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Characterizing the conformality in a Minkowski space

by

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ABSTRACT. — Given a Minkowski space M of dimension $m \geqslant 3$, we prove the following theorem: let $F : U \to M$ be an injective map defined on a connected open set U in M; for any closed segment σ which is contained in U and lies in a light ray, let us suppose that $F(\sigma)$ is a closed segment which lies in a light ray; then F is the restriction to U of a conformal transformation in M. The condition $m \geqslant 3$ and the connectivity of U are essential. The physical meaning of this theorem consists in the fact that every local transformation in M which preserves the special relativistic law of light propagation is a conformal transformation. We derive a generalization of Zeeman's theorem [7] concerning the luminal case. We also obtain a local characterization of Weyl transformations (*i. e.* Lorentz transformations composed with translations and dilatations).

1. INTRODUCTION

In the m-dimensional real space $\mathbb{R}^m$, $m \geqslant 2$, we consider the Minkowski quadratic form Q given by

$$(1.1) \qquad Q(x) = x_1^2 - x_2^2 - \dots - x_m^2,$$

where $(x_1, x_2, \dots, x_m)$ are the canonical coordinates in $\mathbb{R}^m$. We say that pair $M = (\mathbb{R}^m, Q)$ is the *m-dimensional Minkowski space.* We shall also use the canonical affine structure on $\mathbb{R}^m$ and denote by $A(m ; \mathbb{R})$ the corresponding affine group acting on $\mathbb{R}^m$. We note that $A(m ; \mathbb{R})$ is generated by the group $GL(m ; \mathbb{R})$ together with the group $\mathcal{T}$ of all translations of $\mathbb{R}^m$.

The *Lorentz group* $\mathcal{L}$ of the Minkowski space M consists of all linear applications $F \in GL(m ; \mathbb{R})$ such that $Q(F(x)) = Q(x)$ for any $x \in \mathbb{R}^m$. The *Poincaré group* $\mathcal{P}$ of M is the subgroup of $A(m ; \mathbb{R})$ generated by $\mathcal{L}$ and $\mathcal{T}$. The *Weyl group* $\mathcal{W}$ of M is generated in $A(m ; \mathbb{R})$ by $\mathcal{P}$ together with the dilatations of $\mathbb{R}^m$.

We recall that the *light cone* of the Minkowski space M with the vertex at a point $x \in \mathbb{R}^m$ is the degenerate quadric of all points $y \in \mathbb{R}^m$ for which $Q(x - y) = 0$. A *light ray* of M is a straight line which lies in a light cone. A *light segment* of M is a closed segment which lies in a light ray.

Now we briefly indicate the construction of the *conformal group* $\mathcal{C}$ which acts on the *compactified Minkowski space* $\overline{M}$ corresponding to M. To begin with, we consider the quadratic space $(\mathbb{R}^{m+2}, \tilde{Q})$, where

$$\tilde{Q}(y) = y_1^2 - y_2^2 - \dots - y_m^2 - y_{m+1}y_{m+2}\,. \tag{1.2}$$

We introduce the pointed cone

$$C = \{\, y \in \mathbb{R}^{m+2} : \tilde{Q}(y) = 0,\ y \neq 0 \,\}, \tag{1.3}$$

and denote by $\psi : \mathbb{R}^{m+2} - \{0\} \to P^{m+1}$ the canonical projection onto the $(m+1)$-dimensional real projective space P^{m+1}. We identify the compactified Minkowski space $\overline{M}$ with the compact quadric $\psi(C)$. Moreover, we can imbed the Minkowski space M onto a dense open submanifold of $\overline{M}$ by the map $\overline{\psi} : M \to \overline{M}$ given by

$$\overline{\psi}(x) = \psi(x_1, \dots, x_m, Q(x), 1), \quad \forall x \in \mathbb{R}^m. \tag{1.4}$$

Let $P(m + 1 ; \mathbb{R})$ be the projective group acting on P^{m+1}. We note that any transformation $G \in P(m + 1 ; \mathbb{R})$ is given by

$$y'_\mu = G_{\mu\nu} y_\nu, \quad \mu, \quad \nu = 1, \dots, m + 2, \tag{1.5}$$

where the matrix $(G_{\mu\nu})$ is defined by G up to a non-null scalar factor.

On the other hand, we consider the orthogonal group $0(\tilde{Q})$ as the subgroup of $GL(m + 2 ; \mathbb{R})$ which preserves $\tilde{Q}$. Denoting by I the unity matrix of degree $m + 2$ and putting $Z_2 = \{ I, -I \}$, we see that the subgroup $\mathcal{C} = 0(\tilde{Q})/Z_2$ of $P(m + 1 ; \mathbb{R})$ leaves invariant the quadric $\overline{M} = \psi(C)$. We say that $\mathcal{C}$, together with its canonical action on $\overline{M}$, is the conformal group associated to the Minkowski space M.

An element $G \in \mathcal{C}$ corresponds to a matrix $(G_{\mu\nu})$ for which

$$\sum_{\rho,\sigma=1}^{m+2} G_{\mu\rho} G_{\nu\sigma} \eta_{\rho\sigma} = \eta_{\mu\nu}\,, \tag{1.6}$$

where $\eta_{\mu\nu}(\mu, \nu = 1, \ldots, m + 2)$ are given by

$$(1.7) \quad \begin{cases} \eta_{ij} = \varepsilon_i \delta_{ij}, & i,j = 1, \ldots, m, \quad \varepsilon_1 = -\varepsilon_2 = \ldots = -\varepsilon_m = 1 ; \\ \eta_{ri} = \eta_{ir} = 0, & i = 1, \ldots, m, \quad r = m + 1, \quad m + 2 ; \\ \eta_{rs} = \eta_{sr} = -2, \quad \eta_{rr} = \eta_{ss} = 0, & r = m + 1, \quad s = m + 2 . \end{cases}$$

We note that G defines up to the sign the matrix $(G_{\mu\nu})$.

Further, G induces by means of the imbedding $\overline{\psi}$ the following analytic transformation in $\mathbb{R}^m$

$$(1.8) \quad x'_i = \frac{G_{ij}x_j + G_{ir}Q(x) + G_{is}}{G_{sj}x_j + G_{sr}Q(x) + G_{ss}}, \ i, j = 1, \ldots, m, r = m + 1, s = m+2,$$

called a *conformal transformation* in the Minkowski space M. If we take $G_{sj} = G_{sr} = G_{ir} = 0$ in (1.8), we obtain the Weyl group $\mathscr{W}$. For any map G given by (1.8) which is not a Weyl transformation, the set of all singularities of G is either a light cone or a $(m - 1)$-plane tangent to a light cone.

It is a well-known property of every conformal transformation G to map light cones into light cones. Since any light ray appears as the intersection between two tangent light cones, it follows that G maps also light rays into light rays. More precisely, if U is a connected open set in the domain of G, then G maps diffeomorphically U onto the connected open set G(U), and any light segment contained in U onto a light segment contained in G(U).

The aim of this paper is to prove the following converse property:

THEOREM 1.1. — *In a Minkowski space* $M = (\mathbb{R}^m, Q)$ *of dimension* $m \geqslant 3$, *let* $F : U \to \mathbb{R}^m$ *be an injective map defined on a connected open set* U *in* $\mathbb{R}^m$. *If* F *maps any light segment contained in* U *onto a light segment, then* F *is the restriction to* U *of a conformal transformation.*

It is known that Theorem 1.1 holds for F of class C^3 (*see* e. g. [2, p. 377-384] and 6, p. 46-50]). Our contribution is that we eliminated any assumption for F to be differentiable. We require the continuity of F along light rays only.

The theorem of Zeeman [7] concerning the luminal case derives from

COROLLARY 1.2. — *In a Minkowski space* $M = (\mathbb{R}^m, Q)$ *of dimension* $m \geqslant 3$, *let* $F: \mathbb{R}^m \to \mathbb{R}^m$ *be an injective map. If* F *maps any light segment of* M *onto a light segment, then* F *is a Weyl transformation.*

Corollary 1.2 is an immediate consequence of Theorem 1.1. Indeed, the single conformal transformations whose domains coincide with $\mathbb{R}^m$ are Weyl transformations.

Remark. — We cannot derive the theorem of Alexandrov [*1*] concerning the luminal case from Theorem 1.1. On the other hand, Corollary 1.2 is not a consequence of the above mentioned theorem of Alexandrov.

In Section 2 some auxiliary results are obtained. Since the main theorem of projective geometry cannot be used to prove Theorem 1.1 and Corollary 1.2, we have taken some elements from the original proof of E. C. Zeeman in a modified form, and introduced other new constructions.

In Section 3 we shall prove Theorem 1.1 ; we shall also establish its

COROLLARY 1.3. — *In a Minkowski space* $M = (\mathbb{R}^m, Q)$ *of dimension* $m \geqslant 3$, *let* $F : U \to \mathbb{R}^m$ *be an injective map which is defined on a connected open set* U *in* $\mathbb{R}^m$ *and applies any light segment contained in* U *onto a light segment. We consider m pairs* (R_i, R'_i), $1 \leqslant i \leqslant m$, *of distinct parallel light rays which intersect* U, *such that the lines* R_i *pass through a point* $a \in U$ *and generate affinely* $\mathbb{R}^m$. *For any* $i = 1, \ldots, m$, *let us suppose that* $F(U \cap R_i)$ *and* $F(U \cap R'_i)$ *are contained in two parallel light rays, respectively. Then* F *is the restriction to* U *of a Weyl transformation.*

We remark that Corollary 1.3 generalizes the theorem given in [5].

By a counterexample we shall show thet the hypothesis from Theorem 1.1 concerning the connectivity of U is essential for F to be a conformal transformation. We note that the hypothesis $m \geqslant 3$ is also essential (*see* e. g. the example from [7]).

In Section 4 we shall discuss the physical meaning of Theorem 1.1 and Corollary 1.2.

2. SOME AUXILIARY RESULTS

Let $M = (\mathbb{R}^m, Q)$ be a Minkowski space. A coordinate system on $\mathbb{R}^m$ is called an *inertial system* (or a *Lorentz system*) if it is obtained from the canonical coordinate system of $\mathbb{R}^m$ by a Poincaré transformation (by a Lorentz transformation, respectively). We note that there exists a canonical one-to-one map between the set of all Lorentz systems of M and the set of all frames of the vector space $\mathbb{R}^m$ which are orthonormal with respect to Q. For such a frame $f = (f_1, \ldots, f_m)$, we have

$$(2.1) \qquad f_i f_j = \varepsilon_i \delta_{ij},\ i, j = 1, \ldots, m,\ \varepsilon_1 = -\varepsilon_2 = \ldots = -\varepsilon_m = 1\,.$$

Let v be an arbitrary vector in $\mathbb{R}^m$. If $Q(v) = 0$ we say that v is a *light vector*. If $Q(v) > 0$ [or $Q(v) < 0$] we say that the vector v is *time-like* [*space-like*, respectively].

LEMMA 2.1. — *Let* $M = (\mathbb{R}^m, Q)$ *be a Minkowski space and let* V *be the vector subspace of* $\mathbb{R}^m$ *generated by* $n \geqslant 2$ *light vectors* $e_1, \ldots, e_n$ *of* M, *which are linearly independent. Then the restriction of* Q *to* V *is a Lorentz quadratic form* (*i. e.* of signature $+, -, \ldots, -$).

Proof. — For $n = 2$, we choose the orthonormal frame f such that $e_1 = \lambda(f_1 + f_2)$, put $e_2 = \sum_{i=1}^{m} \mu_i f_i$, and we have $e_1 e_2 = \lambda(\mu_1 - \mu_2)$. It follows that $e_1 e_2 \neq 0$; otherwise e_1 and e_2 should be proportional.

If $n > 2$, let us suppose that Lemma 2.1 is true for the vector subspace V′ generated by $e_1, \ldots, e_{n-1}$. We choose the orthonormal frame f such that V′ be generated by $f_1, \ldots, f_{n-1}$, and

$$e_n = \sum_{i=1}^{m} v_i f_i, \qquad v_3 = \ldots = v_{n-1} = 0.$$

By using the frame $(f_1, \ldots, f_{n-1}, e_n)$ of V, it can be shown that the restriction of Q to V must be again a Lorentz form. Q. E. D.

Any translation $T_a : x \mapsto x + a$ of $\mathbb{R}^m$, which is defined by the vector a, induces canonically a new structure of vector space on $\mathbb{R}^m$, having a as origin and which will be denoted by the pair $(\mathbb{R}^m, a)$. Moreover, Q induces the Lorentz quadratic form Q_a on $(\mathbb{R}^m, a)$ given by $Q_a(x) = Q(T_{-a}(x))$. The triplet $(\mathbb{R}^m, a, Q_a)$ will be denoted by (M, a) and named a *pointed Minkowski space*. The Minkowski scalar product corresponding to Q_a of any two vectors $x, y \in (\mathbb{R}^m, a)$ will be also denoted by xy and we put $xx = x^2$.

Let $\mathscr{L}_a$ be the group of all Poincaré transformations of M which leave the point a invariant. We note that $\mathscr{L}_a$ can be regarded as the Lorentz group of the pointed Minkowski space (M, a). We identify $\mathbb{R}^m$ and M with $(\mathbb{R}^m, 0)$ and (M, 0), respectively.

Let a be an arbitrary point in $\mathbb{R}^m$ and let $e_1, \ldots, e_n$ be $n \geqslant 1$ light vectors of (M, a), which are linearly independent. The prism

$$(2.2)\quad \begin{cases} \pi = \left\{ x \in (\mathbb{R}^m, a) : x = \sum_{i=1}^{n} \lambda_i e_i, \ -1 \leqslant \lambda_i \leqslant 1 \right\} \\ \quad = \left\{ x \in \mathbb{R}^m : x = \sum_{i=1}^{n} \lambda_i e_i + \left(1 - \sum_{i=1}^{n} \lambda_i\right) a, \ -1 \leqslant \lambda_i \leqslant 1 \right\}, \end{cases}$$

together with the topology induced by $\mathbb{R}^m$ on π, is said to be a *light n-prism* or, more precisely, the light n-prism centred at a and generated by the vectors $e_1, \ldots, e_n$. The vector n-subspace V of $(\mathbb{R}^m, a)$ which contains the prism π is called a *light n-plane* or a *light hyperplane*.

For $n \geqslant 2$, every subset π_j, $1 \leqslant j \leqslant n$, of π given by $\lambda_j = 0$ in (2.2) is named a *median* $(n - 1)$*-prism* of π. We note that π_j is the light $(n - 1)$-prism centred at a and generated by $e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$.

Each light 2-prism (or each light 2-plane) is said to be a *light paralle-*

logram (a *light plane*, respectively). Each light 1-prism (or each light 1-plane) is a light segment (a light ray, respectively).

LEMMA 2.2. — *In a Minkowski space* $M = (\mathbb{R}^m, Q)$, $m \geq 3$, *let* $F : U \to \mathbb{R}^m$ *be an injective map defined on a convex open set* U *in* $\mathbb{R}^m$. *We suppose that* F *maps any light segment contained in* U *onto a light segment. Let* $\pi \subset U$ *be a light n-prism,* $n \geq 3$, *centred at a and generated by the light vectors* $e_1, \ldots, e_n$. *We consider the median* $(n-1)$*-prism* π_3 *centred at a and generated by* e_1, $e_2, e_4, \ldots, e_n$, *and the light parallelograms* π_{13} *and* π_{23} *which have a as centre and are generated by* e_1, e_3 *and* e_2, e_3, *respectively. If the restriction of* F *to* $\pi_3 \cup \pi_{13} \cup \pi_{23}$ *is the corresponding inclusion map in* $\mathbb{R}^m$, *then there exists an open neighbourhood* U_a *of a in* π *such that the restriction of* F *to* U_a *be the inclusion map of* U_a *in* $\mathbb{R}^m$.

Proof. — Firstly we remark that, for any light segment $\sigma \subset U$, the end points of σ are mapped by F on the end points of the light segment $F(\sigma)$ and the interior of σ is mapped by F onto the interior of $F(\sigma)$.

Let V and W be the light hyperplanes which are generated by the prisms π and π_3, respectively. We put $a' = \frac{1}{2}e_3$ and $a'' = -\frac{1}{2}e_3$. Since the restriction of the quadratic form Q_a to V is a Lorentz one (*see* Lemma 2.1), we can choose an orthonormal frame $f = (f_1, \ldots, f_n)$ of $V \subset (\mathbb{R}^m, a)$ for which (2.1) holds with $m = n$ and such that a'' be anterior to the point a with respect to f (*).

Let L_1 and L_2 be two time straight lines passing through the point a'' (i. e. the restrictions of $Q_{a''}$ to L_1 and L_2 are positive), such that L_h, $h = 1, 2$, be contained in the light plane determined by the parallelogram π_{h3}, respectively. We choose in the line L_h an open interval $\eta_h \subset \pi_{h3} - W$ for which $a'' \in \eta_h$. We also choose two distinct parallel $(n-1)$-planes W' and W'' in V passing through a' and a'', respectively, such that $L_h \subset W''$, $h = 1, 2$. We denote by V' the open set in V whose boundary coincides with $W' \cup W''$.

For any point $x \in V'$, we denote by $R_h(x)$, $h = 1, 2$, the light ray which passes through x and intersects the line L_h at the point x''_h anterior to x with respect to the orthonormal frame f. It is clear that $R_h(x)$ intersects always W' in a point x'_h and we denote by $\sigma_h(x)$, $h = 1, 2$, the light segment of end points x'_h and x''_h.

Now we consider the following continuous maps

$$\Phi_h : V' \mapsto W',\ x \mapsto x'_h\ ;\ \Psi_h : W' \to L_h,\ x'_h \mapsto x''_h,\ h = 1, 2\,.$$

(*) For two points $x, y \in V$, $x = x_i f_i$, $y = y_i f_i$, we say that x is anterior to y with respect to f if $x_1 < y_1$ and the straight line passing through x and y is a light ray.

Obviously, $\Phi_h(a) = a'$ and $\Psi_h(a') = a''$. Moreover, for any $x \in V'$ such that $\Phi_h(x) \in \pi$ and $\Psi_h(\Phi_h(x)) \in \eta_h$, we have $x \in \pi$. Hence, if we choose an open neighbourhood $O' \subset \pi$ of the point a' in W′, we obtain the open neighbourhoods $O'_h = \Psi_h^{-1}(\eta_h) \cap O'$ of a' in W′ and the open neighbourhoods $O_h = \Phi_h^{-1}(O'_h)$ of a in π.

In addition, if O′ is convex and disjoint with W, then we can take $U_a = O_1 \cap O_2$. Indeed, for any $x \in O_1 \cap O_2$, each light segment $\sigma_h(x)$ intersects the prism π_3 at a point $\tilde{x}_h$, respectively. Since F leaves invariant the distinct points $\tilde{x}_h$ and x''_h, it follows that F maps $\sigma_h(x)$ onto a light segment contained in the light ray $R_h(x)$.

If $R_1(x) = R_2(x)$, then $x \in \pi_{13} \cap \pi_{23}$ and we have by hypothesis $F(x)=x$. If $R_1(x) \neq R_2(x)$, then both x and $F(x)$ coincide with the intersection point of the straight lines $R_1(x)$ and $R_2(x)$. Q. E. D.

Let a be a point in $\mathbb{R}^m$, $m \geqslant 3$, let V be a light 3-plane of M, $a \in V$, and let α be a strictly positive real number. The hyperboloid with one sheet

$$H = \{ x \in (\mathbb{R}^m, a) \cap V : x^2 + \alpha = 0 \}, \tag{2.3}$$

together with the topology induced by $\mathbb{R}^m$ on H, is said to be a *light hyperboloid.* We note that H is a ruled surface with two families of generators which are light rays; H is homeomorphic with the standard cylinder $S^1 \times \mathbb{R}$,

LEMMA 2.3. — *In a Minkowski space* $M = (\mathbb{R}^m, Q), m \geqslant 3$, *let* $F : U \to \mathbb{R}^m$ *be an injective map defined on a convex open set* U *in* $\mathbb{R}^m$. *We suppose that* F *maps any light segment contained in* U *onto a light segment. Let* π *be a light parallelogram contained in* U. *We denote by* $\sigma_1, \ldots, \sigma_4$ *the sides of* π *with* $\sigma_1 \parallel \sigma_3$, *and put* $\sigma'_i = F(\sigma_i)$, $1 \leqslant i \leqslant 4$. *Then* $F(\pi)$ *is either a light parallelogram, or the compact domain* δ *of a light hyperboloid, whose boundary is* $\varphi = \sigma'_1 \cup \sigma'_2 \cup \sigma'_3 \cup \sigma'_4$, *according as the light segments* σ''_1 *and* σ''_3 *are parallel or not.*

Proof. — Firstly it is easy to show that, if σ'_1 and σ'_3 lay in the same light ray, then F would not be an injective map. Therefore we must consider two cases, according as σ'_1 and σ'_3 are parallel or not.

If $\sigma'_1 \parallel \sigma'_3$, then σ'_1, $1 \leqslant i \leqslant 4$, are contained in a light plane V of $\mathbb{R}^m$. It results that $\sigma'_2 \parallel \sigma'_4$; otherwise V should contain three distinct directions of light rays. Denoting by π' the light parallelogram of sides $\sigma'_1, \ldots, \sigma'_4$, the following two properties hold

(2.4) each light segment $\sigma \subset \pi$ whose end points belong to opposite sides of π, respectively is mapped by F onto a light segment $\sigma' \subset \pi'$ whose end points belong to the corresponding opposite sides of π' ;

(2.5) each light segment $\sigma' \subset \pi'$ whose end points belong to opposite sides of π', respectively is the image by F of a light segment $\sigma \subset \pi$ whose end points belong to the corresponding opposite sides of π.

Property (2.4) is obvious. To prove (2.5), let $p'_1 \in \sigma'_1$ and $p'_3 \in \sigma'_3$ be the end points of σ'. Then we have $p'_1 = F(p_1)$, $p_1 \in \sigma_1$, and $p'_3 = F(p_3)$, $p_3 \in \sigma_3$. If the segment σ of end points p_1, p_3 were not a light segment, then there should be two distinct light segments which contain the points p_1, p_3, respectively, and which are mapped by F onto σ'. So σ is a light segment and $F(\sigma) = \sigma'$. From (2.4) and (2.5) it follows that $F(\pi) = \pi'$.

If σ'_1 and σ'_3 are not parallel, then σ'_i, $1 \leqslant i \leqslant 4$, are contained in a light 3-plane W; moreover $F(\pi) \subset W$. We put

$$\sigma'_4 \cap \sigma_1 = \{ a \},\ \sigma'_1 \cap \sigma'_2 = \{ b \},\ \sigma'_2 \cap \sigma'_3 = \{ c \},\ \sigma'_3 \cap \sigma'_4 = \{ d \},$$

and consider W as a vector subspace of $(\mathbb{R}^m, a)$.

By using Lemma 2.1, we choose a Lorentz coordinate system (x_1, x_2, x_3) of W such that either (2.6) or (2.7) below holds, according as the segment of end points b, d is space-like or time-like.

$$(2.6) \qquad a(0, 0, 0), \quad b(\lambda, \lambda, 0), \quad d(\lambda, -\lambda, 0);$$

$$(2.7) \qquad a(0, 0, 0), \quad b(\lambda, \lambda, 0), \quad d(-\lambda, \lambda, 0).$$

The point c must belong to the intersection between the light cones with the vertices at b and d, respectively. In case (2.6) we obtain

$$(2.8) \qquad c(\alpha, 0, \beta), \quad (\alpha - \lambda)^2 - \beta^2 = \lambda^2, \quad \beta \neq 0.$$

In case (2.7) we obtain

$$(2.9) \qquad c(0, \alpha, \beta), \quad (\alpha - \lambda)^2 + \beta^2 = \lambda^2, \quad \beta \neq 0.$$

We find that the quadrangle φ is contained in the light hyperboloid

$$(2.10) \quad H = \left\{ x \in W : x_1^2 - x_2^2 - x_3^2 + \frac{2\varepsilon\alpha\lambda}{\beta} x_3 = 0 \right\}, \qquad \varepsilon = \pm 1,$$

where α, β verify (2.8) and $\varepsilon = -1$ [or (2.9) and $\varepsilon = +1$] if we consider case (2.6) [case (2.7), respectively].

Now, for each point $p \in \sigma'_1$, we denote by $R_p \subset H$ the light ray passing through p and which does not contain σ'_1. It is easy to show that R_p lies in the palne determined by the points p, c, d. Moreover, if c is given by (2.8), then all rays R_p, $p \in \sigma'_1$, intersect the segment σ'_3 if and only if $\alpha > 2\lambda$; if c is given by (2.9), then R_p, $p \in \sigma'_1$, intersects always the segment σ'_3. Denoting by δ the compact domain of H whose boundary is φ, it results that π and δ verify properties which are analogous with (2.4) and (2.5). Consequently we obtain $F(\pi) = \delta$. Q. E. D.

LEMMA 2.4. — *Let* $F : U \to \mathbb{R}^m$ *be as in Lemma* 2.3. *Then* F *is continuous on* U.

Proof. — Firstly we choose an Euclidean metric ρ on $\mathbb{R}^m$ which is compatible with the canonical affine structure of $\mathbb{R}^m$.

For any light n-prism $\pi \subset U$, $1 \leqslant n \leqslant m$, we shall prove, by induction on n, the following two properties

(2.11) $F(\pi)$ is a bounded set in $\mathbb{R}^m$;

(2.12) F maps continuously the light prism π onto $F(\pi)$.

Obviously, these properties are true for $n = 1$. Now we assume that (2.11) and (2.12) hold for any light $(n-1)$- prism, $1 < n \leqslant m$, which is contained in U.

Let $\pi \subset U$ be a light n-prism given by (2.2). We choose two opposite $(n-1)$-faces π' and π'' of π as follows

$$\pi' = \{ x \in (\mathbb{R}^m, a) \cap \pi : \lambda_1 = -1 \}, \quad \pi'' = \{ x \in (\mathbb{R}^m, a) \cap \pi : \lambda_1 = +1 \}.$$

We assign to each vector $x \in (\mathbb{R}^m, a) \cap \pi$, $x = \lambda_1 e_1 + \ldots + \lambda_n e_n$, its projections x' and x'' on π' and π'', respectively, given by

$$x' = -e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n, \qquad x'' = e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n.$$

By induction hypothesis, there is a ball B defined with the aid of the metric ρ such that the set $F(\pi'' \cup \pi'')$ is contained in B. Since each point of $F(\pi)$ lies in a light segment whose end points belong to $F(\pi')$ and $F(\pi'')$, respectively, we see that $F(\pi) \subset B$. So property (2.11) is proved.

For any point $p \in \pi$ and for any sequence $\{ p_k \}_{k \geqslant 1}$ of points in π which converges to p, we obtain from the induction hypothesis the sequences $\{ F(p_k') \}$ and $\{ F(p_k'') \}$ which converge to $F(p')$ and $F(p'')$, respectively. We remark that $F(p)$ belongs to the light segment σ of end points $F(p')$, $F(p'')$.

Now we choose a subsequence $\{ q_k \}_{k \geqslant 1}$ of $\{ p_k \}$ for which $\{ F(q_k) \}$ converges to a point $q \in \mathbb{R}^m$; the existence of such a subsequence is ensured by property (2.11). Since

$$Q(F(q_k) - F(q_k')) = 0 \qquad \text{and} \qquad Q(F(q_k) - F(q_k'')) = 0$$

for any k, we have by passing to the limit

$$Q(q - F(p')) = 0 \qquad \text{and} \qquad Q(q - F(p'')) = 0,$$

respectively, i. e. the point q belongs to the light ray R determined by σ.

By using an other pair of opposite $(n-1)$-faces of the prism π, we prove analogously that the point q belongs to a new light ray which passes through $F(p)$ and is distinct of R, i. e. we have $q = F(p)$. Since any accumulation point of the bounded sequence $\{ F(p_k) \}$ must coincide with $F(p)$, it follows that this sequence is a convergent one and its limit is the point $F(p)$. So property (2.12) is also proved, because the point p and the sequence $\{ p_k \}$ have been choosen arbitrarily.

Finally, for any point $x \in U$ and for any sequence $\{ x_k \}_{k \geqslant 1}$ of points in U which converges to x, we choose a light m-prism $\pi \subset U$ which is centred at x. Taking into account that all x_k belong to π for $k > N$, from

property (2.12) it results that the sequence $\{ F(x_k) \}$ converges to $F(x)$.
Q. E. D.

Now we introduce two kinds of conformal transformations in the Minkowski space $M = (\mathbb{R}^m, Q)$. Firstly we consider the inversion $I_0 : x \mapsto x/x^2$, $x^2 \neq 0$, which is centred at the origin. For any point $p \in \mathbb{R}^m$, the *inversion of centre p* is the conformal transformation $I_p = T_p I_0 T_{-p}$, where T_p is the translation $x \mapsto x + p$. To every light vector v of (M, p) we associate the *Kelvin transformation* $K_{p,v} = I_p T_v I_p$. In the pointed Minkowski space (M, p) we can write

$$(2.13) \qquad I_p \ : x \mapsto x/x^2, \quad x \in (\mathbb{R}^m, p), \quad x^2 \neq 0 \, ;$$

$$(2.14) \quad K_{p,v} \ : x \mapsto \frac{x + vx^2}{1 + 2vx}, \quad v, x \in (\mathbb{R}^m, p). \quad v^2 = 0, 1 + 2vx \neq 0 \, .$$

The set of the singularities of I_p coincides with the light cone C_p whose vertex is at p, while the set of the singularities of $K_{p,v}$ is a degenerate $(m - 1)$-plane of M.

Each Weyl transformation of M which is of the form $x \mapsto \lambda x$, $x \in (\mathbb{R}^m, p)$, $0 < \lambda < 1$, is called a *contraction* of centre p.

LEMMA 2.5. — *Let* $F : U \to R^m$ *be as in Lemma* 2.3 *and let* $\tau \subset U$ *be a light* 3*-prism centred at a point* $a \in U$. *Then there exist a contradiction* D_a *of centre a and a conformal transformations* J *whose domain contains* $F(D_a(\tau))$, *such that* J *applies* $F(D_a(\tau))$ *onto a light* 3*-prism.*

Proof. — Let τ_i, $1 \leqslant i \leqslant 3$, be the median parallelograms of τ. For any point $p \in F(\tau_1)$, the inversion I_p maps $F(\tau_1)$-C_p into a light plane V. If we choose the point p such that $F(a) \notin C_p$, then we obtain the open neighbourhood $F(\tau)$-C_p of $F(a)$ in $F(\tau)$. By property (2.12) there is a contraction D_a of centre a, such that $F(D_a(\tau))$ is contained in the domain of I_p. Putting $\pi = I_p(F(D_a(\tau)))$ and $\pi_i = I_p(F(D_a(\tau_i)))$, $1 \leqslant i \leqslant 3$, we see that π_1 is a light parallelogram in V.

Now we construct a conformal transformation J' as follows. If π_2 is a light parallelogram we put $J' = I_p$. If π_2 is contained in a light hyperboloid H, then $H \cap V$ consists of two light rays R', R'' whose intersection is a point p'' and such that $\pi_1 \cap \pi_2 \subset R''$. The points p'' and $I_p(F(a))$ are distinct. Otherwise, there exist two distinct light segments which pass through a, are contained in $D_a(\tau_1)$ and $D_a(\tau_2)$, respectively, and have the same image by I_pF; consequently, F should not be an injective map. It results that we can choose a point $p' \in R'$ for which $I_p(F(a)) \notin C_{p'}$ and we put $J' = I_{p'} I_p$. Moreover, by modifying the contraction D_a, we can assume that π is contained in the domain of $I_{p'}$.

Putting $\pi' = J'(F(D_a(\tau)))$ and $\pi'_i = J'(F(D_a(\tau_i)))$, $1 \leqslant i \leqslant 3$, we see that π'_1 and π'_2 are light parallelograms. If π'_3 is a light parallelogram, then we can take $J = J'$. Indeed, let θ_k, $1 \leqslant k \leqslant 4$, be the consecutive faces of

$D_a(\tau)$ which are parallel to the light segment $\tau_1 \cap \tau_2$. It follows that each light segment $J'(F(\theta_1 \cap \theta_2))$, $J'(F(\theta_2 \cap \theta_3))$, $J'(F(\theta_3 \cap \theta_4))$, $J'(F(\theta_4 \cap \theta_1))$ is parallel to $\pi'_1 \cap \pi'_2$. So the faces $J'(F(\theta_k))$, $1 \leqslant k \leqslant 4$, of π' are light parallelograms. Replacing τ_2 by τ_3 in the above argument, we see that all faces of π' are light parallelograms; therefore π' is a light 3-prism.

The case when π'_3 is contained in a light hyperboloid H′ is reduced to the previous one. Indeed, H′ is contained in the light 3-plane V′ determined by π'. On the other hand, we can regard V′, together with the light ray R which contains the segment $\pi' \cap \pi'_2$, as subspaces of $(\mathbb{R}^m, q)$, $q = J'(F(a))$. The equation of the hyperboloid H′ in V′ is of the form $x^2 + bx = 0$. Since the intersection between R and H′ is the point q, there is a light vector $v \in R$ for which $1 - 2bv = 0$. Denoting by W the $(m-1)$-plane of singularities of the Kelvin transformation $K = K_{q,v}$, it is easy to show that K maps $\pi'_i - W$, $1 \leqslant i \leqslant 3$, into three light planes, respectively. By modifying once more the contraction D_a, we can assume that $\pi'_i \cap W = \emptyset$. So we can take $J = KJ'$. Q. E. D.

LEMMA 2.6. — *Let G be a conformal transformation in a Minkowski space* $M = (\mathbb{R}^m, Q)$. *If G applies identically a light parallelogram π of centre a onto itself, then G is a Lorentz transformation of* (M, a).

Proof. — We denote by V the light plane generated by π, choose an orthonormal frame $f = (f_1, \ldots, f_m)$ of (M, a), for which the vectors f_1, f_2 generate V, and denote by $(x_1, \ldots, x_m)$ the coordinates of the Lorentz system associated to f. Since G maps identically π onto itself, then there is an $\alpha > 0$ such that

$$(2.15) \quad G(x_1 f_1 + x_2 f_2) = x_1 f_1 + x_2 f_2, \quad \forall x_1, x_2 \in \mathbb{R} : |x_1| < \alpha, \quad |x_2| < \alpha .$$

Taking $x_3 = \ldots = x_m = 0$ in (1.8) and using (2.15), we obtain

$$x_1 = \frac{G_{11}x_1 + G_{12}x_2 + G_{1r}x^2 + G_{1s}}{G_{s1}x_1 + G_{s2}x_2 + G_{sr}x^2 + G_{ss}}, \quad r = m+1, \quad s = m+2, \quad x^2 = x_1^2 - x_2^2 .$$

Since this identity is true for $|x_1| < \alpha$, $|x_2| < \alpha$, we find

$$(2.16) \qquad G_{s1} = G_{1r} = G_{sr} = 0, \qquad r = m+1, \qquad s = m+2 .$$

Now (1.6) and (1.7) show that $G_{si} = 0$ for $1 \leqslant i \leqslant m$. From (2.16) we can also obtain $G_{ir} = 0$ for $1 \leqslant i \leqslant m$, i. e. $G \in \mathscr{W}$. Further $G \in \mathscr{L}_a$, because G leaves the points a, f_1 invariant and $Q_a(f_1) = 1$. Q. E. D.

3. PROOF OF THEOREM 1.1 AND COROLLARY 1.3

PROPOSITION 3.1. — *In a Minkowski space* $M = (\mathbb{R}^m, Q)$, $m \geqslant 3$, *let* $F : U \to \mathbb{R}^m$ *be an injective map which is defined on a convex open set U in* $\mathbb{R}^m$

and which applies any light segment contained in U *onto a light segment. We consider an arbitrary light n-prism* $\theta \subset U$, $3 \leqslant n \leqslant m$, *which is centred at a point* $a \in U$. *Then there exist a contraction* D_a *of centre a and a conformal transformation* G, *such that* $D_a(\theta)$ *is contained in the domain of* G *and the restrictions of* F *and* G *to* $D_a(\theta)$ *coincide.*

Proof. — Firstly we construct a « Zeeman figure » [7, Lemma 4] for any light segment $\sigma \subset U$. We denote by p_0, p_1 the end points of the segment σ and by p_2 its midpoint. We choose a light parallelogram whose the consecutive vertices p_1, p_2, p_3, p_4 belong to U. In a light 3-plane which contains this parallelogram we consider a point $p_5 \neq p_k$, $0 \leqslant k \leqslant 4$, such that the straight lines determined by p_5, p_2 and p_5, p_4, respectively, are light rays, and denote by p_6 the point for which p_3, p_4, p_5, p_6 are the consecutive vertices of a new light parallelogram. We can show, as in the proof of Lemma 2.3, that p_5 describes a curve which passes through p_1 and is either a hyperbola of the form (2.8) or a circle of the form (2.9). By an elementary continuity argument, we can fix the points p_5 and p_6 in U. If we also consider the light parallelogram whose consecutive vertices are p_5, p_6, p_7, p_2, we have the following equipollence relation

$$(p_2, p_0) \# (p_1, p_2) \# (p_4, p_3) \# (p_5, p_6) \# (p_2, p_7).$$

So $p_7 = p_0$ and we obtain the third light parallelogram whose the consecutive vertices are p_5, p_6, p_0, p_2. We say that the points $p_h \in U$, $0 \leqslant h \leqslant 6$, form a *Zeeman figure* associated to the light segment σ.

Now we prove Proposition 3.1 for $n = 3$. We consider the contraction D_a and the conformal transformation J given by Lemma 2.5. We denote by π_i, $1 \leqslant i \leqslant 3$, the median parallelograms of the 3-prism $\pi = D_a(\theta)$ and we put

$$\sigma_1 = \pi_2 \cap \pi_3, \qquad \sigma_2 = \pi_3 \cap \pi_1, \qquad \sigma_3 = \pi_1 \cap \pi_2.$$

It results that $F' = JF$ applies π_i, $1 \leqslant i \leqslant 3$, onto the median parallelograms π'_i of the light 3-prism $\pi' = F'(\pi)$, respectively. We also consider the light segments $\sigma'_i = F'(\sigma_i)$, $1 \leqslant i \leqslant 3$.

For any light segment $\sigma \subset \sigma_3$, we can construct the above Zeeman figure such that the light segment θ_1 of end points p_3, p_4 is contained in π_1 and the intersection of the parallelogram $p_3p_4p_5p_6$ with the plane determined by π_2 is a light segment θ_2, which lies in the domain of F'. Since $F'(\theta_1)$ and $F'(\theta_2)$ are both parallel to σ'_3, it results that the points $p'_h = F'(p_h)$, $0 \leqslant h \leqslant 6$, verify the following equipollence relation

$$(p'_1, p'_2) \# (p'_4, p'_3) \# (p'_5, p'_6) \# (p'_2, p'_0).$$

Therefore, the points p'_h form a Zeeman figure associated to the light segment $F'(\sigma)$, the midpoint of which coincides with $F'(p_2)$. So we see that F' *applies the midpoint of any segment* $\sigma \subset \sigma_1$, $1 \leqslant i \leqslant 3$, *on the midpoint of the segment* $F'(\sigma)$.

We denote by b_i, c_i the end points of the segment σ_i, $1 \leqslant i \leqslant 3$, and put $b'_i = F'(b_i)$, $c'_i = F'(c_i)$. For any integer $s > 0$ we define the partition $\mathscr{A}_s$ of the segment σ_i whose division points $a_0 = b_i, a_1, \dots, a_t = c_i$, $t = 2^s$, are given by

$$(a_0, a_1) \# (a_1, a_2) \# \dots \# (a_{t-1}, a_t). \tag{3.1}$$

The points $a'_u = F'(a_u)$, $0 \leqslant u \leqslant t$, define a partition of the segment σ'_i. Since F' applies the midpoint a_{u+1} of the segment whose end points are a_u and a_{u+2} on the midpoint a''_u of the segment whose end points are a'_u and a'_{u+2}, from (3.1) it follows that $a''_u = a'_{u+1}$. So we have

$$(a'_0, a'_1) \# (a'_1, a'_2) \# \dots \# (a'_{t-1}, a'_t). \tag{3.2}$$

Let $\tilde{\sigma}_i \subset \sigma_i$ be the set of all division points of the partitions $\mathscr{A}_s$, $s > 0$. Taking account of (3.1) and (3.2), we obtain the following equality of affine ratios

$$\frac{pb_i}{c_ib_i} = \frac{p'b'_i}{c'_ib'_i}, \qquad p' = F'(p), \tag{3.3}$$

for any $p \in \tilde{\sigma}_i$. Since $\tilde{\sigma}_i$ is a dense subset of σ_i and F' applies continuously σ_i onto σ'_i, it results that (3.3) holds for any $p \in \sigma_i$. In other words, if we consider the light vectors $e_i \in (\mathbb{R}^m, a)$ and $e'_i \in (\mathbb{R}^m, a')$, $a' = F'(a)$, given by the points b_i and b'_i, respectively, we have

$$F'(\lambda e_i) = \lambda e'_i, \qquad \forall \lambda \in [-1, +1], \qquad 1 \leqslant i \leqslant 3. \tag{3.4}$$

For any vector $v \in (\mathbb{R}^m, a) \cap \pi$, we consider the light parallelograms $\pi_i(v)$ which are parallel to π_i, respectively, which pass through v and have the sides on the boundary of π. We define the projection $v_i \in (\mathbb{R}^m, a)$ on σ_i as being the intersection between $\pi_i(v)$ and σ_i. By using the parallelograms π'_i and the segments σ'_i, we introduce analogously, for any vector $v' \in (\mathbb{R}^m, a') \cap \pi'$, the light parallelograms $\pi'_i(v')$ and the projections $v'_i \in (\mathbb{R}^m, a')$. Since $F'(\pi_i(v)) = \pi'_i(F'(v))$, we see that $F'(v_i) = (F'(v))_i$. Taking into account (3.4) and putting $v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, we have

$$F'(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_1 e'_1 + \lambda_2 e'_2 + \lambda_3 e'_3, \tag{3.5}$$

for any $\lambda_1, \lambda_2, \lambda_3 \in [-1, +1]$.

Let $V \subset (\mathbb{R}^m, a)$ and $V' \subset (\mathbb{R}^m, a')$ be the vector 3-spaces generated by π and π', respectively. By (3.5) we extend F' to a unique linear transformation $L : V \to V'$. Since L applies each light segment of π onto a light segment of π', an elementary calculation shows that L is a Weyl transformation between $(V, Q_{a|V})$ and $(V', Q_{a'|V'})$, i. e.

$$Q_a(v) = \alpha Q_{a'}(L(v)), \qquad \forall v \in V,$$

where α is a non-null constant.

Let W and W′ be the orthogonal spaces to V and V′ with respect to Q_a and Q'_a, respectively. Further we extend L to a Weyl transformation L′ of the Minkowski space M as follows: we choose the orthonormal frames $(f_1, \ldots, f_{m-3})$ and $(f'_1, \ldots, f'_{m-3})$ in W and W′, respectively; we put $L'(f_i) = \alpha f'_i$, $1 \leqslant i \leqslant m-3$; for any vector $v \in V$, we put $L'(v) = L(v)$. Taking $G = J^{-1}L'$. Proposition 3.1 is proved for $n = 3$.

Now we prove Proposition 3.1 by induction on n. Let us assume that this is true for $n-1$ with $4 \leqslant n \leqslant m$.

In a light n-prism $\theta \subset U$ centred at a, we consider two median $(n-1)$-prisms π'' and τ''. From the induction hypothesis, we find a contraction D_a of centre a and two conformal transformation A, B such that: $D_a(\pi'')$ is contained in the domain of A: the restrictions of F and A to $D_a(\pi'')$ coincide; $D_a(\tau'')$ is contained in the domain of B; the restrictions of F and B to $D_a(\tau'')$ coincide.

Taking into account the continuity of F given by Lemma 2.4, we find a convex open neighbourhood U′ of the point a in U for which the injective maps

$$A^{-1}F, \qquad B^{-1}F : U' \to \mathbb{R}^m$$

apply any light segment contained in U′ onto two light segments, respectively. In addition, we can modify the contraction D_a such that $D_a(\theta) \subset U'$. So $A^{-1}F$ applies identically $D_a(\pi'')$ onto itself, and $B^{-1}F$ applies identically $D_a(\tau'')$ onto itself. Since the conformal transformations A and B coincide onto the light $(n-2)$-prism $D_a(\pi'' \cap \tau'')$, $n-2 \geqslant 2$, from Lemma 2.6 we obtain $B^{-1} = L_1A^{-1}$, where L_1 is a Lorentz transformation of (M, a), and we have

$$B^{-1}F = L_1A^{-1}F, \qquad F : U' \to \mathbb{R}^m. \tag{3.6}$$

Let $V_1 \subset (\mathbb{R}^m, a)$ be the vector space generated by τ'', and let V_2 be its orthogonal space with respect to Q_a. We can choose the light 3-prism $\pi \subset D_a(\pi'')$ and the light $(n-1)$-prism $\tau \subset D_a(\tau'')$ for which we have: π is generated by the light vectors e_1, e_2, e_3 of (M, a); τ is generated by the light vectors $e_1, e_2, e_4, \ldots, e_n$ of V_1; putting

$$e_3 = e + e', \qquad e \in V_1, \qquad e' \in V_2,$$

the vectors e, e_1, e_2 are coplanar and $e, e' \in \pi$. From (3.6) it follows that $B^{-1}F$ applies linearly the prism π onto $L_1(\pi)$. Therefore we obtain

$$B^{-1}F(e_3) = e + B^{-1}F(e'), \qquad B^{-1}F(e') \in V_2.$$

So we can find a Lorentz transformation L_2 of (M, a) which applies identically V_1 onto itself and maps $B^{-1}F(e_3)$ on e_3. We see that the injective map $L_2B^{-1}F$ applies identically $\pi \cup \tau$ onto itself and any light segment of U′ onto a light segment. From Lemma 2.2 it results that, by a new modification of D_a, the map $L_2B^{-1}F$ applies identically the prism $D_a(\theta)$ onto itself, and so we can take $G = BL_2^{-1}$. Q. E. D.

Proof of Theorem 1.1. — Let $F: U \to \mathbb{R}^m$ be an injective map which is defined on a connected open set U in $\mathbb{R}^m$ and which applies any light segment contained in U onto a light segment. By using Proposition 3.1 for $n = m$, it results that any point $a \in U$ has a convex open neighbourhood U_a in U such that F coincides on U_a with a conformal transformation G_a. Since U is a connected open set and G_a, $a \in U$, are analytic maps, it is easy to show that there is a unique conformal transformation G whose restriction to each U_a coincides with G_a. Therefore F is the restriction to U of the conformal transformation G. Q. E. D.

Examining the proof of Lemma 2.5 and of Proposition 3.1, from Theorem 1.1 we also obtain

COROLLARY 3.2. — *The conformal group* $\mathscr{C}$ *of a Minkowski space* M *of dimension* $m \geqslant 3$ *is generated in the projective group* $P(m+1; \mathbb{R})$ *by the Weyl group* $\mathscr{W}$ *of* M *together with an inversion* I_p, $p \in M$.

Proof of Corollary 1.3. — In the pointed Minkowski space (M, a) we consider a Lorentz coordinate system. Let $e_i \in (\mathbb{R}^m, a)$ be a light vector which generates R_i, $1 \leqslant i \leqslant m$, and let e_{ij} be the components of e_i with respect to the choosen Lorentz system.

In the following we shall identify the Minkowski space M with its image by the imbedding $\overline{\psi}$ given by (1.4). So the point at infinity p_i of each light ray R_i, $1 \leqslant i \leqslant m$, has the homogeneous components $(e_{ij}, 0, 0)$ in the projective space P^{m+1}.

From Theorem 1.1 it results that F is the restriction to U of a conformal transformation G, which can be written in the form (1.5). Since $F(U \cap R_i)$ and $F(U \cap R_i')$ are contained in parallel light rays, it follows that $G(p_i)$ is also a point at infinity, and we have

$$\sum_{j=1}^{m} G_{sj} e_{ij} = 0, \qquad s = m+2, \qquad 1 \leqslant i \leqslant m .$$

But the vectors e_i are linearly independent, because the concurrent light rays R_i generate affinely R^m; hence $G_{si} = 0$ for $s = m+2$, $1 \leqslant i \leqslant m$.

Taking account of (1.6) and (1.7), we obtain either $G_{sr} = 0$ or $G_{ss} = 0$, $r = m+1, s = m+2$. But only the case $G_{sr} = 0$ is possible; otherwise $G(a)$ should be a point at infinity. Q. E. D.

Now we show by a counterexample that the connectivity of U is an essential hypothesis in Theorem 1.1. In a Minkowski space $M = (\mathbb{R}^m, Q)$ let us consider the following two connected open sets

$$(3.7) \quad U_1 = \{ x \in \mathbb{R}^m : Q(x) + 1 < 0 \}, U_2 = \{ x \in \mathbb{R}^m : Q(x) + 1 > 0 \}.$$

We take U as being the open set $U_1 \cup U_2$ with two connected components, and introduce the injective map

$$(3.8) \qquad F : U \to \mathbb{R}^m, \qquad F(x) = \begin{cases} x & \text{if } x \in U_1 \\ -x & \text{if } x \in U_2 . \end{cases}$$

We remark that F leaves invariant each set U_1 and U_2; in addition, the restrictions of F to U_1 and U_2 are given by two Lorentz transformations, respectively. Therefore F maps any light segment contained in U onto a light segment. On the other hand F is not a conformal transformation in the Minkowski space M.

4. PHYSICAL INTERPRETATION

In this section we consider the 4-dimensional Minkowski space $M = (\mathbb{R}^4, Q)$ and use its canonical coordinates $(x_1, \ldots, x_4)$. We say that a point x of M is *anterior* to a point y of M if $x_1 < y_1$ and if the straight line determined by x, y is a light ray. We say that a set $S \subset \mathbb{R}^4$ is *weakly convex* if S is a connected set and if, for any two distinct points $x, y \in S$ which determine a light ray, the segment of end points x, y is contained in S. We note that each light hyperboloid is weakly convex, but it is not convex.

We represent the physical universe corresponding to the special relativity by a non-empty set Ω whose elements are called *events*. The phenomenon of light propagation in Ω is described by a binary relation $\Omega' \subset \Omega \times \Omega$ and by a ternary relation $\Omega'' \subset \Omega \times \Omega \times \Omega$ such that

(4.1) if $(u, v) \in \Omega'$, then $(v, u) \in \Omega$ and $u \neq v$;
(4.2) if $(u, v, w) \in \Omega''$, then $(u, w) \in \Omega'$ and $u \neq v \neq w$.

In a physical language, the membership of a pair (u, v) to Ω' means that *the events u and v can be connected by light propagation.* The membership of a ternary (u, v, w) to Ω'' means that *the events u and w can be connected by light propagation, but the event v can hinder this phenomenon.*

In order to precise completely the relations Ω' and Ω'', firstly we introduce the notion of *luminal reference system.* Such a system is a bijective map $\xi : \omega \to U$, where ω is a subset of Ω and U is a weakly convex open set in $\mathbb{R}^4$, so that

(4.3) for any two distinct events $u, v \in \omega$, the pair (u, v) belongs to Ω' if and only if the points $\xi(u)$, $\xi(v)$ determine a light ray of M;

(4.4) for any pair $(u, w) \in \Omega'$ with $u, w \in \omega$, the set of all events v for which the ternary (u, v, w) belongs to Ω'' is contained in ω and is represented by ξ onto the open segment whose end points are $\xi(u)$, $\xi(w)$.

We say that the luminal reference system ξ is an *inertial* one if its domain ω

coincides with the whole universe Ω. We assume that *there exists an inertial reference system* ξ_0 *which establishes a bijection between* Ω *and* $\mathbb{R}^4$.

The physical meaning of Theorem 1.1 is given by its

COROLLARY 4.1. — *Let* $\xi : \omega \to \mathrm{U}$ *be a bijective map, where* ω *is a subset of* Ω *and* U *is a weakly convex open set in* $\mathbb{R}^4$. *Then* ξ *is a luminal reference system if and only if the map* $\xi_0\xi^{-1} : \mathrm{U} \to \mathbb{R}^4$ *is given by a conformal transformation in the Minkowski space* M ; *in this case* $\xi_0(\omega)$ *is a weakly convex open set in* $\mathbb{R}^4$.

Indeed, from (4.3) and (4.4) it results that ξ is a luminal reference system if and only if $\xi_0\xi^{-1}$ applies any light segment contained in U onto a light segment.

The physical meaning of Corollary 1.2 follows from

COROLLARY 4.2. — *Let* $\xi : \Omega \to \mathrm{U}$ *be a bijective map, where* U *is a weakly convex open set in* $\mathbb{R}^4$. *Then* ξ *is an inertial reference system if and only if the map* $\xi\xi_0^{-1}$ *is a Weyl transformation of the Minkowski space* M ; *in this case we have* $\mathrm{U} = \mathbb{R}^4$.

So we see that the image of each inertial reference system coincides with $\mathbb{R}^4$.

PROPOSITION 4.3. — *Let* $\xi : \omega \to \mathrm{U}$ *and* $\xi' : \omega' \to \mathrm{U}'$ *be two luminal reference systems with* $\omega \cap \omega' \neq \emptyset$. *Then the map* $\xi'\xi^{-1}$ *establishes a conformal diffeomorphism between the open sets* $\xi(\omega \cap \omega')$ *and* $\xi'(\omega \cap \omega')$.

Proof. — We have $\xi(\omega \cap \omega') = \xi\xi_0^{-1}(\xi_0(\omega) \cap \xi_0(\omega'))$. From Corollary 4.1 it results that $\xi_0(\omega) \cap \xi_0(\omega')$ is an open set in $\mathbb{R}^4$ which is mapped by the conformal transformation $\xi\xi_0^{-1}$ onto a set of the same kind. On the other hand, the map $\xi'\xi^{-1}$ is given by the composition between the conformal transformations $\xi'\xi_0^{-1}$ and $\xi_0\xi^{-1}$. We note that $\xi(\omega \cap \omega')$ and $\xi'(\omega \cap \omega')$ are not always connected sets. Q. E. D.

Corollary 4.2 shows that, for any two inertial reference systems ξ and ξ', the map $\xi'\xi^{-1}$ is a Weyl transformation of the Minkowski space M.

Now we introduce a new binary relation $\prec$ on Ω, putting $u \prec v$ if and only if $\xi_0(u)$ is anterior to $\xi_0(v)$. It is easy to prove

PROPOSITION 4.4. — *For two events* $u, v \in \Omega$, *the pair* (u, v) *belongs to* Ω' *if and only if either* $u \prec v$ *or* $v \prec u$. *For three events* $u, v, w \in \Omega$, *the ternary* (u, v, w) *belongs to* Ω'' *if and only if either* $u \prec v \prec w$ *or* $w \prec v \prec u$.

So we see that Ω' and Ω'' are completely characterized. Moreover, Proposition 4.4 suggests the following definition of the *luminal causality* in Ω: we say that the event u *can influence* the event v by light propagation if and only if $u \prec v$. Therefore, we adopted only the luminal connectivity as a basic notion, and introduced the luminal causality as a derived concept.

We mention the following issue [*3*] [*4*]: are the luminal reference systems achievable in a natural way from the physical point of view?

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