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J. GOMIS
J. A. LOBO
J. M. PONS

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Section A :
Physique théorique.

A singular Lagrangian model for two interacting relativistic particles

by

J. GOMIS, J. A. LOBO and J. M. PONS
Departament de Física Teòrica (*), Universitat de Barcelona

ABSTRACT. — A singular Lagrangian with multiplicative potential for two relativistic particles is proposed and the Lagrangian equations of motion are analysed. The connexion with other models also based upon singular Lagrangians for two particles is studied; as a result of this analysis, it is shown how the one-to-one relation between the Hamiltonian and Lagrangian formalism is recovered in this kind of models. Finally a discussion of the world-line invariance of the trajectories is given.

RÉSUMÉ. — Nous proposons un Lagrangien singulier avec un potentiel multiplicatif pour deux particules relativistes et nous analysons ses équations du mouvement. Nous étudions la connexion avec d'autres modèles basés aussi sur des lagrangiens singuliers pour deux particules. Comme résultat de cette analyse nous montrons comment est retrouvée la relation biunivoque entre les formalismes lagrangien et hamiltonien. Nous présentons à la fin, une discussion de l'invariance des « lignes d'univers » des trajectoires.

1. INTRODUCTION

The problem of the dynamics of a relativistic system of interacting particles has received attention from several authors [*1*]-[*11*] during the last years. Part of the works written by these authors are based on the

(*) Postal address: Diagonal, 647, Barcelona-28, Spain, may 1980.

theory of constrained systems originally developed by Dirac [12]. In a theory of this kind there are first-class and second-class constraints, their presence providing a simple method to avoid the consequences of the zero-interaction theorem, because the position variables are not canonical with respect to Dirac brackets.

The construction of explicit models depends on the choice of the set of constraints, and this can be made in several ways. For example, Komar [16] and Todorov [17] have proposed one in which there is a first-class constraint associated with each particle and no second-class constraints. The presence of the former guarantees that we can perform N independent non-trivial reparametrizations on the solution of the equations of motion. Mukunda [18], however, has shown that these models are dynamically incomplete, for they can only describe world-sheets instead of world-lines; one is then forced to add new constraints—which cannot be interpreted as gauge constraints (they define different dynamical models for different choices). The existence of these new constraints breaks the independent reparametrization invariance of each world-line except for a global one.

There are also models [4]-[7] in which the equations of motion are derived from singular Lagrangians. They carry the interaction on an action-at-a-distance scalar multiplicative potential. The Hamiltonian set of constraints [12] for the two-body case includes one first-class and two second-class constraints; one of the latter is $\langle P, r \rangle = 0$, P^μ being the total four-momentum of the system and r^μ the relative four-separation of the particles. This is always welcome since it is required to eliminate the relative time. It also allows for the exclusion of some unphysical states which appear upon quantization of the model.

On the other hand, if one uses $\langle P, r \rangle = 0$ to complete the Komar-Todorov models for two particles, one obtains the same equations of motion found in the Lagrangian DGL [6] model, which equations also coincide with those given by the multi-temporal model of Droz-Vincent [19], as shown in ref. [20] (there, however, the meaning of the constraint $\langle P, r \rangle = 0$ is qualitatively different), thence following that [19] is the exact predictive extension of [6].

The difference between the models of refs [4] and [5] and those of [6] and [7] is provided by the primary or secondary character of the above mentioned constraint [12]. In this paper we deepen into this difference and demonstrate that there is just one more Lagrangian function which describes the same physical system (for equal masses). In Section II we perform this demonstration and also derive the explicit form of the desired Lagrangian. Section III is devoted to the analysis of the (Lagrangian) constraints and the equations of motion. In Section IV we make a synthesis of the characteristics of all three two-body Lagrangians and see that they reduce to the same function when restricted to the surface of phase-space determined by the constraints, thus recovering the one-to-one classical correspondence between the Lagrangian and Hamiltonian formulations of the dynamical problem. Section V is devoted to discuss the world-line invariance and, in particular, to put the equations of motion given by the new Lagrangian for a constant potential, into three-dimensional instantaneous form. This corresponds to free motion of each particle. Unlike in refs [*5*] and [*6*], we are not allowed to reparametrize independently both world lines, because we must keep the constraint $\langle P, r \rangle = 0$ in this case. Nevertheless, we are still able to construct the trajectories and it, we hope, helps to understand the meaning of the constraints. In Section VI we draw our conclusions and an outlook and devote an appendix to the analysis of the Kalb-van Alstine Lagrangian [*4*] [*5*].

Notation

$\approx$ weak equality sign, i. e., $A \approx B$ means $A = B$ when the constraints are verified

$\langle\, , \rangle$ scalar product: $\langle A, B \rangle = A^\mu B_\mu = A^0 B^0 - \vec{A}.\vec{B}$

$\underset{\sim}{A}^\mu$ vector orthogonal to r^μ: $\underset{\sim}{A}^\mu = A^\mu - \dfrac{\langle A, r \rangle}{r^2} r^\mu$

$\{\, , \}$ Poisson bracket:

$$\{ A, B \} = - \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial P_\mu} + \frac{\partial A}{\partial P_\mu}\frac{\partial B}{\partial x^\mu} - \frac{\partial A}{\partial r^\mu}\frac{\partial B}{\partial q_\mu} + \frac{\partial A}{\partial q_\mu}\frac{\partial B}{\partial r^\mu}$$

x, P and r, q are canonical variables i. e.

$$\{ x^\mu, P^\nu \} = \{ r^\mu, g^\nu \} = - g^{\mu\nu}$$

the rest vanishing.

$$P^\mu = p_1{}^\mu + p_2{}^\mu ; \quad q^\mu = \frac{1}{2}(q_2{}^\mu - q_1{}^\mu) ; \quad x^\mu = \frac{1}{2}(x_1{}^\mu + x_2{}^\mu) ; \quad r^\mu = x_2{}^\mu - x_1{}^\mu .$$

2. THE LAGRANGIAN

Some of the existing models [*4*]-[*7*] which describe two interacting relativistic particles through a singular Lagrangian with a multiplicative scalar potential, have—when the masses of the particles are equal—the following set of stable hamiltonian constraints

$$\phi_0 \equiv P^2 + 4q^2 + V(r^2) \approx 0 \tag{2.1a}$$

$$\phi_1 \equiv \langle P, r \rangle \approx 0 \tag{2.1b}$$

$$\phi_2 \equiv \langle P, q \rangle \approx 0 \tag{2.1c}$$

Among the constraints (2.1) there exist the Poisson-bracket relations

$$\{\phi_0, \phi_1\} = -8\phi_2 \approx 0 ; \quad \{\phi_0, \phi_2\} = 2V'\phi_1 \approx 0 ; \quad \left(V' \equiv \frac{dV}{dr^2}\right) \tag{2.2a}$$

$$\{\phi_1, \phi_2\} = P^2 \tag{2.2b}$$

whence it is seen that ϕ_0 is a first class constraint [3] whereas ϕ_1, ϕ_2 are second class unless $P^2 = 0$, which case will not be considered here. According to Dirac theory (see [12]) the Hamiltonian is then

$$H = \lambda_0\phi_0 \tag{2.3}$$

provided the canonical hamiltonian is zero. We assume it is zero because our theory must be invariant under reparametrization.

It is not difficult to verify that the only primary hamiltonians which generate all the constraints (2.1) are

a) $$H = \lambda_0\phi_0 + \lambda_1\phi_1 \tag{2.4a}$$

b) $$H = \lambda_0\phi_0 + \lambda_2\phi_2 \tag{2.4b}$$

c) $$H = \lambda_0\phi_0 + \lambda_1\phi_1 + \lambda_2\phi_2 \tag{2.4c}$$

λ_0, λ_1, λ_2 being undeterminate functions. The models of refs. [4] [5] and [6] [7] only differ from each other in which of the constraints (2.1) are primary or, in other words, in which of (2.4) is the primary hamiltonian. In [4] [5] it is (2.4*a*) and in [6], it is (2.4*b*).

In this section we are going to investigate which is the Lagrangian model that yields (2.4*c*) as the primary Hamiltonian. We find this interesting because the answer to this question will exhaust the Lagrangian treatment of the two-particle model characterized by (2.1). This, we hope, will throw light onto the comprehension of the constrained systems in which, as we see, there is not a one-to-one correspondence between the Lagrangian and Hamiltonian formulations, as is the case in Classical Newtonian Mechanics.

Let us then look for a Lagrangian having (2.4*c*) as identical relations once the momenta

$$P_\mu = -\frac{\partial \mathscr{L}}{\partial \dot{x}^\mu}, \qquad q_\mu = -\frac{\partial \mathscr{L}}{\partial \dot{r}^\mu} \tag{2.5}$$

are defined. In order to achieve this, we first express the momenta P, q as functions of coordinates and velocities, what can be done after the latter are defined by means of the Hamiltonian equations of motion:

$$\dot{x}^\mu \approx \{x^\mu, H\} \approx -2\lambda_0 P^\mu - \lambda_1 r^\mu - \lambda_2 q^\mu \tag{2.6a}$$

$$\dot{r}^\mu \approx \{r^\mu, H\} \approx -8\lambda_0 q^\mu - \lambda_2 P^\mu \tag{2.6b}$$

According to reparametrization invariance and Euler's theorem it is verified that

$$\mathscr{L} = \dot{x}^\mu \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} + \dot{r}^\mu \frac{\partial \mathscr{L}}{\partial \dot{r}^\mu} = - \langle P, \dot{x} \rangle - \langle q, \dot{r} \rangle \tag{2.7}$$

Now, making use of (2.6) and (2.1)

$$\mathscr{L} = - 2\lambda_0 V(r^2) \tag{2.8}$$

To find $\mathscr{L}$ it is necessary and sufficient to find λ_0. But this can be done because λ_0 is inserted in a system of 11 independent algebraic equations, namely (2.1) and (2.6), for 11 unknowns, namely, P^μ, q^μ, λ_0, λ_1, λ_2 all of them to be expressed in terms of coordinates and velocities. From (2.6) it follows

$$P^\mu = - \frac{1}{2\lambda_0(4 - v^2)} [4(\dot{x}^\mu + \lambda_1 r^\mu) - v\dot{r}^\mu] \tag{2.9a}$$

$$q^\mu = - \frac{1}{2\lambda_0(4 - v^2)} [\dot{r}^\mu - v(\dot{x}^\mu + \lambda_1 r^\mu)] \tag{2.9b}$$

where it has been defined

$$v \equiv \frac{\lambda_2}{2\lambda_0}$$

Imposing (2.1*b*) and (2.1*c*) on (2.9) we obtain a system of two equations for λ_1 and v

$$\lambda_1 = \frac{\langle \dot{r}, r \rangle}{4r^2} v - \frac{\langle \dot{x}, r \rangle}{r^2}$$

$$\langle \dot{\underset{\sim}{x}}, \dot{\underset{\sim}{r}} \rangle v^2 - (4\dot{\underset{\sim}{x}}^2 + \dot{\underset{\sim}{r}}^2)v + 4 \langle \dot{\underset{\sim}{x}}, \dot{\underset{\sim}{r}} \rangle = 0$$

whose solutions are

$$v = 2\frac{a - b}{a + b}, \qquad \lambda_1 = - \frac{1}{r^2} \frac{ab}{a + b} \left\langle \frac{\dot{x}_1}{a} + \frac{\dot{x}_2}{b}, r \right\rangle \tag{2.10a}$$

and

$$v = 2\frac{a + b}{a - b}, \qquad \lambda_1 = \frac{1}{r^2} \frac{ab}{a - b} \left\langle \frac{\dot{x}_1}{a} - \frac{\dot{x}_2}{b}, r \right\rangle \tag{2.10b}$$

where

$$a \equiv + \sqrt{\dot{\underset{\sim}{x}}_1^2}, \qquad b \equiv + \sqrt{\dot{\underset{\sim}{x}}_2^2} \tag{2.11}$$

Let us observe, however, that (2.10*b*) are not physically consistent solutions. Indeed λ_1 and v must both be zero during the motion because the primary hamiltonian (2.4*c*) becomes the first class hamiltonian (2.3) after consistency conditions are imposed on it. We are therefore led to reject (2.10*b*) because v can never be zero in that case. On the other hand, $\lambda_1 = v = 0$ must be Lagrangian constraints.

After substitution of (2.10a) into (2.9) and (2.1c) we find λ_0 and thence $\mathscr{L}$ by (2.8):

$$\mathscr{L} = \pm \frac{1}{2}\sqrt{-\mathrm{V}(r^2)}\left[(a+b)^2 + \frac{\langle \dot{r}, r\rangle^2}{r^2}\right]^{1/2} \tag{2.12}$$

There is a double-sign ambiguity in this solution which does not matter either for the equations of motion or the lagrangian constraints. We take then conventionally the minus sign in order that the extremal of the action be a minimum.

3. ÉQUATIONS OF MOTION

In this section we start from the Lagrangian

$$\mathscr{L} = -\frac{1}{2}\sqrt{-\mathrm{V}(r^2)}\left[(a+b)^2 + \frac{\langle \dot{r}, r\rangle^2}{r^2}\right]^{1/2} \tag{3.1}$$

The canonical momenta are

$$p_{1\mu} = -\frac{\partial \mathscr{L}}{\partial \dot{x}_1{}^{\mu}} = \frac{1}{4}\frac{\mathrm{V}}{\mathscr{L}}\left[\left(1+\frac{b}{a}\right)\dot{\underset{\sim}{x}}_{1\mu} - \frac{\langle \dot{r}, r\rangle}{r^2} r_\mu\right] \tag{3.2a}$$

$$p_{2\mu} = -\frac{\partial \mathscr{L}}{\partial \dot{x}_2{}^{\mu}} = \frac{1}{4}\frac{\mathrm{V}}{\mathscr{L}}\left[\left(1+\frac{a}{b}\right)\dot{\underset{\sim}{x}}_{2\mu} + \frac{\langle \dot{r}, r\rangle}{r^2} r_\mu\right] \tag{3.2b}$$

whence it follows

$$\mathrm{P}_\mu = \frac{1}{4}\frac{\mathrm{V}}{\mathscr{L}}(a+b)\left(\frac{\dot{\underset{\sim}{x}}_{1\mu}}{a} + \frac{\dot{\underset{\sim}{x}}_{2\mu}}{b}\right) \tag{3.3a}$$

$$q_\mu = \frac{1}{8}\frac{\mathrm{V}}{\mathscr{L}}(a+b)\left(\frac{\dot{\underset{\sim}{x}}_{2\mu}}{b} - \frac{\dot{\underset{\sim}{x}}_{1\mu}}{a}\right) + \frac{1}{4}\frac{\mathrm{V}}{\mathscr{L}}\frac{\langle \dot{r}, r\rangle}{r^2} r_\mu \tag{3.3b}$$

It is immediately verified that eqs. (2.1) are identically satisfied. In order to find the Lagrangian constraints we must evaluate the hessian matrix (with respect to $\dot{x}_1$, $\dot{x}_2$) of (3.1). According to the general theory [*14*] it will be a singular matrix with three null vectors corresponding to the three Hamiltonian primary constraints. After some algebra one finds.

$$\mathrm{H}_{\mu\nu ij} \equiv \frac{\partial^2 \mathscr{L}}{\partial \dot{x}_i{}^{\mu} \partial \dot{x}_j{}^{\nu}} = -\frac{1}{4}\frac{\mathrm{V}}{\mathscr{L}}\begin{pmatrix} \mathrm{Q}_{1\mu\nu} + \frac{b}{a}\mathrm{R}_{1\mu\nu} & \mathrm{M}_{\mu\nu} \\ (\mathrm{M}^{\mathrm{T}})_{\mu\nu} & \mathrm{Q}_{2\mu\nu} + \frac{a}{b}\mathrm{R}_{2\mu\nu} \end{pmatrix} \tag{3.4}$$

where Q_i, R_i, M are the following tensors

$$Q_i^{\mu\nu} = g^{\mu\nu} - \frac{p_i^{\mu} p_i^{\nu}}{p_i^2}, \qquad R_i^{\mu\nu} = G^{\mu\nu} - \frac{\dot{\underset{\sim}{x}}_i^{\mu} \dot{\underset{\sim}{x}}_i^{\nu}}{\dot{x}_i^2}, \qquad (i = 1, 2) \tag{3.5}$$

$$M^{\mu\nu} = -\frac{p_1^{\mu} p_2^{\nu}}{p_1^2} + \frac{\dot{\underset{\sim}{x}}_1^{\mu} \dot{\underset{\sim}{x}}_2^{\nu}}{ab} - \frac{r^{\mu} r^{\nu}}{r^2}, \qquad (M^T)^{\mu\nu} = M^{\nu\mu} \tag{3.6}$$

with

$$G^{\mu\nu} = g^{\mu\nu} - \frac{r^{\mu} r^{\nu}}{r^2} \tag{3.7}$$

Q_i is a projector orthogonal to p_i; R_i is also a projector orthogonal to $\dot{\underset{\sim}{x}}_i$ and r, so that it is contained in Q_i owing to (3.2), i. e. $Q_i R_i = R_i Q_i = R_i$; M is not a projector, nor $\mathscr{H}$ is; it can be, however, written so:

$$\mathscr{H} = -\frac{1}{4}\frac{V}{\mathscr{L}}(\mathscr{P} - \mathscr{B}) \tag{3.8}$$

where

$$\mathscr{P} = \begin{pmatrix} Q_1 + R_1 & M \\ M^T & Q_2 + R_2 \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} \left(1 - \frac{b}{a}\right) R_1 & 0 \\ 0 & \left(1 - \frac{a}{b}\right) R_2 \end{pmatrix} \tag{3.9}$$

It is verified that $\mathscr{P}^2 = 2\mathscr{P}$ so that $\frac{1}{2}\mathscr{P}$ is a projector. $\mathscr{B}$ is not itself a projector but is a direct sum of two « except-for-a-factor-projectors ».

The null vectors of $\mathscr{B}$ are

$$v_1 = \begin{pmatrix} p_1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ p_2 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} r \\ r \end{pmatrix} \tag{3.10}$$

and those of $\mathscr{B}$ are v_1, v_2, $\begin{pmatrix} r \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ r \end{pmatrix}$; therefore the null vectors of $\mathscr{H}$ are simply (3.10).

The equations of motion may be written in the form

$$\sum_{j=1}^{2} \mathscr{H}_{\mu\nu ij} \ddot{x}_j^{\nu} = -\frac{1}{4}\frac{V}{\mathscr{L}} \alpha_{i\mu}, \qquad (i = 1, 2) \tag{3.11}$$

where

$$-\frac{1}{4}\frac{V}{\mathscr{L}} \alpha_{i\mu} \equiv \frac{\partial \mathscr{L}}{\partial x_i^{\mu}} - \sum_{j=1}^{2} \frac{\partial^2 \mathscr{L}}{\partial \dot{x}_i^{\mu} \partial x_j^{\nu}} \dot{x}_j^{\nu}, \qquad (i = 1, 2) \tag{3.12}$$

The calculation of α_i is rather tedious. The final result is

$$\alpha_1{}^\mu = \left[\langle \dot{r}, r \rangle \Gamma_1^2 - \frac{k}{a}\left(1 - \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{a^2}\right)\right] \dot{\underset{\sim}{x}}_1{}^\mu - k\left(\frac{1}{a} + \frac{1}{b}\right)\dot{\underset{\sim}{x}}_2{}^\mu - a(a+b)\Gamma_1^2 r^\mu \quad (3.13a)$$

$$\alpha_2{}^\mu = \left[\langle \dot{r}, r \rangle \Gamma_2^2 + \frac{k}{b}\left(1 - \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{b^2}\right)\right] \dot{\underset{\sim}{x}}_2{}^\mu + k\left(\frac{1}{a} + \frac{1}{b}\right)\dot{\underset{\sim}{x}}_1{}^\mu + b(a+b)\Gamma_2^2 r^\mu \quad (3.13b)$$

being

$$k = \frac{1}{r^2}(\langle \dot{x}_2, r \rangle a + \langle \dot{x}_1, r \rangle b) \quad (3.14)$$

and

$$\Gamma_1^2 = \frac{1}{r^2}\left(1 - \frac{b}{a}\right)\left(1 + \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{ab}\right)\frac{\mathrm{V}}{4\mathscr{L}^2}\left[b(a+b) - \frac{\langle \dot{r}_1, r \rangle \langle \dot{x}_2, r \rangle}{r^2}\right] - \left(1 + \frac{b}{a}\right)\frac{\mathrm{V}'}{\mathrm{V}} \quad (3.15a)$$

$$\Gamma_2^2 = \frac{1}{r^2}\left(1 - \frac{a}{b}\right)\left(1 + \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{ab}\right)\frac{\mathrm{V}}{4\mathscr{L}^2}\left[a(a+b) + \frac{\langle \dot{x}_1, r \rangle \langle \dot{r}_2, r \rangle}{r^2}\right] - \left(1 + \frac{a}{b}\right)\frac{\mathrm{V}'}{\mathrm{V}} \quad (3.15b)$$

Now there must be some Lagrangian constraints expressing the fact that the 8-vector $\alpha = \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}$ is orthogonal to the null vectors of $\mathscr{H}$ as is seen from (3.11). This yields the following relations

$$\langle \alpha, v_1 \rangle = \frac{1}{4}\frac{\mathrm{V}}{\mathscr{L}}(a+b)k\left(1 + \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{ab}\right) = 0 \quad (3.16a)$$

$$\langle \alpha, v_2 \rangle = -\langle \alpha, v_1 \rangle = 0 \quad (3.16b)$$

$$\langle \alpha, v_3 \rangle = -(a^2 - b^2)\left(1 + \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{ab}\right) = 0 \quad (3.16c)$$

of which only two are independent and are equivalent to

$$k\left(1 + \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{ab}\right) = 0 \quad (3.17a)$$

$$(a - b)\left(1 + \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{ab}\right) = 0 \quad (3.17b)$$

since $\frac{\mathrm{V}}{\mathscr{L}}(a + b)$ can never be zero. (3.17*a*) and (3.17*b*) can be made equal to zero in two different manners:

1)
$$1 + \frac{\langle \dot{\underset{\sim}{x}}_1, \dot{\underset{\sim}{x}}_2 \rangle}{ab} = 0 \tag{3.18}$$

This is equivalent to demand $\mathrm{P}^2 = 0$ according to (3.3*a*). We do not consider this case since we have assumed $\mathrm{P}^2 \neq 0$ from the outset.

2)
$$a = b \quad \text{and} \quad k = 0 \tag{3.19}$$

These two equations are written in a more explicit form as

$$\dot{x}_1^2 = \dot{x}_2^2, \qquad \langle \dot{x}_1 + \dot{x}_2, r \rangle = 0 \tag{3.20}$$

We recognize in these Lagrangian constraints the necessary conditions for the functions λ_1, v of (2.10*a*) to be null.

This is, therefore, a consistent result.

Let us then impose stability to (3.20); this implies that

$$\langle \ddot{x}_1 + \ddot{x}_2, r \rangle = 0, \qquad \langle \dot{x}_1, \ddot{x}_1 \rangle = \langle \dot{x}_2, \ddot{x}_2 \rangle \tag{3.21}$$

We see it is necessary to know $\ddot{x}_1$, $\ddot{x}_2$ to draw consequences from (3.21) This is fortunately an easy matter provided (3.19) is fulfilled. Indeed in this case the hessian (3.8) simply reduces to $\mathscr{H} = -\frac{1}{4}\frac{\mathrm{V}}{\mathscr{L}}\mathscr{P}$ and therefore the accelerations can be isolated in (3.11) due to the projector character of $\mathscr{P}$. The result is

$$\ddot{x}_1{}^\mu = \frac{1}{2}\alpha_1{}^\mu + \mu_1(\tau)p_1{}^\mu + \mu_3(\tau)r^\mu$$

$$\ddot{x}_2{}^\mu = \frac{1}{2}\alpha_2{}^\mu + \mu_2(\tau)p_2{}^\mu + \mu_3(\tau)r^\mu$$

Now imposing the constraints (3.19) on α_i, these equations take the form

$$\ddot{x}_1{}^\mu = -\left(\frac{\dot{\mathrm{V}}}{2\mathrm{V}} - \frac{\mathrm{V}}{2\mathscr{L}}\mu_1\right)\dot{x}_1{}^\mu + \left(2a^2\frac{\mathrm{V}'}{\mathrm{V}} + \frac{\mathrm{V}}{4\mathscr{L}}\frac{\langle \dot{r}, r \rangle}{r^2}\mu_1 + \mu_3\right)r^\mu \tag{3.22a}$$

$$\ddot{x}_2{}^\mu = -\left(\frac{\dot{\mathrm{V}}}{2\mathrm{V}} - \frac{\mathrm{V}}{2\mathscr{L}}\mu_2\right)\dot{x}_2{}^\mu - \left(2b^2\frac{\mathrm{V}'}{\mathrm{V}} + \frac{\mathrm{V}}{4\mathscr{L}}\frac{\langle \dot{r}, r \rangle}{r^2}\mu_2 - \mu_3\right)r^\mu \tag{3.22b}$$

and substitution of them into (3.21) yields the consistency conditions

$$\mu_1 = \mu_2, \qquad \mu_3 = \frac{1}{8}\frac{\mathrm{V}}{\mathscr{L}}\frac{\langle \dot{r}, r \rangle}{r^2}(\mu_2 - \mu_1) \tag{3.23}$$

which reduce to one the number of arbitrary functions present in the theory; this is a desired result. Redefining

$$\mu(\tau) = \frac{V}{2\mathcal{L}}\,\mu_1(\tau) - \frac{\dot{V}}{2V} \tag{3.24}$$

we obtain the following final equations of motion

$$\ddot{x}_1{}^\mu = 2\frac{\mathcal{L}^2}{V^2}\,V'(r^2)r^\mu + \mu(\tau)\dot{x}_1{}^\mu \tag{3.25a}$$

$$\ddot{x}_2{}^\mu = -\,2\frac{\mathcal{L}^2}{V^2}\,V'(r^2)r^\mu + \mu(\tau)\dot{x}_2{}^\mu \tag{3.25b}$$

These equations can also be written as

$$\ddot{x}^\mu = \mu(\tau)\dot{x}^\mu \tag{3.26a}$$

$$\ddot{r}^\mu = -\,4\frac{\mathcal{L}^2}{V^2}\,V'(r^2)r^\mu + \mu(\tau)\dot{r}^\mu \tag{3.26b}$$

Equations (3.25) or (3.26) together with equations (3.20), which can be as well written as

$$\langle\, \dot{x}, r \,\rangle = 0\,, \qquad \langle\, \dot{x}, \dot{r} \,\rangle = 0\,, \tag{3.27}$$

are the equations of motion of the system. It must be remembered, however, that eqs. (3.27) must only be imposed on the initial conditions and this ensures that they are verified for any other value of τ because of their stability conditions contained in (3.27) and the equations (3.26) themselves.

It is very easy to check that

$$\mu(\tau) = \frac{d}{d\tau}\ln\frac{\mathcal{L}}{V} \tag{3.28}$$

whence it follows the transformation law for the gauge function $\mu(\tau)$ after a change of parameter:

$$\tau \to \tau' = f(\tau) \Rightarrow \mu(\tau) \to \mu'(\tau') = \frac{\dot{f}(\tau)\mu(\tau) - \ddot{f}(\tau)}{\dot{f}^2(\tau)}\,. \tag{3.29}$$

(3.29) also shows explicitly that the eqs. of motion (3.25) or (3.26) are invariant under a change of parameter. The easiest choice of gauge is, of course $\mu(\tau) = 0$ in which case $\frac{\mathcal{L}}{V}$ is constant; let us call

$$\mu(\tau) = 0 \Rightarrow \frac{\mathcal{L}}{V} \equiv \frac{\gamma}{2} \qquad (\gamma = \text{const.}) \tag{3.30}$$

Eqs. (3.26) then read

$$\ddot{x}^{\mu} = 0\,, \qquad \ddot{r}^{\mu} + \gamma^2 \mathrm{V}'(r^2) r^{\mu} = 0 \tag{3.31}$$

and the first of them says that, in this gauge, τ is proportional to the center of mass proper time.

4. CONNECTION WITH OTHER MODELS

The Lagrangian deduced in section 2 exhauts, together with those of ref. [4] (henceforth referred to as KvA) and [6] (DGL), the set of possible Lagrangian functions that give rise to the Hamiltonian constraints (2.1). Each one of them is bound to one of the primary Hamiltonians of (2.4). Once all the stability conditions are imposed on the constraints the process ends up into the same first class Hamiltonian.

There is, at this stage, a one-to-one relation between Lagrangians and primary Hamiltonians. Both carry all the information about the dynamics of the problem as well as about the constraints to which the dynamical variables are subjected. Note, however, that, as pointed out above, all yield the same first class Hamiltonian, thus disappearing the one-to-one relation between the Lagrangian and the Hamiltonian formulation of the problem.

Now, we pose the question: is it possible to restore biunivocity between Lagrangians and first class Hamiltonians, at least on the surface of phase-space determined by the constraints? We indeed expect an affirmative answer; for the Lagrangian constraints are equivalent to making equal to zero those arbitrary functions in the primary Hamiltonian which are coefficients of the second-class constraints, as seen in section II. In this section we are going to show that the lost biunivocity is actually restored in this sense.

To this end let us first see that the surface determined by the Lagrangian constraints is the same in the three cases:

a) The Lagrangian (3.1) produces the constraints

$$\langle \dot{x}, r \rangle = 0\,, \qquad \langle \dot{x}, \dot{r} \rangle = 0 \tag{4.1}$$

according to (3.27).

b) The DGL Lagrangian

$$\mathscr{L}_{\mathrm{DGL}} = -\sqrt{m^2 - \mathrm{V}_{\mathrm{DGL}}(r^2)}\,\big(\sqrt{\dot{x}_1^2} + \sqrt{\dot{x}_2^2}\big) \tag{4.2}$$

gives rise to (see ref. [6])

$$\left\langle \frac{\dot{x}_1}{\sqrt{\dot{x}_1^2}} + \frac{\dot{x}_2}{\sqrt{\dot{x}_2^2}}, r \right\rangle = 0\,; \qquad \left\langle \frac{\dot{x}_1}{\sqrt{\dot{x}_1^2}} + \frac{\dot{x}_2}{\sqrt{\dot{x}_2^2}}, \dot{r} \right\rangle = 0 \tag{4.3}$$

The second of (4.3) can also be written

$$\left\langle \frac{\dot{x}_1}{\sqrt{\dot{x}_1^2}} + \frac{\dot{x}_2}{\sqrt{\dot{x}_2^2}}, \dot{r} \right\rangle = (\sqrt{\dot{x}_2^2} - \sqrt{\dot{x}_1^2})\left(1 + \frac{\langle \dot{x}_1, \dot{x}_2 \rangle}{\sqrt{\dot{x}_1^2}\sqrt{\dot{x}_2^2}}\right)$$

but the second factor is proportional to $P^2 = \left(\frac{\partial \mathscr{L}_{DGL}}{\partial \dot{x}_1} + \frac{\partial \mathscr{L}_{DGL}}{\partial \dot{x}_2}\right)^2$ and thence cannot vanish; so the second of eqs. (4.3) is equivalent to $\sqrt{\dot{x}_1^2} = \sqrt{\dot{x}_2^2}$; now replacing this into (4.3) again and making use of collective coordinates we find that it can be expressed by

$$\langle \dot{x}, r \rangle = 0, \qquad \langle \dot{x}, \dot{r} \rangle = 0 \tag{4.4}$$

just like (4.1).

c) The KvA Lagrangian

$$\mathscr{L}_{KvA} = -\ [-\ V_{KvA}(r^2)(\dot{\underset{\sim}{x}}^2 + \beta^{-2}\dot{r}^2)]^{1/2} \tag{4.5}$$

has exactly the constraints (4.1) as seen in formula (A.11) of the appendix.

We see therefore that the surface determined by the Lagrangian constraints coincides in all cases. We have to show now that all Lagrangians adopt the same functional form when restricted to the surface defined by the constraints. This is

$$\mathscr{F} = -\left[-\ U(r^2)\left(\dot{x}^2 + \frac{1}{4}\dot{r}^2\right)\right]^{1/2} \tag{4.6}$$

where $U(r^2)$ is a multiplicative potential conveniently related to the potentials in each Lagrangian function. Indeed:

a) The Lagrangian (3.1) becomes

$$\mathscr{L} = -\frac{1}{2}\sqrt{-V}\left[(\sqrt{\dot{\underset{\sim}{x}}_1^2} + \sqrt{\dot{\underset{\sim}{x}}_2^2})^2 + \frac{\langle \dot{r}, r \rangle^2}{r^2}\right]^{1/2} \xrightarrow[\langle \dot{x}_1 + \dot{x}_2, r \rangle = 0]{\dot{x}_1^2 = \dot{x}_2^2} \mathscr{F} \tag{4.7}$$

being $U(r^2) = V(r^2)$ simply in this case

b) $\mathscr{L}_{DGL} = -\sqrt{m^2 - V_{DGL}}\,(\sqrt{\dot{x}_1^2} + \sqrt{\dot{x}_2^2}) \xrightarrow[\langle \dot{x}_1 + \dot{x}_2, r \rangle = 0]{\dot{x}_1^2 = \dot{x}_2^2} \mathscr{F}$

by identifying

$$U(r^2) = 4(V_{DGL}(r^2) - m^2) \tag{4.8}$$

c) To recover (4.6) from $\mathscr{L}_{KvA}$ it is necessary to assume $\beta = 2$ in which case the result is obvious. This is not, however less general than $\beta \neq 2$. In fact one can always redefine—within this model—the relative coordinate by $2\beta^{-1}r^\mu = x_2{}^\mu - x_1{}^\mu$ due to the fact that in this case there is no definition of the individual coordinates, the only specification being that r^μ be a Poincaré four-vector, i. e., independent of the origin of coordinates.

We have completed the proof of the desired result: on the surface of the constraints the one-to-one relation between Lagrangian and Hamiltonian formulations is recovered, for all Lagrangian functions reduce to (4.6) when restricted to that surface. In a forthcoming paper we hope to give a generalization of these results to a problem of N interacting particles.

5. DISCUSSION OF WORLD-LINE INVARIANCE

The discussion of the world-line invariance of equations (3.26) for a non-constant potential is just that of the DGL-model for equal masses, already performed in the general case in refs. [*11*] and [*15*]: if t is the time variable for a certain Lorentz observer, it is possible to write the solution of the equations of motion in the form

$$\vec{x}_i(t) = \vec{\psi}_i(t - t_0 ; \vec{x}_j(t_0), \vec{v}_k(t_0)), \qquad (i, j, k = 1, 2) \tag{5.1}$$

where $\vec{x}_j(t_0)$ and $\vec{v}_k(t_0)$ are the positions and velocities of the two particles at the instant t_0 for the considered observer. The number of arbitrary constants in (5.1) is 13 whereas the solution of (3.25) has 16. There are, however, two Lagrangian constraints (3.27) which lower this number to 14. But there is still a gauge freedom which brings in a new (gauge) constraint, once that freedom is fixed, setting finally equal to 13 the number of independent arbitrary constants. The question of whether there is a one-to-one relation between them and the quantities $\vec{x}_j(t_0)$, $\vec{v}_k(t_0)$, t_0 has been analysed in [*11*].

The preceeding discussion can also be made directly in the four-dimensional form by constructing the predictive extension of the model. This has been done in [*20*], where it is shown that such predictive extension is provided by a Hamiltonian model developed by Droz-Vincent [*19*].

In the free case (V = cte), however, the model we have just proposed does not coincide with that of DGL, for in the latter there is a double reparametrization freedom—one per particle—, whereas in the former there remains only one. This is due to the fact that the constraint $\langle P, r \rangle = 0$ does not disappear in this model even when V = cte. Nevertheless, it is still possible to discuss the world-line invariance of the solutions of the free equations of motion and to put them in an explicity predictive three-dimensional form. This is what we are going to do in this section.

In the free case the equations of motion (3.25) are written

$$\ddot{x}_1{}^\mu = \ddot{x}_2{}^\mu = 0 \tag{5.2}$$

in the gauge $\mu = 0$. Their general solution is

$$x_1{}^\mu(\tau) = a_1{}^\mu \tau + b_1{}^\mu, \qquad x_2{}^\mu(\tau) = a_2{}^\mu \tau + b_2{}^\mu \tag{5.3}$$

The constraints (3.20) read here

$$a_1^2 = a_2^2\,, \qquad \langle a_1 + a_2, b_2 - b_1 \rangle = 0 \tag{5.4}$$

and there is also the gauge constraint (3.30); it leads to the new condition

$$a_1^2 = a_2^2 = -\frac{1}{4}\gamma^2 V_0$$

Letting $\gamma^2 = -\frac{4}{V_0}$ the final set of conditions on a_i, b_i is

$$a_1^2 = a_2^2 = 1\,, \qquad \langle a_1 + a_2, b_2 - b_1 \rangle = 0 \tag{5.5}$$

(5.5) also guarantees the velocities of the particles

$$v_i \equiv \frac{\dot{\vec{x}}_i}{\dot{x}_i^0} = \frac{\vec{a}_i}{\sqrt{1 + \vec{a}_i^{\,2}}} \qquad (i = 1, 2) \tag{5.6}$$

are time-like four-vectors as indeed they must be.

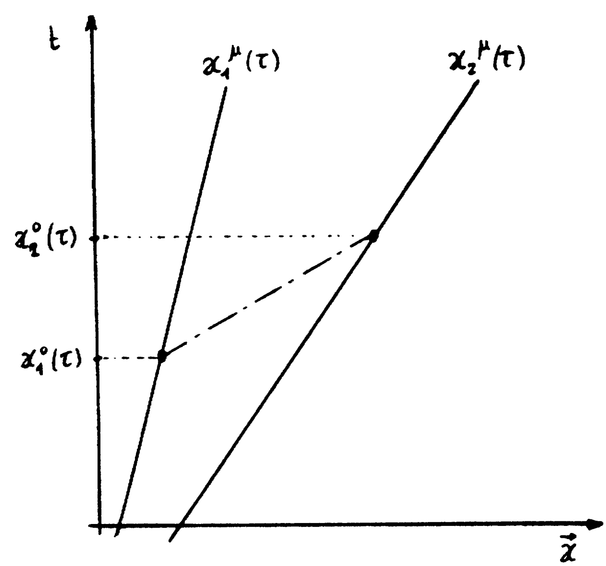
Let us take the following set of independent initial conditions:

$$x_1^0 = b_1^0 \equiv t_0\,, \quad \vec{x}_1(0) = \vec{b}_1\,, \quad \vec{x}_2(0) = \vec{b}_2\,, \quad \dot{\vec{x}}_1(0) = \vec{a}_1\,, \quad \dot{\vec{x}}_2(0) = \vec{a}_2 \tag{5.7}$$

The remaining three are then

$$x_2^0(0) = b_2^0 = t_0 - \frac{(\vec{a}_1 + \vec{a}_2)\cdot(\vec{b}_2 - \vec{b}_1)}{\sqrt{1 + \vec{a}_1^2} + \sqrt{1 + \vec{a}_2^2}}\,,$$

$$a_1^0 = \sqrt{1 + \vec{a}_1^2}\,, \qquad a_2^0 = \sqrt{1 + \vec{a}_2^2} \tag{5.8}$$



(5.7) and (5.8) are to be substituted in (5.3) to give the solution of the problem, but this does not still have the desired form (5.1). In order to achieve this the figure will illustrate the procedure: for a certain value of τ we have a point in each trajectory but it does not correspond to the same instant of time for the considered observer. However there exists a certain value τ' for which $t \equiv x_1^0(\tau) = x_2^0(\tau')$; so

$$\tau = \frac{t - b_1^0}{a_1^0} = \frac{t - t_0}{a_1^0}; \qquad \tau' = \frac{t - b_2^0}{a_2^0} \tag{5.9}$$

Now (5.3) can as well be written

$$x_1{}^\mu(\tau) = a_1{}^\mu \tau + b_1{}^\mu, \qquad x_2{}^\mu(\tau') = a_2{}^\mu \tau' + b_2{}^\mu$$

whence we find, after (5.9) (see also (5.6)-(5.8)),

$$\vec{x}_1(t) = (t - t_0)\vec{v}_1 + \vec{x}_1(t_0), \qquad \vec{x}_2(t) = (t - t_0)\vec{v}_2 + \vec{x}_2(t_0) \tag{5.10}$$

where

$$\vec{x}_1(t_0) = \vec{b}_1, \qquad \vec{x}_2(t_0) = \vec{b}_2 + \frac{(\vec{a}_1 + \vec{a}_2)\cdot(\vec{b}_2 - \vec{b}_1)}{\sqrt{1 + \vec{a}_1^2} + \sqrt{1 + \vec{a}_2^2}} \tag{5.11}$$

Equations (5.10) are in predictive form, and it is the expected one of free motion. But still, is there a one to one correspondence between the quanties $\vec{x}_i(t_0)$, $\vec{v}_i$ and $\vec{a}_i$, $\vec{b}_i$? Yes, and the inversion formulas for (5.6) and (5.11) are

$$\vec{a}_i = \gamma_i \vec{v}_i, \qquad \vec{b}_1 = \vec{x}_1(t_0), \qquad \vec{b}_2 = \vec{x}_2(t_0) + \frac{\vec{V}\cdot\vec{r}(t_0)}{1 - \vec{V}\cdot\vec{v}_2}\vec{v}_2 \tag{5.12}$$

with

$$\gamma_i = (1 - \vec{v}_i^2)^{-1/2}, \qquad \vec{V} = \frac{\gamma_1 \vec{v}_1 + \gamma_2 \vec{v}_2}{\gamma_1 + \gamma_2}, \qquad \vec{r}(t_0) = \vec{x}_2(t_0) - \vec{x}_1(t_0)$$

CONCLUSIONS

We have seen in this work how can one make a Hamiltonian description of a two-body relativistic system and how it may be achieved via three different singular Lagrangians, and only three. They are related to the three possible primary Hamiltonians which give rise to the same first-class Hamiltonian.

The existence of such variety of Lagrangians breaks the one-to-one correspondence between the canonical and the Lagrangian formulations of the same problem, always present in the classical case. Nevertheless biunivocity is recovered after restriction to the surface of phase space in which the motion takes place, i. e., that determined by the constraints.

This means that the Lagrangian description of the model can simply be made giving the Lagrangian function (4.7)

$$\mathcal{F} = - \sqrt{- \mathrm{U}(r^2)} \left(\dot{x}^2 + \frac{1}{4} \dot{r}^2 \right)^{1/2}$$

over the surface

$$\langle \dot{x}, r \rangle = 0 \,, \qquad \langle \dot{x}, \dot{r} \rangle = 0$$

This is a very interesting result since it immediately suggests a generalization to more than two particles. In a forthcoming paper we shall give a more detailed treatment of that problem.

The number of constraints is exactly the one needed to construct physical trajectories from an arbitrary given set of initial three-positions and three-velocities. In particular, we have explicitely shown in section IV how the equations of the free trajectories can be cast in such manifestly predictive form.

APPENDIX

LAGRANGIAN ANALYSIS OF THE KALB-VAN ALSTINE MODEL

The Kalb-van Alstine Lagrangian is [6]

$$\mathscr{L}_{\mathrm{KvA}} = \sqrt{-\mathrm{V}_{\mathrm{KvA}}(r^2)(\dot{\underset{\sim}{x}}^2 + \beta^{-2}\dot{r}^2)} \tag{A.1}$$

where β is a dimensionless constant. In this case

$$\mathrm{P}_\mu = -\frac{\partial \mathscr{L}_{\mathrm{KvA}}}{\partial \dot{x}^\mu} = \frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}_{\mathrm{KvA}}}\dot{\underset{\sim}{x}}_\mu\,; \qquad q_\mu = -\frac{\partial \mathscr{L}_{\mathrm{KvA}}}{\partial \dot{r}^\mu} = \beta^{-2}\frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}_{\mathrm{KvA}}}\dot{r}_\mu \tag{A.2}$$

$$\mathscr{H}_{\mu\nu ij} = -\frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}_{\mathrm{KvA}}}\begin{pmatrix} \mathrm{G}_{\mu\nu} + \dfrac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}^2_{\mathrm{KvA}}}\dot{\underset{\sim}{x}}_\mu \dot{\underset{\sim}{x}}_\nu & \beta^{-2}\dfrac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}^2_{\mathrm{KvA}}}\dot{\underset{\sim}{x}}_\mu \dot{r}_\mu \\ \beta^{-2}\dfrac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}^2_{\mathrm{KvA}}}\dot{r}_\mu \dot{\underset{\sim}{x}}_\nu & \beta^{-2}\left(g_{\mu\nu} + \beta^{-2}\dfrac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}^2_{\mathrm{KvA}}}\dot{r}_\mu \dot{r}_\nu\right) \end{pmatrix} \tag{A.3}$$

The hessian (A.3) has two null vectors

$$v_1 = \begin{pmatrix} \dot{x} \\ \dot{r} \end{pmatrix}, \qquad v_2 = \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{A.4}$$

and the following property: if we define π by

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}\left(-\frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}_{\mathrm{KvA}}}\mathscr{H}\right)\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \tag{A.5}$$

then π is a projector: $\pi^2 = \pi$, $\pi = \pi^{\mathrm{T}}$. The equations of motion are

$$\sum_j \mathscr{H}_{\mu\nu ij}\ddot{x}_j^{\,\nu} = -\frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}_{\mathrm{KvA}}}\mathrm{F}_{i\mu} \qquad (i = 1, 2) \tag{A.6}$$

where the indices i, j refer to x for $i = 1$, and to r for $i = 2$. After some algebra

$$-\frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}_{\mathrm{KvA}}}\mathrm{F}_{1\mu} = -\left(\frac{\dot{\mathrm{V}}_{\mathrm{KvA}}}{2\mathrm{V}_{\mathrm{KvA}}} - \frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}^2_{\mathrm{KvA}}}\frac{\langle \dot{\underset{\sim}{x}}, \dot{r}\rangle\langle \dot{x}, r\rangle}{r^2}\right)\ddot{\underset{\sim}{x}}_\mu + \frac{\langle \dot{\underset{\sim}{x}}, \dot{r}\rangle}{r^2}r_\mu + \frac{\langle \dot{x}, r\rangle}{r^2}\dot{r}_\mu \tag{A.7a}$$

$$-\frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}_{\mathrm{KvA}}}\mathrm{F}_{2\mu} = -\beta^{-2}\left(\frac{\dot{\mathrm{V}}_{\mathrm{KvA}}}{2\mathrm{V}_{\mathrm{KvA}}} - \frac{\mathrm{V}_{\mathrm{KvA}}}{\mathscr{L}^2_{\mathrm{KvA}}}\frac{\langle \dot{\underset{\sim}{x}}, \dot{r}\rangle\langle \dot{x}, r\rangle}{r^2}\right)\dot{r}_\mu - \frac{\langle \dot{x}, r\rangle}{r^2}\dot{\underset{\sim}{x}}_\mu - \frac{\mathscr{L}^2_{\mathrm{KvA}}}{\mathrm{V}^2_{\mathrm{KvA}}}\mathrm{V}'_{\mathrm{KvA}}r_\mu \tag{A.7b}$$

The accelerations in (A.6) can be made explicit thanks to the projector character of π in (A.5). Remembering (A.4) we find

$$\ddot{x}^\mu = \mathrm{F}_1{}^\mu + \lambda_1\dot{x}^\mu + \lambda_2 r^\mu \tag{A.8a}$$

$$\ddot{r}^\mu = \mathrm{F}_2{}^\mu\beta^2 + \lambda_1\dot{r}^\mu \tag{A.8b}$$

λ_1, λ_2 being undetermined functions. Now the Lagrangian constraints appear in the usual way, multiplying the null vectors (A.4) by (A.7); there is only one constraint as a consequence of this:

$$\langle \dot{\underset{\sim}{x}}, \dot{r}\rangle = 0 \tag{A.9}$$

whose stability condition yields

$$\langle \dot{x}, r \rangle \underset{\sim}{\dot{x}^2} = 0 \tag{A.10}$$

Here there are two possibilities: either $\dot{x}^2 = 0$ or $\langle \dot{x}, r \rangle = 0$. In the former we have $P^2 = 0$ according to (A.2); so we take $\langle \dot{x}, r \rangle = 0$. (A.9) and (A.11) are equivalent to

$$\langle \dot{x}, r \rangle = 0\,, \qquad \langle \dot{x}, \dot{r} \rangle = 0 \tag{A.11}$$

Stability of $\langle \dot{x}, r \rangle = 0$ causes λ_2 to vanish, so that, finally, the equations of motion are

$$\ddot{x}^\mu = \lambda(\tau)\dot{x}^\mu \tag{A.12a}$$

$$\ddot{r}^\mu = -\beta^2 \frac{\mathscr{L}_{\mathrm{K}v\mathrm{A}}}{\mathrm{V}^2_{\mathrm{K}v\mathrm{A}}} \mathrm{V}'_{\mathrm{K}v\mathrm{A}}(r^2)r^\mu + \lambda(\tau)\dot{r}^\mu \tag{A.12b}$$

where $\lambda(\tau) \equiv \lambda_1(\tau) - \dfrac{\dot{\mathrm{V}}_{\mathrm{K}v\mathrm{A}}}{2\mathrm{V}_{\mathrm{K}v\mathrm{A}}}$. It is also verified that

$$\lambda(\tau) = \frac{d}{d\tau}\left(\ln \frac{\mathscr{L}_{\mathrm{K}v\mathrm{A}}}{\mathrm{V}_{\mathrm{K}v\mathrm{A}}}\right) \tag{A.13}$$

whence the equations of motion (A.11) and (A.12) coincide with (3.26) and (3.27) of section 3 except for an irrelevant scale factor in the gauge function.

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