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The Quantum Equivalence Principle, and Spin 1/2 massive particles

by

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ABSTRACT. — Using Quantum Equivalence Principle, a self consistent Quantum Field Theory for spin 1/2 massive particles in Curve Space-Time is developed. The particle density and energy density created by a time dependent gravitational field in an Expanding Universe turns out to be finite and well defined.

INTRODUCTION

In the two proceeding papers, [*4*] and [*5*], we have shown that Quantum Equivalence Principle (i. e. to define particle model in curve space-time through $G_1(x, x')$ equal to $\Delta_1(x, x')$ written in normal coordinates) yields to an implementable Bogolyubov transformation and therefore to the creation of a finite number of particles and to a reasonable Quantum Field Theory for spin 0 particles.

In this paper we shall show that exactly the same thing happens with spin 1/2 massive particles. In fact. We can repeat *mutatis mutandi* all the theory of the former papers and we arrive to practically the same results.

In paragraph I we introduce, for the sake of completeness, the spinors field in curve space-time, in the simplest and most general way i. e. defining at each point a set of four Dirac's matrices that satisfy an adequate anti-commutation relation. This method was introduced in paper [*1*] and, in our opinion, is the most adequate to face the problem.

In paragraph 2 we study Dirac's equation in a generic curve space-time and its main properties. The expansion of the propagator, the anticommutation relations and the way to define the positive and negative frequency solutions are presented in paragraph 3.

We restrict ourselves to the case of an expanding universe in paragraph 4.

We introduce the Quantum Equivalence Principle in paragraph 5 and find the corresponding initial condition for the particle model, neglecting terms in the square of Hubble coefficient and higher ones as in work [5].

In paragraph 6 we shall see how one can find an approximative solution of Dirac's equation.

We shall compute the Bogolyubov transformation and we shall demostrate that it turns out to be implementable in paragraph 7.

In paragraph 8 we shall prove that also the created energy is convergent.

In this way all the main features at the spin 0 case are reproduced and we have also a reasonable Quantum Field Theory in Curve Space-Time for spin 1/2 Fields.

1. NOTATION, DIRAC'S MATRICES

a) Let V_4 be an orientable Riemannian manyfold of C^∞ class. Let g_{ij} be the metric tensor of V_4 ; g_{ij} has the diagonal $(1, -1, -1, -1)$ in the diagonal form.

The Dirac's Matrices are four 4×4 matrices on $\mathbb{C}$: $\{ \gamma_i{}^a{}_b(x) \}$ ($i, j, k, \ldots = 0, 1, 2, 3$, will be the ordinary vector indices and $a, b, c, \ldots = 1, 2, 3, 4$, the spinor indices). These γ are C^∞ point functions of $x \in V_4$ that satisfy the following anticommutation relations at every point:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2g_{ij}I, \tag{1.1}$$

where I is the unit matrix, i. e.:

$$\gamma_i{}^a{}_b\gamma_j{}^b{}_c + \gamma_j{}^a{}_b\gamma_i{}^b{}_c = -2g_{ij}\delta^a{}_c. \tag{1.2}$$

The first spinor indices label the rows of the matrices γ, the second one the columns and the product is made « column by row ».

Dirac's Matrices satisfy, among others, the following properties:

i) A 4×4 matrix on $\mathbb{C}$ that commutes with all the γ_i is necessarily a matrix aI where $a \in \mathbb{C}$ and I is the unit matrix.

ii) If $\{ \dot{\gamma}_i \}$ is another set of Dirac's Matrices, a matrix S exists, such that:

$$\dot{\gamma}_i = S\gamma_iS^{-1}\ (^1), \tag{1.3}$$

the matrix S is unique but it has an undeterminated constant factor.

(1) We shall call γ^i to $\gamma^i = g^{ij}\gamma_j$.

b) The matrices $S = (S^{a'}{}_a)$, $S^{-1} = (S^{-1a}{}_{a'})$ can be considered as change of base matrices in the spinor space, therefore eq. (1.3) can be considered as an equation of change of coordinates where one goes from on spinorial base to another.

c) We define the following symbols:

— $*$ is the ordinary conjugation in $\mathbb{C}$
— T is the 4×4 matrix transposition
— $+$ is a conjugation and a transposition i. e. the *adjunction.*

Let us define the following types of spinors according to the way they change coordinates under a change of the spinor base. We shall call contravariant spinors the column spinor and covariant spinors the row ones:

Contravariants.

Type I $\quad \psi' = S\psi, \quad \psi^{a'} = S^{a'}{}_a\psi^a,$

Type II $\quad \psi' = S^*\psi, \quad \psi^{a'} = \overset{*}{S}{}^{a'}{}_a\psi^a,$

Type III $\quad \psi' = \overset{-1}{S}{}^{T}\psi, \quad \psi^{a'} = \overset{-1}{S}{}^{Ta'}{}_a\psi^a,$

Type IV $\quad \psi' = \overset{-1}{S}{}^{+}\psi, \quad \psi^{a'} = \overset{-1}{S}{}^{+a'}{}_a\psi^a.$

Covariants.

Type I $\quad \varphi' = \varphi S^{-1}, \quad \varphi_{a'} = \varphi_a \overset{-1}{S}{}^a{}_{a'},$

Type II $\quad \varphi' = \varphi\overset{*}{S}{}^{-1}, \quad \varphi_{a'} = \varphi_a\overset{*}{S}{}^{-1a}{}_{a'},$

Type III $\quad \varphi' = \varphi S^T, \quad \varphi_{a'} = \varphi_a\overset{T}{S}{}^a{}_{a'},$

Type IV $\quad \varphi' = \varphi S^+ \quad \varphi_{a'} = \varphi_a\overset{+}{S}{}^a{}_{a'},$

Where

(1.4) $$S^{Ta}{}_{a'} = S^{a'}{}_a.$$

(1.5) $$S^{+a}{}_{a'} = \overset{*}{S}{}^{a'}{}_a.$$

d) If ψ is a contravariant spinor of Type I we have

(1.6) $$\psi' = S\psi, \qquad \overset{*}{\psi}{}' = S^*\psi^*,$$
$$\overset{T}{\psi}{}' = \overset{T}{\psi}S^T, \qquad \overset{T}{\psi}{}' = \psi^+S^+.$$

so ψ^* is a contravariant spinor of type II and ψ^T and ψ^+ are covariant spinors of type III and IV respectively.

Analogously if φ is a covariant spinor of Type I we have:

(1.7) $$\varphi' = \varphi S^{-1}, \qquad \varphi^{*\prime} = \varphi^*\overset{*}{S}{}^{-1},$$
$$\overset{T}{\varphi}{}' = \overset{-1}{S}{}^T\varphi^T, \qquad \overset{+}{\varphi}{}' = \overset{-1}{S}{}^+\varphi^+$$

so φ^* is a contravariant spinor of type II and φ^T and φ^+ are covariant spinors of type III and IV respectively, etc.

e) If we change the coordinates of *Dirac's Matrices* γ^i to another coordinates system $\{ x^{i'} \}$ of V_4 and to another spinor base we have the following equation of change of coordinates:

$$\gamma^{i'} = A^{i'}{}_i S \gamma^i S^{-1} . \tag{1.8}$$

where $A^{i'}{}_i = \partial x^{i'}/\partial x^i$ and where $S(x)$ is the C^∞ matrix that changes the spinor base. We usually normalize S with the condition:

$$\det (S) = \det (S^{-1}) = 1 \tag{1.9}$$

so we can consider the γ^i as a contravariant vector-contravariant spinor type I-covariant spinor type I.

But

$$\overset{*}{\gamma}{}^{i'} = A^{i'}{}_i S^* \overset{*}{\gamma}{}^i \overset{*}{S}{}^{-1} , \tag{1.10}$$

$$\overset{T}{\gamma}{}^{i'} = A^{i'}{}_i \overset{-1}{S}{}^T \overset{T}{\gamma}{}^i S^T \tag{1.11}$$

$$\overset{+}{\gamma}{}^{i'} = A^{i'}{}_i \overset{-1}{S}{}^+ \overset{+}{\gamma}{}^i S^+ , \tag{1.12}$$

so the $\overset{*}{\gamma}{}^i$ (resp. $\overset{T}{\gamma}{}^i$, resp. $\overset{+}{\gamma}{}^i$) are a contravariant vector—contravariants spinor of type II (resp. III, resp. IV)—covariant spinor of type II (resp. III, resp. IV).

f) The matrices :

$$-\gamma^i ; \qquad \overset{*}{\gamma}{}^i ; \qquad -\overset{+}{\gamma}{}^i ; \qquad -\overset{T}{\gamma}{}^i$$

satisfy also the anticommutation rules (1.1) or (1.2), therefore from property (*ii*) we deduce that the matrices ξ, α, β, Γ exist such that:

$$\begin{aligned} -\gamma^i &= \xi \gamma^i \xi^{-1} , \\ \overset{*}{\gamma}{}^i &= \alpha \gamma^i \alpha^{-1} , \\ -\overset{+}{\gamma}{}^i &= \beta \gamma^i \beta^{-1} , \\ -\overset{T}{\gamma}{}^i &= \Gamma \gamma^i \Gamma^{-1} , \end{aligned} \tag{1.13}$$

all these matrices are unique except for an arbitrary constant factor.

ξ can be defined in a covariant way as:

$$\xi = \frac{i}{4\,!} \gamma^i \gamma^j \gamma^k \gamma^l \eta_{ijkl} , \tag{1.14}$$

where η is the element of volume pseudo-tensor i. e.:

$$\eta_{ijkl} = \frac{1}{\sqrt{|g|}} \varepsilon_{ijkl} , \tag{1.15}$$

where ε is the Levi-Civita symbol and $g = \det (g_{ij})$. Using at every point of V_4 an orthonormal base, where g_{ij} = diagonal matrix and the relations (1.1) both eq. (1.13_1) and $\xi^2 = 1$ can be verified.

From eqs. $(1.13_{2,3,4})$ we can deduce that:

(1.16)
$$\begin{aligned} \gamma^i &= \alpha^* \overset{*}{\gamma}{}^i \overset{-1}{\alpha}{}^* &&= \alpha^*\alpha\gamma^i\alpha^{-1}\overset{*}{\alpha}{}^{-1}, \\ -\gamma^i &= \overset{+}{\beta}{}^{-1}\overset{+}{\gamma}{}^i\beta^+ &&= -\overset{-1}{\beta}{}^+\beta\gamma^i\beta^{-1}\beta^+, \\ -\gamma^i &= \overset{-1}{\Gamma}{}^{\mathrm{T}}\overset{\mathrm{T}}{\gamma}{}^i\Gamma^{\mathrm{T}} &&= -\overset{-1}{\Gamma}{}^{\mathrm{T}}\Gamma\gamma^i\overset{-1}{\Gamma}\Gamma^{\mathrm{T}}, \end{aligned}$$

therefore from property i we have:

(1.17)
$$\alpha^*\alpha = a\mathrm{I}, \qquad \overset{-1}{\beta}\beta^+ = b\mathrm{I}, \qquad \overset{-1}{\Gamma}\Gamma^{\mathrm{T}} = c\mathrm{I},$$

If we want to find out how matrices α, β, Γ change under a change of spinor base, from eq. (1.13) we can have:

(1.18)
$$\mathrm{S}^*\overset{*}{\gamma}{}^i\overset{*}{\mathrm{S}}{}^{-1} = \acute{\alpha}\mathrm{S}\gamma^i\mathrm{S}^{-1}\acute{\alpha}^{-1},$$

(1.19)
$$-\overset{\mathrm{T}}{\mathrm{S}}{}^{-1}\overset{\mathrm{T}}{\gamma}{}^i\mathrm{S}^{\mathrm{T}} = \acute{\Gamma}\mathrm{S}\gamma^i\mathrm{S}^{-1}\Gamma^{-1},$$

(1.20)
$$-\overset{+}{\mathrm{S}}{}^{-1}\overset{+}{\gamma}{}^i\mathrm{S}^+ = \beta'\mathrm{S}\gamma^i\mathrm{S}^{-1}\beta^{-1},$$

therefore:

(1.21)
$$\begin{aligned} \overset{*}{\gamma}{}^i &= \overset{*}{\mathrm{S}}{}^{-1}\alpha'\mathrm{S}\gamma^i\overset{-1}{\mathrm{S}}\acute{\alpha}^{-1}\mathrm{S}^*, \\ -\overset{\mathrm{T}}{\gamma}{}^i &= \mathrm{S}^{\mathrm{T}}\acute{\Gamma}\mathrm{S}\gamma^i\overset{-1}{\mathrm{S}}\acute{\Gamma}^{-1}\overset{\mathrm{T}}{\mathrm{S}}{}^{-1}, \\ -\overset{+}{\gamma}{}^i &= \mathrm{S}^+\beta'\mathrm{S}\gamma^i\mathrm{S}^{-1}\overset{*}{\beta}{}^{-1}\overset{+}{\mathrm{S}}{}^{-1}. \end{aligned}$$

Therefore

(1.22)
$$\alpha = \overset{*}{\mathrm{S}}{}^{-1}\alpha'\mathrm{S}, \qquad \Gamma = \mathrm{S}^{\mathrm{T}}\acute{\Gamma}\mathrm{S}, \qquad \beta = \mathrm{S}^+\beta'\mathrm{S},$$

but in all these equations we can put an arbitrary constant factor of modulus 1 on account of eq. (1.19).

But from (1.22_1) we have $\alpha^*\alpha = \mathrm{S}^{-1}\acute{\alpha}^*\mathrm{S}^*\overset{*}{\mathrm{S}}{}^{-1}\acute{\alpha}\mathrm{S} = \overset{-1}{\mathrm{S}}\acute{\alpha}^*\acute{\alpha}\mathrm{S} = \acute{\alpha}^*\alpha'$. Therefore from (1.17_1) we have $a' = a$ i. e. a is a scalar. Then it is sufficient to compute a in a particular representation of the γ^i (we shall use these Dirac matrices every time we shall need an explicit representation):

(1.23)
$$\gamma_0 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_\alpha = i\begin{pmatrix} 0 & \tau_\alpha \\ -\tau_\alpha & 0 \end{pmatrix} \qquad \alpha = 1, 2, 3,$$
$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & -0 \end{pmatrix}; \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Then $\alpha = \gamma_2$ and $\alpha^*\alpha = \mathrm{I}$ therefore $a = 1$ in general.

In similar way we can have $b = 1$, $c = -1$ so eq. (1.17) becomes:

(1.24)
$$\alpha\alpha^* = \mathrm{I}, \qquad \beta = \beta^+, \qquad \Gamma^{\mathrm{T}} = -\Gamma.$$

Let us finally observe that

$$(1.25)\qquad -\overset{+}{\gamma}{}^i = -(\gamma^{Ti})^* = (\Gamma\gamma^i\Gamma^{-1})^* = \Gamma^*\overset{*}{\gamma}{}^i\overset{*}{\Gamma}{}^{-1} = \overset{*}{\Gamma}\alpha\gamma^i\alpha^{-1}\overset{*}{\Gamma}{}^{-1}.$$

Therefore

$$(1.26)\qquad \beta = \Gamma^*\alpha.$$

g) If we would like to build, from contravariant spinor type I, ψ, a covariant spinor type I, $\overline{\psi}$, we can write

$$(1.27)\qquad \overline{\psi} = \psi^+\beta,$$

$\overline{\psi}$ is called the Dirac's adjoint of ψ

In fact : (cfr 1.22_5)

$$(1.29)\qquad \overline{\psi}' = \psi'^+\beta' = \psi^+S^+\beta' = \psi^+\beta S^{-1} = \overline{\psi}S^{-1}.$$

Analogously from covariant spinor type I φ we can build a contravariant spinor of type I as

$$(1.29)\qquad \overline{\varphi} = \beta^{-1}\varphi^+.$$

Also from the contravariant spinor of type I ψ we can build a contravariant spinor of type I

$$(1.30)\qquad \tilde{\psi} = \psi^T\Gamma,$$

and a contravariant spinor of type I :

$$(1.31)\qquad \psi^c = \alpha^*\psi^*.$$

And from a covariant spinor of type I, φ we can construct a contravariant spinor of type I

$$(1.32)\qquad \tilde{\varphi} = \Gamma^{-1}\varphi^T$$

and a covariant spinor of type I

$$(1.33)\qquad \varphi^c = \varphi^*\alpha$$

Therefore we can introduce three mappings over the space of contravariant spinors of type I :

— *The Dirac Adjunction:* « A »

$$(1.34)\qquad A\psi = \overline{\psi} = \psi^+\beta$$

— *The Charge Conjugation:* « C »

$$(1.35)\qquad C\psi = \psi^c = \alpha^*\psi^*$$

— and the *Metric Mapping:* « M »

$$(1.36)\qquad M\psi = \tilde{\psi} = \psi^T\Gamma$$

A and M give covariant spinors and C contravariant spinors, all of type I ([2]).

Using eq. (1.29), (1.32), (1.33) similar definitions can be given for covariant spinors.

Let us observe that:

$$\mathrm{AC}\psi = \mathrm{A}\alpha^*\psi^* = \psi^{\mathrm{T}}\alpha^{\mathrm{T}}\beta\,, \tag{1.37}$$

$$\mathrm{CA}\psi = \mathrm{C}\psi^+\beta = \psi^{\mathrm{T}}\beta^*\alpha\,, \tag{1.38}$$

but

$$(\alpha^{\mathrm{T}}\beta)^{\mathrm{T}} = \beta^{\mathrm{T}}\alpha = \beta^*\alpha\,, \tag{1.39}$$

and from eq. (1.26) we have

$$\beta^* = \Gamma\alpha^* = \Gamma\alpha^{-1}\,, \tag{1.40}$$

therefore

$$\beta^*\alpha = \Gamma\,, \tag{1.41}$$

so eq. (1.32) can be written as

$$\begin{aligned} \mathrm{CA}\psi &= \psi^{\mathrm{T}}\Gamma = \mathrm{M}\psi\,, \\ \mathrm{AC}\psi &= \psi^{\mathrm{T}}\Gamma^{\mathrm{T}} = -\,\psi^{\mathrm{T}}\Gamma = -\,\mathrm{M}\psi\,, \end{aligned} \tag{1.42}$$

i. e.

$$\mathrm{AC} = -\,\mathrm{CA} = -\,\mathrm{M} \tag{1.43}$$

Finally let us observe that all these mappings can be generalized in a natural way to higher order spinors e. g. we can define

$$\overline{\gamma}^i = \beta^{-1}\gamma^+\beta, \qquad \overset{c}{\gamma}{}^i = \overset{*}{\alpha}\overset{*}{\gamma}{}^i\alpha\,, \text{ etc.} \tag{1.44}$$

using (1.27) and (1.29), (1.31) and (1.33), but from $(1.13_{2,3})$ we have

$$\overline{\gamma}^i = -\,\gamma^i\,, \qquad \overset{c}{\gamma}{}^i = \gamma^i\,, \tag{1.45}$$

two useful equations, as we shall see.

h) We want now to define covariant derivatives for different types of spinors; we shall do so using four matrices σ_i, that we shall choose later on, in such a way that some natural properties should be fulfilled.

We define :

Contravariant spinors derivatives.

Type I $\qquad \nabla_i\psi = \partial_i\psi + \sigma_i\psi\,,$

Type II $\qquad \nabla_i\psi = \partial_i\psi + \overset{*}{\sigma}_i\psi\,,$

([2]) Mapping C is very important. If $\psi(x)$ is the field of a particle and it is a type I spinor, $\psi^*(x)$ cannot be the field of its antiparticle because it is a type II spinor. But $\psi^c(x) = \alpha^*\psi^*(x)$ is a type I spinor so it is the best candidate to represent the antiparticle, as it is well known in the flat space-time case.

Type III $\nabla_i \psi = \partial_i \psi - \sigma_i^{\mathrm{T}} \psi$,

Type IV $\nabla_i \psi = \partial_i \psi - \sigma_i^{+} \psi$.

Covariant spinors derivatives.

Type I $\nabla_i \varphi = \partial_i \varphi - \varphi \sigma_i$,

Type II $\nabla_i \varphi = \partial_i \varphi - \varphi \sigma_i^{*}$,

Type III $\nabla_i \varphi = \partial_i \varphi + \varphi \sigma_i^{\mathrm{T}}$,

Type IV $\nabla_i \varphi = \partial_i \varphi + \varphi \sigma_i^{+}$,

This definitions can be generalized in a natural way to higher spinors or tensor-spinors. e. g. the derivatives of γ_i, γ_i^*, γ_i^{T}, and γ_i^+ are

(1.46)
$$\begin{aligned} \nabla_i \gamma_j &= \gamma_{j//i} + \sigma_i \gamma_j - \gamma_j \sigma_i , \\ \overset{*}{\nabla}_i \overset{*}{\gamma}_j &= \overset{*}{\gamma}_{j//i} + \overset{*}{\sigma}_i \overset{*}{\gamma}_j - \overset{*}{\gamma}_j \overset{*}{\sigma}_i , \\ \overset{\mathrm{T}}{\nabla}_i \overset{\mathrm{T}}{\gamma}_j &= \overset{\mathrm{T}}{\gamma}_{j//i} - \sigma_i^{\mathrm{T}} \gamma_j^{\mathrm{T}} + \gamma_j^{\mathrm{T}} \sigma_i^{\mathrm{T}} , \\ \nabla_i \gamma_j^{+} &= \gamma_{j//i}^{+} - \sigma_i^{+} \gamma_j^{+} + \gamma_j^{+} \sigma_i^{+} , \end{aligned}$$

where the symbol // is the covariant derivative of γ_i, etc. « as if it were only a vector ».

With this definition the rule of product derivation is also valid for the covariant derivation, as it can be easily proved.

We shall fix matrices σ_i, that we shall call the *spinor connection* in such a way that the covariant derivative would fulfil some useful properties. The first one is:

(1.47)
$$\nabla_i \gamma_j = \gamma_{j//i} + \sigma_i \gamma_j - \gamma_j \sigma_i = 0 .$$

This equation partially determinates σ_i. In fact, from a paper of Loos [9] we know that σ_i must then be:

(1.48)
$$\sigma_i = \frac{1}{2}\left(\mathrm{I} - \frac{7}{6}\mathscr{A} + \frac{4}{3}\mathscr{A}^2 - \frac{2}{3}\mathscr{A}^3 \right) \gamma_{j//i} \gamma^j + v_i \mathrm{I} = \hat{\sigma}_i + v_i \mathrm{I}$$

where $\mathscr{A}$ is the following linear operator on 4×4 matrices $\mathscr{M}$:

(1.49)
$$\mathscr{A} \mathscr{M} = \frac{1}{4} \gamma_i \mathscr{M} \gamma^i ,$$

and v_i is an arbitrary vector. Besides matrix $\hat{\sigma}_i$ has null trace. From (1.46) and (1.47) we deduce that

(1.50)
$$\begin{aligned} (\nabla_i \gamma_j^*) &= (\nabla_i \gamma_j)^* = 0 , \\ \nabla_i \gamma_j^{\mathrm{T}} &= (\nabla_i \gamma_j)^{\mathrm{T}} = 0 , \\ \nabla_i \gamma_j^{+} &= (\nabla_i \gamma_j)^{+} = 0 . \end{aligned}$$

Then from equations (1.13) and (1.22) we have

$$\begin{aligned} \overset{*}{\gamma}{}^i \nabla_j \alpha &= (\nabla_j \alpha)\gamma^i , \\ - \overset{\mathrm{T}}{\gamma}{}^i \nabla_j \Gamma &= (\nabla_j \Gamma)\gamma^i , \\ - \overset{+}{\gamma}{}^i \nabla_j \beta &= (\nabla_j \beta)\gamma^i . \end{aligned} \tag{1.51}$$

i. e.

$$\begin{aligned} \alpha\gamma^i\alpha^{-1}\nabla_j\alpha &= (\nabla_j\alpha)\gamma^i , \\ \Gamma\gamma^i\Gamma^{-1}\nabla_j\Gamma &= (\nabla_j\Gamma)\gamma^i , \\ \beta\gamma^i\beta^{-1}\nabla_j\beta &= (\nabla_j\beta)\gamma^i , \end{aligned} \tag{1.52}$$

therefore from the property i we have

$$\begin{aligned} \nabla_j\alpha &= a_j\alpha , \\ \nabla_j\Gamma &= c_j\Gamma, \\ \nabla_j\beta &= b_j\beta . \end{aligned} \tag{1.53}$$

The unit spinor $\mathrm{I} = \mathrm{I}^a{}_b = \delta^a{}_b$ has null covariant derivatives because

$$\nabla_j \mathrm{I} = \partial_j \mathrm{I} + \sigma_j \mathrm{I} - \mathrm{I}\sigma_j = 0 . \tag{1.54}$$

Therefore from eq. $(1.24_{1,2})$ it can be deduced that:

$$(\nabla_j\alpha)\alpha^* + \alpha(\nabla_j\alpha^*) = 0 , \qquad \nabla_j\beta = \nabla_j\beta^+ , \tag{1.55}$$

and using $(1.53_{1,2})$ we have

$$\begin{aligned} a_j + a_j^* &= 0 , \\ b_j - b_j^* &= 0 , \end{aligned} \tag{1.56}$$

so vector b_j is real and vector a_j imaginary. On the other hand, from (1.26) we have

$$\nabla_j\beta = (\nabla_j\Gamma^*)\alpha + \Gamma^*(\nabla_j\alpha) , \tag{1.57}$$

now using the eq. (1.53) we have:

$$b_j = c_j^* + a_j \tag{1.58}$$

and by conjugation

$$b_j = c_j - a_j \tag{1.59}$$

therefore

$$\begin{aligned} a_j &= \frac{1}{2}(c_j - c_j^*) , \\ b_j &= \frac{1}{2}(c_j + c_j^*) . \end{aligned} \tag{1.60}$$

Now we can use the arbitrary vector v_i, from eq. (1.48), to get a set of formulas even simpler. Let us call $\hat{\nabla}_i$ the covariant derivative with connexion $\hat{\sigma}_i$. In all the formulas, from eq. (1.43) we can put $\hat{\nabla}_i$ instead of ∇_i (i. e. we shall consider the particular case $v_i = 0$). Eq. (1.53$_2$) will then be:

$$\hat{\nabla}_j \Gamma = \hat{c}_j \Gamma. \tag{1.61}$$

If we observe eq. (1.22$_3$) we can deduce the covariant derivative of Γ and we have:

$$\nabla_j \Gamma = \hat{\nabla}_j \Gamma - (v_j \mathrm{I})^{\mathrm{T}} \Gamma - \Gamma(v_j \mathrm{I}) = (\hat{c}_j - 2v_j)\Gamma = c_j \Gamma . \tag{1.62}$$

Then if we take v_j as:

$$v_j = \frac{1}{2} \hat{c}_j \tag{1.63}$$

from (1.60) and (1.62) we have the very useful property

$$a_j = b_j = c_j = 0 \tag{1.64}$$

We choose σ_i such that both eq. (1.47) and (1.64) would be fulfilled. The covariant derivative will be therefore defined by the two properties:

$$\nabla_j \gamma_i = 0 , \qquad \nabla_j \Gamma = 0 , \tag{1.65}$$

and it will satisfy also the equations:

$$\begin{aligned} \nabla_j \overset{*}{\gamma}{}^i &= \nabla_j \gamma^{\mathrm{T}i} = \nabla_j \overset{+}{\gamma}{}^i = 0 , \\ \nabla_j \alpha &= \nabla_j \beta \quad = \nabla_j \xi = 0 . \end{aligned} \tag{1.66}$$

i) We shall also need the equations of change of σ_i under a change of coordines of V_4 $x^i \rightarrow x^{i'}$ and a change of the spinorial base with matrix S i. e. $\gamma^i \rightarrow \grave{\gamma}^i = S\gamma^i S^{-1}$:

$$\grave{\sigma}_{i'} = A^i_{i'} [S\sigma_i S^{-1} + S(\partial_i S^{-1})] , \tag{1.70}$$

as can be directly shown (cf. [*1*]).

j) Covariant derivatives on spinors are not conmutative, in fact it can be proved that (cf. [*7*])

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\psi = -\frac{1}{4} R_{klij} \gamma^k \gamma^l \psi , \tag{1.67}$$

where R_{klij} is the curvature tensor of V_4. From (1.1) and from the symmetry properties of R_{klij} we also have:

$$R_{ijkl} \gamma^j \gamma^k \gamma^l = 2R_{ij} \gamma^j , \tag{1.68}$$

$$R_{ijkl} \gamma^i \gamma^j \gamma^k \gamma^l = -2RI . \tag{1.69}$$

2. THE DIRAC'S EQUATION IN V_4

a) Let V_4 be a globally hyperbolic Riemannian manyfold (3). A spinor field $\psi(x)$ satisfies Dirac's equation if:

$$(2.1) \qquad (P - \mu)\psi = (\gamma^i(x)\nabla_i - \mu)\psi(x) = 0,$$

where $\mu \neq 0$ is the mass of the field and $P = \gamma^i\nabla_i$. If we take the adjoint, the equation will be:

$$(2.2) \qquad \psi^+(x)(\gamma^{+i}\overleftarrow{\nabla}_i - \mu) = 0,$$

i. e.

$$(2.3) \qquad \psi^+[-\beta\gamma^i\beta^{-1}\overleftarrow{\nabla}_i - \mu\beta\beta^{-1}] = 0.$$

if we post-multiplied by β and use (1.66_5) we have:

$$(2.4) \qquad (\overline{P} - \mu)\overline{\psi}(x) = \overline{\psi}(x)[-\gamma^i\overleftarrow{\nabla}_i - \mu] = 0,$$

where $\overline{P} = -\gamma_i\overleftarrow{\nabla}_i$.

This is the equation that satisfies the Dirac's adjoint field.

b) Let us suppose that $u(x)$ and $v(x)$ are two spinor fields that satisfy Dirac's equation (2.1) and let us compute:

$$(2.5) \quad \nabla_i(\overline{u}\gamma^i v) = (\nabla_i\overline{u})\gamma^i v + \overline{u}\gamma^i\nabla_i v = \overline{u}[\gamma^i\overleftarrow{\nabla}_i + \mu]v + \overline{u}[\gamma^i\nabla_i - \mu]v = 0.$$

Therefore we have:

$$(2.6) \quad \nabla_i(\overline{u}\gamma^i v) = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\,\overline{u}\gamma^i v) = 0,$$

so we can define an hermitian inner product $\langle\langle\, ; \rangle\rangle$ in the vector space of the solution of Dirac's equation

$$(2.7) \qquad \langle\langle u, v \rangle\rangle = -i\int_\Sigma \overline{u}\gamma^i v d\sigma^i = \langle\langle v, u \rangle\rangle^*,$$

where Σ is a Cauchy surface of V_4. On account of equation (2.6) the inner product (2.7) is independent of the Cauchy surface used to perform the integration.

An important and well known property of the inner product $\langle\langle\, , \rangle\rangle$ is that it is positive defined. In fact we can use a system of coordinates such that the normal vector to Σ would have coordinates $n^i = (1, 0, 0, 0)$.

(3) As in paragraph 1 V_4 is of class C^∞ and all the functions that we shall consider are also of class C^∞.

i. e. n^i is the coordinate time-axis, then we get an orthonormal tetrade using three space-like axis, then the metric tensor would be diagonal.

We could use Dirac's matrices (1.23) therefore we shall have: (cf. [7], p. 36)

$$\beta = -i\gamma_0, \qquad \gamma_0^2 = -1. \tag{2.8}$$

and

$$\langle\langle u, u \rangle\rangle = -i\int_\Sigma \bar{u}\gamma_i u d\sigma^i = -i\int_\Sigma \bar{u}\gamma_i u n^i d\sigma = -i\int_\Sigma u^+ \beta\gamma_0 u d\sigma \tag{2.9}$$
$$= \int_\Sigma u^+ u d\sigma = \int_\Sigma \sum_{a=1}^{4} | u^a |^2 d\sigma^0 \geqslant 0$$

As the scalar $\bar{u}\gamma_i u n^i$ is invariant under coordinates and spinor base transformations we have proved that $\langle\langle\,,\,\rangle\rangle$ is positive defined.

c) Let us compute:

$$\text{(2.10)}\quad P^2\psi = (\gamma^i\nabla_i)(\gamma^j\nabla_j)\psi = \gamma^i\gamma^j\nabla_i\nabla_j\psi$$
$$= \frac{1}{2}(\gamma^i\gamma^j + \gamma^j\gamma^i)\nabla_i\nabla_j\psi + \frac{1}{2}\gamma^i\gamma^j(\nabla_i\nabla_j - \nabla_j\nabla_i)\psi = -\nabla_i\nabla^i\psi - \frac{1}{8}R_{klij}\gamma^k\gamma^l\gamma^i\gamma^j\psi.$$

therefore from eq. (1.69) we have:

$$(\gamma^i\nabla_i)(\gamma^j\nabla_j)\psi = -\nabla_i\nabla^i\psi + \frac{1}{4}R\psi \tag{2.11}$$

Analogously:

$$\bar{P}^2\varphi = \varphi(\gamma^i\overleftarrow{\nabla}_i)(\gamma^j\overleftarrow{\nabla}_j) = -\nabla_i\nabla^i\varphi + \frac{1}{4}R\varphi. \tag{2.12}$$

Therefore, we shall define it as the D'Alembert operator i. e.:

$$\Delta = -\nabla_i\nabla^i + \frac{1}{4}R \tag{2.13}$$

If ψ satisfies Dirac's equation we have

$$(\gamma^i\nabla_i + \mu)(\gamma^j\nabla_j - \mu)\psi = 0, \tag{2.14}$$

i. e.

$$[(\gamma^i\nabla_i)(\gamma^j\nabla_j) - \mu^2]\psi = 0. \tag{2.15}$$

Therefore

$$(\Delta - \mu^2)\psi = 0. \tag{2.16}$$

So every solution of Dirac's equation is also a solution of Klein-Gordon equation with a D'Alembert operator defined by eq. (2.13).

d) If $S^a{}_{r'}(x, x')$ is a bi-spinor field with the property:

$$S^a{}_r(x, x) = \delta^a{}_r, \tag{2.17}$$

we define the Dirac's bi-spinor as

$$\Sigma(x, x') = S(x, x')\delta(x, x'), \tag{2.18}$$

where $\delta(x, x')$ is the Dirac's bi-scalar distribution in V_4 (we can use the transport bi-scalar as $S(x, x')$ but is not necessary (cf. [6])).

From Lichnerowicz paper [7] we know that there exists two unique kernels $G^{\pm}(x, x')$, i. e. two bi-spinor distributions, that satisfy Klein-Gordon inhomogeneous equation

$$(\Delta_x - \mu^2)G^{\pm}(x, x') = \Sigma(x, x'), \tag{2.19}$$

and such that for a fix x' they would have their support (x) in $E^+(x')$ and $E^-(x')$, the future and the past of x', respectively.

We can define the adjoint of an arbitrary bi-spinor as:

$$[S^a{}_{r'}(x, x')]^+ = \overset{*}{S}{}^{r'}{}_a(x', x), \tag{2.20}$$

and we can also define the Dirac's adjoint of this arbitrary bi-spinor as:

$$\overline{S(x, x')} = \beta^{-1}(x')S^+(x', x)\beta(x). \tag{2.21}$$

Therefore using eq. (2.17) and (2.18) we have

$$\overline{\Sigma}(x, x') = \Sigma(x', x). \tag{2.22}$$

Therefore if we take the Dirac's adjoint of eq. (2.19) we have:

$$(\Delta_x - \mu^2)\overline{G}^{\pm}(x, x') = \Sigma(x', x), \tag{2.23}$$

and as their kernels $G^{\pm}(x, x')$ are unique we have:

$$\overline{G}^{\pm}(x, x') = G^{\mp}(x', x). \tag{2.24}$$

Then, let us define the *propagator*:

$$G(x, x') = G^-(x, x') - G^+(x, x'). \tag{2.25}$$

If we take into account eq. (2.19) we can prove that it satisfies

$$(\Delta_x - \mu^2)G(x, x') = 0, \tag{2.26}$$

and using (2.24) we can also prove that it has the property:

$$\overline{G}(x, x') = - G(x, x'). \tag{2.27}$$

e) Let us introduce two new Kernels:

$$S^{\pm}(x, x') = (\gamma^i\nabla_i + \mu)_x G^{\pm}(x, x'). \tag{2.28}$$

Using eq. (2.15) we can see that these kernels satisfy the equation

$$(\gamma^i\nabla_i - \mu)_x S^{\pm}(x, x') = \Sigma(x, x'). \tag{2.29}$$

If we fix x', $S^{\pm}(x, x')$ have its support (x) in $E^+(x')$ and $E^-(x')$ respectively.

Analogously: (cf. [7], p. 60)

(2.30) $$G^{\pm}(x, x')[-\gamma^{i'}\overleftarrow{\nabla}_{i'} + \mu]_{x'} = S^{\pm}(x, x'),$$

(2.31) $$S^{\pm}(x, x')[-\gamma^{i'}\overleftarrow{\nabla}_{i'} - \mu]_{x'} = \Sigma(x, x').$$

Let us now define the propagator:

(2.32) $$S(x, x') = S^{-}(x, x') - S^{+}(x, x'),$$

that satisfies the equation:

(2.33) $$(\gamma^i\nabla_i - \mu)_x S(x, x') = 0.$$

In order to deduce the symmetry properties of Kernel $S(x, x')$ let us observe that:

(2.34) $$AP\psi = A^i\gamma^i\nabla_i\psi = (\gamma^i\nabla_i\psi)^+\beta = \psi^+\overset{+}{\overleftarrow{\nabla}}_i\overset{+}{\gamma}{}^i\beta = -\overset{+}{\psi}\overleftarrow{\nabla}_i\beta\gamma^i = -\overset{+}{\psi}\beta\gamma^i\overleftarrow{\nabla}_i = \overline{P}A\psi.$$

In a similar way we can prove that:

(2.35) $$A\overline{P}\varphi = PA\varphi,$$

i. e.

(2.36) $$\overline{P\psi} = \overline{P}\overline{\psi},$$

(2.37) $$\overline{\overline{P}\varphi} = P\overline{\varphi}.$$

From equations (2.32), (2.28), (2.30) and (2.25) we have:

(2.38) $$S(x, x') = (P + \mu)_x G(x, x') = (\overline{P} + \mu)_{x'} G(x, x').$$

Therefore

(2.39) $$\overline{S}(x, x') = (\overline{P} + \mu)_x \overline{G}(x, x') = -(\overline{P} + \mu)_x G(x', x) = -S(x', x).$$

so we have

(2.40) $$\overline{S}(x, x') = -S(x', x),$$

and

(2.41) $$\overline{S}^{\pm}(x, x') = S^{\mp}(x', x).$$

f) We can now solve the Cauchy problem for the Dirac's equation using the inner product $\langle\langle\, , \rangle\rangle$. If in the equation (2.5) we put

(2.42) $$\overline{u}(x) \rightarrow \overline{S}^{\pm}(x, x'),$$

(2.43) $$v(x) \rightarrow \psi(x)$$

we have:

(2.44) $\nabla_i[\overline{S}^{\pm}(x, x')\gamma^i\psi(x)] = S^{\mp}(x', x)[\gamma^i\overleftarrow{\nabla}_i + \mu]\psi(x) = -\Sigma(x', x)\psi(x)$

If we integrate in a volume V bounded by a Cauchy surface Σ to the future and a surface Σ' to the past (4) such that $x' \in V$ we have

$$(2.45)\quad -\psi(x') = -\int_V \Sigma(x', x)\psi(x)\eta = \int_V \frac{1}{\sqrt{|g|}}\partial_i[\sqrt{|g|}\,\overline{S}^{\pm}(x, x')\gamma^i\psi(x)]\eta$$

$$= \int_{\Sigma\cup\Sigma'} \overline{S}^{\pm}(x, x')\gamma_i\psi(x)d\sigma^i = \int_{\Sigma\cup\Sigma'} S^{\mp}(x', x)\gamma_i\psi(x)d\sigma^i$$

$$= \begin{cases} \displaystyle\int_\Sigma S^-(x', x)\gamma_i\psi(x)d\sigma^i = \int_\Sigma \overline{S}^+(x, x')\gamma_i\psi(x)d\sigma^i \\ \displaystyle\int_{\Sigma'} S^+(x', x)\gamma_i\psi(x)d\sigma^i = -\int_{\Sigma'} \overline{S}^-(x, x')\gamma_i\psi(x)d\sigma^i, \end{cases}$$

on account of the support of S^- and S^+, and the orientation of Σ and Σ'. Therefore

$$(2.46)\quad -\psi(x') = -\int_\Sigma \overline{S}(x, x')\gamma_i\psi(x)d\sigma^i$$
$$= -i\langle\langle S(x, x'), \psi(x)\rangle\rangle = i\langle\langle \overline{S}(x, x'), \psi(x)\rangle\rangle .$$

If $x, x' \in \Sigma$ equation (2.46) states:

$$(2.47)\quad \overline{S}(x, x')\gamma_i n^i = -\underset{\Sigma}{\Sigma}(x', x),$$

where $\underset{\Sigma}{\Sigma}$ is the Dirac's bi-spinors distribution on Cauchy surface Σ. i. e.

$$(2.48)\quad S(x', x)\gamma_i(x)n^i(x) = \underset{\Sigma}{\Sigma}(x', x).$$

In the particular case $n^i = (1, 0, 0, 0)$ we have

$$(2.49)\quad S(x', x)\gamma_0(x) = \underset{\Sigma}{\Sigma}(x', x),$$

and if we use a system of Dirac's matrices as in eq. (2.8) we shall have

$$(2.50)\quad S(x', x) = -\underset{\Sigma}{\Sigma}(x', x)\gamma_0(x).$$

(4) With their normals pointing to the future.

3. EXPANSION OF KERNEL $S(x, x')$, ANTICOMMUTATION RELATIONS, POSITIVE AND NEGATIVE FREQUENCY SOLUTIONS

a) We shall try to find a base for the vector space of class C^∞ solution $\psi(x)$ of the Dirac's equation.

To solve the problem let us take the case of V_4 endowed with compact Cauchy surfaces. We study this peculiar case for mathematical convenience but it will be very easy to go to the non compact case when we have solved the problem.

Let $\psi(x)$ be an arbitrary C^∞ solution of the Dirac's equation. From eq. (2.44) we know that $\psi(x)$ is defined by its *Cauchy datum* $\psi_\Sigma(x)$ on the compact Cauchy surface Σ i. e. the restriction of $\psi(x)$ to Σ. $\psi_\Sigma(x)$ is an spinor with components $\psi_\Sigma^a(x)$.

Let $\{ Y_A(x) \}$ be a complex base of the vector space of all functions of class C^∞ on Σ, such that it were orthonormal in the inner product.

$$(u, v)_\Sigma = \int_\Sigma u^* v d\sigma , \tag{3.1}$$

such a base exists (cf. [*2*]).

As $\psi_\Sigma^a(x)$ is a C^∞ function on Σ we have

$$\psi_\Sigma^a(x) = \sum_A C_A^a Y_A(x) . \tag{3.2}$$

Let us define the four spinors on Σ.

$$\underset{\Sigma}{\psi_A^{(1)}} = \begin{pmatrix} Y_A \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underset{\Sigma}{\psi_A^{(2)}} = \begin{pmatrix} 0 \\ Y_A \\ 0 \\ 0 \end{pmatrix}, \quad \underset{\Sigma}{\psi_A^{(3)}} = \begin{pmatrix} 0 \\ 0 \\ Y_A \\ 0 \end{pmatrix}, \quad \underset{\Sigma}{\psi_A^{(A)}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ Y_A \end{pmatrix}. \tag{3.3}$$

Then we can write

$$\psi_\Sigma(x) = \sum_{A,a} C_A^a \psi_A^{(a)}(x) . \tag{3.4}$$

But if now we take $\underset{\Sigma}{\psi_A^{(a)}}$ as a Cauchy datum on Σ and we find the C^∞ functions $\psi_A^{(a)}$ on V_4 that satisfy the Dirac's equation and the Cauchy datum $\underset{\Sigma}{\psi_A^{(a)}}$ we can find the expansion of $\psi(x)$ as:

$$\psi(x) = \sum_{A,a} C_A^a \psi_A^{(a)} . \tag{3.5}$$

The base $\{ \psi_A^{(a)} \}$ is orthonormal in the positive defined inner product $\langle\langle\, , \rangle\rangle$, in fact:

(3.6) $$(Y_A, Y_B)_\Sigma = \int_\Sigma Y_A^* Y_B d\sigma = \delta_{AB},$$

and if we take the same tetrade and the same Dirac's matrices as in eq. (2.8) we can easily compute:

(3.7) $$\langle\langle \psi_A^{(a)} ; \psi_B^{(a)} \rangle\rangle$$
$$= - i \int_\Sigma \overline{\psi}{}_{\Sigma A}^{(a)} \gamma_i \psi_{\Sigma B}^{(b)} n^i d\sigma = - \int_\Sigma \psi_{\Sigma A}^{+(a)} \gamma_0^2 \psi_{\Sigma B}^{(b)} d\sigma = \delta^{ab} \int_\Sigma Y_A^* Y_B d\sigma = \delta^{ab} \delta_{AB}$$

Once we have find an orthonormal base $\{ \psi_A^{(a)} \}$ we can go to a generic one via a unitary transformation. So we have found the general solution of our problem.

Now if $\{ \psi_A^{(a)} \}$ is an arbitrary orthonormal base we can compute the coefficients C_A^a as usual and we have

(3.8) $$C_A^a = \langle\langle \psi_A^{(a)}, \psi \rangle\rangle .$$

b) Now we want to find the expansion of $S(x, x')$ in an arbitrary orthonormal base.

First we shall prove that $S(x, x')$ is completely defined through equation (2.4) i. e. if there is a Kernel $S(x, x')$ such that it would satisfy

(3.9) $$\psi(x') = i \langle\langle S(x, x'), \psi(x) \rangle\rangle ,$$

for every $\psi(x)$ solution of the Dirac's equation, this $S(x, x')$ is unique and therefore it is the propagator. In fact; suppose that there are two kernels $S(x, x')$ and $\mathring{S}(x, x')$ and the last one satisfies eq. (3.8) too:

(3.10) $$\psi(x') = i \langle\langle \mathring{S}(x, x'), \psi(x) \rangle\rangle ,$$

substracting (3.9) from (3.10) we have

(3.11) $$\langle\langle \mathring{S}(x, x') - S(x, x'), \psi(x) \rangle\rangle = 0 ,$$

using the same arguments as in eq. (2.7′) we shall obtain

(3.12) $$\int_\Sigma \beta(x') [\mathring{S}(x', x) - S(x', x)] \psi_\Sigma(x) d\sigma_a = 0 .$$

But $\psi_\Sigma(x)$ is an arbitrary C^∞ function that we can choose in the set of test C^∞ functions of compact support, and x' is an arbitrary point of V_4 therefore:

$$\overset{\circ}{S}(x', x) = S(x', x) \tag{3.13}$$ q. e. d.

Now from (3.5) and (3.8) we have

$$\begin{aligned}(3.14)\quad \psi(x') &= \sum_{A,a} C^a_A \psi^{(a)}_A(x') \\ &= \sum_{A,a} \langle\langle \psi^{(a)}_A(x) ; \psi(x) \rangle\rangle_x \psi^{(a)}_A(x') = i \left\langle\left\langle i \sum_{A,a} \psi^{(a)}_A(x)\overline{\psi}^{(a)}_A(x') ; \psi(x) \right\rangle\right\rangle .\end{aligned}$$

Then from (3.9) and the last lemma we have:

$$S(x, x') = i \sum_{A,a} \psi^{(a)}_A(x)\overline{\psi}^{(a)}_A(x'), \tag{3.15}$$

which can be written with spinorial indices as:

$$S^c{}_{b'}(x, x') = i \sum_{A,a} [\psi^{(a)}_A(x)]^c[\overline{\psi}^{(a)}_A(x')]_{b'} \tag{3.16}$$

The order of the factors of equation (3.15) is important because it is a matrix equation and we allways write the matrices as $M^a{}_b$, therefore the column vector $\psi^{(a)}_A(x)$ must be the first one an the row vector $\overline{\psi}^{(a)}_A(x')$ the second one.

Now if we take the Dirac's adjoint we have

$$\overline{S}(x, x') = - i \sum_{A,a} \psi^{(a)}_A(x')\overline{\psi}^{(a)}_A(x) = - S(x', x) \tag{3.17}$$

that coincide with eq. (2.40).

We can also compute:

$$CS(x, x') = S(x, x') \tag{3.17'}$$

and we can say that S is « real ».

As the indices A, a are contracted in equation (3.15) it is obvious that we have an invariant expansion under unitary base transformation. $S(x, x')$ is the unique propagator and we have found its unique expansion.

c) Let's now pass from the classic level to the quantum level. Now field $\psi(x)$ is an operator on an Hilbert space H and the symbol (*) means to take the adjoint operator in space H. As $iS(x, x')$ behaves like

$$\overline{iS(x, x')} = iS(x', x) \tag{3.18}$$

and besides it has all the properties to be considered the anticommutator of the field:

e. g. its support (x) lays in the interior of $E^+(x') \cup E^-(x')$ (cf. eq. (2.32) and the properties of $S^\pm(x, x')$ below eq. (2.29)), $S(x, x')$ satisfies Dirac's equation (2.31), and it has the correct symmetry behaviour.

Therefore we adopt the following anticommutation relation (cf. [8], p. 269).

$$[\psi(x), \bar{\psi}(x')]_+ = \frac{1}{i} S(x, x')E, \tag{3.19}$$

where E is the unit operator of space H that we shall omit from now on.

If we want to compute the anticommutator of the charge conjugated field $\psi^c(x)$ we have (cf. (1.43), (3.17), (3.17'))

$$\begin{aligned}(3.20)\quad & [\psi^c(x), \bar{\psi}^c(x')]_+ \\ &= [C_x\psi(x), A_{x'}C_{x'}\psi(x')]_+ = -[A_xC_x\bar{\psi}(x), A_{x'}C_{x'}\psi(x')]_+ \\ &= -\frac{1}{i}A_xA_{x'}C_xC_{x'}S(x', x) = -\frac{1}{i}\overline{CS(x', x)} = -\frac{1}{i}\overline{S(x', x)} = \frac{1}{i}S(x, x'),\end{aligned}$$

i. e. field $\psi^c(x)$ has the same anticommutation rule. We see that our anticommutation relation has all the properties that it normally has in Special Relativity, in fact $S(x, x')$ becomes the ordinary $S(x, x')$ of Quantum Field Theory if we consider the particular case of flat space-time.

d) To continue we are forced to choose our particle model or models. In fact, in flat space-time we have also a base $\psi_A^{(a)}$ but the four functions have a precise physical meaning, two of them are $u(\boldsymbol{k}, h)e^{-i(\omega kt + \boldsymbol{k}\cdot\boldsymbol{x})}$ (where $h = \pm 1$) and they are the particle model with positive and negative helicity, the other two $v(\boldsymbol{k}, h)e^{+i(\omega kt + \boldsymbol{k}\cdot\boldsymbol{x})}$ are the models of antiparticle with positive and negative helicity. Instead we have a set of completely equivalent orthonormal bases $\{\psi_A^{(a)}\}$ and we have not a criterium to choose the correct one and to decide, in that base, which vectors correspond to particles and which vectors correspond to antiparticles. The Quantum Equivalence Principle will provide us such a criterium.

For the moment let us suppose that, some how, we choose a base $\{U_A^{(s,h)}\}$ as our particle-antiparticle model. We change the four-values index (a) for a pair of two-values indeces (s, h) $s = -$ will correspond to particles, $s = +$ to antiparticle, $h = +$ will correspond to a (not yet clearly defined) positive helicity $h = -$ to a negative helicity.

Besides we will ask a completely new but natural property for our base

$$CU_A^{(s,h)} = U_A^{(-s,-h)} \tag{3.21}$$

in order that particles would become antiparticles under a charge conjugation ([5]).

The base $\{ U_A^{(s,h)} \}$ shall be also orthonormal i. e.

$$\langle\langle U_A^{(s,h)}, U_{A'}^{(s',h')} \rangle\rangle = \delta_{AA'}\delta^{ss'}\delta^{hh'} . \tag{3.22}$$

Now we must see what kind of transformation is adequate to go from base $\{ U_A^{s,h)} \}$ to another base $\{ V_A^{(r,k)} \}$ with the same properties (3.21), (3.22):

$$V_A^{(r,k)} = \sum_{B,h,s} \alpha_{A(s,h)}^{B(r,k)} U_B^{(s,h)} . \tag{3.23}$$

As the new base must be orthonormal we must have

$$\sum_{B,r,k} \overset{*}{\alpha}{}_{A(r,k)}^{B(s,h)} \alpha_{A'(r'k')}^{B\ (s'h')} = \delta_{AA'}\delta^{hh'}\delta^{ss'} , \tag{3.24}$$

so the transformation is *unitary*. We must also have:

$$\begin{aligned} CV_B^{(r,k)} &= \Sigma \overset{*}{\alpha}{}_{B(h,s)}^{A(r,k)} CU_A^{(s,h)} \\ &= \Sigma \overset{*}{\alpha}{}_{B(s,h)}^{A(r,k)} U_A^{(-s,h)} = V_B^{(-r,-k)} = \Sigma \alpha_{B(-s,-h)}^{A(-r,-k)} U_A^{(-s,-h)} \end{aligned} \tag{3.25}$$

thus

$$\overset{*}{\alpha}{}_{B(s,h)}^{A(r,k)} = \alpha_{B(-s,-h)}^{A(-r,-k)} , \tag{3.26}$$

Therefore the transformation must satisfy (3.22) and (3.26).

([5]) In fact this is the case in flat space-time because the u and v functions satisfy the equations

$$(i\gamma^i k_i + \mu)u(\boldsymbol{k}) = 0 ,$$
$$(-i\gamma^i k_i + \mu)v(\boldsymbol{k}) = 0 ,$$

if we charge conjugate the first one and we remind equation (1.45_2) we have

$$(-i\gamma^i k_i + \mu)u^c(\boldsymbol{k}) = 0$$

also

$$\frac{\sigma_\alpha k^\alpha}{|\boldsymbol{k}|} u(\boldsymbol{k}, h) = hu(\boldsymbol{k}, h)$$

therefore

$$\frac{\sigma_\alpha k^\alpha}{|\boldsymbol{k}|} u^c(\boldsymbol{k}, h) = -hu^c(\boldsymbol{k}, h)$$

because $\sigma_\alpha = -\frac{i}{2}(\gamma_\beta\gamma_\gamma - \gamma_\gamma\gamma_\beta)$ is a pure imaginary matrix if we again take into account eq. (1.45_2). Therefore

$$Cu(\boldsymbol{k}, h) = v(\boldsymbol{k}, -h)$$

and

$$Cu(\boldsymbol{k}h)e^{-i(\omega_k t + \boldsymbol{k}\cdot\boldsymbol{x})} = v(\boldsymbol{k}, -h)e^{i(\omega k t + \boldsymbol{k}\cdot\boldsymbol{x})}$$

In this kind of bases we can expand $S(x, x')$ as

$$S(x, x') = i \sum_{A,s,h} U_A^{(s,h)}(x)\overline{U}_A^{(s,h)}(x') . \tag{3.27}$$

We can now define a new kernel:

$$S_1(x, x') = - \sum_{A,s,h} s U_A^{(s,h)}(x)\overline{U}_A^{(s,h)}(x') , \tag{3.28}$$

which has the following symmetry property

$$\overline{S}_1(x, x') = - \sum_{A,s,h} s U_A^{(s,h)}(x')\overline{U}_A^{(s,h)}(x) = S_1(x', x) \tag{3.29}$$

We can also compute $S_1^c(x, x')$. Let us first observe that

$$C\overline{U}_A^{(s,h)} = CAU_A^{(s,h)} = - ACU_A^{(s,h)} = - AU_A^{(-s,-h)} = - \overline{U}_A^{(-s-h)} . \tag{3.40}$$

Therefore, using equations (3.21) and (3.40), we have

$$\begin{aligned} CS_1(x, x') &= \sum_{A,s,h} s U_A^{(-s-h)}(x)\overline{U}_A^{(-s-h)}(x') \\ &= - \sum_{A,s,h} s U_A^{(s,h)}(x)\overline{U}_A^{(s,h)}(x') = S_1(x, x') . \end{aligned} \tag{3.41}$$

and we can say that, as in the case of $S(x, x')$, $S_1(x, x')$ is also « real ».

From its own definition we see that $S_1(x, x')$ is not invariant under a unitary transformation, more over, it cannot be defined in a generic orthonormal base if we do not decide which vectors are particles and which vectors are antiparticles. Therefore the kernel $S_1(x, x')$ contains, in a covariant way, the particle-antiparticle model.

In fact: we can define a mapping $\psi \rightarrow p\psi = \psi_1$, in the vector space of the solution of Dirac's equation:

$$\psi_1(x') = \langle\langle S_1(x, x'), \psi(x) \rangle\rangle_x . \tag{3.42}$$

This map has the following property:

We shall call the function:

$$\psi^{(+)} = \sum_{A,h} b^{*(h)}_A U_A^{(+,h)} \tag{3.43}$$

a positive frequency solution, and it is a generic anti-particle because is an arbitrary linear combination of the $U_A^{(+h)}$ or more precisely an antiparticle creation operator. And

$$(3.44) \qquad \psi^{(-)} = \sum_{A,h} a_A^{(h)} U_A^{(-,h)}$$

is a negative frequency solution, a generic particle or more precisely a particle anihilation operator. Of course for every solution we have the descomposition

$$(3.45) \qquad \psi = \psi^{(+)} + \psi^{(-)}$$

in a unique and canonical way.

Now

$$(3.46) \quad \psi_1^{(+)}(x') = \left\langle\left\langle \sum_{A,s,h} s U_A^{(s,h)}(x)\overline{U}_A^{(s,h)}(x'), \sum_{B,k} b^*{}_B^{(b)} U_B^{(+k)}(x) \right\rangle\right\rangle_x$$

$$= \sum_{A,s,h,B,k} sb^*{}_B^{(k)} \langle\langle U_A^{(s,h)} ; U_B^{(+,k)} \rangle\rangle_x U_A^{(s,h)}(x) = \sum_{A,h,B,k} b^*{}_B^{(k)} \delta^{hk} \delta_{AB} U_A^{(+,h)}(x')$$

$$= \sum_{\Delta,h} b^*{}_B^{(h)} U_A^{(+,h)}(x') = \psi^{(+)}(x') .$$

Analogously:

$$(3.47) \qquad \psi_1^{(-)}(x') = - \psi_1^{(-)}(x') .$$

Therefore the mapping p has the following property

$$(3.48) \qquad p\psi^{(+)} = \psi^{(+)}, \qquad p\psi^{(-)} = - \psi^{(-)},$$

and therefore it defines the positive and negative frequency solutions.

Using this definition we can see that if $\psi^{(+)}$ is a positive frequency solution $C\psi^{(+)}$ is a negative one. In fact

$$(3.49) \quad \psi^{(+)}(x') = \langle\langle S_1(x, x'), \psi^{(+)}(x) \rangle\rangle$$

$$= - i \int_\Sigma \overline{S}_1(x, x')\gamma_i(x)\psi^{(+)}(x)d\sigma^i = - i \int_\Sigma S_1(x', x)\gamma_i(x)\psi^{(+)}(x)d\sigma^i .$$

Therefore from equation (3.41) we have:

$$(3.50) \quad C\psi^{(+)}(x') = i \int_\Sigma S_1^c(x', x)\gamma_i^c(x)\psi^{c(+)}(x)d\sigma^i$$

$$= i \int_\Sigma S_1(x', x)\gamma_i(x)\psi^{c(+)}(x)d\sigma^i = - \langle\langle S_1(x, x'), \psi^{c(+)}(x) \rangle\rangle \quad \text{q. e. d.}$$

Inversely $C\psi^{(-)}$ is a positive frequency solution.

As usual we can see that

$$(3.51) \quad p^2\psi = p.p(\psi^{(+)} + \psi^{(-)}) = p(\psi^{(+)} - \psi^{(-)}) = \psi^{(+)} + \psi^{(-)} = \psi$$

therefore $p^2 = \mathrm{E}$ and p is an involution and we can define the proyectors

$$(3.52) \qquad \oplus = \frac{1}{2}(\mathrm{E} + p), \qquad \ominus = \frac{1}{2}(\mathrm{E} - p),$$

such that $\oplus^2 = \oplus$, $\ominus^2 = \ominus$, $\oplus\ominus = \ominus\oplus = 0$,
and we have

$$(3.53) \qquad \begin{aligned} \oplus\psi^{(+)} &= \psi^{(+)}, & \ominus\psi^{(+)} &= 0, \\ \oplus\psi^{(-)} &= 0, & \ominus\psi^{(-)} &= \psi^{(-)}. \end{aligned}$$

$\oplus$, $\ominus$ are the proyectors into the sub-spaces of positive and negative frequency solutions.

So particle-antiparticle model and splitting in positive and negative frequency are both aspects of the same definition that we can make in a covariant way choosing a particular $\mathrm{S}_1(x, x')$.

e) From (3.43), (3.44), (3.45) we have

$$(3.54) \qquad \psi(x) = \sum_{\mathrm{A},h} a_{\mathrm{A}}^{(h)}\mathrm{U}_{\mathrm{A}}^{(-,h)} + b^{*(h)}_{\mathrm{A}}\mathrm{U}_{\mathrm{A}}^{(+,h)},$$

therefore if we define the anticommutation relations

$$(3.55) \qquad \begin{aligned} \left[a_{\mathrm{A}}^{(h)}, a^{*(k)}_{\mathrm{B}}\right]_+ &= \delta^{hk}\delta_{\mathrm{AB}}, \\ \left[b_{\mathrm{A}}^{(h)}, b^{*(k)}_{\mathrm{B}}\right]_+ &= \delta^{hk}\delta_{\mathrm{AB}}, \end{aligned}$$

and all the other anticommutators vanish, we have:

$$(3.56) \quad [\psi(x), \overline{\psi}(x')]_+ = \left[\sum_{\mathrm{A},h} a_{\mathrm{A}}^{(h)}\mathrm{U}_{\mathrm{A}}^{(+,h)} + b^{*(h)}_{\mathrm{A}}\mathrm{U}_{\mathrm{A}}^{(-,h)} ; \sum_{k,\mathrm{B}} a^{*(k)}_{\mathrm{B}}\overline{\mathrm{U}}_{\mathrm{B}}^{(+,k)} + b_{\mathrm{B}}^{(k)}\overline{\mathrm{U}}_{\mathrm{B}}^{(-,k)}\right]_+$$
$$= \sum_{\mathrm{A},\mathrm{B},h,k} \delta^{hk}\delta_{\mathrm{AB}}\big(\mathrm{U}_{\mathrm{A}}^{(+,h)}(x)\overline{\mathrm{U}}_{\mathrm{B}}^{(+,k)}(x') + \mathrm{U}_{\mathrm{A}}^{(-,h)}(x)\overline{\mathrm{U}}_{\mathrm{B}}^{(-,k)}(x')\big) = \frac{1}{i}\mathrm{S}(x, x'),$$

i. e. eq. (3.19).

With anticommutation relations (3.55) we can find all the usual operators e. g. the particle-number operators:

$$(3.57) \qquad \begin{aligned} \mathrm{N}_{\mathrm{A}}^{(h)} &= a_{\mathrm{A}}^{*(h)}a_{\mathrm{A}}^{(h)}, \\ \tilde{\mathrm{N}}_{\mathrm{A}}^{(h)} &= b_{\mathrm{A}}^{*(h)}b_{\mathrm{A}}^{(h)}, \end{aligned}$$

that only can have eigenvalues 0, 1 and give the number of particles or antiparticles in the model h, A, etc.

f) As in spin-0 case if the space-time is static we can find an unique $S_1(x, x')$ invariant under time displacement. If space-time is not static there is not a good criterium to fix a unique $S_1(x, x')$ so we shall follow the idea of papers [*2*] [*3*] [*4*] [*5*] i. e. to suppose that there is a different $S_1^{(\Sigma)}(x, x')$ for each Cauchy surface Σ and that the variation of the $S_1^{(\Sigma)}(x, x')$ cause either particle creation or anihilation.

In fact, at a Cauchy surface Σ we shall have a $S_1^{\Sigma}(x, x')$, and therefore a base $\{ \overset{\Sigma}{U}{}_A^{(s,h)} \}$ the operators $a_A^{\Sigma(h)}$, $b_A^{\Sigma(h)}$, $\overset{\Sigma}{N}{}_A^{(h)}$, $\overset{\Sigma}{\tilde{N}}{}_A^{(h)}$, the vacuum $|0\rangle_\Sigma$, etc. We shall also have the same objects for Σ'. The field $\psi(x)$ can be expanded in base $\{ \overset{\Sigma}{U}{}_A^{(s,h)} \}$ and $\{ \overset{\Sigma'}{U}{}_B^{(r,k)} \}$ as:

$$\psi(x) = \sum_{A,s,h} \overset{\Sigma}{C}{}_A^{(s,h)} \overset{\Sigma}{U}{}_A^{(s,h)}(x) = \sum_{B,r,k} \overset{\Sigma'}{C}{}_B^{(r,k)} \overset{\Sigma}{U}{}_B^{(r,k)}(x) . \tag{3.58}$$

As we shall see in the cases we are going to study we can choose bases $\{ \overset{\Sigma}{U}{}_A^{(s,h)} \}$ and $\{ \overset{\Sigma'}{U}{}_B^{(r,k)} \}$ such that:

$$\overset{\Sigma}{U}{}_A^{(s,h)} = \sum_r \alpha_{r,k}^{s,h} \overset{\Sigma'}{U}{}_A^{(r,k)} , \tag{3.59}$$

where $\alpha_{r,k}^{s,h}$ is a 4×4 unitary matrix that satisfies

$$\sum_{r,k} \overset{*}{\alpha}{}_{(r,k)}^{(s,h)} \alpha_{(r,k)}^{(s',h')} = \delta^{ss'} \delta^{kk'} , \qquad \sum_{s,h} \overset{*}{\alpha}{}_{(r,k)}^{(s,h)} \alpha_{(r',k')}^{(s,h)} = \delta_{rr'} \delta_{kk'} , \tag{3.60}$$

and

$$\overset{*}{\alpha}{}_{(r,k)}^{(s,h)} = \alpha_{(-r,-k)}^{(-s,-h)} . \tag{3.61}$$

Besides from (3.58) we have that

$$\sum_s \overset{\Sigma}{C}{}_{(s,h)}^A \overset{\Sigma}{U}{}_A^{(s,h)} = \sum_r \overset{\Sigma'}{C}{}_{(r,h)}^A \overset{\Sigma'}{U}{}_A^{(r,h)} = \sum_s \overset{\Sigma}{C}{}_{(s,h)}^A \sum_{r,k} \alpha_{r,k}^{s,h} U_A^{(r,k)} , \tag{3.62}$$

so

$$\overset{\Sigma'}{C}{}_{(s,h)}^A = \sum_s \alpha_{s,h}^{r,k} \overset{\Sigma}{C}{}_{(r,k)}^A , \tag{3.63}$$

or

(3.64)
$$\overset{\Sigma'}{a}{}^{A}_{(h)} = \alpha^{-k}_{-h}\overset{\Sigma}{a}{}^{A}_{(h)} + \alpha^{+k}_{-h}\overset{\Sigma}{b}{}^{*A}_{(k)} ,$$
$$\overset{\Sigma'}{b}{}^{*A}_{(h)} = \alpha^{-k}_{+h}\overset{\Sigma}{a}{}^{A}_{(k)} + \alpha^{+k}_{+h}\overset{\Sigma}{b}{}^{*A}_{(h)} .$$

As we shall work in the mode A we shall write these equations simply as

(3.65)
$$a'_h = \sum_k \alpha^k_h a_k + \overset{*}{\beta}{}^k_h b^*_k ,$$
$$\overset{*}{b}{}'_h = \sum_k \beta^{-k}_{-h} a_k + \overset{*}{\alpha}{}^{-k}_{-h} b^*_k .$$

where the 2×2 matrices α^k_h and β^k_h are related by the following equations.

(3.66)
$$\sum_h \overset{*}{\alpha}{}^k_h \alpha^{k'}_h + \beta^k_h \overset{*}{\beta}{}^{k'}_h = \delta^{kk'} ,$$
$$\sum_h \overset{*}{\alpha}{}^k_h \beta^{-k'}_{-h} + \beta^k_h \overset{*}{\alpha}{}^{-k'}_{-h'} = 0 , \qquad \text{etc.}$$

e. g. we have

(3.67)
$$|\alpha^+_+|^2 + |\alpha^+_-|^2 + |\beta^+_+|^2 + |\beta^+_-|^2 = 1 ,$$
$$|\alpha^-_+|^2 + |\alpha^-_-|^2 + |\beta^-_+|^2 + |\beta^-_-|^2 = 1 , \qquad \text{etc.}$$

therefore all the matrix coefficients have $\leqslant 1$ modulus.

Let us now take, at surface Σ an eigenstate $|\Sigma\rangle$ of the particle and antiparticle number operators $\overset{\Sigma}{\mathrm{N}}_{(h)}$, $\overset{\Sigma}{\tilde{\mathrm{N}}}_{(h)}$ with eigenvalues $n^{(h)}$ and $\tilde{n}^{(h)}$ respectively, let us compute the expectation value of the particles number at Σ' in this eigenstate:

(3.68)
$$\begin{aligned} n'_{(h)} &= \langle \Sigma | \overset{*}{a}{}'_{(h)} a'_{(h)} | \Sigma \rangle \\ &= \langle \Sigma | (\alpha^+_h a_+ + \alpha^-_h a_- + \beta^{*+}_h b^*_+ + \overset{*}{\beta}{}^-_h b^*_-)^* \\ &\qquad (\alpha^+_h a_+ + \alpha^-_h a_- + \overset{*}{\beta}{}^+_h b^*_+ + \beta^{*-}_h b^*_-) | \Sigma \rangle \\ &= \langle \Sigma | (|\alpha^+_h|^2 \mathrm{N}_{(+)} + |\alpha^-_h|^2 \mathrm{N}_{(-)} \\ &\qquad + |\beta^+_h|^2 b_+ b^*_+ + |\beta^-_h|^2 b_- b^*_-) | \Sigma \rangle \\ &= |\alpha^+_h|^2 n_{(+)} + |\alpha^-_h|^2 n_{(-)} - |\beta^+_n|^2 \tilde{n}_{(+)} - |\beta^-_n|^2 \tilde{n}_{(-)} + |\beta^+_h|^2 + |\beta^-_h|^2 , \end{aligned}$$

where we have use the orthonormality of the eigenstate base $|\Sigma\rangle$ and the anticommutation rules of the b.

If the eigenstate at Σ is the vacuum $|0\rangle_\Sigma$ the particles number expectation will be

(3.69)
$$\overset{\circ}{n}_{(h)} = \overset{\circ}{\tilde{n}}_{(h)} = |\beta^+_h|^2 + |\beta^-_h|^2 .$$

Therefore the theory will be implementable if

$$\sum_{\mathrm{A}} |\beta^{+}_{\mathrm{A}h}|^2 + |\beta^{-}_{\mathrm{A}h}|^2 < +\infty\,, \tag{3.70}$$

for $h = \pm$ i. e. we shall have a finite particle creation.

4. THE PARTICULAR CASE OF AN EXPANDING UNIVERSE

a) Let us now suppose that V_4 has the metric

$$dS^2 = (dx^0)^2 - a^2(x^0)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]\,. \tag{4.1}$$

We shall call V_4 an « Expanding Universe » even in the case where $a(t)$ is not monotonically increasing with time $t = x^0$.

If $\hat{\gamma}^i$ is an arbitrary representation of the constant Dirac's matrices of flat space-time, e. g. those of eq. (1.23), we can define the γ^i of space-time V_4 as follows:

$$\begin{aligned} \gamma_0 &= \hat{\gamma}_0\,, & \gamma_\alpha &= a(t)\hat{\gamma}_\alpha\,, \\ \gamma^0 &= \hat{\gamma}^0\,, & \gamma^\alpha &= \overset{-1}{a}(t)\hat{\gamma}^\alpha\,. \end{aligned} \tag{4.2}$$

Then as the $\hat{\gamma}^i$ satisfies the anticommutation relation $\hat{\gamma}^i\hat{\gamma}^j + \hat{\gamma}^j\gamma^i = -2\eta^{ij}$, where η^{ij} is the usual Minkowski metric we have:

$$\gamma^i\gamma^j + \gamma^j\gamma^i = -2g^{ij}\,, \tag{4.3}$$

where g^{ij} is the inverse of the metric tensor of eq. (4.1).

If $\hat{\Gamma}$ is the Γ matrix of the $\hat{\gamma}^i$ and Γ the one for the γ^i, we obviously have

$$\Gamma = \hat{\Gamma} \tag{4.4}$$

The coefficients of the Riemannian connexion in metric (4.1) are

$$\Gamma^0_{\alpha\alpha} = a\dot{a}\,, \qquad \Gamma^\alpha_{0\alpha} = \Gamma^\alpha_{\alpha 0} = \frac{\dot{a}}{a} = \mathrm{H}\,, \tag{4.5}$$

(in $\Gamma^\alpha_{0\alpha}$ and $\Gamma^\alpha_{\alpha 0}$ we do not use Einstein convention) all the other Γ^i_{jk} are zero and H is the Hubble coefficient.

The spinorial connexion is:

$$\sigma_0 = 0\,, \qquad \sigma_\alpha = -\frac{1}{2}\dot{a}\hat{\gamma}^0\hat{\gamma}^\alpha = \frac{1}{2}\mathrm{H}\gamma^0\gamma_\alpha\,, \tag{4.6}$$

in fact, it can be shown by direct computation that:

$$\nabla_i\gamma_j = 0\,, \qquad \nabla_i\Gamma = 0\,. \tag{4.7}$$

b) Dirac's equation in metric (4.1) is:

$$(4.8)\quad (\gamma^i\nabla_i - \mu)\psi = [\gamma^i(\partial_i + \sigma_i) - \mu]\psi = \left[\gamma^0\partial_0 + \gamma^\alpha\left(\partial_\alpha + \frac{1}{2}\mathrm{H}\gamma^0\gamma_\alpha\right) - \mu\right]\varphi = 0$$

But from the anticommutation relation we know that $\gamma^i\gamma_i = -4$ and in our metric $\gamma^0\gamma_0 = -1$, thus $\gamma^\alpha\gamma_\alpha = -3$, so we have:

$$(4.9)\quad \left(\gamma^i\partial_i + \frac{3}{2}\mathrm{H}\gamma^0 - \mu\right)\psi = 0$$

This equation can be solved by variable separation:

$$(4.10)\quad \psi(t, x^\alpha) = \frac{1}{a^{3/2}(t)} f_k(t)e^{ik_\alpha x^\alpha}.$$

Then $f_k(t)$ satisfies the equation

$$(4.11)\quad \gamma_0 \dot{f}_k + ik_\alpha\gamma^\alpha f_k - \mu f_k = 0,$$

or using definitions (4.2)

$$(4.12)\quad \hat{\gamma}^0 \dot{f}_k + i\frac{k_\alpha}{a}\hat{\gamma}^\alpha f_k - \mu f_k = 0.$$

This equation will be studied in paragraph 6 using Olver's method [*10*].

c) Now we shall write the main formulas of paragraph 3 for this particular case. Of course in this case Σ is not compact but $(\sqrt{x\pi})^{-\frac{3}{2}}e^{ik_\alpha x^\alpha}$ take the role of the $\mathrm{Y_A}$, with the only substitution of the δ_{AB} by a $\delta(\boldsymbol{k} - \boldsymbol{h})$.

The orthonormal base $\{\mathrm{U}_\mathrm{A}^{(s,h)}\}$, with the model of particle and antiparticle, will be

$$(4.13)\quad \mathrm{U}_{\boldsymbol{k}}^{(-,h)}(t, x) = \frac{\mathrm{A}_k}{(2\pi a)^{3/2}} u(\boldsymbol{k}, h, t)e^{-ik_\alpha x^\alpha},$$

$$\mathrm{U}_{\boldsymbol{k}}^{(+,h)}(t, x) = \frac{\mathrm{A}_k}{(2\pi a)^{3/2}} v(\boldsymbol{k}, h, t)e^{ik_\alpha x^\alpha}$$

where u and v are solutions of equation (4.11) with some properties that we must find out and A_k is a normalization constant. This base must satisfy the conditions:

$$(4.14)\quad \langle\langle \mathrm{U}_{\boldsymbol{k}}^{(s,h)}, \mathrm{U}_{\boldsymbol{k}'}^{(s',h')} \rangle\rangle = \delta^{ss'}\delta^{hh'}\delta(\boldsymbol{k} - \boldsymbol{k}'),$$

$$\mathrm{CU}_{\boldsymbol{k}}^{(+,h)} = \mathrm{U}_{\boldsymbol{k}}^{(-,h)}.$$

The first one is:

$$(4.15)\quad \frac{-i}{(2\pi a)^3}|\mathrm{A}_k|^2\bar{u}(\boldsymbol{k}, \boldsymbol{h}, t)\mathring{\gamma}u(\boldsymbol{k}', \boldsymbol{h}', t) \times \int e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}a^3d^3\boldsymbol{x} = \delta^{hh'}\delta(\boldsymbol{k} - \boldsymbol{k}'),$$

computing all others we find the following equations:

$$(4.16)\qquad \begin{aligned} -i\,|A_k|^2\bar{u}(\boldsymbol{k}, h, t)\overset{\circ}{\gamma}u(\boldsymbol{k}, h', t) &= \delta_{hh'}\,,\\ -i\,|A_k|^2\bar{v}(\boldsymbol{k}, h, t)\overset{\circ}{\gamma}v(\boldsymbol{k}, h', t) &= \delta_{hh'}\,,\\ \bar{u}(\boldsymbol{k}, h, t)\overset{\circ}{\gamma}v(-\boldsymbol{k}, h', t) &\\ = \bar{v}(\boldsymbol{k}, h, t)\overset{\circ}{\gamma}u(-\boldsymbol{k}, h', t) &= 0. \end{aligned}$$

The second one of (4.14) is:

$$(4.17)\qquad \alpha^* v^*(\boldsymbol{k}, h, t) = u(\boldsymbol{k}, h, t)\,.$$

All these conditions must be fulfilled.

The Kernel $S_1(x, x')$ taking into account equation (3.28) can be written as:

$$(4.18)\quad S_1(x, x') = -\sum_{s,h}\int_k s U_{\boldsymbol{k}}^{(s,h)}(x)\bar{U}_{\boldsymbol{k}}^{(s,h)}(x')d^3\boldsymbol{k}$$

$$= \frac{|A_k|^2}{(2\pi a)^3}\sum_h \int_k e^{-ik_\alpha(x^\alpha-\dot{x}^\alpha)}[+u(\boldsymbol{k}, h, t)\bar{u}(\boldsymbol{k}, h, t)+v(-\boldsymbol{k}, h, t)\bar{v}(-\boldsymbol{k}, h, t')]d^3\boldsymbol{k}\,,$$

we shall use this formula later on.

5. THE QUANTUM EQUIVALENCE PRINCIPLE

a) Quantum Equivalence Principle was stated for spin-0 fields in works [*5*] and [*4*]. It says that we must take as the Kernel $G_1(x, x')$ the Kernel $\Delta_1(x, x')$ of flat space-time, but this identification must be done only in *normal coordinates* at x (or x'). Normal coordinates have intersting properties that have been discussed in paper [*4*] but if we restrict ourselves only up to the second order of x^α, in the change of variable, its characteristic property is that in such coordinates the Γ^i_{jk} are zero i. e. gravitational forces vanish ([6]). In this work we shall expand the transformation formulas up to this second order, or, what it is the same thing, we shall consider only the first power of H. Under these conditions we can say that Quantum Equivalence Principle consists in the identification of G_1 with Δ_1 in coordinates where gravitational forces vanish.

b) Therefore for spin 1/2 we shall identify S_1 with the flat S_1^{FL} in coordinates $\{x'\}$ where gravitational forces vanish, and in spinorial bases where all gravitational effects vanish too i. e.

$$(5.1)\qquad \Gamma^{i'}_{j'k'} = 0\,,\qquad \sigma_{i'} = 0\,.$$

([6]) So really we use geodesic coordinates. We conjecture that one must use normal coordinates if one take into account higher powers of x^α.

Normal coordinates $\mathring{x}^i$ at point 0 are:

$$\mathring{x}^i = x^i + \frac{1}{2}\Gamma^i_{jk}(0)x^j x^k + \dots \tag{5.2}$$

as we have said we do not take into account the x^3, and following terms. In these coordinates we have that $\mathring{\Gamma}^i_{jk}(0) = 0$. In the case of the manifold V_4 of paragraph 4 we have:

$$\mathring{x}^0 = x^0 + \frac{1}{2}Ha^2 \sum_{k=1}^{3} (x^\alpha)^2 + \dots$$

$$\mathring{x}^\alpha = x^\alpha(1 + Hx^0) + \dots$$

and also

$$A^{0'}{}_0 = \frac{\partial \mathring{x}^{0'}}{\partial x^0} = 1 + \dots \tag{5.4}$$

$$A^{0'}{}_\gamma = \frac{\partial \mathring{x}^{0'}}{\partial x^\gamma} = Ha^2x^2 + \dots$$

$$A^{\alpha'}{}_0 = \frac{\partial \mathring{x}^{\alpha'}}{\partial x^0} = H\delta^{\alpha'}_\alpha x^\alpha + \dots$$

$$A^{\alpha'}{}_\beta = \frac{\partial \mathring{x}^{\alpha'}}{\partial x^\beta} = \delta^{\alpha'}_\beta(1 + Hx^0) + \dots$$

Now we must satisfy (5.1_2), then we must find a matrix S such that (cf. 1.70):

$$\sigma_i S^{-1} + \partial_i S^{-1} = 0\,. \tag{5.5}$$

We can directly see that

$$S = I + \frac{1}{2}H\gamma^0\gamma_\alpha x^\alpha + \dots \tag{5.6}$$

$$S^{-1} = I - \frac{1}{2}H\gamma^0\gamma_\alpha x^\alpha + \dots$$

where we have systematically neglected the term in H^2.

Therefore in normal coordinates and with $\gamma^{i'} = A^{i'}_i S\gamma^i S^{-1}$ we have:

$$\begin{aligned} \mathring{S}_1(x', 0) &= S_1^{FL}(x', 0) \\ &= (\mathring{\gamma}^{i'}\partial_{i'} + \mu)\frac{1}{(2\pi)^3}\int \frac{\cos \omega_p t'}{\omega_p} e^{-iap_\alpha x^\alpha} d^3\boldsymbol{p} \\ &= \frac{1}{(2\pi)^3}\int \frac{e^{-iap_\alpha x^\alpha}}{\omega_p}[-\mathring{\gamma}^{0'}\omega_p \sin \omega_p t' - iap_{\alpha'}\mathring{\gamma}^{\alpha'} \cos \omega_p t' + \mu]d^3\boldsymbol{p}\ (^7)\,, \end{aligned} \tag{5.7}$$

where p_α is the physical momentum and $a(0)x^\alpha$ the physical space coordi-

(7) Strictly speaking the $\mathring{\gamma}$ and the unit spinor matrix of eq. (5.7) must be bispinors on x, 0 but from eqs. (1.47), (1.54) and (5.1) and from the fact that we neglect H^2 and higher powers we can take these bispinors as constants.

nates. Of course $\dot{S}_1(x, 0)$ can be considered the Fourier transform of $\omega_p^{-1}[\ldots]$ of the last member of eq. (5.7) only in distribution sense.

Now we can compute the bispinor $S_1(0, x)$ in the primitive coordinates, we have:

$$(5.8)\quad S_1(x, 0) = S^{-1}(x)S_1'(x', 0)S(0) = \frac{S^{-1}(x)}{(2\pi a)^3}\int \frac{e^{-ik_\alpha \dot{x}^\alpha}}{\omega_k}[-A_i^{0'}\gamma^i\omega_k \sin \omega_k t - ik_{\alpha'}A_i^{\alpha'}\gamma^i \cos \omega_k t + \mu]d^3\boldsymbol{k},$$

where we have changed the integration variable from $\boldsymbol{p} \to \boldsymbol{k} = a\boldsymbol{p}$, and we use the ordinary spatial coordinates x^α. Using eq. (5.3), (5.4) and (5.6) and neglecting all terms in H^2 or the higher powers and making $t = 0$, we have

$$(5.9)\quad S_1(x, 0) = \frac{1}{(2\pi a)^3}\int \frac{e^{-ik_\alpha x^\alpha}}{\omega_k} \times \left\{ \mu - ik_\alpha\gamma^\alpha + H\left[-\omega_k^2\gamma^0\frac{a^2}{2}\sum_{\alpha=1}^{3}(x^\alpha)^2 - i\gamma^0 k_\alpha x^\alpha + \frac{1}{2}(\mu - ik_\alpha\gamma^\alpha)\gamma^0\gamma_\alpha x^\alpha\right]\right\} d^3\boldsymbol{k}.$$

Now we want to find the base $\{ U_{\boldsymbol{k}}^{(s,h)} \}$, therefore we must Fourier analyse (5.9) and compare the result with (4.18) i. e. to find the Fourier transform of

$$\omega_k \sum_{\alpha=1}^{3}(x^\alpha)^2, \quad \frac{k_\alpha x^\alpha}{\omega_k}, \quad \frac{\gamma_\alpha x^\alpha}{\omega_k}, \quad \frac{\gamma^\alpha\gamma_\beta k_\alpha x^\beta}{\omega_k},$$

where x^α and $\sum_{\alpha=1}^{3}(x^\alpha)^2$ must be considered as the distribution, whose Fourier-transforms are

$$(5.10)\quad \hat{x}^\alpha = -i\frac{\partial}{\partial k_\alpha}\delta(\boldsymbol{k}), \qquad \widehat{\sum_{\alpha=1}^{3}(x^\alpha)^2} = -\nabla^2\delta(\boldsymbol{k}),$$

but the transform of a product of one function φ and of one distribution u is:

$$(5.11)\quad (u\varphi)\hat{} = \hat{u} * \hat{\varphi}$$

where the symbol * means the convolution therefore:

$$(5.12)\quad \left(\sum_{\alpha=1}^{3}(x^\alpha)^2\omega_k\right)\hat{} = -\int \nabla^2\delta(\boldsymbol{k} - \boldsymbol{h})\omega_h d^3\boldsymbol{h} = -\frac{1}{a^2}\left(\frac{2}{\omega_k} + \frac{\mu^2}{\omega_k^0}\right),$$

$$(5.13)\quad \left(\frac{x^\alpha k_\alpha}{\omega_k}\right)^{\wedge} = i\int \frac{\partial}{\partial k_\alpha}\delta(\boldsymbol{k}-\boldsymbol{h})\frac{h_\alpha}{\omega_h}d^3\boldsymbol{h} = -i\left(\frac{2}{\omega_k}+\frac{\mu^2}{\omega_k^3}\right),$$

$$(5.14)\quad \left(\gamma_\alpha x^\alpha\frac{1}{\omega_k}\right)^{\wedge} = -i\gamma_\alpha\int \frac{\partial}{\partial k_\alpha}(\boldsymbol{k}-\boldsymbol{h})\frac{1}{\omega_h}d^3\boldsymbol{h} = i\frac{\gamma_\alpha k_\alpha}{a^2\omega_k^3} = -i\frac{\gamma^\alpha k_\alpha}{\omega_k^3},$$

$$(5.15)\quad \left(\frac{x^\beta\gamma^\alpha\gamma_\beta k_\alpha}{\omega_k}\right)^{\wedge} = i\gamma^\alpha\gamma_\beta\int \frac{\partial}{\partial k_\beta}\delta(\boldsymbol{k}-\boldsymbol{h})\frac{h_\alpha}{\omega_k}d^3\boldsymbol{h} = i\left(\frac{2}{\omega_k}+\frac{\mu^2}{\omega_k^3}\right).$$

Substituting these equations in eq. (5.9) we have

$$(5.16)\quad S_1(x,0)$$
$$= \frac{1}{(2\pi a)^3}\int \frac{e^{-ik_\alpha x^\alpha}}{\omega_k}\times\left[\mu - ik_\alpha\gamma^\alpha + \frac{iH\mu}{2\omega_k^2}\gamma^0\gamma^\alpha k_\alpha\right]d^3\boldsymbol{k}$$
$$= \frac{1}{(2\pi a)^3}\int \frac{e^{-ik_\alpha x^\alpha}}{\omega_k}\left(I - \frac{\mu H\gamma^0}{4\omega_k^2}\right)(\mu - ik_\alpha\gamma^\alpha)\left(I + \frac{\mu\cdot H\gamma^0}{4\omega_k^2}\right)d^3\boldsymbol{k}.$$

Comparing to (4.18) and observing that we have changed $x' \to 0$ we can see that we must have

$$(5.17)\quad |A_k|^2\sum_h\left[\overset{a}{u}(\boldsymbol{k},h,0)\underset{b}{\bar{u}}(\boldsymbol{k},h,0) + \overset{a}{v}(-\boldsymbol{k},h,0)\underset{b}{\bar{v}}(-\boldsymbol{k},h,0)\right]$$
$$= \frac{1}{\omega_k}\left[\left(I - \frac{\mu H\gamma^0}{4\omega_k^2}\right)(\mu - ik_\alpha\gamma^\alpha)\left(I + \frac{\mu H\gamma^0}{4\omega_k^2}\right)\right]^a_b.$$

Contracting with $\overset{\circ}{\gamma}{}^b{}_c u^0(\boldsymbol{k}',h')$ and using eq. (4.16) we have

$$(5.18)\quad iu(\boldsymbol{k},h,0) = \frac{1}{\omega_k}\left(I - \frac{\mu H\gamma^0}{4\omega_k^2}\right)(\mu - ik_\alpha\gamma^\alpha)\left(I + \frac{\mu H\gamma^0}{4\omega}\right)\gamma^0 u(\boldsymbol{k},h,0).$$

In flat space-time H = 0, we shall call $\overset{\circ}{u}$, the u of flat space-time and it must satisfy:

$$(5.19)\quad i\overset{\circ}{u} = \frac{1}{\omega_k}(\mu - ik_\alpha\gamma^\alpha)\overset{\circ}{\gamma}\overset{\circ}{u} = \frac{\overset{\circ}{\gamma}}{\omega_k}(\mu + ik_\alpha\gamma^\alpha)\overset{\circ}{u},$$

i. e.

$$(i\omega_k\gamma^0 + ik_\alpha\gamma^\alpha + \mu)\overset{\circ}{u} = 0,$$

therefore $\overset{\circ}{u}$ is the ordinary $u\left(\frac{k}{a}\pm\right)$ a well known spinor.

We can write the last two equations as

$$(5.20)\quad M^{-1}(\mu - ik_\alpha\gamma^\alpha)\overset{\circ}{\gamma}Mu = i\omega_k u,$$
$$(\mu - ik_\alpha\gamma^\alpha)\overset{\circ}{\gamma}\overset{\circ}{u} = i\omega_k\overset{\circ}{u}.$$

Therefore $\overset{\circ}{u} = \mathrm{M}u$ and $u = \mathrm{M}^{-1}\overset{\circ}{u}$, i. e.

$$u(\boldsymbol{k}, h, 0) = \left(\mathrm{I} - \frac{\mu \mathrm{H}\gamma^0}{4\omega_k^2}\right)\overset{\circ}{u}\left(\frac{k}{a}, h\right). \tag{5.21}$$

Analogously

$$v(\boldsymbol{k}, h, 0) = \left(\mathrm{I} - \frac{\mu \mathrm{H}\gamma^0}{4\omega_k^2}\right)\overset{\circ}{v}\left(\frac{k}{a}, h\right). \tag{5.22}$$

The usual $\overset{\circ}{u}$ and $\overset{\circ}{v}$ satisfy the following equations:

$$\overset{\circ}{\bar{u}}(\boldsymbol{k}, h)\gamma^0\overset{\circ}{u}(\boldsymbol{k}, h') = \overset{\circ}{\bar{v}}(\boldsymbol{k}, h)\gamma^0\overset{\circ}{v}(\boldsymbol{k}, h') = i\frac{\omega_k}{\mu}\delta_{hh'} . \tag{5.23}$$

$$\overset{\circ}{\bar{u}}(\boldsymbol{k}, h)\gamma^0\overset{\circ}{v}(-\boldsymbol{k}, h') = \overset{\circ}{\bar{v}}(\boldsymbol{k}, h)\gamma^0\overset{\circ}{u}(-\boldsymbol{k}, h') = 0 ,$$

and also

$$\overset{\circ}{\bar{u}}(\boldsymbol{k}, h)\overset{\circ}{u}(\boldsymbol{k}, h') = -\overset{\circ}{\bar{v}}(\boldsymbol{k}, h)\overset{\circ}{v}(\boldsymbol{k}, h') = \delta_{ss'} \tag{5.24}$$

$$\overset{\circ}{\bar{u}}(\boldsymbol{k}, h)\overset{\circ}{v}(\boldsymbol{k}, h') = \overset{\circ}{\bar{v}}(\boldsymbol{k}, h)\overset{\circ}{u}(\boldsymbol{k}, h') = 0$$

Therefore to satisfy (4.16) we need

$$\mathrm{M}\overline{\mathrm{M}} = \mathrm{I} \qquad |\mathrm{A}_k|^2 = \frac{\mu}{\omega_k} \tag{5.25}$$

the first one is true if we take into account the equation (1.45_1). We can use the equation (1.45_2) to prove that also equations (4.17) hold true.
Therefore we have

$$\mathrm{U}_{\boldsymbol{k}}^{(-,h)}(0, x) = \frac{1}{(2\pi a)^{3/2}}\left(\frac{\mu}{\omega_k}\right)^{1/2}\left(\mathrm{I} - \frac{\mu \mathrm{H}\gamma^0}{4\omega_k^2}\right)\overset{\circ}{u}\left(\frac{k}{a}, h\right)e^{-ik_\alpha x^\alpha}, \tag{5.26}$$

$$\mathrm{U}_{\boldsymbol{k}}^{(+,h)}(0, x) = \frac{1}{(2\pi a)^{3/2}}\left(\frac{\mu}{\omega_k}\right)^{1/2}\left(\mathrm{I} - \frac{\mu \mathrm{H}\gamma^0}{4\omega_k^2}\right)\overset{\circ}{v}\left(\frac{k}{a}, h\right)e^{ik_\alpha x^\alpha}.$$

These are the initial conditions issued from the Quantum Equivalence Principle. The base function $\mathrm{U}_{\underline{k}}^{(-,h)}(t, x)$ and $\mathrm{U}_{\underline{k}}^{(+,h)}(t, x)$ can be obtained solving Dirac's Equation with these Cauchy data.

6. DIRAC'S EQUATION APPROXIMATIVE SOLUTION

a) Now we can continue the solution of Dirac's Equation using Olver paper [*10*]. We can write eq. (4.12) as

$$\hat{\gamma}^0 a\frac{d}{dt}f_k + ik_\alpha\hat{\gamma}^\alpha f_k - a\mu f_k = 0 , \tag{6.1}$$

and introduce a new time parameter

$$\tau = \int_0^t \frac{dt}{a(b)} \tag{6.2}$$

Thus eq. (6.1) becomes:

$$\left[\hat{\gamma}^0 \frac{d}{d\tau} + ik_\alpha \hat{\gamma}^\alpha - \mu a(\tau) \right] f_k = 0 . \tag{6.3}$$

Let us introduce the operator $\hat{\gamma}^0 \dfrac{d}{d\gamma} + ik_\alpha \hat{\gamma}^\alpha + \mu a(\tau)$ so

$$\left[\hat{\gamma}^0 \frac{d}{d\tau} + ik_\alpha \hat{\gamma}^\alpha + \mu a(\tau) \right]\left[\hat{\gamma}^0 \frac{d}{d\tau} + ik_\alpha \hat{\gamma}^\alpha - \mu a(\tau) \right] f_k = 0 , \tag{6.4}$$

i. e.

$$\left(\frac{d^2}{d\tau^2} + \overline{k}^2 + \mu^2 a^2 + \hat{\mu}\gamma^0 \frac{da}{d\tau} \right) f = 0 . \tag{6.5}$$

This equation belongs to the class of equations that can be sloved by Olver's Theorem 2. All the solutions of eq. (6.3) are solutions of eq. (6.5) but the inverse is not true. So we must, somehow, single out among the set of solutions of (6.5) the solutions of (6.3).

Let us consider a function g that satisfies the equation:

$$\left[\hat{\gamma}^0 \frac{d}{d\tau} + ik_\alpha \hat{\gamma}^\alpha + \mu a(\tau) \right] g = 0 . \tag{6.6}$$

If for $\tau = 0$ is $g(0) = 0$, the solution of the equation (6.6) with this boundary condition is $g \equiv 0$. Now if we call

$$g = \left[\hat{\gamma}^0 \frac{d}{d\tau} + ik_\alpha \hat{\gamma}^\alpha - \mu a(\tau) \right] f_k , \tag{6.7}$$

we see that if f_k satisfies eq. (6.5) then g satisfies eq. (6.6), therefore if $g(0) = 0$, f_k satisfies eq. (6.3). Therefore we must only find the solutions of eq. (6.5) that would satisfy (6.3) at $\tau = 0$, these are the solutions of eq. (6.3).

b) Now we want to pass from Olver's Theorem 2 to Theorem 4, noticing that in our case as the function f of Theorem 2 is complex so it will be the function p. Olver's formulas (4.2) to (4.5) show that equation (4.18) to (4.21) are also valid in this case.

So let us write (6.5) as:

$$\left[\frac{d^2}{d\tau^2} + k^2 p(k, \tau) \right] f_k = 0 , \tag{6.8}$$

where

$$p(k, \tau) = \frac{a^2}{k^2}\,\omega_k^2\left(1 + \frac{\mu \mathrm{H}}{\omega_k^2}\,\hat{\gamma}_0\right), \tag{6.9}$$

f_k is a spinor with components that we shall call f_k^1, f_k^2, $f_k^{\dot 1}$, $f_k^{\dot 2}$, these components satisfy the equations

$$\left[\frac{d^2}{d\tau^2} + k^2 p(k, \tau)\right] f_k^{\alpha} = 0\,, \tag{6.10}$$
$$\left[\frac{d^2}{d\tau^2} + k^2 p^*(k, \tau)\right] f_k^{\dot\alpha} = 0\,, \qquad \alpha = 1, 2\,,$$

where

$$p(k, \tau) = \frac{a^2}{k^2}\,\omega_k^2\left(1 + i\frac{\mu \mathrm{H}}{\omega_k^2}\right). \tag{6.11}$$

Therefore from Theorem 4 we have the solutions

$$f_k^{\alpha} = \frac{a^{\alpha} k^{1/2}}{a^{1/2}\omega_k^{1/2}\left(1 + i\dfrac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4}}\left\{\exp i\int \omega_k\sqrt{1 + i\frac{\mu \mathrm{H}}{\omega_k^2}}\,dt + \varepsilon\right\} \tag{6.12}$$
$$+ \frac{b^{\alpha} k^{1/2}}{a^{1/2}\omega_k^{1/2}\left(1 + i\dfrac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4}}\left\{\exp\left(-i\int \omega_k\sqrt{1 + i\frac{\mu \mathrm{H}}{\omega_k^2}}\,dt\right) + \varepsilon\right\},$$
$$f_k^{\dot\alpha} = \frac{a^{\dot\alpha} k^{1/2}}{a^{1/2}\omega_k^{1/2}\left(1 - i\dfrac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4}}\left\{\exp i\int \omega_k\sqrt{1 - i\frac{\mu \mathrm{H}}{\omega_k^2}}\,dt + \varepsilon\right\}$$
$$+ \frac{b^{\dot\alpha} k^{1/2}}{a^{1/2}\omega_k^{1/2}\left(1 - i\dfrac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4}}\left\{\exp\left(-i\int \omega_k\sqrt{1 - i\frac{\mu \mathrm{H}}{\omega_k^2}}\,dt\right) + \varepsilon\right\}.$$

As $p(k, \tau)$ can be written as:

$$p(k, \tau) = 1 + k^{-2}\tilde{p}(\tau)\,, \tag{6.13}$$
$$\tilde{p}(\tau) = \mu^2 a^2 + i\mu a^2 \mathrm{H}\,,$$

using Olver's results it can be proved (as in paper [5]) that

$$|\,\varepsilon\,| \sim 0(k^{-3}) \tag{6.14}$$

when $k \to +\infty$.

f_k of eq. (6.12) is the general solution of eq. (6.5). To verify that $g(0) = 0$ we just need the $\frac{d}{d\tau} f_k$ at $\tau = 0$ i. e.

$$(6.15)\qquad \left(\frac{df^\alpha}{d\tau}\right)_0 = ika^\alpha p^{1/4}\left(1 + \frac{i}{4kp^{3/2}}\frac{dp}{d\tau}\right) - ikb^\alpha p^{1/4}\left(1 + \frac{i}{4kp^{3/2}}\frac{dp}{d\tau}\right),$$

$$\left(\frac{df^{\dot\alpha}}{d\tau}\right)_0 = ika^{\dot\alpha} p^{*1/4}\left(1 + \frac{i}{4kp^{*3/2}}\frac{dp^*}{d\tau}\right) - ikb^{\dot\alpha} p^{*1/4}\left(1 + \frac{i}{4kp^{*3/2}}\frac{dp^*}{d\tau}\right).$$

but

$$(6.16)\qquad \frac{i}{4kp^{3/2}}\frac{dp}{d\tau} = i\,\frac{\dfrac{2\mu a^2}{k^2}\mathrm{H} + \dfrac{i\mu}{k^2}\dfrac{d^2a}{d\tau^2}}{4k\dfrac{a^2\omega_k^3}{k^3}\left(1 + i\dfrac{\mu\mathrm{H}}{\omega_k^2}\right)^{3/2}} \sim 0(k^{-3}),$$

when $k \to +\infty$.

We shall follow the computation neglecting all the $0(k^{-3})$, as we are forced by eq. (6.14).

Therefore at $\tau = 0$ we shall have:

$$(6.17)\qquad \left(\frac{df^\alpha}{d\tau}\right)_0 = ik^{1/2}a^{1/2}\omega_k^{1/2}\left[a^\alpha\left(1 + i\frac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4} - b^\alpha\left(1 + i\frac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4}\right],$$

$$\left(\frac{df^{\dot\alpha}}{d\tau}\right)_0 = ik^{1/2}a^{1/2}\omega_k^{1/2}\left[a^{\dot\alpha}\left(1 - i\frac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4} - b^{\dot\alpha}\left(1 - i\frac{\mu\mathrm{H}}{\omega_k^2}\right)^{1/4}\right].$$

But as we shall neglect the terms $0(k^{-3})$ and $0(\mathrm{H}^2)$ we compute

$$(1 + i\mu\mathrm{H}\omega_k^{-2})^{1/4} = 1 + \frac{1}{4}i\mu\mathrm{H}\omega_k^{-2}.$$

From (6.12) at $\tau = 0$ we have:

$$(6.18)\qquad (f_k^\alpha)_0 = \frac{k^{1/2}}{a^{1/2}\omega_k^{1/2}}\left[a^\alpha\left(1 - \frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right) + b^\alpha\left(1 - \frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right)\right],$$

$$(f_k^{\dot\alpha})_0 = \frac{k^{1/2}}{a^{1/2}\omega_k^{1/2}}\left[a^{\dot\alpha}\left(1 + \frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right) + b^{\dot\alpha}\left(1 + \frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right)\right].$$

and from (6.17)

$$(6.19)\quad \left(\frac{df^{\alpha}}{d\tau}\right)_0 = ik^{1/2}a^{1/2}\omega_k^{1/2}\left[a^{\alpha}\left(1+\frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right)-b^{\alpha}\left(1+\frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right)\right],$$

$$\left(\frac{df^{\dot\alpha}}{d\tau}\right)_0 = ik^{1/2}a^{1/2}\omega_k^{1/2}\left[a^{\dot\alpha}\left(1-\frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right)-b^{\dot\alpha}\left(1-\frac{i}{4}\frac{\mu\mathrm{H}}{\omega_k^2}\right)\right].$$

i. e.

$$(6.20)\quad (f_k)_0 = \left(\frac{k}{a\omega_k}\right)^{1/2}\left(1-\frac{\mu\mathrm{H}\hat\gamma_0}{4\omega_k^2}\right)(\mathrm{A}+\mathrm{B}),$$

$$(6.21)\quad \left(\frac{df_k}{d\tau}\right)_0 = i(ak\omega_k)^{1/2}\left(1+\frac{\mu\mathrm{H}\hat\gamma_0}{4\omega_k^2}\right)(\mathrm{A}-\mathrm{B}),$$

where we have introduced the two column spinors

$$\mathrm{A} = \begin{pmatrix} a^{\alpha} \\ a^{\dot\alpha}\end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} b^{\alpha} \\ b^{\dot\alpha}\end{pmatrix}.$$

Now we must compute this arbitrary constant spinor A and B in such a way that at $t = \tau = 0$ equation (6.7) with $g = 0$ should be fulfilled and also the initial conditions (5.26) for the $\mathrm{U}_{\underline{k}}^{(\pm,h)}$, in this way we shall find a solution of Dirac's equation that satisfies equivalence principle at $t=\tau=0$.

From equations (4.10) and (5.26$_1$) we have that the initial condition for the $(f_k)_0$ of $\mathrm{U}_k^{(-,h)}$ is:

$$(6.22)\quad (f_k)_0 = \frac{1}{(2\pi)^{3/2}}\frac{\mu^{1/2}}{\omega_k^{1/2}}\left(1-\frac{\mu\mathrm{H}\gamma^0}{4\omega_k^2}\right)\mathring{u}\left(\frac{k}{a}, h\right).$$

Therefore we can satisfy this equation if we make:

$$(6.23)\quad \mathrm{A}+\mathrm{B} = \frac{1}{(2\pi)^{3/2}}\left(\frac{a\mu}{k}\right)^{1/2}\mathring{u}\left(\frac{k}{a}, h\right).$$

Equation (6.7) with $g = 0$ and with the substitution $k_\alpha \to -k_\alpha$, because we are computing the constant for $\mathrm{U}_{\underline{k}}^{(-,h)}$, is:

$$(6.24)\quad \hat\gamma^0\left(\frac{df_k}{d\tau}\right)_0 + (-ik\hat\gamma_0 - \mu a(0))(f_k)_0 = 0.$$

If we put equation (6.20), (6.21) and (6.23) in (6.24) we can find:

$$(6.25)\quad \mathrm{A}-\mathrm{B} = -\frac{i\hat\gamma_0}{(2\pi)^{3/2}\omega_k}\left(\frac{\mu}{ka}\right)^{1/2}\left[-(ik_\alpha\hat\gamma^\alpha+\mu a)+\frac{\mu^2\mathrm{H}a}{2\omega_k^2}\hat\gamma_0\right]\mathring{u},$$

but $\mathring{u}$ satisfies equation (5.19) therefore we have:

$$(6.26)\quad \mathrm{A}-\mathrm{B} = \frac{1}{(2\pi)^{3/2}}\left(\frac{a\mu}{k}\right)^{1/2}\left(-1+\frac{i\mu^2\mathrm{H}a}{2\omega_k^3}\right)\mathring{u}$$

if we neglect the term $0(k^{-3})$ as usual we find that

$$(6.27) \qquad A = 0\,, \qquad B = \frac{1}{(2\pi)^{3/2}}\left(\frac{a\mu}{k}\right)^{1/2} \overset{\circ}{u}\left(\frac{k}{a}, h\right).$$

We can repeat the computation for $U_k^{(+,h)}$ and we shall find:

$$(6.28) \qquad A = \frac{1}{(2\pi)^{3/2}}\left(\frac{a\mu}{k}\right)^{1/2} \overset{\circ}{v}\left(\frac{k}{a}, h\right), \qquad B = 0\,.$$

Therefore the base functions that satisfy Quantum Equivalence Principle at $t = 0$ are:

$$(6.29) \quad U_k^{(-,h)}(t, x) = \frac{1}{[2\pi a(t)]^{3/2}}\left(\frac{a(0)\mu}{a(t)\omega_k(t)}\right)^{1/2}\left(I - \frac{\mu H\hat{\gamma}_0}{4\omega_k^2(t)}\right) \times \left[\exp - i\int_0^t \omega_k\left(1 + \frac{\mu H\hat{\gamma}_0}{2\omega_k^2}\right)dt + \varepsilon\right] \overset{\circ}{u}\left(\frac{k}{a(0)}, h\right)e^{-ik_\alpha x^\alpha},$$

$$U_k^{(+,h)}(t, x) = \frac{1}{[2\pi a(t)]^{3/2}}\left(\frac{a(0)\mu}{a(t)\omega_k(t)}\right)^{1/2}\left(I - \frac{\mu H\hat{\gamma}_0}{4\omega_k^2(t)}\right) \times \left[\exp i\int_0^t \omega_k\left(1 + \frac{\mu H\hat{\gamma}_0}{2\omega_k^2}\right)dt + \varepsilon'\right] \overset{\circ}{v}\left(\frac{k}{a(0)}, h\right)e^{ik_\alpha x^\alpha}.$$

7. COMPUTATION OF α AND β, IMPLEMENTABILITY

If we take Σ to be the Cauchy surface at time τ and Σ' the one at time τ' and we use the equation (6.29) the equation (3.59) becomes:

$$(7.1) \quad a(\tau)^{1/2}\left\{\exp - i\int_\tau^t \omega_k\left(1 + \frac{\mu H\hat{\gamma}_0}{2\omega_k^2}\right)dt + \varepsilon\right\} \overset{\circ}{u}\left(\frac{k}{a(\tau)}, h\right) = \sum_l \alpha_{kl}^h a(\tau')^{1/2}\left\{\exp - i\int_{\tau'}^t \omega_k\left(1 + \frac{\mu H\hat{\gamma}_0}{2\omega_k^2}\right)dt + \varepsilon\right\} \overset{\circ}{u}\left(\frac{k}{a(\tau')}, l\right) + \overset{*}{\beta}{}_{kl}^h a(\tau')^{1/2}\left\{\exp i\int_{\tau'}^t \omega_k\left(1 + \frac{\mu H\hat{\gamma}_0}{2\omega_k^2}\right)dt + \varepsilon\right\} \overset{\circ}{v}\left(-\frac{k}{a(0')}, l\right).$$

In order to compute β_{kl}^h, let us take $t = \tau' = 0$, so we have:

$$(7.2) \quad a(\tau)^{1/2}\left\{\exp i\int_0^\tau \omega_k\left(1 + \frac{\mu H\hat{\gamma}_0}{2\omega_k^2}\right)dt + \varepsilon\right\} \overset{\circ}{u}\left(\frac{k}{a(\tau)}, h\right) = \sum_l \alpha_{kl}^h a(0)^{1/2} \overset{\circ}{u}\left(\frac{k}{a(0)}, l\right) + \overset{*}{\beta}{}_{kl}^h a(0)^{1/2} \overset{\circ}{v}\left(-\frac{k}{a(0)}, l\right)$$

as $\overset{\circ}{u}$ and $\overset{\circ}{v}$ are orthonormal, cf. (5.23), therefore if we pre multiply by $\overset{\circ}{\bar{v}}\left(-\frac{k}{a(0)}, l\right)\hat{\gamma}_0$ we have

$$(7.3)\quad \overset{\circ}{\bar{v}}\left(-\frac{k}{a(0)}, l\right)\hat{\gamma}_0 a(\tau)^{1/2}\left\{\exp -i\int_0^\tau \omega_k\left(1+\frac{\mu H\hat{\gamma}_0}{2\omega_k^{1/2}}\right)dt+\varepsilon\right\}\overset{\circ}{u}\left(\frac{k}{a(\tau)}, h\right) = \overset{*}{\beta}{}_{kl}^{h} i\frac{\omega_k(0)}{\mu}a(0)^{1/2}\ (^8).$$

In a similar way we can have the α.

Therefore we have:

$$(7.4)\quad \overset{*}{\beta}{}_{kl}^{h} = -i\frac{\mu}{\omega_k(0)}\overset{\circ}{\bar{v}}\left(-\frac{k}{a(0)}, l\right)\hat{\gamma}_0\left(\frac{a(\tau)}{a(0)}\right)^{1/2} \times\left[\exp i\int_0^\tau \omega_k\left(1+\frac{\mu H\hat{\gamma}_0}{2\omega_k^2}\right)dt+\varepsilon\right]\overset{\circ}{u}\left(\frac{k}{a(\tau)}, h\right).$$

As we are really only interested in $|\beta_{kl}^h|^2$ we can neglect the modulus 1 scalar factor $\exp i\int_0^\tau \omega_k dt$, and we can also neglect the $0(k^{-3})$ term ε as usual, therefore we have

$$(7.5)\quad \beta_{kl}^{*h} = -\frac{i\mu}{\omega_k(0)}\bar{v}^0\left(-\frac{k}{a(0)}, l\right)\hat{\gamma}_0\left(\frac{a(\tau)}{a(0)}\right)^{1/2} \times\left(\exp\frac{i\mu\hat{\gamma}_0}{2}\int_0^\tau \frac{H}{\omega_k}dt\right)\overset{\circ}{u}\left(\frac{k}{a(0)}, h\right).$$

Now we can compute:

$$(7.6)\quad \int_0^\tau \frac{H dt}{\omega_k} = \frac{1}{k}\int_{a(0)}^{a(\tau)}\frac{da}{\left(1+\frac{a^2\mu^2}{k^2}\right)^{1/2}} = \frac{1}{\mu}\log\frac{\frac{\mu}{k}a(\tau)+\left(1+\frac{\mu^2}{k^2}a(\tau)^2\right)^{1/2}}{\frac{\mu}{k}a(0)+\left(1+\frac{\mu^2}{k^2}a(0)^2\right)^{1/2}}.$$

As $i\hat{\gamma}_0$ component are ∓ 1 [cf. (1.23)] ($-$ for the two first components $a = 1, 2$, $+$ for the other two $a = 1, 2$) we have that the matrix

$$\exp\frac{i\mu\hat{\gamma}_0}{2}\int_0^\tau \frac{H}{\omega_k}dt$$

(8) Where $\omega_k(0) = (\mu^2 + k^2/a^2(0))^{1/2}$.

is a diagonal matrix with diagonal elements

$$\left[\frac{\frac{\mu}{k}a(\tau) + \left(1 + \frac{\mu^2}{k^2}a(\tau)^2\right)^{1/2}}{\frac{\mu}{k}a(0) + \left(1 + \frac{\mu^2}{k^2}a(0)^2\right)^{1/2}}\right]^{\mp\frac{1}{2}}$$

with $-$ for the two first elements and $+$ for the others two.

As we are interested in the case $k \to \infty$ we may expand the last formula in powers of k^{-1} and we obtain:

$$1 \mp \frac{1}{2}\frac{a(\tau) - a(0)}{k}\mu + \frac{1}{8}\frac{(a(\tau) - a(0))^2}{k^2}\mu^2 + \dots$$

Now we can compute:

$$(7.6) \quad \beta_{k\,+}^{*\,+} = \frac{\mu}{\omega_k(0)}\left(\frac{a(\tau)}{a(0)}\right)^{1/2}\left(\frac{\omega_k(0) + \mu}{2\mu}\right)^{1/2}\left(\frac{\omega_k(\tau) + \mu}{2\mu}\right)^{1/2}$$
$$\times\left(\frac{-k_3}{a(0)(\omega_k(0) + \mu)}, \frac{-k_1 + ik_2}{a(0)(\omega_k(0) + \mu)}, 1, 0\right)$$
$$\times\left(\mathrm{I} + \frac{1}{2}\frac{a(\tau) - a(0)}{k}\mu i\hat{\gamma}_0 + \frac{1}{2}\frac{(a(\tau) - a(0))^2}{k^2}\mu^2 + \dots\right)$$
$$\times\left(1, 0, \frac{k_3}{a(\tau)(\omega_k(\tau) + \mu)}, \frac{k_1 + ik_2}{a(\tau)(\omega_k(\tau) + \mu)}\right)$$

we can also expand:

$$(7.7) \qquad \frac{1}{a(\omega_k + \mu)} = \frac{1}{k}\left(1 - \frac{a\mu}{k} \pm \frac{1}{2}\frac{a^2\mu^2}{k^2} + \dots\right).$$

Substituting in (7.6) we have

$$(7.8) \qquad \beta_{k\,+}^{*\,+} = 0(k^{-3}).$$

The same thing happens for the other β_{kl}^{h}. Turning back to equation (3.70) we have that

$$(7.9) \quad \int(|\beta_{kh}^{+}|^2 + |\beta_{kh}^{-}|^2)d^3k$$
$$= 4\pi\int_0^\infty 0(k^{-6})k^2dk = 4\pi\int_0^\infty 0(k^{-4})dk < +\infty,$$

and therefore the transformation for one Cauchy surface to the other is implementable, and the density of particles created is finite.

8. THE ENERGY

We can obtain the symmetric momentum energy tensor of the field from [8] (equations (14.6) and (13.3)), and we have

$$\theta_{00} = \frac{1}{2}\,[(\nabla_0\overline{\psi})\gamma_0\psi - \overline{\psi}\gamma_0\nabla_0\psi]\,, \tag{8.1}$$

but $\nabla_0\psi = \partial_0\psi + \sigma_0\psi = \partial_0\psi$ because $\sigma_0 = 0$ so

$$\mathscr{H} = \int \theta_{00} d\sigma = \frac{1}{2}\int [\partial_0\overline{\psi}\gamma_0\psi - \overline{\psi}\gamma_0\partial_0\psi] d\sigma\,. \tag{8.2}$$

Of course this $\mathscr{H}$ is not a constant with time and we must compute it at each Cauchy surface.

We can do so because we have

$$\psi = \sum_h \int [b_{\boldsymbol{k}}^{*(h)} \mathrm{U}_{\boldsymbol{k}}^{(+,h)} + a_{\boldsymbol{k}}^{(h)} \mathrm{U}_{\boldsymbol{k}}^{(-,h)}] d^3\boldsymbol{k}\,, \tag{8.3}$$

Then we can compute $\mathscr{H}$ at e. g. $t = 0$ where

$$\begin{aligned}
\mathrm{U}_{\boldsymbol{k}}^{(+,h)} &= \frac{1}{(2\pi a)^{3/2}}\left(\frac{\mu}{\omega_k}\right)^{1/2}\left(\mathrm{I} - \frac{\mu\mathrm{H}\hat{\gamma}_0}{4\omega_k^2}\right)v^0\left(\frac{k}{a}, h\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \\
\mathrm{U}_{\boldsymbol{k}}^{(-,h)} &= \frac{1}{(2\pi a)^{3/2}}\left(\frac{\mu}{\omega_k}\right)^{1/2}\left(\mathrm{I} - \frac{\mu\mathrm{H}\hat{\gamma}_0}{4\omega_k^2}\right)u^0\left(\frac{k}{a}, h\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \\
\frac{\partial \mathrm{U}_{\boldsymbol{k}}^{(+,h)}}{\partial t} &= \frac{i}{(2\pi a)^{3/2}}(\mu\omega_k)^{1/2}\left(\mathrm{I} + \frac{\mu\mathrm{H}\hat{\gamma}_0}{4\omega_k^2}\right)\mathring{v}\left(\frac{k}{a}, h\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \\
\frac{\partial \mathrm{U}_{\boldsymbol{k}}^{(-,h)}}{\partial t} &= \frac{i}{(2\pi a)^{3/2}}(\mu\omega_k)^{1/2}\left(\mathrm{I} + \frac{\mu\mathrm{H}\hat{\gamma}_0}{4\omega_k^2}\right)\mathring{u}\left(\frac{k}{a}, h\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,.
\end{aligned} \tag{8.4}$$

as we can see from eq. (6.21).

Therefore neglecting the term in $0(k^{-1})$ and using the orthonormality properties of the $\mathring{u}$ and $\mathring{v}$ (5.23) and (5.24) we have

$$\begin{aligned}
\int \overline{\mathrm{U}}_{\boldsymbol{k}}^{(+,h)}\gamma_0\partial_0\mathrm{U}_{\boldsymbol{k}}^{(+,h')}d\sigma &= \left(-\,\omega_k + i\,\frac{\mu^2\mathrm{H}}{2\omega_k^2}\right)\delta_{hh'}\,, \\
\int \overline{\mathrm{U}}_{\boldsymbol{k}}^{(-,h)}\gamma_0\partial_0\mathrm{U}_{\boldsymbol{k}}^{(-,h')}d\sigma &= \left(-\,\omega_k - i\,\frac{\mu^2\mathrm{H}}{2\omega_k^2}\right)\delta_{hh'}\,, \\
\int \partial_0\overline{\mathrm{U}}_{\boldsymbol{k}}^{(+,h)}\gamma_0\mathrm{U}_{\boldsymbol{k}}^{(+,h')}d\sigma &= \left(\omega_k + i\,\frac{\mu^2\mathrm{H}}{2\omega_k^2}\right)\delta_{hh'}\,, \\
\int \partial_0\overline{\mathrm{U}}_{\boldsymbol{k}}^{(-,h)}\gamma_0\mathrm{U}_{\boldsymbol{k}}^{(-,h')}d\sigma &= \left(\omega_k - i\,\frac{\mu^2\mathrm{H}}{2\omega_k^2}\right)\delta_{hh'}\,.
\end{aligned} \tag{8.5}$$

Substituting (8.3) and (8.5) in (8.2) we have

$$\mathcal{H} = \int d^3k \sum_h \omega_k \left(b_k^{*(h)} b_k^{(h)} + a_k^{*(h)} a_k^{(h)}\right) \tag{8.6}$$
$$= \int d^3k \omega_k \left(\mathrm{N}_k^{(+)} + \mathrm{N}_k^{(-)} + \tilde{\mathrm{N}}_k^{(+)} + \tilde{\mathrm{N}}_k^{(-)}\right)$$

i. e. the usual expansion of the energy.

If we take the vacuum as the initial state, as we do in equation (3.69) all the N behave like $0(k^{-6})$ (cf. (7.9)) so the integrand behaves like $0(k^{-3})$ and the energy is convergent.

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