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Relativistic spin particles (*)

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ABSTRACT. — We construct a three dimensional classical formalism in order to describe the dynamics of a system of interacting spin particles within the predictive relativistic mechanics framework. The integral formulation we present is suitable for a perturbative treatment.

We also construct a modified version of this formalism in order to allow explicit calculations by restricting ourselves to the center of mass frame in the two-particle case.

We present the complete lowest order approximation to the problem of the electromagnetic interaction between two relativistic spin particles in the framework of the predictive relativistic mechanics. This is done in the three dimensional formalism by imposing the center of mass condition which greatly simplifies the algebraic calculations.

RÉSUMÉ. — Nous établissons un formalisme classique tridimensionnel valable pour décrire la dynamique d'un système de particules interagissantes avec spin dans le cadre de la Mécanique Relativiste Prédictive. La formulation intégrale que nous présentons se révèle très utile dans le contexte des méthodes perturbatives.

Nous établissons aussi une autre version de ce formalisme laquelle facilite les calculs explicites quand on se borne à étudier le problème des deux corps dans le référentiel du centre de masse.

On présente la première solution complète (au sens du premier ordre en g) du problème de l'interaction électromagnétique entre deux particules

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relativistes avec spin dans le cadre de la Mécanique Relativiste Prédictive. Ceci est fait dans le formalisme tridimensionnel en imposant la condition du centre de masse pour simplifier les calculs algébriques.

INTRODUCTION

The study of the classical dynamics of spinning particles is an important tool for the quantization of such systems within the predictive relativistic mechanics framework. When studying these systems, computational problems are by no means the less important ones. This paper is devoted to the explicit construction of a set of coordinates that have proved to be of great usefulness in the study of such systems both at the classical and quantum levels.

Predictive relativistic mechanics has first been developped in the framework of a three dimensional description [*1*]. This formalism is very clear and has a direct physical interpretation. Nevertheless, the difficulty of performing explicit calculations is usually greater than in the equivalent manifestly covariant formalism [*2*], especially in its integral version [*3*] (suitable for perturbative calculations).

Recently, there has been developped an analogous integral version [*4*] for the three dimensional formalism which allows one to perform explicit calculations in the case of structureless particles, though the expressions that one obtains are very much longer than the corresponding four-dimensional ones. Our purpose is to extend this approach to the case of spin particles, where many simplifications are needed to avoid lengthy or infeasible computations.

The complexity of the three dimensional formalism comes in part from its form invariance: the expressions are valid for any intertial frame. Our purpose is to restrict ourselves to the two-particle case and to take advantage of the simplifications coming from the election of a particular frame: the center of mass one. We adapt then the formalism [*4*] both to the spin case and to the imposition of the center of mass condition.

In the second part of the paper (sections 5 to 8) we apply this formalism to the study of the electromagnetic interaction of two spinning particles, where we need as an input the generalisation of the Maxwell-Lorentz equations to take into account the spin. Many authors [*5*] [*6*] have used a model of spin particles describing them by means of a collapse of a system of structureless particles.

We have rather adopted the approach of ref. [*7*] [*8*] in which the spin particle has been considered elementary from the beginning. In addition, we have also assumed (following closely ref. [*8*]) that in a linear theory, as it is the case with electromagnetism, the field and force equations are logically independent one from each other; for alternative approaches, see ref. [6] and references quoted therein.

The results we obtain are, at the best of our knowledge, the first complete lowest order classical solution for the electromagnetic spin case. That is, our explicit results up to the first order in the electromagnetic coupling constant g are exact in c (althoug only valid in the center-mass frame) and not only up to $1/c^2$ as it is the case in ref. [8]. This shows the power of the three dimensional framework, when supplemented by the center of mass (or, eventually, other invariance-breaking) condition.

1. CLASSICAL SPIN PARTICLES

In the framework of predictive relativistic mechanics the interaction among spin particles can be described [7] by means of the system of equations:

$$\frac{d\vec{X}_a}{dt} = \vec{V}_a \qquad \frac{d\vec{V}_a}{dt} = \vec{a}_a(X, V, \alpha) \qquad (a, b, \ldots = 1, 2, 3) \qquad (1.1)$$
$$\frac{d\vec{\alpha}_a}{dt} = \vec{\beta}_a(X, V, \alpha) \qquad \vec{\alpha}_a . \vec{\beta}_a = 0$$

where $\vec{\alpha}_a$ ($\vec{\alpha}_a^2 = 1$) stands for the instantaneous spin orientation of the particle a. We may then define the intrinsic spin orientation γ_a^λ as the four-vector which coincides with $(0, \vec{\alpha}_a)$ in the instantaneous rest frame of the particle a, that is:

$$\gamma^0 = \vec{u} . \vec{\alpha} \qquad \vec{\gamma} = \vec{\alpha} + [1 + (\vec{u}^2 + 1)^{1/2}]^{-1}(\vec{u} . \vec{\alpha})\vec{u} \qquad (1.2)$$

where we have taken $\vec{u} \equiv [1 - \vec{V}^2]^{-1/2}\vec{V}$ $(c = 1)$.

The transformation properties of the variables $\vec{X}$, $\vec{V}$, $\vec{\alpha}$ under the Poincaré group determine the infinitesimal generators

$$\underline{P_k} = - \Sigma_a \frac{\partial}{\partial X_a^k} \qquad \underline{H} = - V_a^i \frac{\partial}{\partial X_a^i} - a_a^i \frac{\partial}{\partial V_a^i} - \beta_a^i \frac{\partial}{\partial \alpha_a^i}$$
$$\underline{J_k} = - \varepsilon_{ki}{}^j \left(X_a^i \frac{\partial}{\partial X_a^j} + V_a^i \frac{\partial}{\partial V_a^j} + \alpha_a^i \frac{\partial}{\partial \alpha_a^j} \right)$$
$$\underline{K_j} = - X_{aj} V_a^k \frac{\partial}{\partial X_a^k} + \Sigma_a(\delta_j^k - V_{aj}V_a^k - X_{aj}a_a^k) \frac{\partial}{\partial V_a^k} \qquad (1.3)$$
$$+ [1 + (\vec{u}_a^2 + 1)^{1/2}]^{-1}(\vec{u}_a . \vec{\alpha}_a \delta_j^k - \alpha_{aj} V_a^k) \frac{\partial}{\partial \alpha_a^k} - X_{aj}\beta_a^k \frac{\partial}{\partial \alpha_a^k}$$

where the Einstein convention applies to both space and particle indices and ε_{ijk} is the totally antisymmetric tensor. Taking now into account (1.3) and the Lie algebra commutation relations associated to this representation of the Poincaré group:

$$
\begin{array}{lll}
[\underline{P}_i, \underline{P}_j] = 0 & [\underline{J}_i, \underline{P}_j] = \varepsilon_{ij}{}^k \underline{P}_k & [\underline{H}, \underline{P}_i] = 0 \\
[\underline{K}_i, \underline{P}_j] = \delta_{ij} \underline{H} & [\underline{J}_i, \underline{J}_j] = \varepsilon_{ij}{}^k \underline{J}_k & [\underline{H}, \underline{J}_i] = 0 \\
[\underline{K}_i, \underline{J}_j] = \varepsilon_{ij}{}^k \underline{K}_k & [\underline{K}_i, \underline{H}] = \underline{P}_i & [\underline{K}_i, \underline{K}_j] = -\varepsilon_{ij}{}^k \underline{J}_k
\end{array}
\tag{1.4}
$$

the following restrictions of the functions $\vec{a}$, $\vec{\beta}$ are obtained

$$
\begin{array}{ll}
L(\underline{P}_k)\vec{a} = \vec{0} & L(\underline{J}_k)a^i = \varepsilon_{k.j}^{\ \ i} a^j \\
L(\underline{P}_k)\vec{\beta} = \vec{0} & L(\underline{J}_k)\beta^i = \varepsilon_{k.j}^{\ \ i} \beta^j \\
\multicolumn{2}{l}{L(\underline{K}_j)a^i = X_j L(\underline{H})a^i - 2V_j a^i - V^i a_j} \\
\multicolumn{2}{l}{L(\underline{K}_j)\vec{\beta} = X_j L(\underline{H})\vec{\beta}}
\end{array}
\tag{1.5}
$$

The symbol L( ) stands for the Lie derivative. These conditions are the natural extension of the Currie-Hill equations [*1*] [*7*] [*8*] which takes spin into account.

2. CONDENSED DESCRIPTION OF THE DYNAMICAL SYSTEM

The knowledge of the solution of (1.1) corresponding to the initial conditions $\vec{X}_0$, $\vec{V}_0$, $\vec{\alpha}_0$ is equivalent to that of the trajectories:

$$
\begin{aligned}
\vec{q}_a(X, V, \alpha) &= \vec{q}_a(X_0, V_0, \alpha_0) + t\vec{p}_a/E_a \\
\vec{p}_a(X, V, \alpha) &= \vec{p}_a(X_0, V_0, \alpha_0) = \text{const.} \\
\vec{\omega}_a(X, V, \alpha) &= \vec{\omega}_a(X_0, V_0, \alpha_0) = \text{const.}
\end{aligned}
\tag{2.1}
$$

in the neighborhood of every point $(\vec{X}, \vec{V}, \vec{\alpha})$ in which $\vec{q}$, $\vec{p}$, $\vec{\omega}$ are continuous and differentiable functions such that

$$
\frac{\partial(q^i_a, p^j_b, \omega_c)}{\partial(X^m_d, V^n_e, \alpha^r_f)} \neq 0
$$

and verify the differential equations

$$
\begin{aligned}
L(\underline{H})\vec{q}(X, V, \alpha) &= -\vec{p}/E \\
L(\underline{H})\vec{p}(X, V, \alpha) &= \vec{0} \\
L(\underline{H})\vec{\omega}(X, V, \alpha) &= \vec{0}
\end{aligned}
\tag{2.2}
$$

where $E \equiv [\vec{p}^{\,2} + m^2]^{1/2}$.

We now demand to the solutions of (2.2) that they reduce to $\vec{X}$, $m\vec{u}$, $\vec{\gamma}$ when the particles are far apart in the past:

$$\lim_{\lambda \to -\infty} R'(\lambda)(\vec{q}_a - \vec{X}_a) = \vec{0} \qquad \lim_{\lambda \to -\infty} R'(\lambda)(\vec{\omega}_a - \vec{\gamma}_a) = \vec{0}$$
$$\lim_{\lambda \to -\infty} R'(\lambda)(\vec{p}_a - m_a\vec{u}_a) = \vec{0} \tag{2.3}$$

where use has been made of the shift operator $R'(\lambda)$

$$R'(\lambda) f(\vec{X}_a, \vec{V}_b, \vec{\alpha}_c) \equiv f(\vec{X}_a + \lambda\vec{V}_a, \vec{V}_b, \vec{\alpha}_c)$$

We shall see later that, at least in the framework of perturbation theory the differential system (2.2) and the assymptotic conditions (2.3) determine uniquely defined expressions for $\vec{q}$, $\vec{p}$, $\vec{\omega}$. The set of equations (2.2) and (2.3) give us then the dynamical behaviour of the system in the implicit form (2.1).

In order to be able to use perturbation theory, it is best to set up (2.2) and (2.3) in an equivalent integral form:

$$\begin{aligned}
\vec{q} &= \vec{X} - \int_{-\infty}^{0} d\lambda R'(\lambda)\left[a_b^i \frac{\partial \vec{q}}{\partial V_b^i} + \beta_b^i \frac{\partial \vec{q}}{\partial \alpha_b^i} + \vec{V} - \vec{p}/E \right] \\
\vec{p} &= m\vec{u} - \int_{-\infty}^{0} d\lambda R'(\lambda)\left[a_b^i \frac{\partial \vec{p}}{\partial V_b^i} + \beta_b^i \frac{\partial \vec{p}}{\partial \alpha_b^i} \right] \\
\vec{\omega} &= \vec{\gamma} - \int_{-\infty}^{0} d\lambda R'(\lambda)\left[a_b^i \frac{\partial \vec{\omega}}{\partial V_b^i} + \beta_b^i \frac{\partial \vec{\omega}}{\partial \alpha_b^i} \right]
\end{aligned} \tag{2.4}$$

The proof of the equivalence between (2.4) and the eqs. (2.2) and the assymptotic conditions (2.3) is a straightforward extension of the analogous proof in the scalar case [*4*] based on the following easy to check properties of $R'(\lambda)$:

$$V_a^i \frac{\partial}{\partial X_a^i} R'(\lambda) = R'(\lambda) V_a^i \frac{\partial}{\partial X_a^i} = \frac{d}{d\lambda} R'(\lambda)$$
$$R'(\lambda)R'(\tau) = R'(\lambda + \tau)$$

Now, if we know $\vec{\alpha}$, $\vec{\beta}$ as power series of some coupling constant, we may impose to the solutions of (2.4) to be of the form

$$\vec{q}_a = \sum_{n=0}^{\infty} \overset{(n)}{\vec{q}_a} \qquad \vec{p}_a = \sum_{n=0}^{\infty} \overset{(n)}{\vec{p}_a} \qquad \vec{\omega}_a = \sum_{n=0}^{\infty} \overset{(n)}{\vec{\omega}_a}$$
$$\overset{(0)}{\vec{q}_a} = \vec{X}_a \qquad \overset{(0)}{\vec{p}_a} = m_a\vec{u}_a \qquad \overset{(0)}{\vec{\omega}_a} = \vec{\gamma}_a \tag{2.5}$$

and, substituting in (2.4), one obtains the recurrent algorithm:

$$\begin{aligned}
\overset{(n)}{\vec{q}} &= \int_{-\infty}^{0} d\lambda \mathrm{R}'(\lambda)\left[\overset{(n)}{(\vec{p}/\mathrm{E})} - \sum_{r+s=n}\left(\overset{(r)}{a^i_b}\frac{\partial \overset{(s)}{\vec{q}}}{\partial \mathrm{V}^i_b} + \overset{(r)}{\beta^i_b}\frac{\partial \overset{(s)}{\vec{q}}}{\partial \alpha^i_b}\right)\right] \\
\overset{(n)}{\vec{p}} &= -\int_{-\infty}^{0} d\lambda \mathrm{R}'(\lambda)\sum_{r+s=n}\left(\overset{(r)}{a^i_b}\frac{\partial \overset{(s)}{\vec{p}}}{\partial \mathrm{V}^i_b} + \overset{(r)}{\beta^i_b}\frac{\partial \overset{(s)}{\vec{p}}}{\partial \alpha^i_b}\right) \qquad (2.6) \\
\overset{(n)}{\vec{\omega}} &= -\int_{-\infty}^{0} d\lambda \mathrm{R}'(\lambda)\sum_{r+s=n}\left(\overset{(r)}{a^i_b}\frac{\partial \overset{(s)}{\vec{\omega}}}{\partial \mathrm{V}^i_b} + \overset{(r)}{\beta^i_b}\frac{\partial \overset{(s)}{\vec{\omega}}}{\partial \alpha^i_b}\right)
\end{aligned}$$

where it is evident that (2.5) and (2.6) determine unique expressions for $\vec{q}$, $\vec{p}$, $\vec{\omega}$ as we have stated before.

3. TRANSFORMATION PROPERTIES OF THE NEW COORDINATES

Starting from the expressions (1.3) for the infinitesimal generators $\underline{\mathrm{P}}_k$, $\underline{\mathrm{J}}_k$, the well known transformation properties of $\vec{\mathrm{X}}$, $\vec{u}$, $\vec{\gamma}$, the structure of the algorithm (2.6) and the conditions (1.5) on $\vec{a}$, $\vec{\beta}$, it is easy to see that $\vec{q}$ transforms as a point and $\vec{p}$, $\vec{\omega}$ as three vectors under space translations and rotations, that is:

$$\begin{aligned}
\mathrm{L}(\underline{\mathrm{P}}_k)q^i &= -\delta^i_k & \mathrm{L}(\underline{\mathrm{J}}_k)q^i &= \varepsilon_{k.j}^{\ i}q^j \\
\mathrm{L}(\underline{\mathrm{P}}_k)\vec{p} &= \vec{0} & \mathrm{L}(\underline{\mathrm{J}}_k)p^i &= \varepsilon_{k.j}^{\ i}p^j \qquad (3.1) \\
\mathrm{L}(\underline{\mathrm{P}}_k)\vec{\omega} &= \vec{0} & \mathrm{L}(\underline{\mathrm{J}}_k)\omega^i &= \varepsilon_{k.j}^{\ i}\omega^j
\end{aligned}$$

We are going to show now that the following relationships hold in the framework of perturbation theory:

$$\mathrm{L}(\underline{\mathrm{K}}_j)\vec{q} = q_j\mathrm{L}(\underline{\mathrm{H}})\vec{q} = -q_j\vec{p}/\mathrm{E} \qquad (3.2)$$

$$\begin{aligned}
\mathrm{L}(\underline{\mathrm{K}}_j)p^i &= \delta^i_j\mathrm{E} \\
\mathrm{L}(\underline{\mathrm{K}}_j)\omega^i &= \delta^i_j\vec{\omega}\cdot\vec{p}/\mathrm{E}
\end{aligned} \qquad (3.2)'$$

To show this, we apply $\mathrm{L}(\underline{\mathrm{H}})$ to both members of (3.2) and take into account the commutation relations (1.4) to obtain, using (2.2) and (3.1):

$$\mathrm{L}(\underline{\mathrm{H}})[\mathrm{L}(\underline{\mathrm{K}}_j)q^i + q_jp^i/\mathrm{E}] = (\delta^i_k - p^ip_k\mathrm{E}^{-2})[\delta^k_j - \mathrm{E}^{-1}\mathrm{L}(\underline{\mathrm{K}}_j)p^k] \qquad (3.3)$$

$$\begin{aligned}
\mathrm{L}(\underline{\mathrm{H}})[\mathrm{L}(\underline{\mathrm{K}}_j)p^i - \delta^i_j\mathrm{E}] &= 0 \\
\mathrm{L}(\underline{\mathrm{H}})[\mathrm{L}(\underline{\mathrm{K}}_j)\omega^i - \delta^i_j\vec{\omega}.\vec{p}/\mathrm{E}] &= 0
\end{aligned} \qquad (3.3)'$$

In addition, it is straightforward from the expression (1.3) for $\underline{\mathrm{K}}_j$ and the definition of $R'(\lambda)$ to see that, in the cases when the following conditions hold:

$$\lim_{\lambda \to -\infty} [\lambda R'(\lambda)\vec{\beta}] = \vec{0} \qquad \lim_{\lambda \to -\infty} [\lambda^2 R'(\lambda)\vec{a}] = \vec{0} \tag{3.4}$$

we obtain, using (2.3), the asymptotic conditions:

$$\lim_{\lambda \to -\infty} R'(\lambda)[L(\underline{K_j})\vec{q} + q_j\vec{p}/E] = \vec{0} \tag{3.5}$$

$$\left.\begin{aligned} &\lim_{\lambda \to -\infty} R'(\lambda)[L(\underline{K_j})p^i - \delta^i_j E] = 0 \\ &\lim_{\lambda \to -\infty} R'(\lambda)[L(\underline{K_j})\omega^i - \delta^i_j \vec{\omega}.\vec{p}/E] = 0 \end{aligned}\right. \tag{3.5)'$$

Now we must note that the equations (3.3)' and the conditions (3.5)' are just of the same class than the second and third relationships (2.2) and (2.3), respectively. Then we know that there must exist an algorithm similar to (2.6) which gives us uniquely defined solutions for the expressions appearing within the brackets in (3.3)' corresponding to every set of zeroth order terms. This guarantees that if equations (3.2)' hold at zeroth order they must hold then at any order. This puts (3.3) and (3.5) at the same level of the corresponding primed relationships and guarantees also the validity of (3.2) at any order in perturbation theory.

We have shown that $\vec{q}$, $\vec{p}$, $\vec{\omega}$ transform under the Poincaré group exactly in the same form as $\vec{X}$, $m\vec{u}$, $\vec{\gamma}$ in the no interaction case. If the particles have no spin [*4*], $\vec{q}$, $\vec{p}$ may be used as a canonical set of phase-space coordinates (the Poincaré group being implemented by canonical transformations). On the contrary we know [*9*] that when the particles have spin these transformation properties are not compatible with the canonical character of the coordinates [*10*].

4. THE CENTER OF MASS FORMALISM

In what follows, we will restrict ourselves to the case of two particles. Let us recall the expressions for the new coordinates

$$\vec{q}_a(X, V, \alpha) \qquad \vec{p}_a(X, V, \alpha) \qquad \vec{\omega}_a(X, V, \alpha) \tag{4.1}$$

and remark that, according to (2.1) $\vec{p}_a$, $\vec{\omega}_a$ are constants of the motion. We note then that $\vec{p}_a + \vec{p}_{a'}$ $(a' \neq a)$ is also a constant of the motion and that, when the particles are far apart, this constant coincides with the three momentum of the system.

If, as indicated in the introduction, we want to fix our reference frame to coincide with the center of mass one, then we must demand $\vec{p}_a = -\vec{p}_{a'} \equiv \vec{p}$. The essential point is that it is very difficult to impose directly this condition due to the fact that we know $\vec{p}_a$ only as a power series. Nevertheless it is possible, at least in the framework of perturbation theory, to invert the second and third expressions in (4.1) to obtain

$$\vec{V}_a(X_b, p_c, \omega_d) \qquad \vec{\alpha}_a(X_b, p_c, \omega_d) \tag{4.2}$$

and substitute in the expression for $\vec{q}_a$ to have

$$\vec{q}_a(X_b, p_c, \omega_d) \tag{4.2'}$$

Now using these new variables it is trivial to impose the center of mass condition. Let us reconsider the preceding formalism with the new variables: the form of the infinitesimal generators (1.3) is now simpler, as it follows from the preceding section:

$$\begin{aligned}
&\underline{P}_K = -\Sigma_a \frac{\partial}{\partial X_a^k} \qquad \underline{H} = -V_a^i(X, p, \omega)\frac{\partial}{\partial X_a^i} \\
&\underline{J}_K = -\varepsilon_{ki}{}^j\left(X_a^i \frac{\partial}{\partial X_a^j} + p_a^i \frac{\partial}{\partial p_a^j} + \omega_a^i \frac{\partial}{\partial \omega_a^j}\right) \\
&\underline{K}_j = -X_{aj}V_a^i(X, p, \omega)\frac{\partial}{\partial X_a^i} - \delta_j^i\left(E_a \frac{\partial}{\partial p_a^i} + \vec{\omega}_a \cdot \vec{p}_a/E_a \frac{\partial}{\partial \omega_a^i}\right)
\end{aligned} \tag{4.3}$$

The dynamical information is contained in the functions (4.2) and (4.2)′ which can be obtained as the solutions of the equations:

$$\begin{aligned}
L(\underline{H})\vec{q}_a = -\vec{p}_a/E_a \qquad L(\underline{H})\vec{V}_a = -\vec{a}_a(X, p, \omega) \\
L(\underline{H})\vec{\alpha}_a = -\vec{\beta}_a(X, p, \omega)
\end{aligned} \tag{4.4}$$

which satisfy the assymptotic conditions

$$\begin{aligned}
&\lim_{\lambda \to -\infty} R(\lambda)(\vec{q}_a - \vec{X}_a) = 0 \qquad \lim_{\lambda \to -\infty} R(\lambda)(\vec{V}_a - \vec{p}_a/E_a) = 0 \\
&\lim_{\lambda \to -\infty} R(\lambda)[\vec{\alpha}_a - \vec{\omega}_a + (E_a + m_a)^{-1}(\vec{\omega}_a \cdot \vec{p}_a)\vec{p}_a/E_a] = 0
\end{aligned} \tag{4.5}$$

where the shift operator $R(\lambda)$ is defined now by the identity:

$$R(\lambda)f(\vec{X}_a, \vec{p}_b, \vec{\omega}_c) \equiv f\left(\vec{X}_a + \lambda \frac{\vec{p}_a}{E_a}, \vec{p}_b, \vec{\omega}_a\right)$$

It is of course desirable to recast (4.4) and (4.5) in its corresponding integral form, that is

$$\begin{aligned}
&\vec{q} = \vec{X} - \int_{-\infty}^{0} d\lambda R(\lambda)\Sigma_b(V_b^i - p_b^i/E_b)\frac{\partial \vec{q}}{\partial x_b^i} \\
&\vec{V} = \vec{p}/E + \int_{-\infty}^{0} d\lambda R(\lambda)\left[\vec{a} - \Sigma_b\left(V_b^i - \frac{p_b^i}{E_b}\right)\frac{\partial \vec{V}}{\partial X_b^i}\right] \\
&\vec{\alpha} = \vec{\omega} - \frac{\vec{\omega} \cdot \vec{p}}{E + m}\vec{p}/E + \int_{-\infty}^{0} d\lambda R(\lambda)\left[\vec{\beta} - \Sigma_b\left(V_b^i - \frac{p_b^i}{E_b}\right)\frac{\partial \vec{\alpha}}{\partial X_b^i}\right].
\end{aligned} \tag{4.6}$$

Now, by imposing the center of mass condition and using the new variables $\vec{X}$, $\vec{p}$ and $\vec{\omega}$ and the expressions (4.6), it is possible to determine (with a reasonable amount of elementary calculations) the functions which give us the dynamics of the system in a condensed form. In the following sections, we will study a particular case of physical interest namely the electromagnetic interaction, and calculate these functions at first order in perturbation theory.

5. FIELD OF A SPINNING CHARGE

The field arising from a point charge of spin s which moves along the world-line $\phi^\mu(\tau)$ has to be obtained by substituting into the Maxwell equations the corresponding current density:

$$j^\mu(x)=e\int_{-\infty}^{\infty}\dot{\phi}^\mu(\tau)\delta^4(x-\phi(\tau))d\tau + \frac{\lambda es}{2mc}\varepsilon^{\mu\nu}{}_{\cdot\cdot\alpha\beta}\frac{\partial}{\partial x^\nu}\int_{-\infty}^{\infty}\dot{\phi}^\beta(\tau)\gamma^\alpha(\tau)\delta^4(x-\phi(\tau))d\tau \quad (5.1)$$

where λ is the giromagnetic ratio, γ^α the intrinsic spin orientation and ε stands for the totally antisymmetric four-tensor with $\varepsilon^{0123}=1$.

The resulting retarded potential is, in the Lorentz gauge:

$$A^\mu(x) = e\dot{\phi}^\mu(\hat{\tau})\hat{r}^{-1} + \frac{\lambda es}{2mc}\varepsilon^{\mu}_{\cdot\,\nu\alpha\beta}\hat{l}^\nu\hat{r}^{-2}\left[\frac{d}{d\tau}(\gamma^\alpha\dot{\phi}^\beta)_{\tau=\hat{\tau}} + \hat{r}^{-1}\gamma^\alpha(\hat{\tau})\dot{\phi}^\beta(\hat{\tau})(1+\ddot{\phi}_\sigma(\hat{\tau})\hat{l}^\sigma)\right] \quad (5.2)$$

where $\hat{r}\equiv-\dot{\phi}^\mu(\hat{\tau})\hat{l}_\mu$, $\hat{l}^\mu\equiv x^\mu-\phi^\mu(\hat{\tau})$ and $\hat{\tau}$ is to be found as the solution of $[x-\phi(\hat{\tau})]^2=0$, $x^0>\phi^0(\hat{\tau})$.

If we are only interested in the first order contribution to (5.2), we can assume that the source of the field is a free particle, that is:

$$\dot{\phi}^\beta(\hat{\tau}) = u'^\beta = \text{const.} \qquad \hat{l}^\mu = (x-x')^\mu - [(x-x')^\sigma u'_\sigma + \tilde{r}\,]u'^\mu$$
$$\hat{r}=\tilde{r}\equiv[(x-x')^2+((x-x')^\sigma u'_\sigma)^2]^{1/2} \qquad \gamma^\alpha(\hat{\tau})=\gamma'^\alpha=\text{const.}$$

and then we have at the lowest order:

$$F^{\mu\nu}(x)\equiv\partial^\mu A^\nu-\partial^\nu A^\mu = -e\tilde{r}^{-3}(\delta^\mu_\alpha\delta^\nu_\beta-\delta^\nu_\alpha\delta^\mu_\beta)(x-x')^\alpha u'^\beta + \frac{\lambda es}{2mc}\tilde{r}^{-3}\varepsilon^{\mu\nu}{}_{\cdot\cdot\alpha\beta}[\gamma^\alpha-3\tilde{r}^{-2}(x-x')^\alpha\gamma_\sigma(x-x')^\sigma]u'^\beta \quad (5.3)$$

Let us note that the same result would also be obtained if we had used advanced or time symmetric potentials.

6. THE LIENARD-WIECHERT CONDITIONS

The generalization of the Lorentz force equation to particles with spin [8] can be obtained by a procedure similar to that used by Bargmann-

Michel-Telegdi [11] (BMT) to obtain the equation governing the time evolution of a magnetic dipole in an external field F. The result which has been obtained is the following:

$$\frac{du^{\mu}}{d\tau} = \frac{e}{mc^2}\,\mathrm{F}^{\mu\nu}u_{\nu} + \frac{\lambda es}{2m^2c^3}\,(\eta^{\mu\sigma} + u^{\mu}u^{\sigma})\mathrm{F}^{*}_{\alpha\beta|\sigma}u^{\alpha}\gamma^{\beta} \tag{6.1}$$

where η stands for the Minkowski metric and F^* is the tensor density dual of F. The BMT equation gives us the precession of the intrinsic spin orientation:

$$\frac{d\gamma^{\mu}}{d\tau} = \frac{\lambda e}{2mc^2}\,(\eta^{\mu\sigma} + u^{\mu}u^{\sigma})\mathrm{F}_{\sigma\nu}\gamma^{\nu} + \gamma_{\sigma}\frac{du^{\sigma}}{d\tau}\,u^{\mu} \tag{6.2}$$

Now, if we suppose that every charge moves according to (6.1) and (6.2), where F is the total field arising from the others (no self-interaction), we obtain the generalization of the Liénard-Wiechert conditions; at the lowest order we have [12]:

$$\begin{aligned} mc^2\frac{du^{\mu}}{d\tau} = \sum_{a'\neq a} & ee'\tilde{r}^{-3} \\ & \Big\{ -(u.u')\mathrm{X}^{\mu} + (\mathrm{X}.u)u'^{\mu} - \frac{\lambda's'}{2m'c}\,\varepsilon^{\mu}{}_{\nu\alpha\beta}u^{\alpha}u'^{\beta}[\gamma'^{\nu} - 3\tilde{r}^{-2}\,\mathrm{X}^{\nu}(\mathrm{X}.\gamma')] \\ & - \frac{\lambda s}{2mc}\,(\eta^{\mu\sigma} + u^{\mu}u^{\sigma})\varepsilon_{\nu\rho\alpha\beta}u^{\nu}\gamma^{\rho}\big[\delta^{\alpha}_{\sigma} - 3\tilde{r}^{-2}\mathrm{X}^{\alpha}(\mathrm{X}_{\sigma} + (\mathrm{X}.u')u'_{\sigma})\big]u'^{\beta} \\ & + 3\,\frac{\lambda s}{2mc}\,\frac{\lambda's'}{2m'c}\,\tilde{r}^{-2}(\eta^{\mu\sigma} + u^{\mu}u^{\sigma})(u_{\alpha}\gamma_{\beta} - \gamma_{\alpha}u_{\beta})u'^{\beta}\big[\delta^{\alpha}_{\sigma}(\mathrm{X}.\gamma') + \mathrm{X}^{\alpha}\gamma'_{\sigma} \\ & + (\gamma'^{\alpha} - 5\tilde{r}^{-2}\mathrm{X}^{\alpha}(\mathrm{X}.\gamma'))(\mathrm{X}_{\sigma} + (\mathrm{X}.u')u'_{\sigma})\big]\Big\} \end{aligned} \tag{6.3}$$

$$\begin{aligned} c\,\frac{d\gamma^{\mu}}{d\tau} = \frac{\lambda e}{2mc}\,(\eta^{\mu\nu} + u^{\mu}u^{\nu})\sum_{a'\neq a} e'\tilde{r}^{-3}\Big[& (\delta^{\beta}_{\nu}\gamma^{\alpha} - \gamma^{\beta}\delta_{\nu})\mathrm{X}_{\alpha}u'_{\beta} + \frac{\lambda's'}{2m'c}\,\varepsilon_{\nu\sigma\alpha\beta}\gamma^{\sigma}u'^{\beta} \\ & \times(\gamma'^{\alpha} - 3\tilde{r}^{-2}\mathrm{X}^{\alpha}(\mathrm{X}.\gamma'))\Big] + c\left(\gamma.\frac{du}{d\tau}\right)u^{\mu} \end{aligned} \tag{6.4}$$

where $(\cdot)$ stands for the contraction of two fourvectors and $\mathrm{X} \equiv \mathrm{X}_a - \mathrm{X}_{a'}$.

7. THREE DIMENSIONAL CENTER OF MASS FORMULATION

Let us restrict ourselves to the two particle case and let us rewrite the equations (6.3) and (6.4) in the framework presented in sect. 4. First of all, we obtain the three dimensional version of (6.3) and (6.4) by using the well known relationships:

$$\frac{d\vec{V}}{dt} = c^2(1-\vec{V}^2/c^2)\left[\frac{d\vec{u}}{d\tau} - \frac{du^0}{d\tau}\vec{V}/c\right]_{X^0=0}$$
$$\frac{d\vec{\gamma}}{dt} = c\left(1-\frac{\vec{V}^2}{c^2}\right)^{1/2}\left[\frac{d\vec{\gamma}}{d\tau}\right]_{X^0=0} \qquad (7.1)$$

Second, we must use in the right hand terms of (7.1) the functions $\vec{v}/c(X, p, \omega)$, $\vec{\gamma}(\vec{X}, p, \omega)$ (at first order, this only means to substitute $\vec{V}/c$ and $\vec{\gamma}$ by $\vec{p}/E$ *and* $\vec{\omega}$) and impose the center of mass condition $\vec{p}_a = -\vec{p}_{a'} \equiv \vec{p}$. The resulting equations are:

$$\frac{d\vec{V}}{dt} = \frac{g}{m'}E^{-2}\tilde{r}^{-3}\left\{ k\vec{X} - \vec{X}.\vec{p}M\frac{\vec{p}}{E} + \frac{\lambda s}{2mc}M\right.$$
$$\left[\vec{\omega}\times\vec{p} + 3\vec{\omega}.\vec{n}\tilde{r}^{-2}\left(\vec{X}+M\frac{\vec{X}.\vec{p}}{m'^2}\frac{\vec{p}}{E}\right)\right] - \frac{\lambda' s'}{2m'c}M(\vec{\omega}'\times\vec{p} - 3\vec{\omega}'.\vec{X}\tilde{r}^{-2}\vec{n})$$
$$+ 3\frac{\lambda s}{2mc}\frac{\lambda' s'}{2m'c}\tilde{r}^{-2}\left[k\vec{\omega}.\vec{X}\left(\vec{\omega}-\omega^0\frac{\vec{p}}{E}\right) + \left(\vec{X}+M\frac{\vec{X}.\vec{p}}{m'^2}\frac{\vec{p}}{E}\right)\right.$$
$$\times\left(\frac{1}{2}\omega^0\omega'^0(M^2+m^2+m'^2)+k\vec{\omega}.\vec{\omega}'\right) + \left(\vec{\omega}'-\omega'^0\frac{\vec{p}}{E}\right)(k\vec{X}.\vec{\omega}-M\omega^0\vec{X}.\vec{p})$$
$$\left.\left. - 5\tilde{r}^{-2}\vec{X}.\vec{\omega}'\left(\vec{X}+M\frac{\vec{X}.\vec{p}}{m'^2}\frac{\vec{p}}{E}\right)(k\vec{X}.\vec{\omega}-M\omega^0\vec{X}.\vec{p})\right]\right\} \qquad (7.2)$$

$$\frac{d\vec{\gamma}}{dt} = \frac{\lambda}{2c}\frac{g}{m'}E^{-1}\tilde{r}^{-3}\left\{ -\vec{X}.\vec{\omega}ME\frac{\vec{p}}{m^2} + M\omega^0\left(\vec{X}+\frac{\vec{X}.\vec{p}}{m^2}\vec{p}\right)\right.$$
$$+ \frac{\lambda' s'}{2m'c}\left[E'(\vec{\omega}\times\vec{\omega}') + \vec{p}.(\vec{\omega}\times\vec{\omega}')M\frac{\vec{p}}{m^2} + (\omega'^0\vec{\omega}+\omega^0\vec{\omega}')\right.$$
$$\left.\left.\times\vec{p} - 3\tilde{r}^{-2}\vec{X}.\vec{\omega}'\left(\omega^0\vec{n} + E'\vec{\omega}\times\vec{X} + \vec{\omega}.\vec{n}M\frac{\vec{p}}{m^2}\right)\right]\right\} + \frac{\vec{p}}{m^2}\vec{\omega}.\frac{d\vec{V}}{cdt}E \qquad (7.3)$$

where $\vec{n}\equiv\vec{X}\times\vec{p}$, $M\equiv E+E'$, $\omega^0=\vec{\omega}.\vec{p}/E$, $\omega'^0 = -\vec{\omega}.\vec{p}/E'$, $k\equiv EE'+\vec{p}^2$. We can easily see that eqs. (7.2) and (7.3) are not longer than the corresponding four-dimensional ones (6.3), (6.4). This is a direct consequence of the form invariance breaking due to the center of mass condition.

In order to explicitly solve these equations in condensed form by using the integral formalism constructed in sect. 4 it is convenient to use the following variables:

$$Z \equiv \vec{X}.\hat{\vec{p}} \qquad \hat{\vec{p}} \equiv \vec{p}/p \qquad \vec{b} \equiv \vec{X} - Z\hat{\vec{p}}$$

because their behavior under the shift operator $R(\lambda)$ is very simple

$$R(\lambda)Z = Z + \lambda \frac{Mc}{EE'} p \qquad R(\lambda)\vec{b} = \vec{b}$$

(remember that $R(\lambda)f(\vec{X}, \vec{p}) = f(\vec{X} + \lambda c\vec{p}/E, p))$ and then the integrations of section 4 can be easily performed:

$$\int_{-\infty}^{0} d\lambda R(\lambda) f(Z, \vec{b}, \vec{p}, \vec{\omega}) = \frac{EE'}{Mcp} \int_{-\infty}^{Z} dZ' f(Z', \vec{b}, \vec{p}, \vec{\omega}) \tag{7.4}$$

Rearranging terms in (7.2), (7.3) we have

$$\frac{d\vec{V}}{dt} = \frac{g}{m'} E^{-2} \Bigg[\tilde{r}^{-3}(\vec{A}_1 + ZE'\vec{A}_2) + 3\tilde{r}^{-5}(\vec{A}_3 + ZE'\vec{A}_4) - 15\tilde{r}^{-7} \frac{\lambda s}{2mc} \frac{\lambda' s'}{2m'c} \left(\vec{A}_5 + ZE'\vec{A}_6 + \left(Z\frac{E'}{m'}\right)^2 \vec{A}_7 + Z^3 \frac{E'^3}{m'^2} \vec{A}_8 \right) \Bigg] \tag{7,5}$$

$$\frac{d\vec{\varsigma}}{dt} = \frac{\lambda g}{2m'Ec} \left[\tilde{r}^{-3}\vec{B}_1 - 3\tilde{r}^{-5} \frac{\lambda' s'}{2m'c} \left(\vec{B}_2 + ZE'\vec{B}_3 + \left(Z\frac{E'}{m'}\right)^2 \vec{B}_4 \right) \right] + \frac{E\vec{p}}{m^2} \vec{\omega} . \frac{d\vec{V}}{cdt} \tag{7.6}$$

where $\tilde{r} = [\vec{b}^2 + (ZE'/m')^2]^{1/2}$ and where we have introduced the following notation:

$$\vec{A}_1 = k\vec{b} + \frac{\lambda s}{2mc} M\vec{\omega} \times \vec{p} - \frac{\lambda' s'}{2m'c} M\vec{\omega}' \times \vec{p}$$

$$\vec{A}_2 = m^2 \frac{\vec{p}}{E}$$

$$\vec{A}_3 = \frac{\lambda s}{2mc} M\vec{\omega}.\vec{n}\vec{b} + \frac{\lambda' s'}{2m'c} M\vec{\omega}'.\vec{b}\vec{n} + \frac{\lambda s}{2mc} \frac{\lambda' s'}{2m'c} \left[\left(\frac{1}{2} \omega^0 \omega'^0 (M^2 + m^2 + m'^2) + k\vec{\omega}.\vec{\omega}' \right) \vec{b} + k\vec{\omega}'.\vec{b} \left(\vec{\omega} - \omega^0 \frac{\vec{p}}{E} \right) + k\vec{\omega}.\vec{b} \left(\vec{\omega}' - \omega'^0 \frac{\vec{p}}{E} \right) \right]$$

$$\vec{A}_4 = \frac{\lambda s}{2mc} M\vec{\omega}.\vec{n}k/m'^2\hat{\vec{p}}/E + \frac{\lambda' s'}{2m'c} M \frac{\omega'^0}{p} \vec{n} + \frac{\lambda s}{2mc} \frac{\lambda' s'}{2m'c} \left[\left(\frac{1}{2} \omega^0 \omega'^0 (M^2 + m^2 + m'^2) + k\vec{\omega}.\vec{\omega}' \right) \frac{k}{m'^2} \frac{\hat{\vec{p}}}{E} + p^{-1}(k\omega'^0\vec{\omega} + m^2\omega^0\vec{\omega}') - M\omega^0\omega'^0\hat{\vec{p}} \right]$$

$$\vec{A}_5 = k\vec{b}.\vec{\omega}\vec{b}.\vec{\omega}'\vec{b}$$

$$\vec{A}_6 = \vec{b}.\vec{\omega}\vec{b}.\vec{\omega}'\frac{k^2}{m^2}\frac{\hat{\vec{p}}}{E} + p^{-1}(k\vec{b}.\vec{\omega}\omega'^0 + m^2\omega^0\vec{b}.\vec{\omega}')\vec{b}$$

$$\vec{A}_7 = (-k\vec{\omega}.\vec{b}\omega'^0 + m^2\vec{\omega}'.\vec{b}\omega^0)\,kp^{-1}\frac{\hat{\vec{p}}}{E} - \omega^0\omega'^0 m^2 m'^2 p^{-2}\vec{b}$$

$$\vec{A}_8 = -\omega^0\omega'^0\frac{km^2}{Ep^2}\hat{\vec{p}}$$

$$\vec{B}_1 = -\vec{\omega}.\vec{b}\mathrm{ME}\frac{\vec{p}}{m^2} + \mathrm{M}\omega^0\vec{b} + \frac{\lambda's'}{2m'c}\left[E'\vec{\omega}\times\vec{\omega}' + (\omega'^0\vec{\omega} + \omega^0\vec{\omega}')\times\vec{p} + \vec{p}(\vec{\omega}\times\vec{\omega}')\mathrm{M}\frac{\vec{p}}{m^2}\right]$$

$$\vec{B}_2 = \vec{b}.\vec{\omega}'\left(\omega^0\vec{n} + E'\vec{\omega}\times\vec{b} + \vec{\omega}.\vec{n}\mathrm{M}\frac{\vec{p}}{m^2}\right)$$

$$\vec{B}_3 = -\omega'^0 p^{-1}\left(\omega^0\vec{n} + E'\vec{\omega}\times\vec{b} + \vec{\omega}.\vec{n}\mathrm{M}\frac{\vec{p}}{m^2}\right) + \vec{b}.\vec{\omega}'\vec{\omega}\times\hat{\vec{p}}$$

$$\vec{B}_4 = -\omega'^0 p^{-1}m'^2\vec{\omega}\times\hat{\vec{p}}\,.$$

If we apply the results of sect. 4 and take into account (7.4), we obtain up to first order in the coupling constant g:

$$\vec{V}(X, p, \omega) = c\frac{\vec{p}}{E} + \frac{EE'}{Mcp}\int_{-\infty}^{Z} dZ'\frac{d\overset{(1)}{\vec{V}}}{dt}(Z') + \ldots = c\frac{\vec{p}}{E} + (\mathrm{EM}cp)^{-1}\times g$$
$$\times\left\{\frac{1}{b^2}\left(\frac{ZE'}{m'\tilde{r}} + 1\right)\vec{A}_1 - m'\tilde{r}^{-1}\vec{A}_2 + \frac{1}{b^2}\left[Z\frac{E'}{m'}\tilde{r}^{-3} + \frac{2}{b^2}\left(\frac{ZE'}{m'\tilde{r}} + 1\right)\right]\right.$$
$$\left(\vec{A}_3 - 3\frac{\lambda s}{2mc}\frac{\lambda's'}{2m'c}\vec{A}_7\right) - m'\tilde{r}^{-3}\left(\vec{A}_4 - 5\frac{\lambda s}{2mc}\frac{\lambda's'}{2m'c}\vec{A}_8\right)$$
$$-3\frac{\lambda s}{2mc}\frac{\lambda's'}{2m'c}\frac{1}{b^2}\left[Z\frac{E'}{m'}\tilde{r}^{-5} + \frac{4}{3b^2}\left(Z\frac{E'}{m'}\tilde{r}^{-3} + \frac{2}{b^2}\left(\frac{ZE'}{m'\tilde{r}} + 1\right)\right)\right]$$
$$\left.\times(\vec{A}_5 - b^2\vec{A}_7) - 3\frac{\lambda s}{2mc}\frac{\lambda's'}{2m'c}m'\tilde{r}^{-5}[\vec{A}_6 - b^2\vec{A}_8]\right\} + \ldots \qquad (7.7)$$

$$\vec{\gamma}(X, p, \omega) = \vec{\omega} + \frac{EE'}{Mcp}\int_{-\infty}^{Z} dZ'\left[\frac{d\overset{(1)}{\vec{\gamma}}}{dt}(Z')\right] + \ldots = \vec{\omega}$$
$$+ \lambda(2\mathrm{M}cp)^{-1}g\left\{\frac{1}{b^2}\left(\frac{ZE'}{m'\tilde{r}} + 1\right)\left(\vec{B}_1 + 3\frac{\lambda's'}{2m'c}\vec{B}_4\right)\right.$$
$$\left.-\frac{1}{b^2}\left[Z\frac{E'}{m'}\tilde{r}^{-3} + \frac{2}{b^2}\left(\frac{ZE'}{m'\tilde{r}} + 1\right)\right]\frac{\lambda's'}{2m'c}(\vec{B}_2 - b^2\vec{B}_4) + \tilde{r}^{-3}\frac{\lambda's'}{2c}\vec{B}_3\right\}$$
$$+ \frac{E\vec{p}}{m^2}\vec{\omega}.\overset{(1)}{\vec{V}}/c + \ldots \qquad (7.8)$$

8. COMPLETE INTEGRATION

At this point, we find the first difficulty: the integral (4.6) we need to calculate $\vec{q}$ diverges because of the $\vec{A}_2$ term in (7.7). This means that we can not impose in the electromagnetic case the assymptotic condition:

$$\lim_{\lambda \to -\infty} R(\lambda)(\vec{q}_a - \vec{X}_a) = \vec{0} \qquad \forall a \tag{8.1}$$

We shall impose, instead of (8.1), the less restrictive condition [*13*]:

$$\lim_{\lambda \to -\infty} R(\lambda) \frac{\partial}{\partial X_b{}^i} (\vec{q}_a - \vec{X}_a) = \vec{0} \qquad \forall a, b, i \tag{8.2}$$

The breakdown of the integral formalism due to the fact that we can not enforce the conditions (8.1) opens the door to some non-unicity for $\vec{q}$. To see to what extent (8.2) restricts the solutions, we decompose $\vec{q} - \vec{X}$ in the cylindrical basis with axis in the direction of $\vec{p}$:

$$\vec{q} - \vec{X} = \Delta_b \vec{b} - \Delta_\phi \vec{n} + \Delta_Z \vec{p}$$

where symbols with a hat are unit vectors. It is easy to see that (8.2) is equivalent to:

$$\lim_{\lambda \to -\infty} R(\lambda)\Delta_b = \lim_{\lambda \to -\infty} R(\lambda)\Delta_\phi = 0 \qquad \lim_{\lambda \to -\infty} R(\lambda)\frac{\partial \Delta_Z}{\partial X^i} = 0 \qquad \forall i \tag{8.3}$$

therefore (8.2) determine uniquely Δ_b, Δ_ϕ, and, up to an arbitrary function independent of $\vec{X}$, Δ_Z. This allows us to propose the following procedure:

i) to perform the indefinite integration for the $\vec{A}_2$ term taking into account (8.3) and add an arbitrary function (independent of $\vec{X}$) to Δ_Z,

ii) to perform the definite (between $-\infty$ and z) integration for the leading terms on (7.7) in order to complete the result.

The first step involves the integration:

$$- E'(Mcp)^{-2} \int dZ(-m'\tilde{r}^{-1}A_2) \\ = - m'^2(Mcp)^{-2} \ln(\tilde{r} - ZE'/m')\vec{A}_2 + \vec{F}(b, p, m) \tag{8.4}$$

where, taking into account (8.3), we have:

$$\vec{F}(b, p, m) = F(p, m)\hat{\vec{p}}$$

Equivalently (8.4) can be written in the form

$$- m^2 m'^2 (Mcp)^{-2} \ln\,[(\tilde{r} - ZE'/m')L^{-1}]\hat{\vec{p}}/E \tag{8.5}$$

where L must be an arbitrary function (independent of $\vec{x}$) with units of length [*14*].

Finally, going to the second step, we arrive at the final result:

$$\begin{aligned}
\vec{q}(\mathrm{X}, p, \omega) \\
= \vec{\mathrm{X}} - g \frac{m^2 m'^2}{(\mathrm{M}cp)^2} \ln \left[(\tilde{r} - \mathrm{ZE}'/m')\mathrm{L}^{-1}\right]\hat{\vec{p}}/\mathrm{E} - \frac{m'g}{b^2(\mathrm{M}cp)^2} \Bigg\{ (\tilde{r} + \mathrm{ZE}'/m')\vec{\mathrm{A}}_1 \\
- \left[\tilde{r}^{-1} - \frac{2}{b^2}(\tilde{r} + \mathrm{ZE}'/m')\right]\left(\vec{\mathrm{A}}_3 - 3\frac{\lambda s}{2mc}\frac{\lambda' s'}{2m'c}\vec{\mathrm{A}}_7\right) \\
- \left(\frac{\mathrm{ZE}'}{\tilde{r}} + m'\right)\left(\vec{\mathrm{A}}_4 - 5\frac{\lambda s}{2mc}\frac{\lambda' s'}{2m'c}\vec{\mathrm{A}}_8\right) \\
+ \left[\tilde{r}^{-3} + \frac{4}{b^2}\left(\tilde{r}^{-1} - \frac{2}{b^2}(\tilde{r} + \mathrm{ZE}'/m')\right)\right] \times \frac{\lambda s}{2mc}\frac{\lambda' s'}{2m'c}(\vec{\mathrm{A}}_5 - b^2\vec{\mathrm{A}}_7) \\
+ \left[\mathrm{ZE}'\tilde{r}^{-3} + \frac{2}{b^2}\left(\frac{\mathrm{ZE}'}{\tilde{r}} + m'\right)\right] \times \frac{\lambda s}{2mc}\frac{\lambda' s'}{2m'c}(\vec{\mathrm{A}}_6 - b^2\vec{\mathrm{A}}_8)\Bigg\} + \dots \qquad (8.6)
\end{aligned}$$

which completes the integration procedure and gives us, together with (7.8), all the dynamical information relative to the two charges system.

9. CONCLUSIONS AND ACKNOWLEDGMENTS

The main conclusion of this work is that, in the two particles case, explicit three-dimensional calculations are as feasible as fourdimensional ones even in the more involved situations (for instance, when spin is accounted) by breaking the form invariance of the three-dimensional formalism of predictive relativistic mechanics, that is, by fixing the reference frame (in our case by means of the center of mass condition). This statement is verified by presenting the first (at the best of our knowledge) complete lowest order classical solution for the electromagnetic relativistic spin case.

Let us discuss briefly the possibility of invert the changes of coordinates that we have made in order to study the dynamics of the system. We shall take into account the transformation properties (3.1) to see that:

$$\frac{\partial(q_a, p_b, \omega_c)}{\partial(x_d, v_e, \alpha_f)} = \frac{\partial(q, p_b, \omega_c)}{\partial(x, v_e, \alpha_f)} \qquad (9.1)$$

The time evolution of the relative coordinates $\vec{q}$ is given by the expression: $\vec{q} = \vec{q}_0 + t\vec{p}$, $\vec{p} = \vec{p}_a/\mathrm{E}_a - \vec{p}_{a'}/\mathrm{E}_{a'}$ and any point of the space spanned by $(\vec{q}, \vec{p}_a, \vec{\omega}_b)$ is then determined by the values of the set of constants $\vec{p}_a/\mathrm{E}_a$, $\vec{\omega}_b$, $\vec{b} = \vec{q} - \vec{q}.\vec{p}/\rho^2\vec{p}$ and the parameter $\vec{q}.\vec{p} = \vec{q}_0.\vec{p} + t\rho^2$.

The time evolution of relative coordinates $\vec{x}$ is given by the solution of the system of equations:

$$\frac{dx}{dt} = \vec{v}_b - \vec{v}_{b'} \qquad \frac{d\overrightarrow{\alpha_b}}{dt} = \vec{\beta}_b(x, v_c, \alpha_d)$$
$$\frac{d\overrightarrow{v_b}}{dt} = \vec{a}_b(x, v_c, \alpha_d) \qquad \vec{\alpha}^2 = 1 \tag{9.2}$$

where we know that every set of Cauchy data $(\vec{x}, \vec{v}_b, \vec{\alpha}_c)$ determines an unique trajectory parametrized by the time t.

In a scattering situation (when the particles actually reach spatial infinity) the assymptotic conditions (2.3) enable us to interpret the constants of the motion $\vec{p}_a/E_a$, $\vec{\omega}_b$, and $\vec{b}$ as the incident velocities, intrinsic spins and impact parameter, respectively. These constants constitute a set of Cauchy data of the system (9.2) in the infinite past (that is, in the limit when $\vec{q}.\vec{\rho} \to t\rho^2 \to -\infty$) because in this limit the conditions (2.3) amount that $\vec{q} \to \vec{x}$, $\vec{p}_a/E_a \to \vec{v}_a$, $\vec{\omega}_a \to \vec{\gamma}_a$. We have then a one-to-one smooth correspondence between trajectories in both spaces and also between the respective parameters ($\vec{q}.\vec{\rho}$ and t) along these lines. We must conclude therefore that the Jacobi determinants (9.1) are everywhere different from zero in the scattering case.

If we take into account $\dfrac{\partial(q, p, \omega)}{\partial(x, v, \alpha)} = \dfrac{\partial(q, p, \omega)}{\partial(x, p, \omega)} \dfrac{\partial(x, p, \omega)}{\partial(x, v, \alpha)} \neq 0$ we must conclude that the only regions in which we can not easily impose the center of mass condition $\left(\text{because } \dfrac{\partial(x, p, \omega)}{\partial(x, v, \alpha)} = 0\right)$ are those in which $\dfrac{\partial(q, p, \omega)}{\partial(x, p, \omega)}$ diverges. A direct inspection of our expression (8.6) for $\vec{q}(x, p, \omega)$ shows that, when the perturbative treatment is valid (b is not small), the only trouble some region is the infinite futur ($z \to +\infty$). Just in that region we can solve the problem by substituting the assymptotic conditions (4.5) and (8.2) by a set of similar ones with the limit $\lambda \to -\infty$ replaced by a limit $\lambda \to +\infty$, this change amounting to report the singularity to the infinite past, and then match the regular parts of the two complementary expressions.

It is easy to see that this change of the assymptotic conditions amounts only to replace $-\infty$ by $+\infty$ in the lower limits of the integrals in equations (4.6) and this leads to few minor modifications in the first order expressions (7.7), (7.8) and (8.6) when writen in terms of the outgoing speed $\vec{p}'/E'$ and intrinsic spin $\vec{\omega}'$.

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APPENDIX

We have applied our results to the computation of the differential cross-section. We have used a procedure analogous to that of ref. [*15*], the only difference arising from the fact that the particles do not move in a plane. This is because of the spin is taked into account and the result will show an explicit dependence in the polar angle ϕ.

We obtain for large impact parameters ($\sin^3 \theta/2 \ll 1$) the following result in the center-of-mass frame (we take $c = 1$, $\varepsilon = g/|g|$)

$$\frac{d\sigma}{d\Omega} = \frac{g^2k^2}{4M^2p^4} \operatorname{cosec}^4 \frac{\theta}{2} - \varepsilon \frac{g^2\pi}{16mMp^2} \operatorname{cosec}^3 \frac{\theta}{2} - \frac{M^2p}{2k^2} \operatorname{cosec}^2 \frac{\theta}{2}$$
$$\times \left[\Sigma_a \left(\frac{\mu_a}{g}\right)^2 \sin^2 \alpha_a \cos 2(\phi - \omega_a) + 2\left(1 - \frac{2k^3}{M^3p^3}\right) \times \frac{\mu_a\mu_{a'}}{g^2} \sin \alpha_a \sin \alpha_{a'} \cos (2\phi - \omega_a - \omega_{a'}) \right]$$

where m is the reduced mass, $\mu_a/g = \frac{\lambda_a}{2} \frac{s_a}{m_a}$ for $g \neq 0$ and $\mu_a/g = 0$ when $g = 0$. ω_a and α_a are the polar and azimuthal angles, respectively, of the orientation of the spin of the particle a.

This result needs some explanation. The first term is the well known contribution from the first order in g scalar terms in eq. (7.7). The second one is the contribution of second order in g scalar terms, as it is accounted for in ref. [*15*]; the contribution at this order ($\operatorname{cosec}^3 \theta/2$) of the spin-orbit terms (linear in μ) of eq. (7.7) cancels out. In the last term, one can have three kinds of new contributions:

i) from scalar third order in g terms: this gives zero as it can be seen *a priori*, without computing them,

ii) from spin-orbit second order terms (like $\mu_a g/\sin^2 \theta/2$): we do not compute them. They are numerically negligeable with respect to the following contribution if we consider particles like the electron ($\lambda s/g \sim 137$),

iii) from the spin-spin first order in g terms appearing in eq. (7.7): their contribution to this order ($\operatorname{cosec}^2 \theta/2$) is completely accounted for in our expression.

If we average over the orientations of the spins or over the polar angle φ, we recover the same result that in the scalar case.

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[*8*] L. Bel, J. Martin, « Pred. Rel. Mech. of systems of N part. with spin II.: the electromagnetic interaction » (submitted to *Ann. Inst. H. Poincaré*).

[9] See for instance E. C. G. SUDARSHAN, N. MUKUNDA, *Classical Dynamics*, Wiley, 1974.

[10] It is possible, of course, to preserve the canonicity of the coordinates in spite of their transformation properties. To make one or another choice is a matter of taste at this classical level.

[11] V. BARGMANN, L. MICHEL and V. L. TELEGDI, *Phys. Rev. Lett.*, t. **2**, 1959, p. 435.

[12] The equations coincide with the ones previously obtained by L. BEL and J. MARTIN (8).

[13] This choice is justified because this restriction is equivalent to the assymptotic condition on the symplectic form $\lim_{\lambda \to -\infty} R(\lambda) dq \wedge dp = dx \wedge dp$ and guarantees the unicity of the Hamiltonian formulation based, in the scalar case, on this symplectic form (See for details the ref. cited below.)

[14] This dimensional condition requires the presence of some new constant with dimensions of a length. This result is a direct consequence of (8.2): no such additional constant is required in the paper by L. BEL and J. MARTIN (*Ann. Inst. H. Poincaré*, t. **22**, 1975, p. 173), which deals with structureless particles because their corresponding fourdimensional assymptotic condition is even weaker than (8.2).

[15] R. LAPIEDRA, F. MARQUÉS and A. MOLINA, *J. Math. Phys.*, t. **20**, 1979, p. 1316.

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