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Differential pseudoconnections and field theories

by

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ABSTRACT. — We define a « differential pseudoconnection of order k » on a bundle $p : E \to M$ as a translation morphism $\Gamma : J^kE \to \bigvee_k T^* \otimes VE$ on the affine bundle $J^kE \to J^{k-1}E$. Such concept is a generalization of usual connections. Then we study in the framework of jet spaces several important differential operators used in physics. In this context an interest arises naturally for the second order affine differential equations, called « special », given by $\varepsilon^2 \equiv \ker(G \circ H) \hookrightarrow J^2E$, where $H : J^2E \to \bigvee_2 T^* \otimes VE$ is a differential pseudoconnection and $G : \bigvee_2 T^* \otimes VE \to VE$ is the linear submersion induced by a metric $g : \bigvee_2 T^* \to \mathbb{R}$. Particular cases of special equations are both the geodesics equation (an ordinary equation) and any kind of Laplace equation (a partial equation) even modified by the addition of physical terms. So special equations are candidate to fit a lot of fundamental physical fields. At the present state of the theory we can emphasize several common features of physical fields. Further right developments will be the object of next papers.

INTRODUCTION

Several general field theories have been successful in describing fundamental physical fields by a unique schema. Among the main approaches we mention the Lagrangian-Hamiltonian theories, which are based on symplectic structures, and the gauge theories, which are based on connections on principal bundles.

Our purpose is to present the first step of a different attempt based on differential pseudoconnections on jet bundles. In this paper we are dealing with the essential elements of such an approach and with the testing of a certain number of important examples. Further right developments will be treated in subsequent works. Now we can account and emphasize some common, but not evident, analogies among several fields and we can put new problems.

Why jet bundles and why connections ?

Jet bundles are the natural geometrical framework for differential equations. They account the clear distinction between the point-wise point of view of structural properties of the equation itself and the functional point of view of the sheaf of solutions. Important theories have been developed in such direction. We mention the fundamental theoretical work of Spencer-Kumpera [*12*] and the research of Pommaret [*10*] extending the theory so that applications, which seem very promising for theoretical physics, become possible. In such works geometrical properties of given differential equations are studied.

Our aim is first to exhibit a differential equation which could be proposed to rule physical fields. Moreover we try to make a heuristic use of jet spaces in order to get suggestions for the formal structure of field theories from their geometrical structure. If we claim a physical field to be represented by a section $s : \mathrm{M} \to \mathrm{E}$ of a bundle $\mathrm{E} \to \mathrm{M}$, which is the solution of a differential equation ε^k of order k, then a natural way of expressing ε^k is to consider the k-jet $j^k s : \mathrm{M} \to \mathrm{J}^k\mathrm{E}$ and to impose some condition on it. Most of physical equations have tensorial form. Then we need some geometrical constitutive law which enables us to make $j^k s$ into a tensor. To this purpose, let us consider a fundamental property of jet spaces: $\mathrm{J}^k\mathrm{E}$ is an affine bundle over $\mathrm{J}^{k-1}\mathrm{E}$ and its vector bundle is $\underset{k}{\bigvee} \mathrm{T}^* \otimes \mathrm{VE}$, where $\underset{k}{\bigvee} \mathrm{T}^*$ is the symmetrized tensor product of the cotangent bundle of the basis M and VE is the vertical tangent bundle of E. Hence the idea of choosing an origin on each affine fibre of $\mathrm{J}^k\mathrm{E}$. Such a choice is called a « differential pseudoconnection of order k » and is obtained by a section $\urcorner : \mathrm{J}^{k-1}\mathrm{E} \to \mathrm{J}^k\mathrm{E}$ or equivalently by a translation $\ulcorner : \mathrm{J}^k\mathrm{E} \to \underset{k}{\bigvee} \mathrm{T}^* \otimes \mathrm{VE}$ (i. e. an affine morphism over $\mathrm{J}^{k-1}\mathrm{E} \to \mathrm{E}$, such that the fibre derivative is $\mathrm{D}\ulcorner = 1$). Usual connections are a particular kind of differential pseudoconnections of first order. Moreover we remark that our approach does not involve principal bundles.

Now the simplest way to obtain a global and quite regular differential equation of order k, such that the number of scalar equations equals the freedom degree of the field, is to take the kernel of an affine submersion

$J^kE \to VE$. If we combine such requirement with the idea of differential pseudoconnection, we are led to consider the second order affine differential equation $\varepsilon^2 \equiv \ker(G \circ H) \hookrightarrow J^2E$, called « special », where

$$H : J^2E \to \underset{2}{\vee} T^* \otimes VE$$

is a differential pseudoconnection and $G : \underset{2}{\vee} T^* \otimes VE \to VE$ is induced by a metric $g : \underset{2}{\vee} T^* \to \mathbb{R}$.

Deep reasons concerned with the Pommaret's theory on differential equations could perhaps attach importance to special equations. Moreover such type of equation includes any kind of geodesics equations (Newton, Einstein, ...) and any kind of Laplace equations (Laplace, De Rham, Lichnerowicz, ...).

The problem of writing a special equation on a given bundle E is global. The main point is to look for the differential pseudoconnection H. In some cases the bundle itself exhibits a canonical $\hat{H}$, in some we have to choose a « gauge » $\hat{H}$. Then we can write $H = \hat{H} + F$, where $F : JE \to \underset{2}{\vee} T^* \otimes VE$ is a tensorial term expressing the specific physical problem.

The requirement that a physical theory can be ruled by a special equation is an effective global condition. The factorization of a morphism $J^2E \to VE$ through $\underset{2}{\vee} T^* \otimes VE$ by the composition $J^2E \xrightarrow{H} \underset{2}{\vee} T^* \otimes VE \xrightarrow{G} VE$ creates some nontrivial problems. Some fundamental physical equations can be written easily as special equations (classical n-body dynamics, Klein-Gordon field). Some others are factorizable by an H which is affine but not a pseudoconnection, as $DH \neq 1$ (Maxwell and Einstein fields). For good chance, such equations can be transformed into new special equations by imposing « gauge conditions » on initial data, which are propagated by the new equation (Lichnerowicz [*19*]). It is remarkable that the conditions, which are introduced classically to remove ambiguities in the solutions, are just what we need to obtain a special theory.

There are some analogies between the special schemas and the Lagrangian ones. One has to look in the first case for a pseudoconnection

$$H : J^2E \to \underset{2}{\vee} T^* \otimes VE$$

and in the second one for a function $L : JE \to \mathbb{R}$. Then in both cases one has a standard rule to write the field equation ε^2, which is the kernel of a morphism $J^2E \to VE$, in the first case, or $J^2E \to VE^*$, in the second one. Moreover many important Lagrange equations can be translated into special equations and conversely many important special equations can be written in the Lagrangian form. However it is not always true. For instance any classical force leads to a differential pseudoconnection, but not to a Lagrangian.

In conclusion, many other features of special equations must be studied in order to confirm the interest for such approach. For instance we are expecting to find conservation laws and some kind of cohomology. It will be the subject of further works.

CHAPTER ONE

PSEUDOCONNECTIONS ON JET SPACES

1. The tangent space to a bundle

We start with basic definitions and notations, in order to make the paper selfcontained.

All the manifolds and maps considered are assumed to be C^∞ and paracompact, even if it is not said explicitly.

Let M be a m-dimensional manifold. We denote the tangent bundle by $\pi_M : TM \to M$; when the context is clear we replace TM by T. If $\{ x^\lambda \}$ is a local chart of M, then the induced chart of T is $\{ x^\lambda, \dot{x}^\lambda \}$ and the induced basis is $\partial x_\lambda : M \to T$, $1 \leq \lambda \leq m$. One has $x^\lambda \circ \pi_M = x^\lambda$ (by abuse of notation we write x^λ at the place of $x^\lambda \circ \pi_M$). Let N be a further manifold and let $f : M \to N$ be a map. Then $Tf : TM \to TN$ is the tangent map. If $\{ y^\mu \}$ is a chart of N, then $\{ y^\mu, \dot{y}^\mu \} \circ Tf = \{ f^\mu, \dot{x}^\lambda \partial x_\lambda \cdot f^\mu \}$, where $f^\mu \equiv y^\mu \circ f$.

Let $p : E \to M$ be a bundle of dimension $m + 1$. Let $\{ x^\lambda, y^i \}$ be a chart of E. Then one has $x^\lambda \circ p = x^\lambda$. If $s : M \to E$ is a section, then

$$\{ x^\lambda, y^i \} \circ s \equiv \{ x^\lambda, s^i \} .$$

The chart induced on TE is $\{ x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i \}$. The vertical tangent space of E is the vector subspace $i : VE \hookrightarrow TE$ of vectors tangent to the fibres of E and characterized by $\{ x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i \} \circ i = \{ x^\lambda, y^i, 0, \dot{y}^i \}$.

If $f : N \to M$ is a map, then $f^*p : f^*E \to N$ is the pull-back bundle. When the context is clear, we do not always express explicitly all the pullbacks and we sometimes write E at the place of f^*E.

The bundle $Tp : TE \to T$ is given by $\{ x^\lambda, \dot{x}^\lambda \} \circ Tp = \{ x^\lambda, \dot{x}^\lambda \}$. The horizontal tangent space of E is p^*T; its induced chart is $\{ x^\lambda, y^i, \dot{x}^\lambda \}$. One has a natural submersion $\sqcup_E : TE \to p^*T$, which is a linear morphism over E; its expression is given by $\{ x^\lambda, y^i, \dot{x}^\lambda \} \circ \sqcup_E = \{ x^\lambda, y^i, \dot{x}^\lambda \}$. Moreover $\sqcup_E : TE \to p^*T$ is an effine bundle, whose vector bundle is $p^*T \times_E VE$.

If $p : E \to M$ is endowed with an algebraic structure on its fibres, then the tangent functor induces an algebraic structure on the bundle $Tp : TE \to T$. For instance, if E is a vector bundle, then TE is a vector bundle on T.

Moreover, if E is an affine bundle whose bundle is $\bar{E}$, then $VE = E \times \bar{E}$.

If E and F are vector bundles on M and $\{x^\lambda, y^i\}$ and $\{x^\lambda, z^j\}$ are charts on E and F, then $\{x^\lambda, y^i \otimes z^j\}$ denotes the chart induced on the bundle $E \otimes F \to M$.

We denote by $T^*M \to M$ the cotangent bundle of M; when the context is clear we replace T^*M by T^*. The induced chart on T^* is $\{x^\lambda, \dot{x}_\lambda\}$.

We put $T_{(r,s)} \equiv \underset{r}{\otimes} T \underset{s}{\otimes} T^*$ and we denote by $\{x^\lambda, \dot{x}^{j_1 \dots j_r}_{i_1 \dots i_s}\}$ the induced chart on $T_{(r,s)}$. Moreover $\vee$ and $\wedge$ denote the symmetrized and the anti-symmetrized tensor products.

2. Jet bundles

Let $p : E \to M$ be a given bundle.

We denote by J^kE, with $k \geq 0$, the space of k-jets of E. Namely J^kE is the disjoint union $J^kE \equiv \bigsqcup_{x \in M} [s]^k_x$ of the equivalence classes of sections $s : M \to E$ given by $s \sim^k_x s'$ iff

$$0 = (s^i - s'^i)(x) = \dots = \partial x_{\lambda_1} \cdot \dots \cdot \partial x_{\lambda_k} \cdot (s^i - s'^i)(x).$$

In particular $J^0E = E$. The natural chart induced on J^kE is

$$\{x^\lambda, y^i, y^i_\lambda, \dots, y^i_{\lambda_1 \dots \lambda_k}\}, \qquad 1 \leq \lambda_1 \leq \dots \leq \lambda_k \leq m,$$

(if we replace y^i by y_i, then we replace y^i_λ by $y_{i,\lambda}, \dots$).

One has the natural bundle $p^k : J^kE \to M$ and $x^\lambda \circ p^k = x^\lambda$.

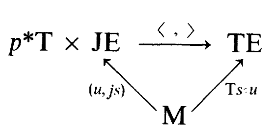
The k-jet of a section $s : M \to E$ is the section given by

$$j^k s \equiv [s]^k : M \to J^kE.$$

Its expression is

$$\{x^\lambda, y^i, \dots, y^i_{\lambda_1 \dots \lambda_k}\} \circ j^k s = \{x^\lambda, s^i, \dots, \partial x_{\lambda_1} \cdot \dots \cdot \partial x_{\lambda_k} \cdot s^i\}.$$

We can relate the tangent map and the 1-jet of s : there is a unique morphism over E $\langle\ ,\ \rangle : p^*T \times JE \to TE$ which makes the following diagram commutative, for each section $s : M \to E$ and $u : M \to T$,



Its expression is $\{x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i\} \circ \langle\ ,\ \rangle = \{x^\lambda, y^i, \dot{x}^\lambda, y^i_\lambda \dot{x}^\lambda\}$.

For $0 \leq h < k$ one has a natural bundle $p^{hk} : J^kE \to J^hE$ and $p^{lh} \circ p^{hk} = p^{lk}$, $p^h \circ p^{hk} = p^k$. Its expression is

$$\{x^\lambda, y^i, \dots, y^i_{\lambda_1 \dots \lambda_h}\} \circ p^{hk} = \{x^\lambda, y^i, \dots, y^i_{\lambda_1 \dots \lambda_h}\}.$$

One can show (taking into account the transformation law of $y^i_{\lambda_1 \dots \lambda_k}$) that $J^k E$ is an affine bundle over $J^{k-1}E$, whose vector bundle is

$$\overline{J^k E} \equiv p^{(0,k-1)*} \bigvee_k p^* T^* \otimes VE .$$

By abuse of notations, we write $\overline{J^k E} = \bigvee_k T^* \otimes VE$. This property will play an important role in the following. The natural chart induced on $\overline{J^k E}$ is $\{ x^\lambda, \dots, y^i_{\lambda_1 \dots \lambda_{k-1}} ; \dot{x}_{\lambda_1 \dots \lambda_k} \otimes y^i \}$.

There is a unique map $i : J^{k+h}E \to J^k J^h E$ which makes the following diagram commutative for each section $s : M \to E$

$$\begin{array}{ccc} J^{k+h}E & \xrightarrow{\ i\ } & J^k J^h E \\ {\scriptstyle j^{k+h}s} \nwarrow & & \nearrow {\scriptstyle j^k j^h s} \\ & M & \end{array}$$

i is a monomorphism over $J^h E$ and its expression is

$$\{ x^\lambda, y^i, \dots, y^i_{\lambda_1 \dots \lambda_h} ; \dots, y^i_{\mu_1 \dots \mu_k}, \dots, y^i_{\lambda_1 \dots \lambda_h, \mu_1 \dots \mu_k} \} \circ i$$
$$= \{ x^\lambda, y^i, \dots, y^i_{\lambda_1 \dots \lambda_h} ; \dots, y^i_{\mu_1 \dots \mu_k}, \dots, y^i_{\lambda_1 \dots \lambda_h \mu_1 \dots \mu_k} \} .$$

Let $q : F \to M$ be a further bundle and let $f : E \to F$ be a morphism over M. Then there is a unique morphism over M $J^k f : J^k E \to J^k F$ which makes the following diagram commutative for each section $s : M \to E$

$$\begin{array}{ccc} J^k E & \xrightarrow{\ J^k f\ } & J^k F \\ {\scriptstyle j^k s} \nwarrow & & \nearrow {\scriptstyle j^k(f \circ s)} \\ & M & \end{array}$$

Then J^k is a covariant functor and its expression is

$$\{ x^\lambda, z^j, z^j_\lambda, \dots \} \circ J^k f = \{ x^\lambda, f^j, \partial x_\lambda . f^j + y^i_\lambda \partial y_i . f^j, \dots \} .$$

There is a unique isomorphism $i : J^k(E \times F) \to J^k E \times J^k F$ over M which makes the following diagram commutative for each section $(s, t) : M \to E \times F$

$$\begin{array}{ccc} J^k(E \times F) & \xrightarrow{\ i\ } & J^k E \times J^k F \\ {\scriptstyle j^k(s,t)} \nwarrow & & \nearrow {\scriptstyle (j^k s, j^k t)} \\ & M & \end{array}$$

Its expression is

$$\{ x^\lambda ; \dots, y^i_{\lambda_1 \dots \lambda_k} ; \dots, z^j_{\lambda_1 \dots \lambda_k} \} \circ i = \{ x^\lambda ; \dots, y^i_{\lambda_1 \dots \lambda_k}, z^j_{\lambda_1 \dots \lambda_k} \} .$$

Let the bundle $p : E \to M$ be endowed with an algebraic structure on its fibres, determined by morphisms over M such as $f : E \to E$, or $f' : E \times E \to E$, or $f'' : F \times E \to E$, ... and by privileged sections as $e : M \to E$. Then we can endow $p^k : J^k E \to M$ with the algebraic structure on its fibres determined by the morphisms over M $J^k f : J^k E \to J^k E$, $J^k f' : J^k E \times J^k E \to J^k E$, $J^k f'' : J^k F \times J^k E \to J^k E, \dots$ and by the section $j^k e : M \to J^k E$.

So if $p : \mathrm{E} \to \mathrm{M}$ is respectively a vector, or an affine (with associated vector bundle $\bar{p} : \bar{\mathrm{E}} \to \mathrm{M}$), or an algebra (with unity), or a group bundle, then $p^k : \mathrm{J}^k\mathrm{E} \to \mathrm{M}$ is a vector, or an affine (with associated vector bundle $\bar{p}^k : \mathrm{J}^k\bar{\mathrm{E}} \to \mathrm{M}$), or an algebra (with unity), or a group bundle.

For instance, let $p : \mathrm{E} \to \mathrm{M}$ be an algebra bundle, with addition $a : \mathrm{E} \times \mathrm{E} \to \mathrm{E}$, scalar multiplication $m : \mathbb{R} \times \mathrm{E} \to \mathrm{E}$ and multiplication $c : \mathrm{E} \times \mathrm{E} \to \mathrm{E}$. Let $\{ x, y^i \}$ a natural chart of E, such that

$$y^i \circ a = y^i \circ \pi^1 + y^i \circ \pi^2 , \qquad y^i \circ m = y^0 y^i , \qquad y^i \circ c = c^i_{lm} y^l \circ \pi^1 \ y^m \circ \pi^2 ,$$

where c^i_{lm} is factorizable through M. Then

$$\begin{aligned} \{ x^\lambda, y^i, y^i_\lambda, \dots \} \circ \mathrm{J}^k a &= \{ x^\lambda, y^i \circ \pi^1 + y^i \circ \pi^2, y^i_\lambda \circ \pi^1 + y^i_\lambda \circ \pi^2, \dots \} \\ \{ x^\lambda, y^i, y^i_\lambda, \dots \} \circ \mathrm{J}^k m &= \{ x^\lambda, y^0 y^i, y^0 y^i_\lambda, \dots \} \\ \{ x^\lambda, y^i, y^i_\lambda, \dots \} \circ \mathrm{J}^k c &= \{ x^\lambda, c^i_{lm} y^l \circ \pi^1 \ y^m \circ \pi^2, \partial x_\lambda . c^i_{lm} y^l \circ \pi^1 \ y^m \circ \pi^2 \\ &\qquad + c^i_{lm}(y^l_\lambda \circ \pi^1 \ y^m \circ \pi^2 + y^l \circ \pi^1 \ y^m_\lambda \circ \pi^2), \dots \} . \end{aligned}$$

Let $p : \mathrm{E} \to \mathrm{M}$ and $q : \mathrm{F} \to \mathrm{M}$ be vector bundles and let $t : \mathrm{E} \times \mathrm{F} \to \mathrm{E} \otimes \mathrm{F}$ be the tensor multiplication. Then $\mathrm{J}^k t : \mathrm{J}^k\mathrm{E} \times \mathrm{J}^k\mathrm{F} \to \mathrm{J}^k(\mathrm{E} \otimes \mathrm{F})$ is a bilinear morphism over M and one gets a canonical linear morphism over M $\tilde{\mathrm{J}}^k t : \mathrm{J}^k\mathrm{E} \otimes \mathrm{J}^k\mathrm{F} \to \mathrm{J}^k(\mathrm{E} \otimes \mathrm{F})$, taking into account the universal property of the tensor product. If $\{ x^\lambda, y^i, z^j \}$ is a linear chart of $\mathrm{E} \times \mathrm{F}$, then

$$\{ x^\lambda, y^i \otimes z^j, (y^i \otimes z^j)_\lambda, \dots \} \circ \tilde{\mathrm{J}}^k t = \{ x^\lambda, y^i \otimes z^j, y^i_\lambda \otimes z^j + y^i \otimes z^j_\lambda, \dots \} .$$

We can express the exterior derivative of forms by means of jets. In fact we can prove, by induction with respect to $1 \leq p \leq m$, that there is a canonical subset $\Sigma^p \hookrightarrow \mathrm{J} \overset{p}{\wedge} \mathrm{T}^*$, which is a vector subbundle over M and an affine subbundle over $\overset{p}{\wedge} \mathrm{T}^*$, whose associated vector bundle is

$$\bar{\Sigma}^p \equiv \mathrm{T}^* \vee \overset{p}{\wedge} \mathrm{T}^* \hookrightarrow \mathrm{T}^* \otimes \overset{p}{\wedge} \mathrm{T}^* .$$

Moreover $\mathrm{J} \overset{p}{\wedge} \mathrm{T}^*$ is decomposable in the direct sum over $\overset{p}{\wedge} \mathrm{T}^*$ of the affine subspace Σ^p and the vector subspace $\overset{p+1}{\wedge} \mathrm{T}^*$ (it determines a trivial bundle over $\overset{p}{\wedge} \mathrm{T}^*$) $\mathrm{J} \overset{p}{\wedge} \mathrm{T}^* = \Sigma^p \oplus \overset{p+1}{\wedge} \mathrm{T}^*$. Then the exterior derivative is the affine morphism over $\overset{p}{\wedge} \mathrm{T}^* \to \mathrm{M}$ given by the projection

$$d : \mathrm{J} \overset{p}{\wedge} \mathrm{T}^* \to \overset{p+1}{\wedge} \mathrm{T}^* .$$

If $s : \mathrm{M} \to \overset{p}{\wedge} \mathrm{T}^*$ is a section then we write $ds \equiv d \circ js : \mathrm{M} \to \overset{p+1}{\wedge} \mathrm{T}^*$. There is a link between this definition and the Spencer operator.

3. Differential connections

DEFINITION. — Let $q : \mathrm{F} \to \mathrm{N}$ be an affine vector bundle, whose vector bundle is $\bar{q} : \bar{\mathrm{F}} \to \mathrm{N}$. *A pseudoconnection* is a translation morphism $\ulcorner : \mathrm{F} \to \bar{\mathrm{F}}$ over N.

Namely $\ulcorner : \mathrm{F} \to \bar{\mathrm{F}}$ is an affine morphism over N, whose fibre derivative is $\mathrm{D}\ulcorner = id_{\bar{\mathrm{F}}} : \mathrm{N} \to \bar{\mathrm{F}}^* \otimes \bar{\mathrm{F}}$ (with rough notation we write $\mathrm{D}\ulcorner = 1$). The following maps determine a natural bijection between differential pseudoconnections $\ulcorner : \mathrm{F} \to \bar{\mathrm{F}}$ and sections $\urcorner : \mathrm{N} \to \mathrm{F}$:

a) $\ulcorner \mapsto \urcorner$, where $\urcorner$ is the unique section such that $\ulcorner \circ \urcorner = 0$;
b) $\urcorner \mapsto \ulcorner = id_{\mathrm{F}} - \urcorner \circ q$.

We will often identify these two objects $\ulcorner$ and $\urcorner$. Let $\{ y^\lambda, z^j \}$ be an affine chart of F and let $\{ y^\lambda, \bar{z}^j \}$ be the naturally associated linear chart of $\bar{\mathrm{F}}$. Then one has

$$\{ y^\lambda, z^j \} \circ \ulcorner = \{ y^\lambda, z^j + \ulcorner^j \} \quad \text{and} \quad \{ y^\lambda, z^j \} \circ \urcorner = \{ y^\lambda, - \ulcorner^j \}$$

where $\ulcorner^j : \mathrm{F} \to \mathbb{R}$ is factorizable through N (we write also, by abuse of notation, $\ulcorner^j : \mathrm{N} \to \mathbb{R}$).

DEFINITION. — Let $p : \mathrm{E} \to \mathrm{M}$ be a bundle. *A differential pseudoconnection* of order $k \geq 1$ on E is a pseudoconnection on the affine bundle $p^{k-1,k} : \mathrm{J}^k\mathrm{E} \to \mathrm{J}^{k-1}\mathrm{E}$.

Namely a differential pseudoconnection of order k is a translation

$$\ulcorner : \mathrm{J}^k\mathrm{E} \to \overline{\mathrm{J}^k\mathrm{E}} \equiv p^{(k-1)*} \bigvee_k \mathrm{T}^* \otimes \mathrm{VE}$$

or, equivalently, a section $\urcorner : \mathrm{J}^{k-1} \to \mathrm{J}^k\mathrm{E}$. We write also, by abuse of notation, $\ulcorner : \mathrm{J}^k\mathrm{E} \to \bigvee_k \mathrm{T}^* \otimes \mathrm{VE}$. One has the following expression

$$\{ x^\lambda, \ldots, y^i_{\lambda_1 \ldots \lambda_{k-1}}; \dot{x}_{\lambda_1 \ldots \lambda_k} \otimes y^i \} \circ \ulcorner = \{ x^\lambda, \ldots, y^i_{\lambda_1 \ldots \lambda_{k-1}}; y^i_{\lambda_1 \ldots \lambda_k} + \ulcorner^i_{\lambda_1 \ldots \lambda_k} \}$$
$$\{ x^\lambda, \ldots, y^i_{\lambda_1 \ldots \lambda_{k-1}}, y^i_{\lambda_1 \ldots \lambda_k} \} \circ \urcorner = \{ x^\lambda, \ldots, y^i_{\lambda_1 \ldots \lambda_{k-1}}; - \ulcorner^i_{\lambda_1 \ldots \lambda_k} \},$$

where $\ulcorner^i_{\lambda_1 \ldots \lambda_k} : \mathrm{J}^{k-1}\mathrm{E} \to \mathbb{R}$.

DEFINITION. — Let $p : \mathrm{E} \to \mathrm{M}$ be a bundle endowed with an algebraic structure on its fibres. *A differential connection* of order $k \geq 1$ on E is a differential pseudoconnection $\urcorner : \mathrm{J}^{k-1}\mathrm{E} \to \mathrm{J}^k\mathrm{E}$, which is a morphism over M with respect to the algebraic structures induced on $\mathrm{J}^{k-1}\mathrm{E}$ and $\mathrm{J}^k\mathrm{E}$ over M.

For instance, let $p : \mathrm{E} \to \mathrm{M}$ be an affine (linear) bundle. Then $\urcorner$ is an affine (linear) differential connection iff $\urcorner$ is affine (linear) over M. Moreover $\urcorner$ is affine (linear) iff

$$\Gamma^i_{\lambda_1 \ldots \lambda_k} = \Gamma^i_{\lambda_1 \ldots \lambda_k j} y^j + \ldots + \Gamma^i_{\lambda_1 \ldots \lambda_k j}{}^{\mu_1 \ldots \mu_{k-1}} y^j_{\mu_1 \ldots \mu_{k-1}},$$

where $\Gamma^i_{\lambda_1 \ldots \lambda_k j} : M \to \mathbb{R}, \ldots, \Gamma^i_{\lambda_1 \ldots \lambda_k j}{}^{\mu_1 \ldots \mu_{k-1}} : M \to \mathbb{R}$.

Let $a^k : A^k E \to M$ be the bundle constituted by the disjoint union $A^k E \equiv \bigsqcup_{x \in M} \{ f_x \}$ of the algebraic morphisms $f_x : J^{k-1}_x E \to J^k_x E$ such that $p^{k-1,k} \circ f_x = id_{J^{k-1}_x E}$. Then the bundle $A^k E$ is endowed with an algebraic structure on its fibers. For instance, if E is an affine (linear) bundle, then $A^k E$ is an affine bundle. In such a case we denote the naturally induced chart of $A^k E$ by $\{ x^\lambda, a^i_{\lambda_1 \ldots \lambda_k j}, \ldots, a^i_{\lambda_1 \ldots \lambda_k j}{}^{\mu_1 \ldots \mu_{k-1}} \}$.

One has a canonical morphism over M $\mathscr{A} : A^k E \times J^{k-1} E \to J^k E$. Moreover $\mathscr{A}$ induces a natural bijection between the sections $\tilde{\urcorner} : M \to A^k E$ and the sections $\urcorner : J^{k-1} E \to J^k E$ which are differential connections. We will often identify these two objects writing also $\urcorner : M \to A^k E$.

Let $\Gamma : J^k E \to \bigvee_k T^* \otimes VE$ be a differential pseudoconnection and let $s : M \to E$ be a section. Then the *covariant derivative* of order k of s is the section $\Gamma \circ j^k s : M \to \bigvee_k T^* \otimes VE$. If E is an affine bundle, whose vector bundle is $\bar{E}$, then we often replace $\Gamma : J^k E \to \bigvee_k T^* \otimes VE$ with the simpler map $\nabla : J^k E \to \bigvee_k T^* \otimes \bar{E}$ and write $\nabla s \equiv \nabla \circ j^k s : M \to \bigvee_k T^* \otimes \bar{E}$. Its expression is

$$\nabla_{\lambda_1 \ldots \lambda_k} s^i = \partial x_{\lambda_1} . \ldots . \partial x_{\lambda_k} . s^i + \Gamma^i_{\lambda_1 \ldots \lambda_k j} s^j + \ldots + \Gamma^i_{\lambda_1 \ldots \lambda_k j}{}^{\mu_1 \ldots \mu_{k-1}} \partial x_{\mu_1} . \ldots . \partial x_{\mu_{k-1}} s^j .$$

The notion of differential pseudoconnection of order k is a generalization of the usual notion of connection. In fact a usual connection on a bundle $p : E \to M$ is a pseudoconnection $\tilde{\urcorner} : p^*T \to TE$ on the affine bundle $TE \to p^*T$, which is linear over E; it determines a splitting $TE = VE \oplus p^*T$ of the vector bundle $TE \to E$. Such a usual connection is algebraic (for instance, affine, linear) if E is endowed with an algebraic structure on its fibres and $\tilde{\urcorner} : p^*T \to TE$ is a morphism over $T \to M$ with respect to the algebraic structures induced naturally on $p^*T \to T$ and on $TE \to M$. In particular we can view any principal bundle as an affine (in the sense of groups) bundle $p : E \to M$, whose associated group bundle is a trivial bundle $E = M \times G$, where G is a Lie group. Then a usual connection on the principal bundle is an algebraic connection in the previous sense. Let us remark that our approach does not need principal bundles. More generally, we can express the link between the usual connections and the first order differential pseudoconnections as follows. Let $p : E \to M$ be a bundle.

Let $\tilde{\urcorner} : p^*T \to TE$ be a usual linear connection on E. Then there is a unique differential pseudoconnection of order 1 $\urcorner : E \to JE$, which makes the following diagram commutative

$$\begin{array}{ccc} p^*T & \xrightarrow{(\urcorner, id)} & JE \times p^*T \\ & {\scriptstyle\tilde{\urcorner}}\searrow \quad \swarrow{\scriptstyle\langle\,.\,\rangle} & \\ & TE & \end{array}$$

Moreover $\tilde{\urcorner}$ is algebraic iff $\urcorner$ is algebraic.

4. Linear differential connections of order one

We are looking for the jet expression of usual notions on linear differential connections of order one, in order to state necessary notations and to show, by the way, how the jet language works suitably. In this section $p : E \to M$ is a vector bundle.

Let $\urcorner : E \to JE$ be a linear differential connection. Then there is a unique linear differential connection $\urcorner^* : E^* \to JE^*$ which makes the following diagram commutative

$$\begin{array}{ccc} E \times E^* & \xrightarrow{\urcorner \times \urcorner^*} & JE \times JE^* \\ & {\scriptstyle 0}\searrow \quad \swarrow{\scriptstyle J\langle\,.\,\rangle} & \\ & J(M \times \mathbb{R}) & \end{array}$$

Its expression is $\urcorner^{*i}_{\lambda j} = - \urcorner^{i}_{\lambda j}$. Then one gets an affine isomorphism $AE \to AE^*$ and we will often write $\mathscr{A} : AE \times E^* \to JE^*$.

Let $p' : E' \to M$ be a further vector bundle and let $\urcorner : E \to JE$ and $\urcorner' : E' \to JE'$ be linear differential connections. Then there is a unique linear differential connection $\urcorner \otimes \urcorner' : E \otimes E' \to J(E \otimes E')$ which makes the following diagram commutative

$$\begin{array}{ccc} E \times E' & \xrightarrow{\urcorner \times \urcorner'} & JE \times JE' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle Jt} \\ E \otimes E' & \xrightarrow{\urcorner \otimes \urcorner'} & J(E \otimes E') \end{array}$$

Its expression is $(\ulcorner \otimes \ulcorner')^{ii'}_{\lambda jj'} = \ulcorner^{i}_{\lambda j}\delta^{i'}_{j'} + \ulcorner'^{i'}_{\lambda j'}\delta^{i}_{j}$. Then one gets a morphism $AE \times AE' \to A(E \otimes E')$ and we will often write

$$\mathscr{A} : AE \times (\bigotimes_r E \bigotimes_s E^*) \to J(\bigotimes_r E \bigotimes_s E^*)$$

(and analogously by replacing $\otimes$ with $\vee$ or $\wedge$).

Let $\urcorner : E \to JE$ be a linear differential connection. Let us consider the bundle $\overset{p}{\wedge} T^* \otimes E$ of p-forms valued on E. We define the affine subbundle $\Omega \equiv Jt(\Sigma^p \times \urcorner(E)) + T^* \vee \overset{p}{\wedge} T^* \otimes E \hookrightarrow J(\overset{p}{\wedge} T^* \otimes E)$, whose associated vector bundle is $\bar{\Omega} \equiv T^* \vee \overset{p}{\wedge} T^* \otimes E$. Then one can prove that $J(\overset{p}{\wedge} T^* \otimes E)$ is decomposable into the direct sum over $\overset{p}{\wedge} T^* \otimes E$ of the affine subspace Ω and the vector subspace $\overset{p+1}{\wedge} T^* \otimes E$

$$J(\overset{p}{\wedge} T^* \otimes E) = \Omega \oplus (\overset{p+1}{\wedge} T^* \otimes E).$$

Then the exterior derivative with respect to $\urcorner$ is the affine morphism over $\overset{p}{\wedge} T^* \otimes E \to M$ given by the projection $d : J(\overset{p}{\wedge} T^* \otimes E) \to \overset{p+1}{\wedge} T^* \otimes E$. Hence one gets a morphism over M $d : AE \times J(\overset{p}{\wedge} T^* \otimes E) \to \overset{p+1}{\wedge} T^* \otimes E$. Its expression is $\dot{x}_{\lambda\lambda_1 \ldots \lambda_p} \otimes y^i \circ d = \dot{x}_{[\lambda_1 \ldots \lambda_p} \otimes y^i_{\lambda]} + a^i_{[\lambda j} \dot{x}_{\lambda_1] \ldots \lambda_p} \otimes y^j$. If $s : M \to \overset{p}{\wedge} T^* \otimes E$ is a section, then we write $d\, s \equiv d \circ (\urcorner, js)$.

The morphism over M

$$JAE \times J^2E \xrightarrow{(a^{01}, J\mathscr{A})} AE \times J(T^* \otimes E) \xrightarrow{\ d\ } \overset{2}{\wedge} T^* \otimes E$$

is factorizable through a morphism over M linear on the second factor $\mathscr{C} : JAE \times E \to \overset{2}{\wedge} T^* \otimes E$. Taking into account the linearity of $\mathscr{C}$ we write also $\mathscr{C} : JAE \to \overset{2}{\wedge} T^* \otimes E \otimes E^*$. The curvature of $\urcorner : E \to JE$ is the section $\mathscr{C} \circ j\urcorner : M \to \overset{2}{\wedge} T^* \otimes E \otimes E^*$. Its expression

$$\mathscr{C}_{\lambda\mu j}{}^i = a^i_{[\mu j, \lambda]} + a^i_{[\lambda k} a^k_{\mu] j}.$$

The previous definition of $\mathscr{C}$ is a version on jet spaces of the Cartan's formulas of structure.

5. Riemannian connections

The torsion morphism can be defined as the morphism given by the composition $AT \times T^* \xrightarrow{\mathscr{A}} JT^* \xrightarrow{d} \overset{2}{\wedge} T^*$. Taking into account its linearity on T^*, we denote also it by $\mathscr{T} : AT \to \overset{2}{\wedge} T^* \otimes T$. Its expression is $\mathscr{T}^i_{\lambda\mu} = a^i_{[\lambda\mu]}$.

The subbundle $B \hookrightarrow AT$ constituted by the torsion free connections is an affine subbundle, whose associated vector bundle is $\bar{B} = \underset{2}{\vee} T^* \otimes T$.

Henceforth we will denote by $l : \mathrm{L} \to \mathrm{M}$ an open subbundle of $\overset{2}{\vee} \mathrm{T} \to \mathrm{M}$ constituted by nondegenerate tensors. For instance, L may be the bundle of (contravariant) metrics with a given signature. L is locally a vector bundle and one can extend many results stated for vector bundles to this case. We denote by $\{ x^\lambda, l^{ij} \}$ the naturally induced chart of L and by $l_{ij} : \mathrm{L} \to \mathbb{R}$ the functions obtained taking the inverse matrix of (l^{ij}). One has the natural linear isomorphism over L

$$\mathrm{L} \times (\underset{r}{\otimes} \mathrm{T} \underset{s}{\otimes} \mathrm{T}^*) \to \mathrm{L} \times (\underset{r-1}{\otimes} \mathrm{T} \underset{s+1}{\otimes} \mathrm{T}^*)$$

and

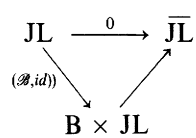
$$\mathrm{L} \times (\underset{r}{\otimes} \mathrm{T} \underset{s}{\otimes} \mathrm{T}^*) \to \mathrm{L} \times (\underset{r+1}{\otimes} \mathrm{T} \underset{s-1}{\otimes} \mathrm{T}^*).$$

We assume the usual notations for lowering and raising indices.

Jet spaces provide a suitable definition of Riemannian connection. We can prove that the morphism over L $\mathscr{A} : \mathrm{L} \times \mathrm{B} \to \mathrm{JL}$ is an affine isomorphism and its inverse is the unique affine morphism over L

$$\mathscr{B} : \mathrm{JL} \to \mathrm{L} \times \mathrm{B},$$

which makes the following diagram commutative



The expressions of $\mathscr{A}$ and $\mathscr{B}$ are $l^{ij}_{\lambda} \circ \mathscr{A} = - l^{(ik}a^{j)}_{k\lambda}$ and

$$a^{i}_{\lambda\mu} \circ \mathscr{B} = \frac{1}{2}\left(l_{\lambda h} l_{\mu k} l^{i\nu} l^{hk}_{\nu} - l_{(\lambda h} l^{ih}_{\mu)}\right).$$

If $g : \mathrm{M} \to \mathrm{L}$ is a section, then $\mathscr{B} \circ jg : \mathrm{M} \to \mathrm{B}$ is the Riemannian connection of g.

Let $\mathscr{R}$ be the affine morphism over $\mathrm{JL} \to \mathrm{L}$ given by the composition

$$\mathrm{J}^2\mathrm{L} \xrightarrow{\mathrm{J}\mathscr{B}} \mathrm{J}(\mathrm{L} \times \mathrm{B}) \to \mathrm{L} \times \mathrm{JB} \xrightarrow{id \times \mathscr{C}} \mathrm{L} \times \overset{2}{\wedge} \mathrm{T}^* \otimes \mathrm{T} \otimes \mathrm{T}^*.$$

Then we can prove that $\mathscr{R} : \mathrm{J}^2\mathrm{L} \to \mathrm{Q} \hookrightarrow \mathrm{L} \times \overset{2}{\wedge} \mathrm{T}^* \otimes \mathrm{T} \otimes \mathrm{T}^*$, where Q is the vector subbundle characterized by $\dot{x}_{ijhk} + \dot{x}_{ijkh} = 0$ and

$$\dot{x}_{ijhk} + \dot{x}_{hijk} + \dot{x}_{jhik} = 0.$$

The expression of the fibre derivative of $\mathscr{R}$ is

$$(\mathrm{D}\mathscr{R})^{\lambda\mu\alpha\beta}{}_{\varepsilon\eta}{}^{\rho\sigma} = \frac{1}{2}\left(\delta^{[\lambda}_{\varepsilon} l^{\mu]\rho} \delta^{[\alpha}_{\eta} l^{\beta]\sigma}\right).$$

If $g : M \to L$ is a section, then $R \equiv \mathscr{R} \circ j^2 g : M \to Q$ is the Riemannian connection of g.

Let $\hat{\mathscr{R}}$ be the affine morphism over $JL \to M$ given by the composition $J^2L \xrightarrow{\mathscr{R}} Q \xrightarrow{c} \bigvee_2 T$, where c is the linear morphism which contracts the first and third indices and then raises the two remaining covariant indices, by means of the metric exhibited by L. If $g : M \to L$ is a section, then $\hat{R} \equiv \hat{\mathscr{R}} \circ j^2 g : M \to \bigvee_2 T$ is the Ricci tensor of g.

In the following it will be useful to factorize the Ricci tensor through J^2L. For this purpose, we denote by $s : Q \to L \times \bigvee_2 T^* \otimes \bigvee_2 T$ the linear morphism over L given by $\dot{x}^{ij}_{\lambda\mu} \circ s \equiv l^{ih} l^{jk} \dot{x}_{(\lambda h \mu)k}$ and we denote by $\mathscr{S}$ the affine morphism over $JL \to L$ given by the composition

$$J^2L \xrightarrow{\mathscr{R}} Q \xrightarrow{s} L \times \bigvee_2 T^* \otimes \bigvee_2 T.$$

Then one gets the following commutative diagram

$$\begin{array}{ccc} J^2L & \xrightarrow{\mathscr{S}} & L \times \bigvee_2 T^* \otimes \bigvee_2 T \\ & \hat{\mathscr{R}} \searrow \quad \swarrow \tfrac{1}{2}\langle\, , \,\rangle & \\ & \bigvee_2 T & \end{array}$$

where $\langle\, , \,\rangle$ is the contraction exhibited by L. We remark that $\mathscr{S}$ is not a differential pseudoconnection as $D\mathscr{S} \neq 1$.

6. Riemannian differential operators

Let $g : M \to L$ be a given metric and let $\otimes_r \ulcorner : J \otimes_r T \to T^* \otimes_r T$ be the Riemannian connection. Then the affine morphism over $\otimes_r T \to M$ $\mathrm{div} \equiv \langle\, , \,\rangle \circ \otimes_r \ulcorner : J \otimes_r T \to \otimes_{r-1} T$ is the divergence operator. If $s : M \to \otimes_r T$ is a section, then we write $\mathrm{div}\, s \equiv \mathrm{div} \circ js : M \to \otimes_{r-1} T$. By considering the metric as a variable, one obtains naturally a morphism $JL \times J \otimes_r T \to \otimes_{r-1} T$. Analogous formulas hold for covariant tensors.

Let $p : E \to M$ be a vector bundle. A first order linear differential connection $\ulcorner^* : JT^* \to T^* \otimes T^*$ and a first order linear differential connection $H : JE \to T^* \otimes E$ induce a second order linear differential connection $H^2 : J^2E \to \bigvee_2 T^* \otimes E$ as follows. There is a unique morphism over M, linear on J^2E, $\mathscr{A}^2 : AT \times JAE \times J^2E \to \bigvee_2 T^* \otimes E$ which makes the following diagram commutative, for each section $s : \mathrm{M} \to \mathrm{E}$, $\urcorner : \mathrm{M} \to \mathrm{AT}$, $\mathrm{H} : \mathrm{M} \to \mathrm{AE}$,

$$\begin{array}{ccc} \mathrm{AT} \times \mathrm{JAE} \times \mathrm{J}^2\mathrm{E} & \xrightarrow{\mathscr{A}^2} & \bigvee_2 \mathrm{T}^* \otimes \mathrm{E} \\ & {\scriptstyle (\urcorner, j\mathrm{H}, j^2 s)} \nwarrow \quad \nearrow {\scriptstyle \mathrm{sym}(\underset{\urcorner^* \otimes \mathrm{H}}{\nabla} \underset{\mathrm{H}}{\nabla} s)} & \\ & \mathrm{M} & \end{array}$$

Its expression is

$$\dot{x}_{\lambda\mu} \otimes y^i \circ \mathscr{A}^2 = y^i_{\lambda\mu} + \left(-\frac{1}{2} a^{\nu}_{(\lambda\mu)} \delta^i_j + h^i_{(\lambda j} \delta^{\nu}_{\mu)} \right) y^j_{\nu}$$

$$+ \frac{1}{2} (h^i_{(\lambda j,\mu)} - a^{\nu}_{(\lambda\mu)} h^i_{\nu j} - h^i_{(\lambda k} h^k_{\mu) j}) y^j$$

In particular, if $\mathrm{E} \equiv \bigotimes_r \mathrm{T} \bigotimes_s \mathrm{T}^*$, then one obtains a morphism

$$\mathscr{A}^2 : \mathrm{JB} \times \mathrm{J}^2 \bigotimes_r \mathrm{T} \bigotimes_s \mathrm{T} \to \bigvee_2 \mathrm{T}^* \bigotimes_r \mathrm{T} \bigotimes_s \mathrm{T}^* .$$

We will denote by $\urcorner^2$ the second order linear differential connection induced by a first order linear connection $\urcorner : \mathrm{M} \to \mathrm{B}$.

REMARK. — In the present work we are not concerned with the general problem of existence of differential pseudoconnections, as we are mostly interested in specific and physically relevant ones. However the previous proposition gives a hint for solving this problem for linear differential connections of order k on a vector bundle E over a paracompact manifold M.

Let $g : \mathrm{M} \to \mathrm{L}$ be a given metric and let $\mathrm{R} : \mathrm{M} \to \mathrm{Q}$ be the Riemann tensor and $(\overset{p}{\wedge} \ulcorner^*)^2 : \mathrm{J}^2 \overset{p}{\wedge} \mathrm{T}^* \to \bigvee_2 \mathrm{T}^* \otimes \overset{p}{\wedge} \mathrm{T}^*$ be the second order Riemannian differential connection. Then the Laplace second order differential pseudoconnection is

$$\ulcorner_{\mathscr{L}} \equiv (\overset{p}{\wedge} \ulcorner^*)^2 + \theta : \mathrm{J}^2 \overset{p}{\wedge} \mathrm{T}^* \to \bigvee_2 \mathrm{T}^* \otimes \overset{p}{\wedge} \mathrm{T}^*$$

where $\theta : \overset{p}{\wedge} \mathrm{T}^* \to \bigvee_2 \mathrm{T}^* \otimes \overset{p}{\wedge} \mathrm{T}^*$ is the linear morphism given by

$$\theta_{\lambda\mu i_1 \dots i_p} \equiv - \sum_{1 \le q \le p} (-1)^q \mathrm{R}_{\lambda i_q \mu}{}^j \dot{x}_{j i_1 \dots \hat{i}_q \dots i_p}$$

$$+ \frac{2}{m} g_{\lambda\mu} (-1)^{r+s} \mathrm{R}^h{}_{i_r}{}^k{}_{i_s} \dot{x}_{hk i_1 \dots \hat{i}_r \dots \hat{i}_s \dots i_p} .$$

Hence the Laplace operator is given by the composition

$$\Delta \equiv g \circ ((\overset{p}{\wedge} \ulcorner^*)^2 + \theta) : \mathrm{J}^2 \overset{p}{\wedge} \mathrm{T}^* \to \overset{p}{\wedge} \mathrm{T}^* .$$

If $s : \mathrm{M} \to \overset{p}{\wedge} \mathrm{T}^*$ is a section, then we write

$$-(d\delta + \delta d)s = \Delta s \equiv \Delta \circ j^2 s : \mathrm{M} \to \overset{p}{\wedge} \mathrm{T}^* .$$

By considering the metric as a variable, one obtains naturally a morphism $\mathrm{J}^2\mathrm{L} \times \mathrm{J}^2 \overset{p}{\wedge} \mathrm{T}^* \to \overset{p}{\wedge} \mathrm{T}^*$.

7. **Clifford connection**

We are dealing now with a selfconsistent formulation of the spinorial connection in terms of jet spaces, which does not involve principal bundles.

Let $g : \mathrm{M} \to \mathrm{L}$ be a given metric. Here T denotes the complexified tangent space of M.

First we introduce the Clifford bundle and the Riemannian differential connection on it by means of a universal property. The Clifford bundle is a couple (γ, C), where $\mathrm{C} \to \mathrm{M}$ is a bundle whose fibre is a complex algebra with unity and where $\gamma : \mathrm{T} \to \mathrm{C}$ is a linear morphism over M, such that the following universal property holds: « If $\mathrm{A} \to \mathrm{M}$ is a bundle whose fibre is a complex algebra with unity and if $\varphi : \mathrm{T} \to \mathrm{A}$ is a linear morphism over M such that $\varphi(x)\varphi(x) = g(x, x)1_{\mathrm{A}}$ for each $x \in \mathrm{T}$, then there is a unique algebraic morphism over M $\mathrm{C}\varphi : \mathrm{C} \to \mathrm{A}$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{T} & \xrightarrow{\ \gamma\ } & \mathrm{C} \\ & {}_{\varphi}\searrow \quad \swarrow_{\mathrm{C}\varphi} & \\ & \mathrm{A} & \end{array} \quad . »$$

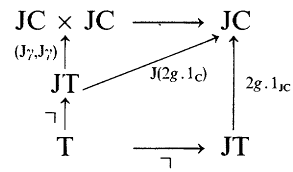
One can see by tensorial way that such bundle exists and one can deduce the following consequences of the universal property:

a) C is defined up to a unique isomorphism;
b) C is a covariant functor;
c) γ is injective and C is generated by $\gamma(\mathrm{T})$;
d) $\gamma(x)\gamma(y) + \gamma(y)\gamma(x) = 2g(x, y)$;
e) $\{ 1, \gamma_i, \ldots, \gamma_{i_1 \ldots i_p}, \ldots, \gamma_{1 \ldots m} \}_{1 \leq i_1 < \ldots < i_p \leq m}$ is a basis of C, where $\gamma_i \equiv \gamma(\partial x_i)$ and $\gamma_{i_1 \ldots i_p} \equiv \gamma_{i_1} \cdots \gamma_{i_p}$;
f) $\gamma_{i_1 \ldots i_p}\gamma_j = (-1)^{r-1} 2 g_{j i_r} \gamma_{i_1 \ldots \hat{i}_r \ldots i_p} + (-1)^{p-1} \gamma_j \gamma_{i_1 \ldots i_p}$;
g) $\dim \mathrm{C} = 2^m$.

PROPOSITION. — There is a unique algebraic differential connection $\urcorner_{\mathrm{C}} : \mathrm{C} \to \mathrm{JC}$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{\ \urcorner_{\mathrm{C}}\ } & \mathrm{JC} \\ & {}^{\gamma}\nwarrow \quad \nearrow^{\mathrm{J}\gamma \circ \urcorner} & \\ & \mathrm{T} & \end{array} \qquad (+)$$

Proof. — We can show that the linear morphism $\varphi \equiv J\gamma \circ \urcorner : T \to JC$ satisfies the requirement of the universal property, taking into account the commutative diagram



Hence there is a unique algebraic morphism $C\varphi \equiv \urcorner_C : C \to JC$ which makes the diagram (+) commutative and which is a section. The expression of $\urcorner_C$ is given by $\nabla_\lambda \gamma_i = \Gamma^k_{\lambda i}\gamma_k$.

It is well known that if $m \equiv 2n$, then C is locally simple. Moreover, if the second Stiefel-Witney class is zero, then there is a vector bundle $s : S \to M$ with dimension 2^n, called the spinors bundle, and an algebraic isomorphism over M $C \to \mathrm{End}\, S$. S and the previous isomorphism are not canonically determined. Henceforth we assume a spinorial representation of C to exist, we choose one and make the identification $C \equiv \mathrm{End}\, S$. Then we have a bilinear morphism over M $C \times S \to S$.

Let M be time oriented. Then we denote by α the unique (up to a complex factor) bilinear antisymmetric nondegenerate form of S such that, $\forall u : M \to T$, $\Lambda : M \to \mathrm{Spin}$, *a)* $\alpha(\psi, \gamma(u)\varphi) = -\alpha(\gamma(u)\psi, \varphi)$, *b)* $\alpha(\Lambda\psi, \Lambda\varphi) = \rho(\bar{\Lambda})\alpha(\psi, \varphi)$; β the unique (up to a positive factor) sesquilinear symmetric nondegenerate form of S such that, $\forall u : M \to T$, $\Lambda : M \to \mathrm{Spin}$, *a)* $\beta(\psi, \gamma(u)\varphi) = -\beta(\gamma(u)\psi, \varphi)$, *b)* $\beta(\Lambda\psi, \Lambda\varphi) = \rho(\bar{\Lambda})\beta(\psi, \varphi)$, *c)* $\psi \mapsto i\beta(\psi, \gamma(u)\psi)$ is positive definite for u time oriented, where $\rho(\bar{\Lambda}) = \pm 1$ is the time signature of $\bar{\Lambda}$ and $\bar{\Lambda}$ is the Lorentz isomorphism associated with Λ. The assumed representation $C \to \mathrm{End}\, S$ induces a natural chart on S, which accounts γ, α and β by constant coefficients.

PROPOSITION. — There is a unique linear differential connection $\urcorner_S : S \to JS$ such that *a)* $\urcorner_C = \urcorner_S \otimes \urcorner_S^*$; $\nabla\alpha = 0$.

Proof. — In a natural chart of S the two conditions read (by matricial notation) *a)* $\Gamma_{\lambda S}\gamma_i - \gamma_i\Gamma_{\lambda S} = \Gamma^j_{\lambda i}\gamma_j$; *b)* $\Gamma^t_{\lambda S} = -\Gamma_{\lambda S}$. This system has the unique solution $\Gamma_{\lambda S} = \frac{1}{4}\Gamma^{[ij]}_{\lambda}\gamma_i\gamma_j$. In fact it is a solution and the kernel is constituted by antisymmetric elements which commute with each γ_i, hence it is zero. Then one obtains $\nabla\beta = 0$.

The Dirac operator is the morphism $\mathrm{Dir} \equiv G' \circ \Gamma_S : JS \to S$, where G' is a linear morphism over M given by the composition

$$T^* \otimes S \xrightarrow{g \otimes id} T \otimes S \xrightarrow{\gamma \otimes id} C \otimes S \to \mathrm{End}\, S \otimes S \to S\,.$$

The spinorial Laplace operator (Lichnerowicz) is defined on sections of S as minus the square of the Dirac operator. We can express it in terms of jet spaces as follows $\Delta_S = g \circ \left(\Gamma^2_S - \frac{1}{m} \hat{\underline{R}} \otimes s^{02} \right) : J^2S \to S$, where $\Gamma^2_S : J^2S \to \bigvee_2 T^* \otimes S$ is the second order spinorial connection, $\hat{\underline{R}} : M \to \bigvee_2 T^*$ is the Ricci tensor and $\hat{\underline{R}} \otimes s^{02} : J^2S \to \bigvee_2 T^* \otimes S$.

8. Systems of differential equations on jet spaces

The following definition of system of differential equations (more briefly « differential equation »), both of ordinary or partial derivatives system, is suitable for our purposes.

DEFINITION. — Let $p : E \to M$ be a bundle. *A differential equation*, of order $k \geq 1$, on E is a subbundle $\varepsilon^k \hookrightarrow J^kE$ over E. A (local) *solution* of ε^k is any (local) section $s : M \to E$ such that $j^k s : M \to \varepsilon^k$. If dim M $\equiv 1$, then one has an ordinary differential equation; if dim M > 1, then one has a partial differential equation. We can find a chart of J^kE of the form

$$\{ x^\lambda, y^i, f^1, \ldots, f^r, \ldots, f^s \},$$

such that the local expression of the equation ε^k is $f^1 = \ldots = f^r = 0$.

Let $h > k$; then the h-prolongation of the differential equation ε^k is $\varepsilon^h \equiv J^{h-k}\varepsilon^k \cap J^hE$. If ε^h itself is a differential equation, then ε^k and ε^h have the same (local) solutions.

The following definition of datum is suitable for our purposes. Let ε^k be a differential equation. Let $i : \Sigma \hookrightarrow M$ be a hypersurface. A datum of ε^k on Σ is a section $\sigma : \Sigma \to i^*J^kE$ which makes the following diagram commutative

$$\begin{array}{ccc} i^*\varepsilon^k & \hookrightarrow & i^*J^kE \\ \uparrow{\scriptstyle\sigma} & & \downarrow \\ \Sigma & \xrightarrow{j^{k-h}(p^{hk}\circ\sigma)} & J^k i^*E \end{array} \qquad 0 \leq h < k$$

We say that a (local) solution $s : M \to E$ of ε^k agrees with the datum σ on Σ if $i^* j^k s = \sigma$.

Moreover we say that ε^k is a Cauchy differential equation if, for each $x \in M$, there exist data hypersurfaces Σ, passing through x, there exist data on Σ and for each datum σ there is locally a unique solution s agreeing with σ. We could give a more general definition, but the previous one is sufficient for our purposes.

An affine differential equation ε^k is an affine subbundle of J^kE over $J^{k-1}E$. Its expression is $f^{\lambda_1\ldots\lambda_k j}_{\mu_1\ldots\mu_r i} y^i_{\lambda_1\ldots\lambda_k} = c^j_{\mu_1\ldots\mu_r}$, where the f's and the c's are maps $J^{k-1}E \to \mathbb{R}$.

CHAPTER TWO

FIELD THEORIES

1. Field theories

First we consider a scheme which fits a very large class of field theories.

DEFINITION. — *A field theory* $\mathscr{F}^k$ is constituted by *a*) a bundle $p : E \to M$, called the *field bundle*; *b*) a Cauchy differential equation (I, 8) ε^k, called the *field equation*. A (local) *field* of $\mathscr{F}^k$ is a (local) solution (I, 8) of ε^k. We denote by I the family of data hypersurfaces of ε^k.

DEFINITION. — A *gauged field theory* is a field theory $\mathscr{F}^k$, with moreover, *c*) a differential equation $\mathscr{G}^h$, with $1 \leq h < k$, called the *gauge equation*, such that 1) $\mathscr{G}^k$ is a differential equation (I, 8); 2) $\forall \Sigma \in I$, there exist data σ of ε^k which are also data of $\mathscr{G}^k$; 3) the local solution of ε^k related to σ is also a local solution of $\mathscr{G}^h$. A *gauge* is a datum σ as above. A (local) *gauged field* is a (local) field related to a gauge.

DEFINITION. — Let $(\mathscr{F}^k, \mathscr{G}^h)$ be a gauged field theory. A *dynamical equation* is a differential equation $\mathscr{D}^k$, such that $\varepsilon^k \cap \mathscr{G}^k = \mathscr{D}^k \cap \mathscr{G}^k = \mathscr{D}^k \cap \varepsilon^k$.

Of course, any gauged field is a solution of $\mathscr{D}^k$.

Let $\mathscr{F}^k$ be a gauged field theory and $\mathscr{F}'^k$ be a field theory, such that 1) $(E, p, M) = (E', p', M')$; 2) $\varepsilon'^k \subset \varepsilon^k$; 3) $I = I'$. Then we say that $\mathscr{F}^k$ and $\mathscr{F}'^k$ are *equivalent* if each gauged field of $\mathscr{F}^k$ is also a field of $\mathscr{F}'^k$.

In particular, let $(\mathscr{F}^k, \mathscr{G}^h)$ be a gauged field theory. Then it is equivalent to the field theory $\mathscr{F}'^k$ characterized by the field equation $\varepsilon'^k \equiv \varepsilon^k \cap \mathscr{G}^k$. Moreover, if $\mathscr{F}^k$ has a dynamical equation $\mathscr{D}^k$, then $\varepsilon'^k = \mathscr{D}^k \cap \mathscr{G}^k$.

We can state a general meaning for some usual physical concepts.

DEFINITION. — We say that a field theory $\mathscr{F}'^k$ is a *subtheory* of a field theory $\mathscr{F}^k$ if 1) there is a monomorphism $c : E' \hookrightarrow E$ over $M = M'$, called the *constraint*; 2) $J^k c : \varepsilon'^k \hookrightarrow \varepsilon^k$; 3) $I = I'$. We can prove that *a*) if $\sigma' : \Sigma \to E'$ is a datum of ε'^k, then $\sigma \equiv c \circ \sigma' : \Sigma \to E$ is a datum of ε^k; *b*) if $s : M \to E$ is the (local) field of ε^k related to σ, then s is also the (local) field of ε'^k related to σ'.

DEFINITION. — Let $\mathscr{F}^k$ be a field theory. We say that $\mathscr{F}^k$ is an *interaction theory* if E is the product $E \equiv E_1 \times \ldots \times E_r$ over M induced by the epimorphisms $\pi_i : E \to E_i$. Then each field $s : M \to E$ is uniquely decomposed as $s = (s_1, \ldots, s_r)$, where $s_i \equiv \pi_i \circ s : M \to E_i$. However the theory itself can either be decomposed into a system of r independent theories or not. We say that the *interaction vanishes* if $\varepsilon^k = \varepsilon_1^k \times \ldots \times \varepsilon_r^k$ and ε_i^k are Cauchy equations with $I_1 = \ldots = I_r = I$. In the other case we say that the *interaction is effective*. In the case when the bundles E_i exhibit canonical differential equations $\varepsilon_{\circ i}^k$, we obtain naturally a *free theory*, with vanishing interaction, related to $\mathscr{F}^k$ and characterized by the field equation

$$\varepsilon_\circ^k \equiv \varepsilon_{\circ 1}^k \times \ldots \times \varepsilon_{\circ r}^k .$$

If it does not occur, then we need inclusions $\rho_i : E_i \hookrightarrow E$, such that $\pi_i \circ \rho_i = id_{E_i}$ (besides the given projections $\pi_i : E \to E_i$) in order to obtain a *free theory*, with vanishing interaction, related to $\mathscr{F}^k$. In fact, we can set

$$\varepsilon_{\circ i}^k \equiv (J^k \rho_i)(E_i) \cap \varepsilon^k$$

and the differential equation

$$\varepsilon_\circ^k \equiv \varepsilon_{\circ 1}^k \times \ldots \times \varepsilon_{\circ r}^k$$

works out.

Thus we see that interaction problems in field theory are related to arrows and directions of arrows.

The previous statements concerning interactions can be easily extended to gauged field theories.

2. **Special field theories**

We now are going to consider a special type of field theories, which seems to arise naturally in the context of jet spaces.

DEFINITION. — A *special field theory* is a field theory $\mathscr{F}^2$, whose field equation is the affine equation (I, 8) $\varepsilon^2 \equiv \ker(G \circ H)$, where *a)* $H : J^2E \to \overline{J^2E}$ is a second order differential pseudoconnection (I, 3); *b)* $G : \overline{J^2E} \to VE$ is a surjective submersion over $JE \to E$, which is decomposable as $G = \tilde{G} \otimes id_{VE}$, where $\tilde{G} : p^{1*} \overset{2}{\vee} T^* \to E \times \mathbb{R}$ is a nondegenerate linear morphism over $JE \to E$. Hence a field of the special theory $\mathscr{F}^2$ is a section $s : M \to E$, which makes the following diagram commutative

$$\begin{array}{ccc} J^2E & \xrightarrow{\ H\ } & \overline{J^2E} \\ {\scriptstyle j^2s}\uparrow & & \downarrow{\scriptstyle G} \\ M & \overset{0}{\to}\ s^*VE \hookrightarrow & VE \end{array}$$

The local expression of such an equation is given by the system of l quasi-linear partial differential equations of second order

$$G^{\lambda\mu} \circ js\, (\partial_\lambda \partial_\mu s^i + H^i_{\lambda\mu} \circ js) = 0 , \qquad 1 \leq i \leq l ,$$

where $G^{\lambda\mu} : JE \to \mathbb{R}$, $\det(G^{\lambda\mu}) \neq 0$, $H^i_{\lambda\mu} : JE \to \mathbb{R}$.

In the particular case when $p : \mathrm{E} \to \mathrm{M}$ is a vector bundle, we use the simpler notation $\mathrm{G} : \overline{\mathrm{J}^2\mathrm{E}} \to \overline{\mathrm{E}}$ (I, 3).

Let us remark that, given ε^2, H is determined up to a map $\mathrm{H} \mapsto \mathrm{H} + \mathrm{K}$, where $\mathrm{K} : \mathrm{JE} \to \bigvee_2 \mathrm{T}^* \otimes \mathrm{VE}$ is such that $\mathrm{G} \circ \mathrm{K} = 0$.

We can review the statements of the previous section in the particular framework of special field theories.

Let $\mathscr{F}^2$ and $\mathscr{F}'^2$ be two special field theories such that

a) there is a monomorphism $c : \mathrm{E}' \hookrightarrow \mathrm{E}$ over $\mathrm{M} = \mathrm{M}'$;
b) $\mathrm{H} \circ \mathrm{J}^2 c : \mathrm{J}^2\mathrm{E}' \to \overline{\mathrm{J}^2\mathrm{E}'} \hookrightarrow \overline{\mathrm{J}^2\mathrm{E}}$ and $\mathrm{H}' = \mathrm{H} \circ \mathrm{J}^2 c$;
c) $\mathrm{G} \circ \mathrm{DJ}^2 c : \overline{\mathrm{J}^2\mathrm{E}'} \to \mathrm{VE}' \hookrightarrow \mathrm{VE}$ and $\mathrm{G}' = \mathrm{G} \circ \mathrm{DJ}^2 c$;
d) $\mathrm{I} = \mathrm{I}'$.

Then $\mathscr{F}'^2$ is a subtheory of $\mathscr{F}^2$. In fact the following diagram is commutative

$$\begin{array}{ccccccccccc} & \nearrow & \varepsilon^2 & \hookrightarrow & \mathrm{J}^2\mathrm{E} & \xrightarrow{\mathrm{H}} & \overline{\mathrm{J}^2\mathrm{E}} & \xrightarrow{\mathrm{G}} & \mathrm{VE} & \searrow & \\ 0 & & \uparrow & & \uparrow \scriptstyle{\mathrm{J}^2 c} & & \uparrow \scriptstyle{\mathrm{DJ}^2 c} & & \uparrow \scriptstyle{\mathrm{T}c} & & 0 \\ & \searrow & \varepsilon'^2 & \hookrightarrow & \mathrm{J}^2\mathrm{E}' & \xrightarrow{\mathrm{H}'} & \overline{\mathrm{J}^2\mathrm{E}'} & \xrightarrow{\mathrm{G}'} & \mathrm{VE}' & \nearrow & \end{array}$$

If $\mathscr{F}^2$ is an interaction special field theory such that

a) H and G are factorizable as follows

$$\begin{array}{ccc} \mathrm{J}^2\mathrm{E} & \xrightarrow{\mathrm{H}} & \overline{\mathrm{J}^2\mathrm{E}} \\ \downarrow \scriptstyle{\mathrm{J}^2\pi_i} & & \downarrow \scriptstyle{\mathrm{DJ}^2\pi_i} \\ \mathrm{J}^2\mathrm{E}_i & \xrightarrow{\mathrm{H}_i} & \overline{\mathrm{J}^2\mathrm{E}_i} \end{array} \qquad \begin{array}{ccc} p^{1*} \bigvee_2 \mathrm{T}^* & \xrightarrow{\tilde{\mathrm{G}}} & \mathbb{R} \\ & \searrow \quad \nearrow & \\ & \bigvee_2 \mathrm{T}^* & \end{array}$$

b) $\varepsilon_i^2 \equiv \ker(\mathrm{G}_i \circ \mathrm{H}_i)$ are Cauchy equations,

then the interaction vanishes.

Let $\mathscr{F}^2$ be an interaction special field theory. Moreover, let $\rho_i : \mathrm{E}_i \hookrightarrow \mathrm{E}$ be monomorphisms over M such that $\pi_i \circ \rho_i = id_{\mathrm{E}_i}$. Then we obtain an interaction vanishing special field theory $\mathscr{F}_0^2$ by the setting

$$\mathrm{H}_i \equiv \mathrm{DJ}^2\pi_i \circ \mathrm{H} \circ \mathrm{J}^2\rho_i\,, \qquad \mathrm{G}_i \equiv \mathrm{T}\pi_i \circ \mathrm{G} \circ \mathrm{DJ}^2\rho_i\,,$$
$$\varepsilon_{\circ i}^2 \equiv \ker(\mathrm{G}_i \circ \mathrm{H}_i) = (\mathrm{J}^2\rho_i)(\mathrm{E}_i) \cup \varepsilon^2\,.$$

We can also define the interaction term $\mathrm{H} - (\mathrm{H}_1 \times \ldots \times \mathrm{H}_r) : \mathrm{JE} \to \overline{\mathrm{J}^2\mathrm{E}}$.

We get similar results in the case when the bundles E_i exhibit canonical differential pseudoconnections $\mathrm{H}_i : \mathrm{J}^2\mathrm{E}_i \to \overline{\mathrm{J}^2\mathrm{E}_i}$.

Now we want to show that several fundamental physical fields can be suitably described in the framework of special field theories.

3. Classical *n*-body dynamics

A first example of special field theory is supplied by the classical *n*-body dynamics. We can put any kind of force in the special scheme naturally, even if it does not lead to a Lagrangian. In this context we can expound the concepts of interaction, vanishing interaction, free theory, interaction term and constraint.

The basic physical assumptions are the following:

a) M is a one dimensional affine oriented space, with an Euclidean metric $\tilde{G}$ (absolute time with a unity of measure); *b*) $\overset{\circ}{E}$ is an affine four dimensional space (event space); *c*) $p : \overset{\circ}{E} \to M$ is an affine surjective map (time function); *d*) $\overset{\circ}{g}$ is an Euclidean metric on $\bar{\overset{\circ}{E}}$ (space-like metric); *e*)

$$(m_1, \ldots, m_n) \in \mathbb{R}^{+n} \qquad (n\text{-mass}) ;$$

$\underline{F} : JE \to \bar{E}^*$ is a morphism (for E see later on) (covariant force).

Moreover we set:

1) $E \equiv \overset{\circ}{E} \times \ldots \times \overset{\circ}{E}$ is the n-fibred product over M (n-event space);
2) $p : E \to M$ is the induced surjective map (n-time function);
3) $\overset{\circ}{\Gamma} : J^2\overset{\circ}{E} \to \overline{J^2\overset{\circ}{E}}$ and $\Gamma = (\overset{\circ}{\Gamma} \times \ldots \times \overset{\circ}{\Gamma}) : J^2E \to \overline{J^2E}$ are the affine differential connections induced by the affine structures of the spaces $\overset{\circ}{E}$ and E (inertial connection);
4) $g \equiv (m_1\overset{\circ}{g}, \ldots, m_n\overset{\circ}{g})$ is the product Euclidean metric on $\bar{E}$;
5) $F \equiv g \circ \underline{F} : JE \to \bar{E}$ (contravariant force).

Then we obtain a special field theory in the following way. The field bundle is $p : E \to M$. The force modifies the inertial connection Γ giving the differential pseudoconnection $H \equiv \Gamma - (\underset{\sim}{G} \otimes F) \circ p^{12}$. Then the field equation ε^2 is the kernel of the affine morphism given by the composition

$$J^2E \xrightarrow{H} \bigvee_2 T^* \otimes \bar{E} \xrightarrow{G} \bar{E},$$

where G is the contraction induced by $\tilde{G}$. The solutions of ε^2 are the sections (motions) $s : M \to E$ such that $\tilde{G} \circ \Gamma \circ j^2s = F \circ js$, i. e. locally (in a chart such that $\tilde{G} = 1$)

$$D^2s^i + \Gamma^i_{hk} \circ s\, Ds^hDs^k + 2\Gamma^i_{0k} \circ s\, Ds^k + \Gamma^i_{00} \circ s - F^i \circ js = 0 .$$

Let us remark that $2\Gamma^i_{0k} \circ s\, Ds^k$ and $\Gamma^i_{00} \circ s$ represent the Coriolis and the dragging forces with respect to the frame of reference associated with the chosen chart.

In this example the inertial connection provides a canonical free theory and the force appears as the interaction term with respect to it. Moreover, in the particular case when $F \equiv (F_1 \times \ldots \times F_n)$ with F_i factorizable through $JE_i \to \bar{E}_i$, the theory $\mathscr{F}^2$ itself is vanishing interaction.

Finally we add the following physical assumptions in order to get the constrained dynamics: 6) $c : E' \hookrightarrow E$ is a subbundle (constraint);

7) $(id \underset{2}{\vee} T^* \otimes p^{\perp}) \circ H \circ J^2 c = 0$ (reaction assumption), where $p^{\perp} : c^*\bar{E} \to c^*\bar{E}^{\perp}$ is the orthogonal projection induced by g.

Then we obtain a constrained special field theory. The field bundle is $p' : E' \to M$. The differential pseudoconnection is

$$H' \equiv (id \underset{2}{\vee} T^* \otimes p'') \circ H \circ J^2 c, \qquad \text{where } p'' : c^*\bar{E} \to \bar{E}'$$

is the parallel projection induced by g. The field equation is the kernel of the affine morphism given by the composition $J^2E' \xrightarrow{H'} \underset{2}{\vee} T^* \otimes E' \xrightarrow{G'} E'$.

4. General relativistic one-body dynamics

An example of special gauged field theory is supplied by the general relativistic one-body dynamics. In such case, the gauge, which is introduced in order to remove a physical indeterminacy, does not modify the original special character of the theory.

The basic physical assumptions are the following:

a) M is a one dimensional affine oriented space, with an Euclidean metric $\underset{\sim}{G}$ (proper time with a unity of measure); *b*) U is a four dimensional time-orientable manifold, with a Lorentz metric $\overset{\circ}{g} : U \to \underset{2}{\vee} TU$ (space-time); $m \in \mathbb{R}^+$ (rest mass); $F : JE \to T^*U$ is a morphism such that $\langle \underset{\sim}{F}(u), u \rangle = 0$, $\forall u \in JE$ (for E see later on) (covariant force).

Moreover we set:

1) $E \equiv M \times V$ and $p \equiv \pi^1 : E \to M$; 2) $\Gamma : J^2E \to \underset{2}{\vee} T^* \otimes TU$ is the connection naturally induced by the Riemannian connection (inertial connection); 3) $g \equiv \frac{1}{m}\overset{\circ}{g}$; 4) $F \equiv g \circ \underset{\sim}{F} : JE \to TU$ (contravariant force).

Then we obtain a gauged special field theory in the following way. The field bundle is $p : E \to M$. The force modifies the inertial connection Γ giving the differential pseudoconnection $H \equiv \Gamma - (\underset{\sim}{G} \otimes F) \circ p^{12}$. Then the field equation ε^2 is the kernel of the affine morphism given by the composition $J^2E \xrightarrow{H} \underset{2}{\vee} T^* \otimes TU \xrightarrow{G} TU$, where G is the contraction induced by $\tilde{G}$. The solutions of ε^2 are the sections (motions) $s : M \to E$ such that $\tilde{G} \circ \Gamma \circ j^2 s = F \circ js$, i. e. locally (in a chart such that $\tilde{G} = 1$) $D^2 s^i + \Gamma^i_{hk} \circ s\, Ds^h Ds^k - F^i \circ js = 0$. The gauge equation is the first order differential equation $\mathcal{G}^1 \subset JE$, which is the kernel over E of the morphism $\underset{=}{g} + m : JE \to \mathbb{R}$. The solutions of $\mathcal{G}^1$ are the sections such that

$$g \circ (js, js) = -m,$$

i. e. locally $g_{ij} \circ s \, Ds^i Ds^j = -m$. The requirement on the gauge is satisfied because of the properties of the Riemannian connection and of the force. Let us remark that the equation $\varepsilon^2 \cap \mathcal{G}^2$ is almost a special equation induced by a pseudoconnection, but not properly by a differential pseudoconnection.

5. General relativistic Klein-Gordon field

The Klein-Gordon equation provides a very simple example of special field theory, without any problems.

The basic physical assumptions are the following:

a) M is a four dimensional manifold, with a Lorentz metric $g : M \to \underset{2}{\vee} T$ (space-time); *b*) $m \in \mathbb{R}^+$ (mass).

Moreover we set:

1) $E \equiv M \times \mathbb{R}$ and $p \equiv \pi^1 : E \to M$; 2) $\Gamma_{\mathcal{L}} : J^2E \to \underset{2}{\vee} T^*$ is the Laplace second order differential connection (I, 6).

Then we obtain a special field theory in the following way. The field bundle is $p : E \to M$. The mass modifies the Laplace connection $\Gamma_{\mathcal{L}}$ giving the differential pseudoconnection $H = \Gamma_{\mathcal{L}} - \frac{m^2}{4} g \otimes p^{02}$. Then the field equation (Klein-Gordon) ε^2 is the kernel of the affine morphism given by the composition $J^2E \xrightarrow{H} \underset{2}{\vee} T^* \xrightarrow{G} M \times \mathbb{R}$, where G is the contraction induced by g. The solutions of ε^2 are the sections $s : M \to E$ such that $\Delta s - m^2 s = 0$, i. e. locally

$$g^{ij}\left(\partial_i \partial_j s - \Gamma^h_{ij} \partial_h s - \frac{m^2}{4} g_{ij} s \right) = 0 .$$

6. The general relativistic Maxwell field

The Maxwell field provides an example of gauged special field theory with a dynamical equation. The gauge condition, which is introduced in order to remove a physical indeterminacy, is the best candidate to make the original affine but not special equation into a special one.

The basic physical assumption is the following:

a) M is a four dimensional manifold, with a Lorentz metric $g : M \to \underset{2}{\vee} T$ (space-time).

Moreover we set:

1) $E \equiv T^*$ and $p : T^* \to M$; 2) $\Gamma^* : JE \to T^* \otimes E$ is the Riemannian connection; 3) $d : JE \to \underset{2}{\wedge} E$ is the exterior derivative (I, 2).

Then we obtain a special gauged field theory, with a dynamical equation, in the following way. The field bundle is $p : \mathrm{E} \to \mathrm{M}$. The dynamical equation is the second order affine differential equation (Maxwell) $\mathscr{D}^2 \subset \mathrm{J}^2\mathrm{E}$, which is the kernel over JE $\to$ M of the affine morphism given by the composition (I, 4) $\mathrm{J}^2\mathrm{E} \xrightarrow{\mathrm{J}d} \mathrm{J}^2\mathrm{E} \xrightarrow{\overset{2}{\wedge}\Gamma^*} \mathrm{T}^* \otimes^2 \mathrm{E} \xrightarrow{g} \mathrm{E}$. The solutions of $\mathscr{D}^2$ are the sections $\mathrm{A} : \mathrm{M} \to \mathrm{E}$ such that $\delta d\mathrm{A} = 0$, i. e. locally

$$g^{hk}(\partial_h \partial_{[k}\mathrm{A}_{j]} + \Gamma^{i}_{hk}\partial_{[i}\mathrm{A}_{j]} + \Gamma^{i}_{kj}\partial_{[i}\mathrm{A}_{k]}) = 0 .$$

Let us remark that $\overset{2}{\wedge}\, \Gamma^* \circ \mathrm{J}d$ is not a pseudoconnection, as its derivative is not 1. The field equation is the second order affine differential equation (Laplace) $\varepsilon^2 \subset \mathrm{J}^2\mathrm{E}$, which is the kernel over JE $\to$ M of the affine morphism given by the composition $\mathrm{J}^2\mathrm{E} \xrightarrow{\mathrm{H}} \underset{2}{\vee}\, \mathrm{T}^* \otimes \mathrm{E} \xrightarrow{\mathrm{G}} \mathrm{E}$, where $\mathrm{H} \equiv \Gamma_{\mathscr{L}}$ is the Laplace differential connection (I, 6) and G is the contraction induced by g. The solutions of ε^2 are the sections such that

$$\Delta \mathrm{A} \equiv \delta d\mathrm{A} + d\delta \mathrm{A} = 0 .$$

The gauge equation is the first order affine differential equation (Lorentz) $\mathscr{G}^1 \subset \mathrm{JE}$, which is the kernel over E $\to$ M of the affine morphism given by the composition $\mathrm{JE} \xrightarrow{\Gamma^*} \mathrm{T}^* \otimes \mathrm{E} \longrightarrow \mathrm{M} \times \mathbb{R}$. The solutions of G^1 are the sections such that $\delta\mathrm{A} = 0$, i. e. locally $g^{hk}(\partial_h \mathrm{A}_k - \Gamma^{i}_{hk}\mathrm{A}_i) = 0$. The requirement on the gauge can be deduced from classical results (Lichnerowicz).

7. The general relativistic Dirac field

The Dirac field provides an example of gauge special field theory. To tell the truth, the Dirac equation itself could be considered as a first order almost special equation (it comes from a first order differential connection and a not decomposable contraction). However we can obtain a proper second order equation (for instance, it is suitable for a symmetric treatment of the Maxwell-Dirac interaction): in such case the Dirac equation plays the role of a gauge equation.

The basic physical assumptions are the following:

a) M is a four dimensional time oriented manifold, with a Lorentz metric: $g : \mathrm{M} \to \underset{2}{\vee}\, \mathrm{T}$, admitting a global spinor structure (space-time); *b)* $m \in \mathbb{R}^+$ (electron mass).

Moreover we set:

1) C $\to$ M is the Clifford bundle and $\gamma : \mathrm{T} \to \mathrm{C}$ is the canonical monomorphism (T is the complexified tangent space) (I, 7); 2) $\mathrm{E} \equiv \mathrm{S}$ and $p : \mathrm{S} \to \mathrm{M}$

is the spinor bundle (I, 7); 3) a linear representation $C \cong \operatorname{End} S$ is given; 4) $\Gamma_S : JE \to T^* \otimes E$ is the spinor differential connection (I, 7); 5) $\Gamma_S^2 : J^2E \to \bigvee_2 T^* \otimes E$ is the second order differential connection induced by the Riemannian connection and by Γ_S (I, 6).

Then we obtain a gauged special field theory in the following way. The field bundle is $p : E \to M$. The mass and the Ricci tensor modify the connection Γ_S^2 giving the differential connection

$$H \equiv \Gamma_S^2 + \frac{1}{4}(m^2 g - \hat{R}) \otimes p^{02}.$$

Then the field equation (Lichnerowicz) ε^2 is the kernel of the affine morphism over $JE \to M$ given by the composition $J^2E \xrightarrow{H} \bigvee_2 T^* \otimes E \xrightarrow{G} E$, where G is the contraction induced by g. The solutions of ε^2 are the sections $\psi : M \to E$ such that $\Delta_S\psi + m^2\psi = 0$, i. e. locally

$$g^{\lambda\mu}(\partial_\lambda\partial_\mu\psi^i + \Gamma^{i\nu}_{\lambda\mu j}\partial_\nu\psi^j + \Gamma^{i}_{\lambda\mu j}\psi^j) + \left(m^2 - \frac{r}{4}\right)\psi^i = 0,$$

where r is the scalar curvature. The gauge equation is the first order affine differential equation (Dirac) $\mathcal{G}^1 \subset JE$, which is the kernel over $E \to M$ of the affine morphism given by the composition $JE \xrightarrow{H'} T^* \otimes E \xrightarrow{G'} E$, where H' is the differential connection $H' \equiv \Gamma_S + \frac{m}{4}\gamma(p^{01})$ and G' is the contraction defined in (I, 7). The solutions of $\mathcal{G}^1$ are the sections such that (I, 7) $\operatorname{Dir} j\psi + m\psi = 0$, i. e. locally

$$g^{\lambda\mu}\gamma^i_{j\lambda}\left(\partial_\mu\psi^j + \Gamma^j_{k\mu}\psi^k + \frac{m}{4}\gamma^i_{k\mu}\psi^k\right) \equiv g^{\lambda\mu}\gamma^i_{j\lambda}(\partial_\mu\psi^j + \Gamma^j_{k\mu}\psi^k) + m\psi^i = 0.$$

Let us remark that the above decomposition of $\mathcal{G}^1$ is allowed just by the anticommutation rules of γ (I, 7).

8. The general relativistic Maxwell-Dirac field

The Maxwell-Dirac field provides an example of interaction gauged special theory, without problems. In fact it sufficies just to take $E \equiv T^* \times S$ as the product bundle. Moreover we can write the usual interaction terms in such a way to obtain a modification term of the free product differential connection $H \equiv H_{Max} \times H_{Dir} : J^2E \to \bigvee_2 T^* \otimes E$. Analogous results hold for the gauge connection. The contractions remain unchanged. We leave the computations to the reader.

9$_{\mathrm{I}}$. The Einstein field

The Einstein field provides an example similar to the Maxwell case. The main new aspect is that now we have not a canonical differential pseudoconnection and we need an arbitrary choice.

The basic physical assumptions are the following:

a) M is a four dimensional manifold (space-time); *b*) $\mathcal{T} : \mathrm{JL} \to \underset{2}{\vee} \mathrm{T}$ is a given section (energy stress tensor); *c*) $\hat{\urcorner} : \mathrm{M} \to \mathrm{B}$ is a given connection, arbitrarily chosen (gauge connection) (I, 4).

Moreover we set:

1) $\mathrm{E} \equiv \mathrm{L} \subset \underset{2}{\vee} \mathrm{T}$ is the bundle of the Lorentz metrics and $p : \mathrm{L} \to \mathrm{M}$;

2) $\underline{p}^{0i} : \mathrm{J}^i\mathrm{L} \to \underset{2}{\vee} \mathrm{T}^*$ is the covariant expression of the projection $p^{0i} : \mathrm{J}^i\mathrm{L} \to \mathrm{L}$;

3) $\hat{\Gamma}^2 : \mathrm{J}^2\mathrm{L} \to \mathrm{L} \times \underset{2}{\vee} \mathrm{T}^* \otimes \underset{2}{\vee} \mathrm{T}$ is the second order connection induced by $\hat{\Gamma}$.

Then we obtain a special gauged field theory, with a dynamical equation, in the following way. The field bundle is $p : \mathrm{E} \to \mathrm{M}$. The dynamical equation is the second order affine differential equation (Einstein) $\mathcal{D}^2 \subset \mathrm{J}^2\mathrm{E}$, which is the Kernel over $\mathrm{JE} \to \mathrm{M}$ of the affine morphism given by the composition (I, 5) $\mathrm{J}^2\mathrm{E} \xrightarrow{\mathrm{H}''} \mathrm{L} \times \underset{2}{\vee} \mathrm{T}^* \otimes \underset{2}{\vee} \mathrm{T} \xrightarrow{\mathrm{G}} \underset{2}{\vee} \mathrm{T}$, where H″ is the affine morphism $\mathrm{H}'' \equiv \mathcal{S} - \frac{1}{2}\underline{p}^{02} \otimes \left(\mathcal{T} \circ p^{12} - \frac{1}{2} p^{02} \operatorname{tr} \mathcal{T} \circ p^{12}\right)$ and G is the contraction exhibited by L. Let us remark that H″ is not a pseudoconnection, as its derivative is not 1. The solutions of $\mathcal{D}^2$ are the sections $g : \mathrm{M} \to \mathrm{E}$ such that (I, 5) $\hat{\mathcal{R}} \circ j^2 g = \mathcal{T} \circ jg - \frac{1}{2} g \operatorname{tr} \mathcal{T} \circ jg$, i. e. locally

$$g^{\lambda\mu}(\partial_\lambda\partial_\mu g^{ij} + g^{i\alpha}g^{j\beta}g_{\lambda l}g_{\mu m}\partial_\alpha\partial_\beta g^{lm} - g^{(i\alpha}g_{(\lambda k}\partial_\alpha\partial_{\mu)}g^{kj)} + \dots) = 2\mathcal{T}^{ij} - g^{ij} \operatorname{tr} \mathcal{T},$$

where we have omitted the terms on $\partial_\lambda g^{hk}$ and g^{hk}. The field equation is the second order affine differential equation $\varepsilon^2 \subset \mathrm{J}^2\mathrm{E}$, which is the kernel over $\mathrm{JE} \to \mathrm{M}$ of the affine morphism given by the composition

$$\mathrm{J}^2\mathrm{E} \xrightarrow{\mathrm{H}} \mathrm{L} \times \underset{2}{\vee} \mathrm{T}^* \otimes \underset{2}{\vee} \mathrm{T} \xrightarrow{\mathrm{G}} \underset{2}{\vee} \mathrm{T},$$

where H is the differential pseudoconnection

$$\mathrm{H} \equiv \hat{\Gamma}^2 + \mathcal{S} \circ \hat{\urcorner}^2 - \frac{1}{2} p^{02} \otimes \left(\mathcal{T} \circ p^{12} - \frac{1}{2} p^{02} \operatorname{tr} \mathcal{T} \circ p^{12}\right).$$

The solutions of ε^2 are the sections such that (I, 6)

$$\frac{1}{2}\langle g, \hat{\nabla}^2 g + \mathscr{S} \circ \hat{\urcorner}^2 \circ jg \rangle = \mathscr{C} \circ jg - \frac{1}{2} g \operatorname{tr} \mathscr{C} \circ jg ,$$

i. e. locally

$$\frac{1}{2} g^{\lambda\mu}(\partial_\lambda \partial_\mu g^{ij} + \hat{\Gamma}^{ijv}_{\lambda\mu hk} \partial_v g^{hk} + \hat{\Gamma}^{ij}_{\lambda\mu hk} g^{hk} + \mathscr{S}^{ij} \circ \hat{\urcorner}^2 \circ jg) = \mathscr{C}^{ij} \circ jg - \frac{1}{2} g^{ij} \operatorname{tr} \mathscr{C} \circ jg .$$

The gauge equation is the first order affine differential equation $\mathscr{G}^1 \subset$ JE, which is the kernel over $E \to M$ of the affine morphism given by the composition $JE \xrightarrow{H'} L \times \overset{2}{\vee} T^* \otimes T \xrightarrow{G'} T$, where H′ is the affine morphism (I, 5) $JE \xrightarrow{\mathscr{B}} L \times B \xrightarrow{id \times -\hat{\urcorner}} L \times \underset{2}{\vee} T^* \otimes T$. The solutions of $\mathscr{G}^1$ are the sections such that $\langle g, \mathscr{B} \circ jg - \hat{\urcorner} \rangle = 0$, i. e. locally

$$\partial_\lambda g^{\lambda i} - \frac{1}{2} g_{lm} g^{i\lambda} \partial_\lambda g^{lm} + g^{\lambda\mu} \hat{\Gamma}^{i}_{\lambda\mu} = 0 .$$

The present approach is an intrinsical version of classical formulations (Lichnerowicz) and the requirement on the gauge can be deduced in an analogous way.

10. The Maxwell-Einstein field

The Maxwell-Einstein field provides an example of interaction gauged special theory. In this case we have a main problem: to remove the second order derivatives of the unknown metric in the electromagnetic Laplacian. We can achieve the purpose taking, as the electromagnetic field equation, the special equation given by

$$\left\langle g, \left(\nabla^2 A + \frac{1}{2} \langle \mathscr{S} \circ j^2 g, A \rangle - \frac{1}{2} \underline{g} \otimes \langle \mathscr{C} \circ (g, jA), A \rangle + \langle \nabla(\Gamma - \hat{\Gamma}), A \rangle \right) \right\rangle = 0 ,$$

where $\mathscr{C}$ is the Maxwell stress tensor. We leave the not immediate calculations to the reader.

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REFERENCES

[1] R. ABRAHAM, J. MARSDEN, *Foundations of mechanics*, sec. ed., Benjamin, 1978.
[2] J. DODSON, *Catégories, bundles and spacetime topology*, Shiva Pub., 1980.
[3] C. GODBILLON, *Géométrie différentielle et mécanique analytique*, Hermann, 1969.
[4] R. HERMANN, *Vector bundles in mathematical physics I*, Benjamin, 1970.
[5] PHAM MAU QUAN, *Introduction à la géométrie des variétés différentiables*, Dunod, 1969.
[6] S. SALVIOLI, On the theory of geometric objects, *J. Diff. Geom.*, t. **7**, 1972.
[7] S. STERNBERG, *Lectures on differential geometry*, Prent. Hall, 1964.
[8] P. LIBERMANN, *Connexions d'ordre supérieur et tenseur de structure*, Atti Conv. Geom. Diff., Bologna, 1967.
[9] R. PALAIS, *Foundations of global non linear analysis*, Benjamin, 1968.
[10] J. F. POMMARET, *Systems of partial differential equations and lie-pseudogroups*, Gordon and Breach, 1978.
[11] P. ROUX, Jets and connexions, *Publ. Dép. Math.*, t. **7-4**, Lyon, 1970.
[12] A. KUMPERA, D. C. SPENCER, Lie equations, I, General theory, *Ann. of Math., Stud.*, t. **73**, Princeton Univ. Press, 1972
[13] Y. CHOQUET-BRUHAT, J. W. YORK Jr., *The Cauchy problem in general relativity*, in Gen. Rel. and Grav. edited by Held, Plenum Press, 1980.
[14] A. E. FISHER, J. E. MARSDEN, *The initial value problem and the dynamical formulation of general relativity*, in Gen. Rel. An Einstein Cent. Survey, edited by S. W. Hawking, W. Israel, Cambridge Un. Press, 1979.
[15] F. G. FRIEDLANDER, *The wave equation on a curved space-time*, Cambridge Un. Press, 1975.
[16] S. HAWKING, G. ELLIS, *The large scale structure of space-time*, Cambridge Un. Press, 1973.
[17] A. LICHNEROWICZ, *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson, 1955.
[18] A. LICHNEROWICZ, *Relativistic hydrodynamics and magnetohydrodynamics*, Benjamen, 1967.
[19] A. LICHNEROWICZ, *Ondes de choc en relativité générale*, Centro di An. Glob. C. N. R., Firenze, 1975.
[20] R. K. SACHS, H. WU, *General relativity for mathematicians*, Springer, 1977.
[21] A. CRUMEYROLLE, Structures spinorielles, *Ann. Inst. H. Poincaré*, t. **XI**, 1969.
[22] A. CRUMEYROLLE, Schémas d'Einstein-Dirac en spin 1/2 et généralizations, *Ann. Inst. H. Poincaré*, t. **XXIII**, 1973.
[23] G. KARRER, Darstellungen von Clifford bündeln, *Ann. Ac. Sc. Fenn*, t. **AI**, 1973.
[24] A. KASTLER, Introduction à l'électrodynamique quantique, Dunod, 1961.
[25] A. LICHNEROWICZ, Champs spinoriels et propagateurs en relativité générale, *Bull. Soc. Math. France*, t. **92**, 1964.
[26] A. LICHNEROWICZ, Champs de Dirac, Champ de Neutrino et transformations CPT sur un espace-temps, *Ann. Inst. H. Poincaré I*, 1964.
[27] M. P. APPELL, Sur une forme générale des équations de la dynamique, *Mémor. Sc. Math. Ac. Sc.*, Paris, 1925.
[28] P. L. GARCIA, The Poincaré cartan invariant in the calculus of variations, *Symp. Math.*, t. **XIV**, 1974.
[29] H. GOLDSCHMIDT, S. STERNBERG, The Hamilton cartan formalism in the calculus of variations, *Ann. Inst. Four.*, t. **23**, 1974.
[30] J. KIJOWSKI, W. TULCZYEJEW, A sympletic framework for field theories, *Lect. Not. Phys.*, t. **107**, Springer, 1979.

[*31*] M. MODUGNO, Formulation of analytical mechanics in general relativity, *Ann. Inst. H. Poincaré*, t. **XII**, 1974.

[*32*] R. OUZILOU, Expression symplectique des problèmes variationnelles, *Symp. Math.*, t. **XIV**, 1974.

[*33*] S. STERNBERG, On the role of field theories in our physical conception of geometry, *Diff. Geom. Meth. in Math. Phys.*, Springer, 1977.

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