant 0(1)m^{-1+\alpha} \exp\left[-\varepsilon m(d(\Delta_1, \Delta_2)+ \ldots +d(\Delta_4, \Delta))\right] \\ &\quad\cdot(1 + S^{\beta}(\varphi\Lambda\chi_{\Delta_2}))(1 + S^{\beta}(\varphi\Lambda\chi_{\Delta_4}))S^{\beta}(\varphi\Lambda\chi_{\Delta}),\end{aligned} \tag{271}$$

where we have used (219), Corollary A.I.4 and Lemma V.8. We have replaced the terms $S^\beta((g^\Lambda - h)\chi_{\Delta_i}) \leqslant 0(m)$ by 1 since they appear only when $d(\Delta_i, \Delta)$ is large i. e. when one can extract an extra $e^{-\varepsilon m}$ term. The same bound holds also for $\frac{\partial}{\partial t(\Delta_2)} \mathscr{A}_{\Delta,\mathrm{loc}}(t_{Z_1})$.

$$\begin{aligned} \| \mathscr{E}_{\Delta,\mathrm{loc}}(t_{Z_1}) \| &\leqslant \lambda(\| \chi_{\Delta_1}\chi_{\Delta_2}\chi_\Delta \| + \| \chi_{\Delta_1}\lambda K^{\xi\pm}_{\Delta_2}(\varphi\Lambda)\chi_\Delta \|_4 \\ &\quad + \| B''_{\Delta_1,\Delta_2,\Delta} \|_2) \, \| K^{\xi\pm}_{\Delta}(\varphi\Lambda) \|_4 \\ &\leqslant 0(1) m^{-1/2+\alpha} \exp\left[- \varepsilon m(d(\Delta_1, \Delta_2) + d(\Delta_2, \Delta))\right] \\ &\quad \cdot (1 + S^\beta(\varphi\Lambda\chi_{\Delta_2})) S^\beta(\varphi\Lambda\chi_\Delta) . \end{aligned} \tag{272}$$

Hence (269) and (270) follow. □

By (268) and Lemma V.26

$$| \tau_{K+\alpha}(z^{K+\alpha}(t_{Z_1}) \cdot \text{localized exterior monomial}) |$$

$$\leqslant \mathscr{Z}^{K+a}(t_{Z_1}) \prod_{j=1}^{K} \| Q_j \|_1 0(1)^{a+e} m^{(-1+\alpha)a + \left(-\frac{1}{2}+\alpha\right)e}$$

$$\cdot \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} \left(\exp\left[- C \sum_i d(\Delta_i, \Delta) \right] \left(\prod_i S^\beta(\varphi\Lambda\chi_{\Delta_i})^{\#} \right) S^\beta(\varphi\Lambda\chi_\Delta) \right), \tag{273}$$

where $S^{\beta\#}$ denotes either S^β or 1 and the sum over the respective choices is cared for by an additional $0(1)^{a+e}$ combinatoric factor.

With help of (267) and (273) we obtain for p close to 1

$$\begin{aligned} \| \tau_K(z^K Q^K) \|_{L^p(\tau)} &\leqslant \sum_{Z_1} 0(1)^{K+|Z_1|} \operatorname{Sup} m^{(-1+\alpha)a + \left(-\frac{1}{2}+\alpha\right)e} \\ &\cdot \left(\prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} \exp\left[- \frac{1}{2} C \sum_i d(\Delta_i, \Delta) \right] \right) \left\| \prod_{j=1}^{K} \| Q_j \|_1 \right\|_{L^q(\tau)} \\ &\cdot \left\| \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} \left(\prod_i S^\beta(\varphi\Lambda\chi_{\Delta_i})^{\#} \right) S^\beta(\varphi\Lambda\chi_\Delta) \right\|_{L^q(\tau)} \\ &\cdot \| \mathscr{F}(t_{Z_1}, Z_{1,\mathscr{F}}) \|_{L^q(\tau)} \, \| \mathscr{Z}^{K+a}(t_{Z_1}) \|_{L^{\tilde{p}}(\tau)} . \end{aligned} \tag{274}$$

We have chosen q large and $\tilde{p} > p$ but close to 1 so that Proposition V.1 applies.

For each $\tilde{\Delta} \subset Z$ denote by $y(\tilde{\Delta})$ the number of $\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}$ such that $\tilde{\Delta}$ is one of the localization squares corresponding to Δ. Then

$$\begin{aligned}
&\Big\| \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} \Big(\prod_i S^\beta(\varphi\Lambda\chi_{\Delta_i})^\# \Big) S^\beta(\varphi\Lambda\chi_\Delta) \Big\|_{L^q(\tau)} \\
&\leqslant \Big\| \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} \Big(\prod_i S^\beta(\varphi\Lambda\chi_{\Delta_i})^\# \Big) \Big\|_{L^{2q}(\tau)} \Big\| \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} S^\beta(\varphi\Lambda\chi_\Delta) \Big\|_{L^{2q}(\tau)} \\
&\leqslant \prod_{\tilde{\Delta} \subset Z} \Big\| \prod_{\substack{(\Delta_i,\Delta) \\ \Delta_i = \tilde{\Delta}}} S^\beta(\varphi\Lambda\chi_{\tilde{\Delta}})^\# \Big\|_{L^{q_1}(\tau)} \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} \| S^\beta(\varphi\Lambda\chi_\Delta) \|_{L^{q_1}(\tau)} \\
&\leqslant \prod_{\tilde{\Delta} \subset Z} \prod_{\substack{(\Delta_i,\Delta) \\ \Delta_i = \tilde{\Delta}}} \| S^\beta(\varphi\Lambda\chi_{\tilde{\Delta}}) \|_{L^{4y(\tilde{\Delta})q_1}(\tau)} \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{E}}} \| S^\beta(\varphi\Lambda\chi_\Delta) \|_{L^{q_1}(\tau)}, \qquad (275)
\end{aligned}$$

where in the second step we used the checkerboard estimate and in the first and the third one the Hölder inequality. But in virtue of Corollary A.II.2

$$\| S^\beta(\varphi\Lambda\chi_{\tilde{\Delta}}) \|_{L^{4y(\tilde{\Delta})q_1}(\tau)} \leqslant 0(1) y(\tilde{\Delta})^{1/2}.$$

Thus

$$\Big\| \prod_\Delta \Big(\prod_i S^\beta(\varphi\Lambda\chi_{\Delta_i})^\# \Big) S^\beta(\varphi\Lambda\chi_\Delta) \Big\|_{L^q(\tau)} \leqslant 0(1)^{|Z_1|} \prod_{\tilde{\Delta} \subset Z} y(\tilde{\Delta})\,!. \qquad (276)$$

As far as the $\mathscr{F}$ terms are concerned

$$\| \mathscr{F}(t_{Z_1}, Z_{1,\mathscr{F}}) \|_{L^q(\tau)} \leqslant \prod_{\Delta \subset Z_{1,\mathscr{F}}} \| \mathscr{F}_\Delta \|_{L^{q_1}(\tau)} \qquad (277)$$

by the checkerboard estimate.
But

$$\mathscr{F}_\Delta = (2\pi^2 d^2)^{-1} \lambda^2 \sum_{k \in \frac{2\pi}{d}\mathbb{Z}^2} F_\Delta(k) :|\widehat{(\varphi\Lambda\chi_\Delta)}(k)|^2:_\tau, \qquad (278)$$

where

$$F_\Delta(k) := I_\Delta(k) - G_\Delta(k) - 2\pi^2\lambda^{-2} m_c^2, \qquad (279)$$

see (94) and (146). In the limit $d \to \infty$ $F_\Delta(k) \to F(k)$, where

$$F(k) = I(k) - G(k) - 2\pi^2\lambda^{-2}m_c^2 = I(k) - G(k) - \frac{2\pi\bar{\alpha}}{1+\bar{\alpha}}, \qquad (280)$$

see (95) and (147).

From (97), (98), (148) and (149) it follows that

$$|F(k)| \leqslant 0(1)\ln\left(1+\frac{k^2}{m^2}\right) \leqslant 0(1)m^{-2\nu}(k^2+m_c^2)^\nu \tag{281}$$

for $0 < \nu < 1$. By (99) and (150) also

$$|F_\Delta(k)| \leqslant 0(1)m^{-2\nu}(k^2+m_c^2)^\nu \tag{282}$$

for $0 < \nu < \frac{1}{2}$.

Since by (278)

$$\mathscr{F}_\Delta = (2\pi^2)^{-1}\lambda^2 :(\varphi\Lambda \mid F_\Delta(P)_\Delta\varphi\Lambda)_{L^2} :_\tau \tag{283}$$

it follows that

$$\begin{aligned} \|\mathscr{F}_\Delta\|_{L^2(\tau)} &= \sqrt{2}(2\pi^2)^{-1}\lambda^2 \| C_{m_c}(\tau)^{1/2}\Lambda F_\Delta(P)_\Delta \Lambda C_{m_c}(\tau)^{1/2}\|_2 \\ &\leqslant \sqrt{2}(2\pi^2)^{-1}\lambda^2 \| C_{m_c}(N)^{1/2}\Lambda F_\Delta(P)_\Delta \Lambda C_{m_c}(N)^{1/2}\|_2 \\ &\leqslant 0(1)\lambda^2 m^{-2\nu} \| C_{m_c}(N)^{1/2}\Lambda\delta(P^2+m_c^2)^\nu_\Delta \Lambda C_{m_c}(N)^{1/2}\|_2 , \end{aligned} \tag{284}$$

where we have included the operator δ since $(1-\delta)F_\Delta(P)_\Delta = 0$ as $F_\Delta(0)=0$. Choosing $\nu = \frac{1}{30}$ and using (182) we obtain for $m \geqslant m_0(\lambda)$

$$\|\mathscr{F}_\Delta\|_{L^2(\tau)} \leqslant 0(1)m^{-1/15+\alpha}. \tag{285}$$

Now the hypercontractivity implies for $q < \infty$

$$\|\mathscr{F}_\Delta\|_{L^q(\tau)} \leqslant 0(1)m^{-1/15+\alpha} \tag{286}$$

and

$$\|\mathscr{F}(t_{Z_1}, Z_{1,\mathscr{F}})\|_{L^q(\tau)} \leqslant 0(1)^f m^{(-\frac{1}{15}+\alpha)f}. \tag{287}$$

Using (276) and (287) we may rewrite (274) as

$$\begin{aligned} \|\tau_K(z^K Q^K)\|_{L^p(\tau)} &\leqslant \sum_{Z_1} 0(1)^{K+|Z_1|} \operatorname{Sup} m^{(-1+\alpha)a+(-\frac{1}{2}+\alpha)e+(-\frac{1}{15}+\alpha)f} \\ &\cdot\left(\prod_{\Delta\subset Z_{1,\mathscr{A}}\cup Z_{1,\mathscr{E}}} \exp\left[-\frac{1}{2}C\sum_i d(\Delta_i,\Delta)\right]\prod_{\tilde{\Delta}\subset Z} y(\tilde{\Delta})\,!\right)\left\|\prod_{j=1}^{K}\|Q_j\|_1\right\|_{L^q(\tau)} \\ &\cdot\|\mathscr{Z}^{K+a}(t_{Z_1})\|_{L^{\tilde{p}}(\tau)} \\ &\leqslant \sum_{Z_1} 0(1)^{K+|Z_1|} \operatorname{Sup} m^{(-\frac{1}{15}+\alpha)(a+e+f+r)}\left\|\prod_{j=1}^{K}\|Q_j\|_1\right\|_{L^q(\tau)} \\ &\cdot\|\mathscr{Z}^{K+a}(t_{Z_1})\|_{L^{\tilde{p}}(\tau)}, \end{aligned} \tag{288}$$

where we have used the inequality $a \geqslant \frac{1}{2}a + \frac{1}{2}r$ and

$$\prod_{\tilde{\Delta} \subset Z} y(\tilde{\Delta})\,! \leqslant 0(1)^{a+e} \prod_{\Delta \subset Z_{1,\mathscr{A}} \cup Z_{1,\mathscr{B}}} \exp\left[0(1) \sum_i d(\Delta_i, \Delta)\right], \tag{289}$$

compare (266).

We apply Proposition V.1 to (288) obtaining

$$\begin{aligned} \| \tau_K(z^K Q^K) \|_{L^p(\tau)} &\leqslant 0(1)^K \left(\prod_{\Delta \subset Z} e^{0(1)n_\Delta} (n_\Delta\,!)^{0(1)} \right) \\ &\cdot \exp\left[- 0(1) m^{2-\alpha} | Z^0 \cup Z' | + 0(1) m^{-1/4+\alpha} | Z | \right] \\ &\cdot \sum_{Z_1} \left(m^{-\frac{1}{15}+\alpha} 0(1)^{\lambda^3} \right) d^{-2|Z_1|} \left\| \prod_{j=1}^{K} \| Q_j \|_1 \right\|_{L^q(\tau)} \\ &\leqslant 0(1)^K \left(\prod_{\Delta \subset Z} e^{0(1)n_\Delta} (n_\Delta\,!)^{0(1)} \right) \exp\left[- 0(1) m^{2-\alpha} | Z^0 \cup Z' | \right. \\ &\left. + 0(1) m^{-\frac{1}{20}} | Z | \right] \left\| \prod_{j=1}^{K} \| Q_j \|_1 \right\|_{L^q(\tau)} \end{aligned} \tag{290}$$

provided $m \geqslant m_0(\lambda)$. Indeed,

$$m^{-\frac{1}{15}+\alpha} 0(1)^{\lambda^3} \leqslant m^{-\frac{1}{20}} \qquad \text{for} \qquad m \geqslant m_0(\lambda)$$

and

$$\sum_{Z_1} m^{-\frac{1}{20}} \leqslant \prod_{\Delta \subset Z} \left(1 + m^{-\frac{1}{20}}\right) \leqslant \exp\left[0(1) m^{-\frac{1}{20}} | Z |\right].$$

(290) completes Proof of Theorem III.1. □

CHAPTER VI

UPPER BOUND

The last chapter consists of Proof of Proposition II.2. This will be obtained by adapting Proof of Lemma II.4.2.2 of [*13*]. To this end notice that

$$\rho^0_\Lambda(\tilde{\Delta}, \partial\tilde{\Delta}, \emptyset) = \int z^0 d\mu_{m_c}(\tau) \tag{1}$$

where z^0 is defined by (V.1) and we have to take $Z = \tilde{\Delta}$, s, $\tau = 0$ on $\partial\tilde{\Delta}$ and $\Sigma \equiv +$ (which implies $g \equiv h \equiv \xi_+$) for (1) to hold. We introduce the interpolating parameters t as in Chapter V, see (V.5). By (V.249)

$$z^0 - z^0(0) = \sum_{\substack{Z_1 \subset \tilde{\Delta} \\ Z_1 \neq \emptyset}} \int dt_{Z_1} \partial_t^{Z_1} z^0(t_{Z_1}) . \tag{2}$$

Estimating the right hand side of (2) as in (V.290) we obtain

$$\| z^0 - z^0(0) \|_{L^1(\tau)} \leqslant \exp \left[0(1) m^{-1/4+\alpha} | \tilde{\Delta} | \right] \sum_{Z_1 \neq \emptyset} (m^{-\frac{1}{20}})^{d-2|Z_1|} \leqslant 0(1) m^{-\nu} \exp \left[0(1) m^{-\nu} \right] \tag{3}$$

for some $\nu > 0$ and all $\lambda \geqslant \lambda_0$, $m \geqslant m_0(\lambda)$.

By (V.5)

$$z^0(0) = \prod_{\Delta \subset \tilde{\Delta}} \chi_+((\varphi\Lambda)_\Delta + \xi_+) \det{}_2 (1 - B'') \cdot \exp \left[- \lambda \sum_{\Delta \subset \tilde{\Delta}} \mathrm{Tr} \left(K_{\Delta}^{\xi_+}(\varphi\Lambda) B'' \right) - D_1 - D_2 \right] . \tag{4}$$

Interpolating again define for $t \in [0, 1]$

$$z^0(0, t) := \prod_{\Delta \subset \tilde{\Delta}} \chi_+((\varphi\Lambda)_\Delta + \xi_+) \det{}_2 (1 - tB'') \cdot \exp \left[- t \left(\lambda \sum_{\Delta \subset \tilde{\Delta}} \mathrm{Tr} \left(K_{\Delta}^{\xi_+}(\varphi\Lambda) B'' \right) + D_1 + D_2 \right) \right] . \tag{5}$$

Then

$$\begin{aligned} z^0(0) - z^0(0, 0) &= \int_0^1 \mathrm{dt} \frac{\partial}{\partial t} z^0(0, t) \\ &= - \int_0^1 dt \prod_{\Delta \subset \tilde{\Delta}} \chi_+((\varphi\Lambda)_\Delta + \xi_+) [t\tau_1((1 - tB'')^{-1} B''^2) \\ &\quad + \lambda \sum_{\Delta \subset \tilde{\Delta}} \mathrm{Tr} \left(K_{\Delta}^{\xi_+}(\varphi\Lambda) B'' \right) + D_1 + D_2 \Big] \det{}_2 (1 - tB'') \\ &\quad \cdot \exp \left[- t \left(\lambda \sum_{\Delta \subset \tilde{\Delta}} \mathrm{Tr} \left(K_{\Delta}^{\xi_+}(\varphi\Lambda) B'' \right) + D_1 + D_2 \right) \right] . \end{aligned} \tag{6}$$

Hence

$$| z^0(0) - z^0(0,0) | \leqslant \sup_{t\in[0,1]} \Big(\| B''^2 \|_1 \; \| (1-tB'')^{-1} \det_2 (1-tB'') \|$$

$$+ \Big| \lambda \sum_{\Delta\subset\tilde{\Delta}} \mathrm{Tr}\,(K_{\Delta}^{\xi+}(\varphi\Lambda)B'') + D_1 + D_2 \Big| \, | \det_2 (1 - tB'') | \Big)$$

$$\cdot \exp\left[-t\Big(\lambda \sum_{\Delta\subset\tilde{\Delta}} \mathrm{Tr}\,(K_{\Delta}^{\xi+}(\varphi\Lambda)B'') + D_1 + D_2 \Big)\right]$$

$$\leqslant 0(1) \sup_{t\in[0,1]} \left(\| B''^*B'' \|_1 + \Big| \lambda \sum_{\Delta\subset\tilde{\Delta}} \mathrm{Tr}\,\big(K_{\Delta}^{\xi+}(\varphi\Lambda)B''\big) + D_1 + D_2 \Big| \right)$$

$$\cdot \exp\left[t\Big(\frac{1}{2} t \, \| B''^*B'' \|_1 - \lambda \sum_{\Delta\subset\tilde{\Delta}} \mathrm{Tr}\,\big(K_{\Delta}^{\xi+}(\varphi\Lambda)B''\big) - D_1 - D_2 \Big)\right], \tag{7}$$

where the last inequality is obtained with the help of Lemma V.5 (with $A = 0$ and $B := B''$).

Now

$$\| z^0(0) - z^0(0,0) \|_{L^1(\tau)} \leqslant \| \, \| B''^*B'' \|_1 + | \lambda \sum_{\Delta\subset\tilde{\Delta}} \mathrm{Tr}\,\big(K_{\Delta}^{\xi+}(\varphi\Lambda)B''\big)$$

$$+ D_1 + D_2 | \, \|_{L^{p'}(\tau)} \, \| \exp\left[t\Big(\frac{1}{2} t \, \| B''^*B'' \|_1 - \lambda \sum_{\Delta\subset\tilde{\Delta}} \mathrm{Tr}\,\big(K_{\Delta}^{\xi+}(\varphi\Lambda)B''\big) - D_1 - D_2 \Big)\right] \|_{L^p(\tau)} \,. \tag{8}$$

Proceeding as in Proofs of Lemmas V.9, 10, 20 and 21 we obtain

$$\| z^0(0) - z^0(0,0) \|_{L^1(\tau)} \leqslant 0(1) m^{-\nu} \exp\,[0(1) m^{-\nu}] \tag{9}$$

for some $\nu > 0$ and all $\lambda \geqslant \lambda_0$, $m \geqslant m_0(\lambda)$.

But

$$z^0(0,0) = \prod_{\Delta\subset\tilde{\Delta}} \chi_+((\varphi\Lambda)_\Delta + \xi_+) \tag{10}$$

and

$$0 \leqslant 1 - z^0(0,0) \leqslant \sum_{\Delta\subset\tilde{\Delta}} \chi_-((\varphi\Lambda)_\Delta + \xi_+) \,. \tag{11}$$

Thus

$$\begin{aligned}
\| z^0(0,0) - 1 \|_{L^1(\tau)} &\leqslant \sum_{\Delta \subset \tilde{\Delta}} \int \chi_-((\varphi\Lambda)_\Delta + \xi_+) d\mu_{m_c}(\tau) \\
&\leqslant \sum_{\Delta \subset \tilde{\Delta}} \int \exp\, [-(\varphi\Lambda)_\Delta - \xi_+ - 1] d\mu_{m_c}(\tau) \\
&\leqslant \sum_{\Delta \subset \tilde{\Delta}} e^{-\xi_+ + 1} \exp\left[\frac{1}{2} d^{-4} (\chi_\Delta \mid \Lambda C_{m_c}(\tau) \Lambda \chi_\Delta)_{L^2}\right] \\
&\leqslant 0(1) \sum_{\Delta \subset \tilde{\Delta}} e^{-\xi_+} \leqslant 0(1) m^{-\nu} \qquad (12)
\end{aligned}$$

for some $\nu > 0$ and all $m \geqslant m_0(\lambda)$.

Now (1), (3), (9) and (12) yield

$$| \rho_\Lambda^0(\tilde{\Delta}, \partial\tilde{\Delta}, \emptyset) - 1 | \leqslant 0(1) m^{-\nu} \qquad (13)$$

which gives (II.54). □

APPENDIX I

In this section the information about operators K which was needed in the main text is given. Our operators will be localized in lattice squares and we use various lattices. When the diameter of the lattice does not depend on λ nor m we shall speak of an 0(1)-lattice. We shall also use an $0\left(\frac{1}{m}\right)$-lattice in the present section. The lattices, as always, are supposed to be compatible in the sense that of any two, one is a refinement of the other.

Let us start with

LEMMA A.I.1. — Let $\underline{\Delta}$, $\underline{\Delta}'$ be $0\left(\frac{1}{m}\right)$-lattice squares. Then

$$\| \chi_{\underline{\Delta}} K(s, f) \chi_{\underline{\Delta}'} \| \leqslant 0(1) m^{-1} e^{-\varepsilon m d(\underline{\Delta}, \underline{\Delta}')} \| f \|_{L^\infty} \tag{1}$$

for some $\varepsilon > 0$.

Proof. — First notice that

$$\| \chi_{\underline{\Delta}}((\not{P} + m)D^{-1})_s \chi_{\underline{\Delta}'} \| \leqslant \| \chi_{\underline{\Delta}}(\not{P} + m)D^{-1}\chi_{\underline{\Delta}'} \| \tag{2}$$

and

$$\| \chi_{\underline{\Delta}}(D^{-1/2})_s \chi_{\underline{\Delta}'} \| \leqslant \| \chi_{\underline{\Delta}} D^{-1/2} \chi_{\underline{\Delta}'} \| \tag{3}$$

see (II.10) and (II.11).

For touching $\underline{\Delta}$, $\underline{\Delta}'$

$$\| \chi_{\underline{\Delta}}(\not{P} + m)D^{-1}\chi_{\underline{\Delta}'} \| \leqslant \| (\not{P} + m)D^{-1} \| = 1 , \tag{4}$$

$$\| \chi_{\underline{\Delta}} D^{-1/2} \chi_{\underline{\Delta}'} \| \leqslant \| D^{-1/2} \| = m^{-1/2} . \tag{5}$$

If $\underline{\Delta}$, $\underline{\Delta}'$ are not touching then $\chi_{\underline{\Delta}}(\not{P} + m)D^{-1}\chi_{\underline{\Delta}'}$ $(\chi_{\underline{\Delta}} D^{-1/2}\chi_{\underline{\Delta}'})$ are given by kernels $m^2\chi_{\underline{\Delta}}(x)H(m(x-y))\chi_{\underline{\Delta}'}(y)(m^{3/2}\chi_{\underline{\Delta}}(x)H'(m(x-y))\chi_{\underline{\Delta}'}(y))$ where the function $H(x)$ $(H'(x))$ have exponential fall-off:

$$| H^{(\prime)}(x) | \leqslant 0(1)e^{-\frac{1}{2}|x|} \quad \text{for} \quad |x| \geqslant 1 , \tag{6}$$

see Appendix II.

Since if the operator L is given by its kernel $L(x, y)$ then

$$\| L \| \leqslant \left(\sup_x \int | L(x, y) | dy \right)^{1/2} \left(\sup_y \int | L(x, y) | dx \right)^{1/2} , \tag{7}$$

(6) yields for non-touching $\underline{\Delta}$, $\underline{\Delta}'$

$$\| \chi_{\underline{\Delta}}(\not{P} + m)D^{-1}\chi_{\underline{\Delta}'} \| \leqslant 0(1) \exp [- \varepsilon m d(\underline{\Delta}, \underline{\Delta}')] , \tag{8}$$

$$\| \chi_{\underline{\Delta}} D^{-1/2} \chi_{\underline{\Delta}'} \| \leqslant 0(1) m^{-1/2} \exp [- \varepsilon m d(\underline{\Delta}, \underline{\Delta}')] . \tag{9}$$

using (2)-(5), (8) and (9) we obtain

$$\begin{aligned} \| \chi_{\underline{\Delta}} K(s, f) \chi_{\underline{\Delta}'} \| &= \| \chi_{\underline{\Delta}}((\not{P} + m)D^{-1})_s (D^{-1/2})_s \Gamma f (D^{-1/2})_s \chi_{\underline{\Delta}'} \| \\ &\leqslant \sum_{\underline{\Delta}_1, \underline{\Delta}_2} \| \chi_{\underline{\Delta}}(\not{P} + m)D^{-1}\chi_{\underline{\Delta}_1} \| \| \chi_{\underline{\Delta}_1} D^{-1/2} \chi_{\underline{\Delta}_2} \| \| \chi_{\underline{\Delta}_2} D^{-1/2} \chi_{\underline{\Delta}'} \| \| f \|_{L^\infty} \\ &\leqslant 0(1) m^{-1} \| f \|_{L^\infty} \sum_{\underline{\Delta}_1, \underline{\Delta}_2} \exp [- \varepsilon m (d(\underline{\Delta}, \underline{\Delta}_1) + d(\underline{\Delta}_1, \underline{\Delta}_2) + d(\underline{\Delta}_2, \underline{\Delta}'))] . \end{aligned} \tag{10}$$

LEMMA A.I.2.

$$\sum_{\underline{\Delta}_1,\dots,\underline{\Delta}_{n-1}} \exp\left[-\varepsilon m(d(\underline{\Delta},\underline{\Delta}_1)+d(\underline{\Delta}_1,\underline{\Delta}_2)+\dots+d(\underline{\Delta}_{n-1},\underline{\Delta}'))\right] \leqslant 0(1)^n \exp\left[-\varepsilon' m d(\underline{\Delta},\underline{\Delta}')\right], \quad (11)$$

where 0(1) and ε' do not depend on n.

With Lemma A.I.2 given, (1) easily follows from (10) completing Proof of Lemma A.I.1. □

Proof of Lemma A.I.2. — For $x \in \underline{\Delta}$, $x' \in \underline{\Delta}'$

$$d(\underline{\Delta},\underline{\Delta}') \geqslant |x-x'| - 0\left(\frac{1}{m}\right) \geqslant 2^{-1/2}(|x^0-(x')^0| + |x^1-(x')^1|) - 0\left(\frac{1}{m}\right). \quad (12)$$

Hence

$$\sum_{\underline{\Delta}_1,\dots,\underline{\Delta}_{n-1}} \exp\left[-\varepsilon m(d(\underline{\Delta},\underline{\Delta}_1)+\dots+d(\underline{\Delta}_{n-1},\underline{\Delta}'))\right] \leqslant 0(1)^n m^{2(n-1)} \int dx_1 \dots dx_{n-1}$$

$$\cdot \exp\left[-2^{-1/2}\varepsilon m \sum_{j=0,1} (|x^j - x_1^j| + |x_1^j - x_2^j| + \dots + |x_{n-1}^j - (x')^j|)\right]$$

$$= 0(1)^n \prod_{j=0,1} \int dx_1^j \dots dx_{n-1}^j \exp\left[-2^{-1/2}\varepsilon(|mx^j - x_1^j| + \dots + |x_{n-1}^j - m(x')^j|)\right]$$

$$= 0(1)^n \prod_j \underbrace{F * \dots * F}_{n \text{ factors}}(m(x^j - (x')^j)), \quad (13)$$

where F is a function of one variable,

$$F(t) = \exp\left[2^{-1/2}\varepsilon|t|\right]. \quad (14)$$

Direct calculation gives

$$\int F(t-s)|s|^n F(s)ds = (2(\sqrt{2}\varepsilon)^{-n-1}n! + (n+1)^{-1}|t|^{n+1})F(t). \quad (15)$$

Thus

$$F * \dots * F(t) = \sum_{r=0}^{n-1} a_{n,r}|t|^r F(t). \quad (16)$$

$a_{n,r}$ are defined by the following recurence relations

$$a_{1,0} = 1,$$
$$a_{k+1,0} = 2\sum_{j=0}^{k-1} (\sqrt{2}\varepsilon)^{-j-1} j!\, a_{k,j}, \quad (17)$$
$$a_{k+1,l+1} = \begin{cases} (l+1)^{-1}a_{k,l} & \text{if} \quad l \leqslant k-1, \\ 0 & \text{if} \quad l \geqslant k. \end{cases}$$

We have the following bound

$$0 \leqslant a_{n,r} \leqslant \frac{1}{r!}\left(\frac{1}{3}\sqrt{2}\varepsilon\right)^{-n+r+1}. \quad (18)$$

Indeed, it holds for $a_{1,0}$ and if it holds for $a_{k,l}$ for all l then

$$0 \leqslant a_{k+1,0} \leqslant 2\sum_{j=0}^{k-1}(\sqrt{2\varepsilon})^{-j-1}j\,!\cdot\frac{1}{j\,!}\left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-k+j+1}$$

$$\leqslant \left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-k}\sum_{j=0}^{k-1}2\left(\frac{1}{3}\right)^{j+1} \leqslant \left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-k} \quad (19)$$

and

$$0 \leqslant a_{k+1,l+1} \leqslant (l+1)^{-1}\frac{1}{l\,!}\left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-k+l+1} = \frac{1}{(l+1)\,!}\left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-(k+1)+(l+1)+1}. \quad (20)$$

Thus

$$0 \leqslant \underbrace{\mathrm{F} * \ldots * \mathrm{F}}_{n \text{ factors}}(t) \leqslant \sum_{r=0}^{\infty}\frac{1}{r\,!}\left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-n+r+1}|\,t\,|^r\,\mathrm{F}(t) \leqslant \left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-n+1}\exp\left[-\frac{1}{6}\sqrt{2\varepsilon}\,|\,t\,|\right]. \quad (21)$$

Inserting (21) to (13) we obtain

$$\sum_{\underline{\Delta}_1,\ldots,\underline{\Delta}_{n-1}}\exp\left[-\varepsilon m(d(\underline{\Delta},\underline{\Delta}_1)+\ldots+d(\underline{\Delta}_{n-1},\underline{\Delta}'))\right]$$

$$\leqslant 0(1)^n\left(\frac{1}{3}\sqrt{2\varepsilon}\right)^{-n+1}\exp\left[-\frac{1}{6}\sqrt{2\varepsilon}m\sum_{j=0,1}|\,x^j-(x')^j\,|\right] \leqslant 0(1)^n\exp\left[-\varepsilon' m d(\underline{\Delta},\underline{\Delta}')\right]. \quad \square$$

COROLLARY A.I.1.

$$||\,\mathrm{K}(s,\,f)\,|| \leqslant 0(1)m^{-1}\,||\,f\,||_{\mathrm{L}^\infty} \quad (22)$$

and

$$||\,\chi_\Delta \mathrm{K}(s,\,f)\chi_{\Delta'}\,|| \leqslant 0(1)m^{-1}\exp\left[-\varepsilon m d(\Delta,\Delta')\right]||\,f\,||_{\mathrm{L}^\infty} \quad (23)$$

for 0(1)-lattice squares Δ, Δ'.

Proof. — By a slight modification of (7)

$$||\,\mathrm{K}(s,\,f)\,|| \leqslant \left(\sup_{\underline{\Delta}}\sum_{\underline{\Delta}'}||\,\chi_{\underline{\Delta}}\mathrm{K}(s,\,f)\chi_{\underline{\Delta}'}\,||\right)^{1/2}\left(\sup_{\underline{\Delta}'}\sum_{\underline{\Delta}}||\,\chi_{\underline{\Delta}}\mathrm{K}(s,\,f)\chi_{\underline{\Delta}'}\,||\right)^{1/2}$$

$$\leqslant 0(1)m^{-1}\sum_{\underline{\Delta}'}\exp\left[-\varepsilon m d(\underline{\Delta},\underline{\Delta}')\right]||\,f\,||_{\mathrm{L}^\infty} \leqslant 0(1)m^{-1}\left(m^2\int dx\exp\left[-\varepsilon m\,|\,x\,|\right]\right)||\,f\,||_{\mathrm{L}^\infty}$$

$$= 0(1)m^{-1}\,||\,f\,||_{\mathrm{L}^\infty}.$$

The second assertion of Corollary follows from the first for touching Δ, Δ' and from Lemma A.I.1 for non-touching ones. $\square$

LEMMA A.I.3. — If $||\,f\,||_{\mathrm{L}^\infty} \leqslant \xi_+$ then for $\lambda \geqslant \lambda_0$ $(1-\lambda\mathrm{K}(s,\,f))^{-1}$ exists and

$$||\,(1-\lambda\mathrm{K}(s,\,f))^{-1}\,|| \leqslant 0(1), \quad (24)$$

$$||\,\chi_{\underline{\Delta}}(1-\lambda\mathrm{K}(s,\,f))^{-1}\chi_{\underline{\Delta}'}|| \leqslant 0(1)\exp\left[-\varepsilon m d(\underline{\Delta},\underline{\Delta}')\right]. \quad (25)$$

Proof. — (24) follows from Corollary A.I.1 since $\lambda\xi_+ m^{-1} = (\exp[2\pi^2\mu^2\lambda^{-2}] - 1)^{1/2}$. To prove (25) we use the Neumann series and Lemmas A.I.1 and 2:

$$\| \chi_{\underline{\Delta}}(1 - \lambda K(s, f))^{-1}\chi_{\underline{\Delta}'} \| \leqslant \sum_{n=0}^{\infty} \lambda^n \| \chi_{\underline{\Delta}} K(s, f)^n \chi_{\underline{\Delta}'} \| \leqslant \| \chi_{\underline{\Delta}\cap\underline{\Delta}'} \|$$

$$+ \sum_{n=1}^{\infty} \lambda^n \sum_{\underline{\Delta}_1, \dots, \underline{\Delta}_{n-1}} \| \chi_{\underline{\Delta}} K(s, f)\chi_{\underline{\Delta}_1} \| \| \chi_{\underline{\Delta}_1} K(s, f)\chi_{\underline{\Delta}_2} \| \cdot \ldots \cdot \| \chi_{\underline{\Delta}_{n-1}} K(s, f)\chi_{\underline{\Delta}'} \|$$

$$\leqslant \| \chi_{\underline{\Delta}\cap\underline{\Delta}'} \| + \sum_{n=1}^{\infty} (0(1)\lambda \| f \|_{L^\infty})^n \sum_{\underline{\Delta}_1, \dots, \underline{\Delta}_{n-1}} \exp\,[-\,\varepsilon m(d(\underline{\Delta}, \underline{\Delta}_1) + \ldots + d(\underline{\Delta}_{n-1}, \underline{\Delta}'))]$$

$$\leqslant \sum_{n=0}^{\infty} (0(1)\lambda m^{-1}\xi_+)^n \exp\,[-\,\varepsilon' m d(\underline{\Delta}, \underline{\Delta}')] \leqslant 0(1) \exp\,[-\,\varepsilon' m d(\underline{\Delta}, \underline{\Delta}')]$$

for λ sufficiently large. □

COROLLARY A.I.2. — If Δ, Δ' are 0(1)-lattice squares, $\| f \|_{L^\infty} \leqslant \xi_+$ and λ is large enough then

$$\| \chi_\Delta(1 - \lambda K(s, f))^{-1}\chi_{\Delta'} \| \leqslant 0(1) \exp\,[-\,\varepsilon m d(\Delta, \Delta')]\,. \tag{26}$$

Proof. — This follows immediately from (24) for touching squares and from (25) for non-touching ones. □

The next thing we shall be occupied with is the trace properties of K operators. For 0(1)-lattice squares Δ, Δ', Δ'' define

$$K(\Delta, \Delta', \Delta'', f) := \chi_\Delta(\not{P} + m)D^{-1}\chi_{\Delta'}D^{-1/2} f \chi_{\Delta''}D^{-1/2}\,. \tag{27}$$

Notice that

$$\sum_{\Delta, \Delta', \Delta''} K(\Delta, \Delta', \Delta'', f) = K(f)\,. \tag{28}$$

Denote

$$S^\beta(f) = \| D^{-\beta} f D^{-1/2} \|_4\,. \tag{29}$$

PROPOSITION A.I.1. — Let $\alpha > 0$ and $\beta = \beta(\alpha) > 0$. Let Δ, Δ', Δ'', Δ''' be 0(1)-lattice squares

1.*a*) For $2 < p \leqslant \infty$

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-1+2/p} \| f \chi_{\Delta''} \|_{L^\infty}\,. \tag{30}$$

1.*b*) For $2 < p \leqslant 4$

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-1+2/p+\alpha} S^\beta(f \chi_{\Delta''})\,. \tag{31}$$

Suppose that Δ, Δ' or Δ', Δ'' or Δ'', Δ''' be non-coinciding. Then for $\frac{16}{9} < p \leqslant 2$

2.*a*)
$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-9/4+4/p+\alpha} \| f \chi_{\Delta''} \|_{L^\infty}\,, \tag{32}$$

2.*b*)
$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-9/4+4/p+\alpha} S^\beta(f \chi_{\Delta''}) \tag{33}$$

and for $2 < p \leqslant 4$

2.*c*)
$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-5/4+2/p} \| f \chi_{\Delta''} \|_{L^\infty}\,. \tag{34}$$

Suppose that Δ, Δ' or Δ', Δ'' or Δ'', Δ''' be non-touching. Then

3.*a*)

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_1 \leqslant 0(1) \exp [- \varepsilon m(d(\Delta, \Delta') + d(\Delta', \Delta'') + d(\Delta'', \Delta'''))] \| f\chi_{\Delta''} \|_{L^\infty}, \tag{35}$$

3.*b*)

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_1 \leqslant 0(1) \exp [- \varepsilon m(d(\Delta, \Delta') + d(\Delta', \Delta'') + d(\Delta'', \Delta'''))] S^\beta(f\chi_{\Delta''}). \tag{36}$$

Proof. — 1.*a*)

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant \| \chi_\Delta(\not{P} + m)D^{-1}\chi_{\Delta'} \| \| \chi_{\Delta'}D^{-1/2}\chi_{\Delta''} \|_{2p} \| f\chi_{\Delta''} \|_{L^\infty} \cdot \| \chi_{\Delta''}D^{-1/2}\chi_{\Delta'''} \|_{2p} \leqslant 0(1)m^{-1+2/p} \| f\chi_{\Delta''} \|_{L^\infty} \tag{37}$$

since

$$\| \chi_\Delta(\not{P} + m)D^{-1}\chi_{\Delta'} \| \leqslant \| (P + m)D^{-1} \| = 1, \tag{38}$$

$$\| \chi_{\Delta'}D^{-1/2}\chi_{\Delta''} \|_{2p} \leqslant \| D^{-1/2}\chi_{\Delta''} \|_{2p} \leqslant 0(1) \| (p^2 + m^2)^{-1/4} \|_{L^{2p}} = 0(1)m^{-\frac{1}{2}+\frac{1}{p}}, \tag{39}$$

where the last has been obtained with use of Lemma 2.1 of [*25*].

1.*b*)

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant \| \chi_\Delta(\not{P} + m)D^{-1}\chi_{\Delta'} \| \| \chi_{\Delta'}D^{-1/2+\beta} \|_{p_1} S^\beta(f\chi_{\Delta''}), \tag{40}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{4}$. But using Lemma 2.1 of [*25*] again we obtain

$$\| \chi_{\Delta'}D^{-1/2+\beta} \|_{p_1} \leqslant 0(1) \| (p^2 + m^2)^{-1/4+\beta/2} \|_{p_1} = 0(1)m^{-1/2+\beta+2/p_1} = 0(1)m^{-1+2/p+\beta}. \tag{41}$$

(38) and (41) together with (40) give (31).

2.*a*) Since

$$S^\beta(f\chi_{\Delta''}) \leqslant \| D^{-\beta}\chi_{\Delta''} \|_{p_1} \| f\chi_{\Delta''} \|_{L^\infty} \| \chi_{\Delta''}D^{-1/2} \|_{p_2} \leqslant 0(1) \| f\chi_{\Delta''} \|_{L^\infty} \tag{42}$$

for $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{4}$ and p_1 large enough (Lemma 2.1 of [*25*] !), 2.*a*) follows from 2.*b*).

2.*b*) We shall need the following

LEMMA A.I.4. — (Slightly refined version of Lemma 2.3 of [*25*]). Let Δ be an 0(1)-lattice square. If $0 \leqslant \nu < \frac{1}{4}$ and $\lambda - 2\nu > \frac{1}{4}$ then

$$\| [D^{-\lambda}, \chi_\Delta]D^\nu \|_4 \leqslant 0(1)m^{\frac{1}{4}-\lambda+2\nu}, \tag{43}$$

$$\| [D^\nu, \chi_\Delta]D^{-\lambda} \|_4 \leqslant 0(1)m^{\frac{1}{4}-\lambda+2\nu}. \tag{44}$$

If $0 \leqslant \nu < \frac{1}{8}$, $\lambda - 2\nu > \frac{1}{8}$ then

$$\| [D^{-\lambda}, \chi_\Delta]D^\nu \|_8 \leqslant 0(1)m^{\frac{1}{8}-\lambda+2\nu}, \tag{45}$$

$$\| [D^\nu, \chi_\Delta]D^{-\lambda} \|_8 \leqslant 0(1)m^{\frac{1}{8}-\lambda+2\nu}. \tag{46}$$

The same is true if $D^{-\lambda}$ is replaced by $(\not{P} + m)D^{-\lambda-1}$.

Proof. — By mimicking Proof of Lemma 2.3 of [*25*] with tracing powers of *m*. □

Now suppose that in $K(\Delta, \Delta', \Delta'', f)$ Δ and Δ' are not coinciding. Then for $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{2}$

$$\begin{aligned}\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p &\leqslant \| \chi_\Delta(\not{P} + m)D^{-1}[\chi_{\Delta'}, D^{-1/2}]f\chi_{\Delta''}D^{-1/2}\chi_{\Delta'''} \|_p \\ &+ \| \chi_\Delta[(\not{P} + m)D^{-3/2}, \chi_{\Delta'}]f\chi_{\Delta''}D^{-1/2}\chi_{\Delta'''} \|_p \\ &\leqslant \| \chi_\Delta(\not{P} + m)D^{-1-\nu} \|_{p_1} \| D^\nu[\chi_{\Delta'}, D^{-1/2}]D^\beta \|_4 S^\beta(f\chi_{\Delta''}) \\ &+ \| \chi_\Delta D^{-\nu} \|_{p_1} \| D^\nu[(\not{P} + m)D^{-3/2}, \chi_{\Delta'}]D^\beta \|_4 S^\beta(f\chi_{\Delta''}) . \end{aligned} \tag{47}$$

But

$$\begin{aligned}\| D^\nu[\chi_{\Delta'}, D^{-1/2}]D^\beta \|_4 &\leqslant \| D^\nu[\chi_{\Delta'}, D^{-1/2+\beta}] \|_4 + \| [D^{-1/2+\nu+\beta}, \chi_{\Delta'}] \|_4 \\ &+ \| [\chi_{\Delta'}, D^{-1/2+\nu}]\chi_{\Delta'}D^\beta \|_4 \leqslant 0(1)(m^{-\frac{1}{4}+\beta+2\nu} + m^{-\frac{1}{4}+\beta+\nu} + m^{-\frac{1}{4}+2\beta+\nu}) \end{aligned} \tag{48}$$

if $\frac{1}{4} - \max(\beta + 2\nu, 2\beta + \nu) > 0$, $0 \leqslant \beta, \nu < \frac{1}{4}$ in virtue of Lemma A.I.4. Hence for $\delta \leqslant \nu$

$$\| D^\nu[\chi_{\Delta'}, D^{-1/2}]D^\beta \|_4 \leqslant 0(1)m^{-1/4+\beta+2\nu} \tag{49}$$

provided $\beta + 2\nu < \frac{1}{4}$.

Similarly

$$\| D^\nu[(\not{P} + m)D^{-3/2}, \chi_{\Delta'}]D^\beta \|_4 \leqslant 0(1)m^{-1/4+\beta+2\nu} \tag{50}$$

under the same assumptions.

Moreover

$$\| \chi_\Delta(\not{P} + m)D^{-1+\nu} \|_{p_1} \leqslant 0(1)m^{-\nu+2/p_1} \tag{51}$$

and

$$\| \chi_\Delta D^{-\nu} \|_{p_1} \leqslant 0(1)m^{-\nu+2/p_1} \tag{52}$$

if $\nu > \frac{2}{p_1} = \frac{2}{p} - 1$.

For $\frac{16}{9} < p \leqslant 2$ there exists β such that $\frac{2}{p} - 1 < \nu < \frac{1}{8} - \frac{1}{2}\beta$. Then inserting (49)-(51) to (49) we obtain

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant 0(1)m^{-5/4+2/p+\nu+\beta}S^\beta(f\chi_{\Delta''}) . \tag{53}$$

Choosing ν close to $\frac{2}{p} - 1$ and β small we obtain (33).

Next consider the case of non-coinciding Δ' and Δ''. Then

$$\begin{aligned}\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p &\leqslant \| \chi_\Delta(\not{P} + m)D^{-1-\nu} \|_{p_1} \| D^\nu[\chi_{\Delta'}, D^{-1/2}]D^\beta \|_4 S^\beta(f\chi_{\Delta''}) \\ &\leqslant 0(1)m^{-9/4+4/p+\alpha}S^\beta(f\chi_{\Delta''}) ,\end{aligned}$$

where we have used (49) and (51). This implies (33).

If Δ'', Δ''' are non-coinciding then

$$\begin{aligned}\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p &\leqslant \| \chi_\Delta(\not{P} + m)D^{-1}\chi_{\Delta''} \| \| D^{-1/2}f\chi_{\Delta''}D^{-\beta} \|_4 \\ &\cdot \| D^\beta[\chi_{\Delta''}, D^{-1/2}]D^\nu \| \| D^{-\nu}\chi_{\Delta'''} \|_{p_1} \leqslant 0(1)m^{-9/4+4/p+\alpha}S^\beta(f\chi_{\Delta''}) \end{aligned} \tag{54}$$

where we used (49) and (52) as before.

(54) yields again (33).

2.*c*) For non-coinciding Δ, Δ' and $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{4}$

$$\begin{aligned}\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p &\leqslant \| \chi_\Delta(\not{P} + m)D^{-1} \| \| [\chi_{\Delta'}, D^{-1/2}] \|_4 \\ &\cdot \| f\chi_{\Delta''} \|_{L^\infty} \| D^{-1/2}\chi_{\Delta'''} \|_{p_1} + \| [(\not{P} + m)D^{-3/2}, \chi_{\Delta'}] \| \cdot \| f\chi_{\Delta''} \|_{L^\infty} \| D^{-1/2}\chi_{\Delta'''} \|_{p_1} , \end{aligned} \tag{55}$$

see (47). Using Lemma 2.1 of [25] and Lemma A.I.4 we obtain

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p \leqslant 0(1)m^{-3/4+2/p_1} \| f\chi_{\Delta''} \|_{L^\infty} = 0(1)m^{-5/4+2/p} \| f\chi_{\Delta''} \|_{L^\infty}.$$

For non-coinciding Δ', Δ'' or Δ'', Δ''' (34) is proven similarly.

3.*a*) Again follows from 3.*b*) in virtue of (42).

3.*b*) We shall need

LEMMA A.I.5. — (Essentially Lemma 2.2 of [25]). Let Δ, Δ' be non-touching squares of an 0(1)-lattice. Then

$$\| A\chi_\Delta D^{-\lambda}\chi_{\Delta'}B \|_1 \leqslant 0(1)e^{-\varepsilon md(\Delta,\Delta')} \| A\chi_\Delta D^{-2k} \|_q \| D^{-2k}\chi_{\Delta'}B \|_{q'}, \quad (56)$$

where k is non-negative integer. The same holds if $D^{-\lambda}$ is replaced by $(\not{P} + m)D^{-\lambda-1}$.

Proof. — As in [25]. □

If Δ, Δ' are non-touching then in virtue of Lemma A.I.5

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_1 \leqslant \exp[-\varepsilon md(\Delta, \Delta')] \| \chi_\Delta D^{-2} \|_{4/3} \cdot \| D^{-2}\chi_{\Delta'}D^{-1/2}f\chi_{\Delta''}D^{-1/2}\chi_{\Delta'''} \|_4. \quad (57)$$

If both Δ', Δ'' and Δ'', Δ''' are touching then we estimate

$$\| D^{-2}\chi_{\Delta'}D^{-1/2}f\chi_{\Delta''}D^{-1/2}\chi_{\Delta'''} \|_4 \leqslant \| D^{-2}\chi_{\Delta'}D^{-1/2+\beta} \| S^\beta(f\chi_{\Delta''}) \| \chi_{\Delta'''} \| \leqslant 0(1)S^\beta(f\chi_{\Delta''}). \quad (58)$$

But

$$\| \chi_\Delta D^{-2} \|_{4/3} \leqslant \sum_{\substack{\Delta_1 \\ \Delta_1 \text{ touching } \Delta}} \| \chi_\Delta D^{-2}\chi_{\Delta_1} \|_{4/3} + \sum_{\substack{\Delta_1 \\ \Delta_1, \Delta \text{ non-touching}}} \| \chi_\Delta D^{-2}\chi_{\Delta_1} \|_1$$
$$\leqslant 0(1) \| \chi_\Delta D^{-1} \|_{8/3}^2 + \sum_{\Delta_1} \exp[-\varepsilon md(\Delta, \Delta_1)] \| \chi_\Delta D^{-2} \|_2 \| D^{-2}\chi_{\Delta_1} \|_2, \quad (59)$$

where we have used Lemma A.I.5 and Lemma 2.1 of [25].

Hence for Δ, Δ' non-touching and Δ', Δ'' and Δ'', Δ''' touching we obtain

$$\| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_1 \leqslant 0(1) \exp[-\varepsilon md(\Delta, \Delta')]S^\beta(f\chi_{\Delta''}). \quad (60)$$

For other cases we proceed analogically using Lemma A.I.5 to separate non-touching squares.

This completes Proof of Proposition A.I.1. □

COROLLARY A.I.3. — Proposition A.I.1 holds provided $\Delta = \Delta'$ and $K(\Delta, \Delta', \Delta'', f)$ is replaced by $\chi_\Delta K(s, f\chi_{\Delta''})$.

Proof. — Notice that

$$\| \chi_\Delta K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_p \leqslant \sum_{\Delta_1} \| K(\Delta, \Delta_1, \Delta'', f)\chi_{\Delta'''} \|_p \quad (61)$$

(squares $\Delta, \Delta', \Delta''$ and Δ''' are all contained in the l-lattice squares). We choose Δ_1 from the lattice of diameter by $\frac{1}{2}$ less then that of Δ, Δ'' and Δ''' and use Proposition 1 in slightly modified form admitting squares of two lattices to estimate $\| K(\Delta, \Delta_1, \Delta'', f)\chi_{\Delta'''} \|_p$. Summation over Δ_1 gives the assertion. □

We also need trace properties of $\chi_\Delta(1 - \lambda K(s, f))^{-1}\chi_{\Delta'}$.

Proposition A.I.2. — Let Δ, Δ' be $0(1)$ lattice squares. Suppose that $\| f \|_{L^\infty} \leqslant \xi_+$ and λ is large enough. Let $\alpha > 0$, $\beta = \beta(\alpha) > 0$. Then for non-coinciding Δ, Δ'

$$\| \chi_\Delta (1 - \lambda K(s, f))^{-1} \chi_{\Delta'} \|_p \leqslant \begin{cases} 0(1) m^{-5/4 + 4/p + \alpha} & \text{if } \frac{16}{9} < p \leqslant 2, \quad (62) \\ 0(1) m^{-1/4 + 2/p} & \text{if } 2 < p \leqslant 4. \quad (63) \end{cases}$$

For non-touching Δ, Δ'

$$\| \chi_\Delta (1 - \lambda K(s, f))^{-1} \chi_{\Delta'} \|_1 \leqslant 0(1) \exp\left[- \varepsilon m d(\Delta, \Delta') \right]. \tag{64}$$

Proof. — To prove (62) and (63) it is enough to estimate $\| [\chi_\Delta, (1 - \lambda K(s, f))^{-1}] \|_p$. But for $\frac{16}{9} < p \leqslant 2$

$$\| [\chi_\Delta, (1 - \lambda K(s, f))^{-1}] \|_p \leqslant \sum_{n=1}^{\infty} \lambda^n \| [\chi_\Delta, K(s, f)^n] \|_p$$
$$\leqslant \sum_{n=1}^{\infty} n \lambda^n \| K(s, f) \|^{n-1} \| [\chi_\Delta, K(s, f)] \|_p \leqslant \lambda \sum_{n=1}^{\infty} n (0(1) \lambda m^{-1} \| f \|_{L^\infty})^{n-1}$$
$$\cdot \sum_{\substack{\Delta', \Delta'' \\ \Delta'' \neq \Delta}} (\| \chi_\Delta K(s, f\chi_{\Delta'}) \chi_{\Delta''} \|_p + \| \chi_{\Delta''} K(s, f\chi_{\Delta'}) \chi_\Delta \|_p)$$
$$\leqslant 0(1) \lambda m^{-9/4 + 4/p + \alpha} \sum_{\Delta', \Delta''} \exp\left[- \varepsilon m (d(\Delta, \Delta') + d(\Delta', \Delta'')) \right] \| f \|_{L^\infty} \leqslant 0(1) m^{-5/4 + 4/p + \alpha}.$$

(63) is proven the same way.

To prove (64) we write

$$\| \chi_\Delta (1 - \lambda K(s, f))^{-1} \chi_{\Delta'} \|_1 = \| \lambda \chi_\Delta K(s, f) \chi_{\Delta'} + \lambda^2 \chi_\Delta K(s, f)^2 \chi_{\Delta'}$$
$$+ \lambda^3 \chi_\Delta (1 - \lambda K(s, f))^{-1} K(s, f)^3 \chi_{\Delta'} \|_1 \leqslant \lambda \sum_{\Delta_1} \| \chi_\Delta K(s, f\chi_{\Delta_1}) \chi_{\Delta'} \|_1$$
$$+ \lambda^2 \sum_{\Delta_1, \Delta_2, \Delta_3} \| \chi_\Delta K(s, f\chi_{\Delta_1}) \chi_{\Delta_2} \|_{p_1} \| \chi_{\Delta_2} K(s, f\chi_{\Delta_3}) \chi_{\Delta'} \|_{p_2}$$
$$+ \lambda^3 \sum_{\substack{\Delta_i \\ i = 1, \ldots, 6}} \| \chi_\Delta (1 - \lambda K(s, f))^{-1} \chi_{\Delta_1} \| \| \chi_{\Delta_1} K(s, f\chi_{\Delta_2}) \chi_{\Delta_3} \|_3$$
$$\cdot \| \chi_{\Delta_3} K(s, f\chi_{\Delta_4}) \chi_{\Delta_5} \|_3 \cdot \| \chi_{\Delta_5} K(s, f\chi_{\Delta_6}) \chi_{\Delta'} \|_3, \tag{65}$$

where Δ_i, $i = 1, 2, \ldots, 6$ are taken from the lattice of diameter by 7 less than that of Δ, $\Delta' \cdot \frac{1}{p_1} + \frac{1}{p_2} = 1$. Now

$$\lambda \| \chi_\Delta K(s, f\chi_{\Delta_1}) \chi_{\Delta'} \|_1 \leqslant 0(1) \exp\left[- \varepsilon m (d(\Delta, \Delta_1) + d(\Delta_1, \Delta')) \right]$$

by Corollary A.I.3. Similarly we bound $\lambda^2 \| \chi_\Delta K(s, f\chi_{\Delta_1}) \chi_\Delta \|_{p_1} \cdot \| \chi_{\Delta_2} K(s, f\chi_{\Delta_3}) \chi_{\Delta'} \|_{p_2}$ by $\exp\left[- \varepsilon m (d(\Delta, \Delta_1) + d(\Delta_1, \Delta_2) + d(\Delta_2, \Delta_3) + d(\Delta_3, \Delta')) \right]$ choosing $p_1 = 2$, $p_2 = \infty$ if either Δ, Δ_1 or Δ_1, Δ_2 are not touching and $p_1 = \infty$, $p_2 = 2$ otherwise (then either Δ_2, Δ_3 or Δ_3, Δ' are not touching). Finally, the last term on the right hand side of (65) is bounded by $0(1) \exp\left[- \varepsilon m (d(\Delta, \Delta_1) + d(\Delta_1, \Delta_2) + \ldots + d(\Delta_6, \Delta')) \right]$ by Corollaries A.I.2 and 3 (we have absorbed powers of m into $\exp\left[- \varepsilon m \ldots \right]$ taking smaller ε). Summation over Δ_i yields (64). ☐

COROLLARY A.I.4. — Proposition A.I.1 holds provided $\Delta = \Delta'$ and $K(\Delta, \Delta', \Delta'', f)$ is replaced by $\chi_\Delta K^{f'}(s, f\chi_{\Delta''})$, where

$$K^{f'}(s, f) := (1 - \lambda K(s, f'))^{-1} K(s, f), \tag{66}$$

with additional assumption that $\| f' \|_{L^\infty} \leqslant \xi_+$ and λ is large enough.

Proof. — We have

$$\| \chi_\Delta K^{f'}(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_p \leqslant \sum_{\Delta_1} \| \chi_\Delta (1 - \lambda K(s, f'))^{-1} \chi_{\Delta_1} \|_{p_1} \cdot \| \chi_{\Delta_1} K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_{p_2}, \tag{67}$$

where $\frac{1}{p_1} + \frac{1}{p_2} \geqslant \frac{1}{p}$. To prove 1.*a*) and 1.*b*) case we put $p_1 = \infty$, $p_2 = p$ and use Corollaries A.I.2 and 3. To prove 2.*a*) we put $p_1 = \infty$, $p_2 = p$ and use Corollaries A.I.2 and 3 unless $\Delta_1 = \Delta'' = \Delta'''$. In this case we put $p_1 = p$ and $p_2 = \infty$ and use Proposition A.I.2 and Corollary A.I.3 (1.*a*) for $\| \chi_{\Delta_1} K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|$). 2.*c*) is proven as 2.*a*). To prove 2.*b*) we again take $p_1 = \infty$, $p_2 = p$ and use Corollaries A.I.2, 3 unless $\Delta_1 = \Delta'' = \Delta'''$. In the latter case we take $p_2 = 4$ and use Corollary A.I.3 (1.*b*) for $\| \chi_{\Delta_1} K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|$) and Proposition A.I.2 ((63), (64)). Finally to prove 3.*a*), *b*) we choose Δ_1 from the lattice of diameter by $\frac{1}{2}$ less than that of Δ, Δ'' and Δ'''. If Δ_1, Δ'' or Δ'', Δ''' are not touching then we take $p_1 = \infty$ and use Corollaries A.I.2 and 3 (or rather their easy modifications admitting squares of two lattices) to get the bound. If Δ_1, Δ'' and Δ'', Δ''' are touching then Δ, Δ_1 are not and we take $p_1 = 1$, $p_2 = 4$ using Proposition A.I.2 and Corollary A.I.3 (1.*a*) or 1.*b*) for $\| \chi_{\Delta_1} K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_4$). □

The last information about trace properties of K operators concerns differences $K(s, f) - K(f)$, where $K(f) = K(s \equiv 1, f)$.

PROPOSITION A.I.3. — Let Δ, Δ'', Δ''' be 0(1)-lattice squares. Let $\alpha > 0$, $\beta = \beta(\alpha) > 0$. Then for $\frac{16}{9} < p \leqslant 2$

$$\| \chi_\Delta (K(s, f\chi_{\Delta''}) - K(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-9/4 + 4/p + \alpha} \cdot S^\beta(f\chi_{\Delta''}) \leqslant 0(1) m^{-9/4 + 4/p + \alpha} \| f\chi_{\Delta''} \|_{L^\infty} \tag{68}$$

and for $2 < p \leqslant 4$

$$\| \chi_\Delta (K(s, f\chi_{\Delta''}) - K(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-5/4 + 2/p} \| f\chi_{\Delta''} \|_{L^\infty}. \tag{69}$$

Proof. — (68) and (69) follow easily as Corollary A.I.3 since

$$\| \chi_\Delta (K(s, f\chi_{\Delta''}) - K(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p \leqslant 2 \sum_{\substack{\Delta_1 \\ \Delta_1 \notin \Delta' \cap \Delta'' \cap \Delta'''}} \| K(\Delta, \Delta', \Delta'', f)\chi_{\Delta'''} \|_p. \quad \square \tag{70}$$

Finally we need

PROPOSITION A.I.4. — Let Δ, Δ', Δ'', Δ''' be 0(1)-lattice squares. Let h be a function on $\mathbb{R}^2$, $h = \xi_+ \sum_{\Delta'} \Sigma(\Delta')\chi_{\Delta'}$ with $\Sigma(\Delta') = \pm$. Suppose that λ is large enough. Then

$$\| \chi_\Delta (K^h(s, f\chi_{\Delta''}) - K^{\Sigma(\Delta'')\xi_+}(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p \leqslant 0(1) m^{-9/4 + 4/p + \alpha} S^\beta(f\chi_{\Delta''}) \tag{71}$$

for $\frac{16}{9} < p \leqslant 2$, $\alpha > 0$ and $\beta = \beta(\alpha) > 0$.

Proof.

$$
\begin{aligned}
&\| \chi_\Delta(K^h(s, f\chi_{\Delta''}) - K^{\Sigma(\Delta'')\xi_+}(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p \\
&\quad \leqslant \| \chi_\Delta((1 - \lambda K(s, h))^{-1} - (1 - \lambda K(s, \Sigma(\Delta'')\xi_+))^{-1})K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_p \\
&\quad + \| \chi_\Delta((1 - \lambda K(s, \Sigma(\Delta'')\xi_+))^{-1} - (1 - \lambda K(\Sigma(\Delta'')\xi_+))^{-1})K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_p \\
&\quad + \| \chi_\Delta(1 - \lambda K(\Sigma(\Delta'')\xi_+))^{-1}(K(s, f\chi_{\Delta''}) - K(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p \\
&\quad = \lambda \| \chi_\Delta(1 - \lambda K(s, h))^{-1}K(s, h - \Sigma(\Delta'')\xi_+)(1 - \lambda K(s, \Sigma(\Delta'')\xi_+))^{-1}K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_p \\
&\quad + \lambda \| \chi_\Delta(1 - \lambda K(s, \Sigma(\Delta'')\xi_+))^{-1}(K(s, \Sigma(\Delta'')\xi_+) - K(\Sigma(\Delta'')\xi_+)) \\
&\qquad \cdot (1 - \lambda K(\Sigma(\Delta'')\xi_+))^{-1}K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_p \\
&\quad + \| \chi_\Delta(1 - \lambda K(\Sigma(\Delta'')\xi_+))^{-1}(K(s, f\chi_{\Delta''}) - K(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p \\
&\quad \leqslant \lambda \sum_{\Delta_1,\Delta_2} \| \chi_\Delta K^h(s, (h - \Sigma(\Delta'')\xi_+)\chi_{\Delta_1})\chi_{\Delta_2} \|_{p_1} \| \chi_{\Delta_2} K^{\Sigma(\Delta'')\xi_+}(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_{p_2} \\
&\quad + \lambda \sum_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \| \chi_\Delta(1 - \lambda K(s, \Sigma(\Delta'')\xi_+))^{-1}\chi_{\Delta_1} \| \| \chi_{\Delta_1}(K(s, \Sigma(\Delta'')\xi_+\chi_{\Delta_2}) \\
&\quad - K(\Sigma(\Delta'')\xi_+\chi_{\Delta_2}))\chi_{\Delta_3} \|_{p_3} \| \chi_{\Delta_3}(1 - \lambda K(\Sigma(\Delta'')\xi_+))^{-1}\chi_{\Delta_4} \| \cdot \| \chi_{\Delta_4} K(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_4 \\
&\quad + \sum_{\Delta_1} \| \chi_\Delta(1 - \lambda K(\Sigma(\Delta'')\xi_+))^{-1}\chi_{\Delta_1} \| \| \chi_{\Delta_1}(K(s, f\chi_{\Delta''}) - K(f\chi_{\Delta''}))\chi_{\Delta'''} \|_p ,
\end{aligned}
\tag{72}
$$

where Δ_i are choosen from the lattice of diameter by $\frac{1}{5}$ less than that of Δ, Δ'', Δ''' and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{p_3} + \frac{1}{4} = \frac{1}{p}$. If $\Delta_2 \subset \Delta'' = \Delta'''$ in the first sum then for non-zero terms Δ_1 cannot coincide with Δ_2 since $\Delta_1 \not\subset \Delta''$. We put $p_2 = 4$ *and* $p_1 = p_3$ and use Corollary A.I.4 (2.*c*) for $\| \chi_\Delta K^h(s, (h - \Sigma(\Delta'')\xi_+)\chi_{\Delta_1})\chi_{\Delta_2} \|_{p_3}$ and 1.*b*) or rather its easy modification with squares from two lattices for $\| \chi_{\Delta_2} K^{\Sigma(\Delta'')\xi_+}(s, f\chi_{\Delta''})\chi_{\Delta'''} \|_4)$. Otherwise we take $p_1 = \infty$, $p_2 = p$ and use again (modified) Corollary A.I.4. The second and the third sum is estimated with use of Corollaries A.I.2, 3 and Proposition A.I.3. □

APPENDIX II

Now let us consider the facts about the ultra-violet cut-off operator $R_\varkappa$ used in the text. We repeat the definition of [13] with the only modification that we adapt it to the squares Δ of the d-lattice:

$$(R_\varkappa f)(x) := \sum_{i=1}^{4} \int_\Delta dy \rho_\varkappa(x^{(i)} - y) f(y), \tag{1}$$

where $\rho_\varkappa$ is the usual smearing function and $x^{(i)}$ denotes either x or its subsequent reflections in two nearest walls of Δ.

We have an estimate:

$$\| R_\varkappa f \|_{L^q} \leqslant 4 \| f \|_{L^q}. \tag{2}$$

It follows by applying the Hölder inequality to the integral in (1):

$$\left| \int_\Delta \rho_\varkappa(x^{(i)} - y) f(y) dy \right| \leqslant \left(\int \rho_\varkappa(x^{(i)} - y) dy \right)^{1/q'} \left(\int_\Delta \rho_\varkappa(x^{(i)} - y) \mid f(y) \mid^q dy \right)^{1/q},$$

hence

$$\| R_\varkappa f \|_{L^q} \leqslant \sum_{p=1}^{4} \left(\int_{\Delta \times \Delta} dx dy \rho_\varkappa(x^{(i)} - y) \mid f(y) \mid^q \right)^{1/q} \leqslant 4 \| f \|_{L^q}.$$

Let us denote the kernel of the operator $\delta\chi_\Delta C_{m_c}(\tau)\chi_\Delta\delta$ by

$$(\delta\chi_\Delta C_{m_c}(\tau)\chi_\Delta\delta)(x, y) = \int (\delta\varphi)(x)(\delta\varphi)(y) d\mu_{m_c}(\tau). \tag{3}$$

We have then

$$(\delta\chi_\Delta C_{m_c}(\tau)\chi_\Delta\delta)(x, y) = (\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(x, y) - \frac{1}{|\Delta|} \chi_\Delta(x) \int_\Delta (\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(x', y) dx'$$
$$- \frac{1}{|\Delta|} \int_\Delta (\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(x, y') dy' \chi_\Delta(y) + \frac{1}{|\Delta|^2} \int_{\Delta \times \Delta} (\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(x', y') dx' dy' \chi_\Delta(x)\chi_\Delta(y). \tag{4}$$

LEMMA A.II.1. — There exists $C(q) < \infty$, $1 \leqslant q < \infty$, such that for each τ

1. $\| (R_\varkappa \chi_\Delta C_{m_c}(\tau)\chi_\Delta R_\varkappa)(.,.) \|_{L^q(\mathbb{R}^4)} \leqslant C(q)$,
2. $\| ((R_\varkappa - R_{\varkappa'})\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(.,.) \|_{L^q(\mathbb{R}^4)} \leqslant C(q) \min(\varkappa, \varkappa')^{-\varepsilon(q)}$.

1. and 2. hold also if we replace $\chi_\Delta C_{m_c}(\tau)\chi_\Delta$ by

$$\delta\chi_\Delta C_{m_c}(\tau)\chi_\Delta\delta, \qquad \chi_\Delta f C_{m_c}(\tau) f \chi_\Delta \qquad \text{or} \qquad \chi_\Delta \delta f C_{m_c}(\tau) f \delta \chi_\Delta,$$

provided $\| f \|_{L^\infty}$, $\| \nabla f \|_{L^\infty} \leqslant 0(1)$.

Proof. — We shall prove 1. and 2. Their modifications may be obtained the same way.

1. is obvious by (2) and the estimates of $C_{m_c}(\tau)$.

2. Suppose first that $2 \leqslant q < 4$. Let $\zeta \in C_0^\infty(\Delta)$, $\zeta(x) = 0$ when $d(x, \partial\Delta) \leqslant \min(\varkappa, \varkappa')^{-1}$, $\zeta(x) = 1$ when $d(x, \partial\Delta) \geqslant 2\min(\varkappa, \varkappa')^{-1}$, $0 \leqslant \zeta \leqslant 1$.

We have

$$\| ((R_\varkappa - R_{\varkappa'})\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(.,.) \|_{L^q} \leqslant \| ((R_\varkappa - R_{\varkappa'})\zeta C_{m_c}(\tau)\zeta)(.,.) \|_{L^q}$$
$$+ 8 \| (1 - \zeta \otimes \zeta)\chi_\Delta \otimes \chi_\Delta \|_{L^{2q}} \| (\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(.,.) \|_{L^{2q}}, \tag{5}$$

but

$$((R_\varkappa - R_{\varkappa'})\zeta C_{m_c}(\tau)\zeta)(x, y) = ((\rho_\varkappa - \rho_{\varkappa'}) * (\zeta C_{m_c}(\tau)\zeta)(., y))(x), \tag{6}$$

hence

$$\| ((R_\varkappa - R_{\varkappa'})\zeta C_{m_c}(\tau)\zeta)(.,.) \|_{L^q} \leqslant 0(1)\left(\int | (\hat{\rho}_\varkappa(p) - \hat{\rho}_{\varkappa'}(p))\widehat{(\zeta C_{m_c}(\tau)\zeta)}(p, q) |^{q'} dpdq \right)^{1/q'} . \quad (7)$$

Furthermore we have

$$| \hat{\rho}_\varkappa(p) - \hat{\rho}_{\varkappa'}(p) | \leqslant \int_0^1 dt \left| p \cdot \operatorname{grad} \hat{\rho}_1\left(p\left(\frac{t}{\varkappa} + \frac{1-t}{\varkappa'} \right) \right) \right| \left| \frac{1}{\varkappa} - \frac{1}{\varkappa'} \right| \leqslant 0(1)(1 + p^2)^{\varepsilon/2} \left| \frac{1}{\varkappa^\varepsilon} - \frac{1}{\varkappa'^\varepsilon} \right|$$

so (7) implies

$$\| ((R_\varkappa - R_{\varkappa'})\zeta C_{m_c}(\tau)\zeta)(.,.) \|_{L^q} \leqslant 0(1) \left| \frac{1}{\varkappa^\varepsilon} - \frac{1}{\varkappa'^\varepsilon} \right| \| (1 + p^2)^{\varepsilon/2} \widehat{(\zeta C_{m_c}(\tau)\zeta)}(.,.) \|_{L^{q'}} . \quad (8)$$

But, since $q' > \frac{4}{3}$, $\| (1 + p^2)^{\varepsilon/2}\widehat{(\zeta C_{m_c}(\tau)\zeta)}(.,.) \|_{L^{q'}}$ is bounded uniformly in τ for ε small enough by Proposition 7.4 of [*11*]. Similarly the second term on the right hand side of (5) can be estimated by $0(1) \min (\varkappa, \varkappa')^{-1/2q}$ and finally we get 2. for $1 \leqslant q < 4$. For $4 \leqslant q < \infty$ we write

$$\| ((R_\varkappa - R_{\varkappa'})\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(.,.) \|_{L^q} \leqslant \| ((R_\varkappa - R_{\varkappa'})\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(.,.) \|_{L^2}^\theta \cdot \| ((R_\varkappa - R_{\varkappa'})\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(.,.) \|_{L^{2q}}^{1-\theta} \quad \text{with} \quad \theta = \frac{1}{q-1}. \quad (9)$$

The first factor on the right side above has been estimated already, and the second one can be bounded by $(8 \| (\chi_\Delta C_{m_c}(\tau)\chi_\Delta)(.,.) \|_{L^{2q}})^{1-\theta} \leqslant C(q)$, and this gives 2. in the general case. □

Next we pass to examination of operator valued random variables of the type $(P^2 + m^2)^{-\nu}\chi_\Delta\varphi(P^2 + m^2)^{-\lambda}$ and their trace properties. We shall prove some estimates which are slight refinements of the ones obtained in [*24*] [*25*]. To prove these we need a good control over the functions $H^{m,s}(x)$ given by

$$H^{m,s}(x) = \frac{1}{(2\pi)^2} \int dp e^{ipx}(p^2 + m^2)^{-s} \quad \text{for} \quad 0 < s < 1 . \quad (10)$$

$H^{m,s}(x - y)$ is the kernel of the operator $(-\Delta + m^2)^{-s}$.

We have to estimate also

$$H^{m,s}_\Delta(x) = \frac{1}{|\Delta|} \sum_{p \in \frac{2\pi}{|\Delta|^{1/2}}\mathbb{Z}^2} e^{ipx}(P^2 + m^2)^{-s} = \sum_{y \in |\Delta|^{1/2}\mathbb{Z}^2} H^{m,s}(x + y) . \quad (11)$$

$H^{m,s}_\Delta(x - y)$ is the kernel of the operator $(p^2 + m^2)_\Delta^{-s}$ on $L^2(\Delta)$. We have

$$0 \leqslant H^{1,s}(x) \leqslant C_s | x |^{-2+2s} e^{-\frac{1}{2}|x|}, \quad (12)$$

which is easily seen from the expression

$$H^{1,s}(x) = \frac{1}{2\pi\Gamma(s)} \int_0^\infty t^{-2+s} e^{-\frac{x^2}{4t}} e^{-t} dt . \quad (13)$$

Also

$$H^{m,s}(x) = m^{2-2s} H^{1,s}(mx), \quad (14)$$

hence

$$0 \leqslant H^{m,s}(x) \leqslant C_s | x |^{-2+2s} e^{-\frac{1}{2}m|x|} . \quad (15)$$

Moreover for $x \in \Delta$, Δ centered at the origin,

$$0 \leqslant H^{m,s}_\Delta(x) \leqslant \sum_{y \in |\Delta|^{1/2}\mathbb{Z}^2} C_s | x + y |^{-2+2s} e^{-\frac{1}{2}m|x+y|} \leqslant C'_s | x |^{-2+2s} e^{-\frac{1}{4}mx} . \quad (16)$$

Applying this estimate and scaling we get

$$\| H^{m,s}\chi_\Delta \|_{L^r} \leqslant C_s m^{-\varepsilon(r)} \qquad \text{for} \qquad 1 \leqslant r < (1-s)^{-1} \tag{17}$$

and Δ centered at the origin. The same inequality holds also for $H^{m,s}_\Delta$.

Let now $\mathscr{A}$ be an operator valued random variable on « a support » of $d\mu_{m_c}(\tau)$. We shall denote

$$\| \mathscr{A} \|_{q,p,\tau} = \left(\int \| \mathscr{A} \|^p_q \, d\mu_{m_c}(\tau) \right)^{1/p}, \tag{18}$$

(compare [23]).

LEMMA A.II.2. — Let $2 \leqslant q \leqslant 4$, $(\nu + \lambda)q > 1$,

$$\mathscr{A}_\varkappa := (p^2 + m^2)^{-\nu} \varphi_\varkappa \chi_\Delta (p^2 + m^2)^{-\lambda}, \tag{19}$$

$$\mathscr{A}'_\varkappa := (p^2 + m^2)^{-\nu}_\Delta \varphi_\varkappa (p^2 + m^2)^{-\lambda}_\Delta . \tag{20}$$

There exist $C > 0$ and $\nu_1 > 0$ (independent of m, $\varkappa$ and τ) such that for $p \geqslant 2q$

$$\| \mathscr{A}^\#_\varkappa \|_{q,p,\tau} \leqslant C p^{1/2} m^{-\nu_1}, \tag{21}$$

$$\| \mathscr{A}^\#_\varkappa - \mathscr{A}^\#_{\varkappa'} \|_{q,p,\tau} \leqslant C p^{1/2} m^{-\nu_1} (\min(\varkappa, \varkappa'))^{-\nu_1}. \tag{22}$$

$\mathscr{A}^\#_\varkappa$ denotes either $\mathscr{A}_\varkappa$ or $\mathscr{A}'_\varkappa$.

The same holds if we replace φ by $f\varphi$, $\delta(f\varphi)$, $(f\varphi)_\Delta$ with f smooth, $\| f \|_{L^\infty}$, $\| \nabla f \|_{L^\infty} \leqslant 0(1)$, or finally $\varphi_\varkappa$ by an arbitrary Wick ordered polynomial $:\mathscr{P}(\varphi_\varkappa):_\tau$ or $f :\mathscr{P}(\varphi_\varkappa):_\tau$ and $p^{1/2}$ by $p^{1/2 \deg \mathscr{P}}$.

Proof. — Mimicking Proof of Theorem 3.4 of [25] we consider first the cases $q = 2$, $q = 4$ and $p = 2q$. Take for example the case of $\mathscr{A}'_\varkappa$.

$$\begin{aligned} \| \mathscr{A}'_\varkappa \|^4_{2,4,\tau} = & \int_{\Delta^4} \prod_{i=4}^{4} dx_i H^{m,2\lambda}_\Delta(x_1 - x_2) H^{m,2\nu}_\Delta(x_2 - x_1) \\ & \cdot H^{m,2\lambda}_\Delta(x_3 - x_4) H^{m,2\nu}_\Delta(x_4 - x_3) \sum_{\substack{\text{pairings } \{l\} \\ \text{of } \{1,\ldots,4\}}} \prod_l (R_\varkappa \chi_\Delta C_{m_c}(\tau) \chi_\Delta R_\varkappa)(x_{l_-}, x_{l_+}) \\ & \leqslant 3 |\Delta|^{2/r} \| H^{m,2\lambda}_\Delta \|^2_{L^{r_1}} \| H^{m,2\nu}_\Delta \|^2_{L^{r_2}} \| (R_\varkappa \chi_\Delta C_{m_c}(\tau) \chi_\Delta R_\varkappa)(.,.) \|^2_{L^{r'}} \leqslant 0(1) m^{-\nu_1}, \end{aligned} \tag{23}$$

by Lemma A.II.1 and (17), if one can find r_1, r_2 such that $\frac{1}{r_1} > 1 - 2\lambda$, $\frac{1}{r_2} > 1 - 2\nu$ and $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} < 1$. This is possible if $2(\nu + \lambda) > 1$.

Similarly

$$\| \mathscr{A}'_\varkappa - \mathscr{A}'_{\varkappa'} \|_{2,4,\tau} \leqslant 0(1) m^{-\nu_1} (\min(\varkappa, \varkappa'))^{-\nu_1} \tag{24}$$

and we use Lemma A.II.1, point 2. For $q = 2$ and $p > 4$ we obtain additionally $p^{1/2}$ factor using the Nelson's hypercontractive bound [28].

Similarly for $q = 4$, $p = 8$

$$\begin{aligned} \| \mathscr{A}'_\varkappa \|^8_{4,8,\tau} = & \int_{\Delta^8} \prod_{i=1}^{8} dx_i H^{m,2\lambda}_\Delta(x_1 - x_2) H^{m,2\nu}_\Delta(x_2 - x_3) H^{m,2\lambda}_\Delta(x_3 - x_4) \\ & \cdot H^{m,2\nu}_\Delta(x_4 - x_1) H^{m,2\lambda}_\Delta(x_5 - x_6) H^{m,2\nu}_\Delta(x_6 - x_7) H^{m,2\lambda}_\Delta(x_7 - x_8) H^{m,2\nu}(x_8 - x_5) \\ & \cdot \sum_{\text{pairings } \{l\} \text{ of } \{1,\ldots,8\}} \prod_l (R_\varkappa \chi_\Delta C_{m_c}(\tau) \chi_\Delta R_\varkappa)(x_{l_-}, x_{l_+}) \\ & \leqslant 0(1) \| (H^{m,2\lambda}_\Delta)^r * (H^{m,2\nu}_\Delta)^r * (H^{m,2\lambda}_\Delta)^r * (H^{m,2\nu}_\Delta)^r \|^{2/r}_{L^2} \| (R_\varkappa \chi_\Delta C_{m_c}(\tau) \chi_\Delta R_\varkappa)(.,.) \|^4_{r'} \\ & \leqslant 0(1) \| H^{m,2\lambda}_\Delta \|^4_{r_1} \| H^{m,2\nu}_\Delta \|^4_{r_2} \| (R_\varkappa \chi_\Delta C_{m_c}(\tau) \chi_\Delta R_\varkappa)(.,.) \|^4_{r'} \leqslant 0(1) m^{-\nu_1}, \end{aligned} \tag{25}$$

if we can choose r_1, r_2 such that $r_1 \geqslant r > 1$, $r_2 \geqslant r$, $\frac{2r}{r_1} + \frac{2r}{r_2} - 3 = 0$ and $\frac{1}{r_1} > 1 - 2\lambda$, $\frac{1}{r_2} > 1 - 2\nu$, which is possible if $4(\nu + \lambda) > 1$.

In the inequality (25) $*$ denotes the convolution on Δ with periodic boundary conditions and we have used the periodic version of the Hausdorff-Yang theorem. Again in the case of general p the hypercontractivity gives the factor $p^{1/2}$.

$\| \mathcal{A}'_\varkappa - \mathcal{A}'_{\varkappa'} \|_{4,p,\tau}$ is estimated analogically, as also the expressions with $\mathcal{A}_\varkappa$ instead of $\mathcal{A}'_\varkappa$ and with φ changed for $f\varphi$, $\delta(f\varphi)$ or $(f\varphi)_\Delta$ (we use a version of Lemma A.II.2 with $\chi_\Delta C_{m_c}(\tau)\chi_\Delta$ replaced by $f\chi_\Delta C_{m_c}(\tau)\chi_\Delta f$).

The case of general q follows by interpolation as in proof of Theorem 3.4 in [*25*]. The idea of the proof for a Wick polynomial $:\mathcal{P}(\varphi_\varkappa):_\tau$ is essentially the same, with the obvious modifications of the formulae (23), (25), so we omit the proof. □

LEMMA A.II.3. — The operator norm limit $\lim_{\varkappa\to\infty} \mathcal{A}^\#_\varkappa =: \mathcal{A}^\#$ exists almost everywhere and is an operator valued random variable with values in $\mathcal{C}_q$ and with the $\|\cdot\|_{q,p,\tau}$ norm finite and bounded as in (21), $q > (\nu + \lambda)^{-1}$, $p < \infty$. The same is true if we put $f\varphi$, $\delta(f\varphi)$, $(f\varphi)_\Delta$ or a Wick polynomial instead of φ in $\mathcal{A}^\#$. □

LEMMA A.II.4. — Operator norm limit $\lim_{\varkappa\to\infty} K(\Delta, \Delta', \Delta'', \varphi_\varkappa) =: K(\Delta, \Delta', \Delta'', \varphi_\varkappa)$ exists almost everywhere and is an operator valued random variable with values in $\mathcal{C}_q$, $q > 2$ if $\Delta = \Delta' = \Delta''$, $q > \frac{16}{9}$ if both Δ, Δ' and Δ', Δ'' are touching but either Δ, Δ' or Δ', Δ'' do not coincide, $q \geqslant 1$ if either Δ, Δ' or Δ', Δ'' are not touching. Operator norm limit $\lim_{\varkappa\to\infty} K_\Delta(\varphi_\varkappa) =: K_\Delta(\varphi)$ exists almost everywhere and is an operator valued random variable with values in $\mathcal{C}_q$, $q > 2$.

Proof of this Lemma is straightforward by Proposition A.I.1 and Lemma A.II.2. □

LEMMA A.II.5. — In the notation of Lemma A.II.2 and under the assumptions listed there

$$\int e^{t\|\mathcal{A}^\#\|_q^2} d\mu_{m_c}(\tau) \leqslant e^{O(1)|t|m^{-\nu_1}} \tag{26}$$

for $|t| \leqslant \alpha$ and m sufficiently large. The same holds if we replace φ by $f\varphi$, $\delta(f\varphi)$, $(f\varphi)_\Delta$; f smooth with $\|f\|_{L^\infty} \leqslant O(1)$, $\|\nabla f\|_{L^\infty} \leqslant O(1)$. Also if φ is replaced by $:\varphi^2:_\tau$ or $f:\varphi^2:_\tau$ and $\|\mathcal{A}^\#\|_q^2$ by $\|\mathcal{A}^\#\|_q$, (26) remains true.

Proof. — Integrability of $\exp[t\|\mathcal{A}^\#\|_q^2]$ for $|t| \leqslant \alpha$ and large m together with the uniform bound

$$\int e^{2\alpha\|\mathcal{A}^\#\|_q^2} d\mu_{m_c}(\tau) \leqslant C < \infty, \tag{27}$$

where C does not depend on m, r or τ, follows for $q = 2, 4$ from Theorem 3.1 of [*25*] and its proof and for $2 < q < 4$ by interpolation.

Now

$$\left|\int (e^{t\|\mathcal{A}^\#\|_q^2} - 1) d\mu_{m_c}(\tau)\right| \leqslant |t| \int \|\mathcal{A}^\#\|_q^2 (e^{\alpha\|\mathcal{A}^\#\|_q^2} + 1) d\mu_{m_c}(\tau) \leqslant O(1)|t| \, \|\mathcal{A}^\#\|_{q,4,\tau}^2 \leqslant O(1)|t| m^{-\nu_1}, \tag{28}$$

for m sufficiently large by the inequality $|e^x - 1| \leqslant |x|(e^x + 1)$, the Hölder inequality, Lemma A.II.3 and (27).

Hence

$$\int e^{t\|\mathscr{A}^{\#}\|_q^2} d\mu_{m_c}(\tau) \leqslant 1 + 0(1)\,|t|\,m^{-\nu_1} \leqslant e^{0(1)|t|m^{-\nu_1}}.$$

The other cases are proven the same way. □

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