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R. F. ALVAREZ-ESTRADA

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Section A :
Physique théorique.

Rigorous approach to elastic meson-nucleon scattering in non-relativistic quantum field theory (I): integral equation and Schrödinger ket (*)

by

R. F. ALVAREZ-ESTRADA

Departamento de Física Teórica, Facultad de Ciencias Físicas,
Universidad Complutense, Madrid-3, Spain

RÉSUMÉ. — La diffusion élastique d'un boson par un nucléon habillé au-dessous du seuil pour la production d'un autre boson est étudié rigoureusement dans un modèle des champs quantifiés non-relativistes correspondant, physiquement, a l'interaction meson-nucléon à basse énergie. On présente une équation intégrale exacte et singulière pour la diffusion élastique, laquelle tient compte de l'infinité complète des mesons virtuels autour du nucléon. Pour des petites valeurs de la constante de couplage, et en admettant certaines hypothèses techniques, on établie : *i*) la compacité du noyau de l'équation intégrale singulière et l'existence des solutions de cette dernière dans un espace de Banach, *ii*) l'existence du vecteur de Schrödinger à dimension infinie qui décrit la diffusion élastique. On donne des bornes qui fournissent une solution partielle du problème posé par le nuage infinie de bosons. Afin de contrôler complètement ce dernier problème, des majorations des classes infinies des diagrammes de Feynman sont nécessaires, lesquelles seront présentées ultérieurement, en un deuxième travail

ABSTRACT. — In a non-relativistic field-theoretic model corresponding physically to the low-energy meson-nucleon interaction, the elastic scattering of a boson by the dressed nucleon below the one-boson production

(*) A very short summary of this work (announcing its main results without proofs) has been contributed to the « Ninth International Conference on the Few-Body Problem », Eugene, Oregón, USA, 17-23 August 1980 (Session on Mathematical and Computational Methods).

threshold is studied rigorously. We present an exact and singular (elastic-scattering) integral equation, which includes the whole infinity of virtual bosons around the nucleon. For small coupling constant and under certain technical assumptions, we establish: *i*) the compactness of the kernel of the singular integral equation and the existence of solutions for it in a suitable Banach space, *ii*) the existence of the infinite-dimensional Schrödinger ket describing the elastic scattering. Bounds are given which partly solve the infinite boson cloud problem. In order to control fully the latter, majoration of infinite classes of Feynman diagrams are required, which are presented in a separate paper.

1. INTRODUCTION

A large body of rigorous results exists about scattering in quantum systems of $n(\geqslant 2)$ particles interacting via two-body potentials (see [1-5] and references therein). Far less rigorous information is known about those dynamical models which described $n(\geqslant 1)$ non-relativistic quantum particles interacting with a quantized boson field, and which are associated to physical phenomena like low-energy meson-nucleon scattering [*6-7*], electron-phonon interactions in solids [*8*], non-relativistic Quantum Electrodynamics [*9*], etc., in spite of a vast physical literature regarding them. Mathematical results on scattering in these models appear specifically in [*10*]. For other rigorous studies about them, see [*11-14*]. In particular, rigorous scattering integral equations for such models which could play a role analogous to the Lippmann-Schwinger [*1*] (Faddeev-Yakubovski [*4*] [*15*]) equations in two-(n −, $n \geqslant 3$) body quantum systems, do not exist, to the author's knowledge. On the other hand, rigorous results about scattering in Lee-type models, where the number of bosons in the cloud around the non-relativistic particle is limited, appear in [*12*], [*16*].

This work presents a general study of elastic scattering in a model of the above kind corresponding, physically, to low-energy meson-nucleon interaction [*6-7*]. Since the mesons are massive and a cut-off function is used (like in physical applications to low-energy pion-nucleon scattering), both infrared and ultraviolet divergences are avoided. Nevertheless, two difficulties exist: *i*) one faces an infinite number of equations, associated to the infinite meson cloud around the nucleon, *ii*) elastic scattering singularities are present, which give rise to singular integral equations. Our main result is the rigorous construction of the Schrodinger ket describing the elastic scattering of a meson by a dressed nucleon below the one-meson production threshold, for small coupling constant and under certain assumptions on the cut-off function. Our methods can be generalized, in principle, for increasing values of the coupling constant (f).

This paper is organized as follows. In section 2, we formulate the model (subsection 2.A), summarize and generalize (subsection 2.B) properties of the dressed one-nucleon state and techniques, which will be very useful later in order to solve difficulty *i*). Generalities about the elastic scattering of a meson by the dressed nucleon are presented in section 3. In section 4, we solve part of the difficulty *i*). Section 5) presents an exact singular elastic-scattering integral equation (subsection 5.A) and a compactness proof for it, therely solving difficulty *ii*) (subsection 5.B). Certain problems left open are solved or discussed briefly in section 6. Our construction also requires a careful analysis and majoration of the infinite class of all Feynman diagrams contributing to certain (Greens') functions, in order to solve completely the difficulty *i*) for elastic scattering. Such an analysis, which is rather lengthy, will be presented in a separate paper. For simplicity, we shall consider a model without spin or isospin dependences throughout our work. At the end of the second paper, we shall add the essential remarks in order to include internal degrees of freedom. Our work constitutes a rigorous formulation, to all orders in f, of an approximate (Tamm-Dancoff) approach to low-energy meson-nucleon scattering presented in [*17*].

N-quantum approximations to models of the type studied in this work have been investigated in [*18*]. A formal continued fraction approach to non-relativistic Quantum Electrodynamics (with applications to spontaneous and stimulated emission), which differs considerably from our developments, appears in [*19*].

2. THE MODEL AND THE DRESSED ONE-NUCLEON STATE

2.A. Formulation of the model.

Let a non-relativistic spinless particle (nucleon) have bare mass m_0 and position and threemomentum operators $\bar{x} = (x_L)$, $\bar{p} = (p_j)$, L, $j = 1, 2, 3$ ($[x_L, p_j] = i\delta_{L,j}$) and let $\Psi(\bar{q})$ be a bare one-nucleon state with three-momentum $\bar{q}$. The nucleon interacts with an indefinite number of scalar bosons (mesons). Let $a(\bar{k})$, $a^+(\bar{k})$ be the destruction and creation operators for a boson with threemomentum $\bar{k}$ and strictly positive energy $\omega(k) \geqslant \omega_0 > 0$ ($[a(\bar{k}), a^+(\bar{k}')] = \delta^{(3)}(\bar{k} - \bar{k}')$, $k = |\bar{k}|$) and let $|0\rangle$ be the vacuum. We assume the total hamiltonian to be:

$$\mathrm{H} = \mathrm{H}_0 + \mathrm{H}_\mathrm{I}, \qquad \mathrm{H}_1 = \frac{\bar{p}^2}{2m_0} + \int d^3\bar{k}\,\omega(k) a^+(\bar{k}) a(\bar{k}) \tag{2.A.1}$$

$$\mathrm{H}_\mathrm{I} = f \int d^3\bar{k}\,[v(k) a(\bar{k}) \exp i\bar{k}\bar{x} + v^*(k) a^+(\bar{k}) \exp(-i\bar{k}\bar{x})] \tag{2.A.2}$$

f being a real dimensionless coupling constant and $v(k)$ being a complex cut-off function. The total conserved threemomentum is

$$\bar{P}_{tot} = \bar{p} + \int d^3\bar{k}.\bar{k}.a^+(\bar{k})a(\bar{k}).$$

Let $\mathscr{H}_{\bar{\pi}}$, with $\frac{\bar{\pi}^2}{2m_0} < \omega_0$, be the subspace of Hilbert space formed by all kets Ψ such that $(\bar{P}_{tot} - \bar{\pi})\Psi = 0$. A basis for $\mathscr{H}_{\bar{\pi}}$ is formed by the set of all kets

$$\Psi(\bar{q}_{\bar{\pi}}; \bar{k}_1 \dots \bar{k}_n) = \Psi(\bar{q}_{\bar{\pi}}) \otimes \left[\frac{1}{(n\,!)^{1/2}} a^+(\bar{k}_1) \dots a^+(\bar{k}_n) \,|\, 0 \rangle\right]$$
$$\bar{q}_{\bar{\pi}} = \bar{\pi} - \sum_{i=1}^{n} \bar{k}_i \tag{2.A.3}$$

Throughout this work, we shall assume that

a) $$\eta = f\left[\int d^3\bar{k}\frac{|\,v(k)\,|^2}{\omega(k)}\right]^{1/2} < +\infty .$$

b) f is sufficiently small (we shall state this condition more explicitly later, at the appropriate places), *c)* $v(k)$ and $\nabla_k v(k)$ are continuous and bounded for any $\bar{k}$. Later, we shall have to add further assumptions, namely, assumption *d)* in section 3, assumptions *e)* and *f)* in subsection 4.B, and assumption *g)* in subsection 5.B. Since they are somewhat technical, it seems more convenient to formulate them when they become necessary.

A standard quadratic-form argument yields, for any normalizable Ψ (compare with [*20*]; see also [*10-14*]):

$$(\Psi, H_I\Psi) \leqslant \eta.\left[\omega_0^{-1/2} \,\|\, H_0^{1/2}\Psi \,\|^2 + \omega_0^{1/2} \,\|\, \Psi \,\|^2\right].$$

Then, if $\omega_0^{-1/2}.\eta < 1$, well-known theorems imply that H is self-adjoint and bounded below [*20*].

2.B. The dressed one-nucleon state: summary of useful properties.

We shall expand the dressed one-nucleon state $\Psi_+(\bar{\pi})$ which belongs to $\mathscr{H}_{\bar{\pi}}((\bar{P}_{tot} - \bar{\pi})\Psi_+(\bar{\pi}) = 0)$ and has physical energy E as:

$$\Psi_+(\bar{\pi}) = \Psi(\bar{\pi})$$
$$+ \sum_{n=1}^{+\infty} \int \left[\prod_{i=1}^{n} d^3\bar{k}_i\right] \frac{b_n(\bar{\pi}; \bar{k}_1 \dots \bar{k}_n)}{|\, e_n(E, \bar{\pi}; \bar{k}_1 \dots \bar{k}_n)\,|^{1/2}} \cdot \Psi(\bar{q}_{\bar{\pi}}; \bar{k}_1 \dots \bar{k}_n) \tag{2.B.1}$$

$$e_n(E, \bar{\pi}; \bar{k}_1 \dots \bar{k}_n) = E - \sum_{i=1}^{n} \omega(k_i) - \frac{1}{2m_0}\left(\bar{\pi} - \sum_{i=1}^{n} \bar{k}_i\right)^2, \quad n \geqslant 1 \tag{2.B.2}$$

$\Psi_+(\bar{\pi})$ is normalized so that the coefficient of $\Psi(\bar{\pi})$ equals one. The n-meson amplitude $b_n(\bar{\pi}; \bar{k}_1 \ldots \bar{k}_n)$ is symmetric under interchanges of $\bar{k}_1 \ldots \bar{k}_n$ (its E-dependence is not made explicit). We shall summarize some recurrence relations and bounds regarding the b_n's and the nucleon self-energy which will be quite useful later. For brevity, we shall omit the rigorous construction of both $\Psi_+(\bar{\pi})$ and E for small $|\bar{\pi}|$ and f based upon such recurrence relations and bounds. It can be carried out by extending directly the proofs presented, for the large-polaron model, in [*14*]. Using $(\mathrm{H} - \mathrm{E})\Psi_+(\bar{\pi}) = 0$, one finds $(e_n^{1/2} = e_n / | e_n |^{1/2},\ b_0 / | e_0 |^{1/2} \equiv 1)$:

$$\begin{aligned} b_n(\bar{\pi}; \bar{k}_1 \ldots \bar{k}_n) &= \frac{1}{e_n(\mathrm{E}, \bar{\pi}; \bar{k}_1 \ldots \bar{k}_n)^{1/2}} \left\{ \frac{f}{n^{1/2}} \sum_{i=1}^{n} v(k_i)^* \cdot \frac{b_{n-1}(\bar{\pi}; \bar{k}_1 \ldots \bar{k}_{i-1}\bar{k}_{i+1} \ldots \bar{k}_n)}{| e_{n-1}(\mathrm{E}, \bar{\pi}; \bar{k}_1 \ldots \bar{k}_{i-1}\bar{k}_{i+1} \ldots \bar{k}_n) |^{1/2}} \right. \\ &\quad \left. + f(n+1)^{1/2} \int d^3\bar{k}\, v(k) \frac{b_{n+1}(\bar{\pi}; \bar{k}\bar{k}_1 \ldots \bar{k}_n)}{| e_{n+1}(\mathrm{E}, \bar{\pi}; \bar{k}\bar{k}_1 \ldots \bar{k}_n) |^{1/2}} \right\} \end{aligned} \tag{2.B.3}$$

$$\mathrm{E} - \frac{\bar{\pi}^2}{2m_0} = \mathrm{M}(\mathrm{E}), \qquad \mathrm{M}(\mathrm{E}) = f \int d^3\bar{k}\, v(k) \frac{b_1(\bar{\pi}; \bar{k})}{| e_1(\mathrm{E}, \bar{\pi}; \bar{k}) |^{1/2}} \tag{2.B.4}$$

we shall introduce the L^2-norms

$$\begin{aligned} \| b_n(\bar{k}_1) \|_2 &= \left\{ \int \left[\prod_{i=2}^{n} d^3\bar{k}_i \right] | b_n(\bar{\pi}; \bar{k}_1\bar{k}_2 \ldots \bar{k}_n) |^2 \right\}^{1/2}, \\ &\qquad n \geqslant 2; \quad \| b_1(\bar{k}_1) \|_2 = | b_1(\bar{k}_1) | \\ \| b_n \|_2 &= \left\{ \int d^3\bar{k}_1 [\, \| b_n(\bar{k}_1) \|_2]^2 \right\}^{1/2}, \qquad n \geqslant 1 \end{aligned} \tag{2.B.5}$$

and the following functions and continued fractions:

$$\begin{aligned} \tau_1 &= \left[f^2 \int d^3\bar{k} \frac{| v(k) |^2}{| e_1(\mathrm{E}, \bar{\pi}; \bar{k}) |} \right]^{1\ 2} \\ \tau_n &= \left[\underset{\bar{k}_1 \ldots \bar{k}_{n-1}}{\mathrm{Max}} \frac{f^2 . n}{| e_{n-1}(\mathrm{E}, \bar{\pi}; \bar{k}_1 \ldots \bar{k}_{n-1}) |} \int d^3\bar{k} \frac{| v(k) |^2}{| e_n(\mathrm{E}, \bar{\pi}; \bar{k}\bar{k}_1 \ldots \bar{k}_{n-1}) |} \right]^{1/2}, \\ &\qquad n \geqslant 2 \end{aligned} \tag{2.B.6}$$

$$\begin{aligned} \sigma_1(k_1) &= \frac{f . | v(k_1) |}{\omega(k_1) - \mathrm{E}} \\ \sigma_n(k_1) &= \frac{f . | v(k_1) |}{\{ n[(n-1)\omega_0 + \omega(k_1) - \mathrm{E}][(n-1)\omega_0 - \mathrm{E}] \}^{1/2}}, \quad n \geqslant 2 \end{aligned} \tag{2.B.7}$$

$$Z_n = \cfrac{1}{1 - \cfrac{\tau_{n+1}^2}{1 - \cfrac{\tau_{n+2}^2}{1 - \cfrac{\tau_{n+3}^2}{1 - \ddots}}}}, \qquad n \geqslant 1 \quad (2.B.8)$$

$$Z'_n = \cfrac{1}{1 - \cfrac{(n/n+1).\tau_{n+1}^2}{1 - \cfrac{((n+1)/(n+2)).\tau_{n+2}^2}{1 - \cfrac{((n+2)/(n+3)).\tau_{n+3}^2}{1 - \ddots}}}}, \qquad n \geqslant 1 \quad (2.B.9)$$

At least for small $|\bar{\pi}|$ and $|E|$, one has: *i)* $\tau_n < +\infty$, $n \geqslant 1$, by virtue of assumption *a)*, *ii)* $\tau_n \to 0$ as $n \to \infty$. Then, all Z_n and Z'_n converge and are positive for any $n \geqslant n_0$ (n_0 depending on f), by virtue of some classical theorems about the convergence of continued fractions [*21*]. Here, we shall assume that f is suitably small (recall assumption *b*) in subsection 2.A) so that all Z_n and Z'_n converge and are strictly positive for $n \geqslant 1$. Some direct L^2-majorations of Eq. (2.B.3) yield (they generalize those in section 4 of [*14*]):

$$\| b_n \|_2 \leqslant \tau_n . \| b_{n-1} \|_2 + \tau_{n+1} . \| b_{n+1} \|_2 , \qquad n \geqslant 1 \qquad (2.B.10)$$

$$\| b_n(\bar{k}_1) \|_2 \leqslant \sigma_n(k_1) \| b_{n-1} \|_2 + \frac{n-1}{n} . \tau_n . \| b_{n-1}(\bar{k}_1) \|_2 + \tau_{n+1} . \| b_{n+1}(\bar{k}_1) \|_2 , \qquad n \geqslant 1 \quad (2.B.11)$$

with the conventions $\| b_0 \|_2 \equiv 1$, $\| b_0(\bar{k}_1) \|_2 \equiv 0$. We stress that (2.B.11) is new, as it was unnecessary for the studies carried out in [*14*]. One proves that the three-term recurrences of inequalities (2.B.10) and (2.B.11) are satisfied by the following inequalities (by using techniques sketched in sections 4 and 5 and Appendix C of [*14*])

$$\| b_n \|_2 \leqslant \tau_n . Z_n . \| b_{n-1} \|_2 , \qquad n \geqslant 1 \qquad (2.B.12)$$

$$\| b_n(\bar{k}_1) \|_2 \leqslant Z'_n . \left\{ \frac{n-1}{n} . \tau_n . \| b_{n-1}(\bar{k}_1) \|_2 + \sigma_n(k_1) . \| b_{n-1} \|_2 + \sum_{l=1}^{+\infty} ([(\tau_{n+1} . Z'_{n+1}).(\tau_{n+2} . Z'_{n+2}) \ldots (\tau_{n+l} . Z'_{n+l})] . \sigma_{n+l}(k_1) . \| b_{n+l-1} \|_2) \right\} , \qquad n \geqslant 1 \quad (2.B.13)$$

The bound (2.B.12) implies that $\| b_n \|_2 \to 0$ as $n \to \infty$. By virtue of the above property *ii)* of the τ_n's, the series on the right-hand-side of (2.B.13) converges. Since $\| b_0(\bar{k}_1) \|_2 \equiv 0$, Eqs. (2.B.7) and the bounds (2.B.12-13) imply the existence of some continuous and positive functions $\sigma_n^{(1)}(k_1)$ such that:

a) $\| b_n(\bar{k}_1) \|_2 \leqslant [\,| v(k_1) | \,.\, \sigma_n^{(1)}(k_1)]/\omega(k_1)^{1/2}$, $\quad n \geqslant 1$
b) $\sigma_n^{(1)}(k_1)$ is bounded for any k_1
c) $\sigma_n^{(1)}(k_1) \to 0$ as $n \to \infty$ for given k_1.

The second Eq. (2.B.4), the bound (2.B.12) for $n = 1$ and $\| b_0 \|_2 = 1$ imply the following bound for the self-energy M(E):

$$| \mathrm{M(E)} | \leqslant \tau_1^2 . \mathrm{Z}_1 \qquad (2.\mathrm{B}.14)$$

3. ELASTIC SCATTERING OF A MESON BY A DRESSED NUCLEON

We shall study the low-energy elastic scattering of a boson by the dressed nucleon, which had small threemomenta $\bar{l}$ and $-\bar{l}$, respectively, in the remote past, at infinite relative separation. According to Wick's time-independent formulation [22], the incoming state is $a^+(\bar{l})\Psi_+(-\bar{l})$ and the full (Schrodinger) state is $\Psi_+(\bar{l}; -\bar{l}) = a^+(\bar{l})\Psi_+(-\bar{l}) + \Psi_{sc}(\bar{l}; -\bar{l})$, $\Psi_{sc}(\bar{l}; -\bar{l})$ being an outgoing ket generated by the interaction. The total energy is $\mathrm{E}_+ = \omega(l) + \mathrm{E}(-\bar{l})$. Since $(\mathrm{H} - \mathrm{E}_+)\Psi_+(\bar{l}; -\bar{l}) = 0$, a simple calculation yields (for similar developments in the static Chew-Low model, see Schweber [6]):

$$(\mathrm{H} - \mathrm{E}_+)\Psi_{sc}(\bar{l}; -\bar{l}) = -f . v(l) \exp i\bar{l}.\bar{x} . \Psi_+(-\bar{l}) \qquad (3.1)$$

By generalizing Eq. (2.B.1) and noticing that $\bar{\mathrm{P}}_{tot}\Psi_{sc}(\bar{l}; -\bar{l}) = 0$, we shall expand $\Psi_{sc}(\bar{l}; -\bar{l})$ formally as

$$\Psi_{sc}(\bar{l}; -\bar{l}) = y_0 . \Psi(\bar{\pi} = \bar{0}) + \sum_{n=1}^{+\infty} \int \left[\prod_{i=1}^{n} d^3\bar{k}_i \right] y_n(\bar{k}_1 \dots \bar{k}_n) . \Psi\left(-\sum_{i=1}^{n} \bar{k}_i; \bar{k}_1 \dots \bar{k}_n \right) \qquad (3.2)$$

y_n is the probability amplitude for finding $n \geqslant 0$ bosons in Ψ_{sc} and is symmetric under intercharges of $\bar{k}_1 \dots \bar{k}_n$. It depends on $\bar{l}$ and E_+, but these dependences will not be made explicit. Upon combining Eqs. (3.1-2), (2.A.1-2) and (2.B.1-2), one derives the basic recurrence for the y_n's (compare with Eq. (2.B.3)):

$$e_n(\mathrm{E}_+, \bar{0}; \bar{k}_1 \dots \bar{k}_n) . y_n(\bar{k}_1 \dots \bar{k}_n) = \frac{f . v(l) . b_n(-\bar{l}; \bar{k}_1 \dots \bar{k}_n)}{| e_n(\mathrm{E}(-\bar{l}), -\bar{l}; \bar{k}_1 \dots \bar{k}_n) |^{1/2}}$$

$$+ \frac{f}{n^{1/2}} \cdot \sum_{i=1}^{n} v(k_i)^* . y_{n-1}(\bar{k}_1 \dots \bar{k}_{i-1} \bar{k}_{i+1} \dots \bar{k}_n)$$

$$+ f(n+1)^{1/2} \int d^3\bar{k} . v(k) y_{n+1}(\bar{k}\bar{k}_1 \dots \bar{k}_n), \qquad n \geqslant 0 \quad (3.3)$$

$$e_0(\mathrm{E}_+, \bar{0}) = \mathrm{E}_+ , \qquad y_{-1} = 0$$

$$\frac{b_0(-\bar{l})}{| e_0(\mathrm{E}(-\bar{l}), -\bar{l}) |^{1/2}} = 1 \qquad (3.4)$$

where all b_n's, $n \geqslant 1$, are regarded as known. Unlike the coefficient of $\Psi(\bar{\pi})$ in Eq. (2.B.1), which equals one by virtue of the normalization condition for $\Psi_+(\bar{\pi})$, here y_0 has to be determined from the recurrence (3.3), like all other y_n's, $n \geqslant 1$. Eqs. (3.3) have only a formal sense in principle, since some e_n's could vanish and, hence, the y_n's should exhibit typical scattering singularities.

We shall add the assumption: *d)* f, and $l = |\bar{l}|$ are so small that

$$\mathrm{E}(-\bar{l}) = \mathrm{E}(l) < \omega_0, \ \mathrm{E}_+ \geqslant \omega_0 + \mathrm{E}(-\bar{l}) > 0$$

and $\mathrm{E}_+ < 2\omega_0$ hold. These conditions imply respectively that

$$e_n(\mathrm{E}(-\bar{l}), -\bar{l}; \bar{k}_1 \dots \bar{k}_n) < 0$$

for $n \geqslant 1$ and any $\bar{k}_1 \dots \bar{k}_n$, $e_0(\mathrm{E}_+, \bar{0}) > 0$ and $e_n(\mathrm{E}_+, \bar{0}; \bar{k}_1 \dots \bar{k}_n) < 0$ for $n \geqslant 2$ and any $\bar{k}_1 \dots \bar{k}_n$ and allow for $e_1(\mathrm{E}_+, \bar{0}; \bar{k}_1)$ to change sign and vanish as $\bar{k}_1$ varies. Notice that, at least for small f and l, $\mathrm{E}(0) < 0$ and $\mathrm{E}(-\bar{l})$ increases and becomes less negative (and, perhaps, even positive) as l increases. Then, the elastic scattering threshold is

$$\omega_0 + \mathrm{E}(0)(\mathrm{E}_+ \geqslant \omega_0 + \mathrm{E}(0) > 0) .$$

We are restricting ourselves to small l such that only elastic scattering occurs and boson production is energetically forbidden.

The above statements and some preliminary study of the recurrence (3.3) indicate that all y_n's, $n \geqslant 0$, can be determined rigorously through the following three-step construction, which constitutes the plan of our work:

1) Solve partially the set of all Eqs. (3.3) for $n \geqslant 2$ and obtain y_2 in terms of y_1 and all b_n's, $n \geqslant 2$, treated as known inhomogeneous terms. Since $e_n < 0$ for $n \geqslant 2$, the kernels for such a set are free of scattering singularities, but one has to cope with an infinite number of equations. The recurrence relations and bounds summarized in subsection 2.B will be quite useful. Moreover, a detailed study and majoration of all contributing Feynman diagrams (presented in a separate paper) will be required.

2) Consider Eq. (3.3) for $n = 1$, plug into it the solution for y_2 in terms of y_1 obtained in step 1 and the expression for y_0 implied by Eq. (3.3) for $n = 0$ and solve for y_1. Here, one has to deal with one integral equation displaying elastic scattering singularities.

3) Extend the work started in step 1 so as to complete the construction of y_0 and all y_n's, $n \geqslant 2$, once y_1 has been determined in step 2.

4. PARTIAL SOLUTION OF EQS (3.3) FOR $n \geqslant 2$ IN TERMS OF y_1 (STEP 1)

4.A. An alternative formulation of Eqs (3.3) for $n \geqslant 2$.

For later convenience, let us divide Eq. (3.3) for $n \geqslant 2$ by

$$e_n(\mathrm{E}_+, \bar{0}; \bar{k}_1 \ldots \bar{k}_n)^{1/2} \neq 0 (e_n^{1/2} . \mid e_n \mid^{1/2} = e_n)$$

and introduce

$$y'_n(\bar{k}_1 \ldots \bar{k}_n) = \mid e_n(\mathrm{E}_+, \bar{0}; \bar{k}_1 \ldots \bar{k}_n) \mid^{1/2} . y_n(\bar{k}_1 \ldots \bar{k}_n), \qquad n \geqslant 2 \quad (4.\mathrm{A}.1)$$

$$\mathrm{Y} = \begin{pmatrix} y'_2(\bar{k}_1 \bar{k}_2) \\ \vdots \\ y'_n(\bar{k}_1 \ldots \bar{k}_n) \\ \vdots \end{pmatrix}$$

$$\mathrm{Y}_1^{(0)} = \begin{pmatrix} \dfrac{f . v(l) . b_2(-\bar{l}; \bar{k}_1 \bar{k}_2)}{e_2(\mathrm{E}_+, \bar{0}; \bar{k}_1 \bar{k}_2)^{1/2} . \mid e_2(\mathrm{E}(-\bar{l}), -\bar{l}; \bar{k}_1 \bar{k}_2) \mid^{1/2}} \\ \vdots \\ \dfrac{f . v(l) . b_n(-\bar{l}; \bar{k}_1 \ldots \bar{k}_n)}{e_n(\mathrm{E}_+, \bar{0}; \bar{k}_1 \ldots \bar{k}_n)^{1/2} . \mid e_n(\mathrm{E}(-\bar{l}), -\bar{l}; \bar{k}_1 \ldots \bar{k}_n) \mid^{1/2}} \\ \vdots \end{pmatrix} \quad (4.\mathrm{A}.2)$$

$$\mathrm{Y}_2^{(0)} = \begin{pmatrix} d^{(0)}(\bar{k}_1 \bar{k}_2) \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$$

$$d^{(0)}(\bar{k}_1 \bar{k}_2) = \frac{f}{e_2(\mathrm{E}_+, \bar{0}; \bar{k}_1 \bar{k}_2)^{1/2} . 2^{1/2}} [v(k_1)^* y_1(\bar{k}_2) + v(k_2)^* y_1(\bar{k}_1)] \quad (4.\mathrm{A}.3)$$

Then, the set formed by all Eqs. (3.3) for $n \geqslant 2$ becomes the following inhomogeneous linear equation for Y:

$$\mathrm{Y} = \mathrm{Y}_0^{(1)} + \mathrm{Y}_0^{(2)} + \mathrm{W}.\mathrm{Y} \tag{4.A.4}$$

where $\mathrm{Y}_0^{(1)}$ and $\mathrm{Y}_0^{(2)}$ are regarded as given inhomogeneous terms and W is the corresponding kernel, which is unambiagously defined through the right-hand-side of Eq. (3.3) for $n \geqslant 2$, the division by $e_n^{1/2}$ and Eq. (A.4.1). One has, in a formal sense, at least:

$$\mathrm{Y} = \sum_{i=1}^{2} d^{(i)}; \qquad d^{(i)} = (\mathbb{1} - \mathrm{W})^{-1}.\mathrm{Y}_i^{(0)}, \qquad i = 1,2$$

$$d^{(i)} = \begin{pmatrix} d_2^{(i)}(\bar{k}_1\bar{k}_2) \\ \vdots \\ d_n^{(i)}(\bar{k}_1 \dots \bar{k}_n) \\ \vdots \end{pmatrix} \tag{4.A.5}$$

where $\mathbb{1}$ denotes the corresponding unit operator.

4.B. Rigorous construction of $d^{(1)}$.

Since none of the e_n's, $n \geqslant 2$, appearing in W and $\mathrm{Y}_1^{(0)}$ vanish for any $\bar{k}_1 \dots \bar{k}_n$, a suitable extension of the techniques sketched in subsection 2.B will allow to construct $d^{(1)}$ rigorously. Thus, *a posteriori*, we shall realize the interest of having introduced y'_n (Eq. (4.A.1)). Notice that $d_n^{(1)}$, $n \geqslant 2$, satisfy the following recurrence relations:

$$\begin{aligned} d_n^{(1)}(\bar{k}_1 \dots \bar{k}_n) &= \frac{1}{e_n(\mathrm{E}_+, \bar{0}; \bar{k}_1 \dots \bar{k}_n)^{1/2}} \left\{ \frac{f.v(l)b_n(-\bar{l}; \bar{k}_1 \dots \bar{k}_n)}{| e_n(\mathrm{E}(-\bar{l}), -\bar{l}; \bar{k}_1 \dots \bar{k}_n) |^{1/2}} + \frac{f}{n^{1/2}} \sum_{i=1}^{n} v(k_i)^* \right. \\ &\cdot \frac{d_{n-1}^{(1)}(\bar{k}_1 \dots \bar{k}_{i-1}\bar{k}_{i+1} \dots \bar{k}_n)}{| e_{n-1}(\mathrm{E}_+, \bar{0}; \bar{k}_1 \dots \bar{k}_{i-1}\bar{k}_{i+1} \dots \bar{k}_n) |^{1/2}} + f.(n+1)^{1/2} \int d^3\bar{k}v(k) \\ &\left. \cdot \frac{d_{n+1}^{(1)}(\bar{k}\bar{k}_1 \dots \bar{k}_n)}{| e_{n+1}(\mathrm{E}_+, \bar{0}; \bar{k}, \bar{k}_1 \dots \bar{k}_n) |^{1/2}} \right\} \quad n \geqslant 2,\ d_1^{(1)} \equiv 0 \end{aligned} \tag{4.B.1}$$

In fact, if $d^{(0)} = 0$, then $d^{(2)} = 0$ and one has $\mathrm{Y} = d^{(1)} = \mathrm{Y}_0^{(1)} + \mathrm{W}d^{(1)}$, whose explicit form is (4.B.1). Upon introducing the L^2-norms $\| d_n^{(1)}(\bar{k}_1) \|_2$ and $\| d_n^{(1)} \|_2$, $n \geqslant 2$, through Eqs. (2.B.5) with b_n replaced by $d_n^{(1)}$, and carrying out majorations similar to those leading from Eq. (2.B.4) to (2.B.10) and (2.B.11), one finds

$$\| d_n^{(1)} \|_2 \leqslant \alpha_n \| b_n \|_2 + \tau_n \| d_{n-1}^{(1)} \|_2 + \tau_{n+1} \| d_{n+1}^{(1)} \|_2 , \qquad n \geqslant 2, \ \| d_1^{(1)} \|_2 \equiv 0 \quad (4.B.2)$$

$$\| d_n^{(1)}(\bar{k}_1) \|_2 \leqslant [\alpha_n . \| b_n(\bar{k}_1) \|_2 + \sigma_n(k_1) . \| d_{n-1}^{(1)} \|_2] + \frac{n-1}{n} . \tau_n . \| d_{n-1}^{(1)}(\bar{k}_1) \|_2 + \tau_{n+1} . \| d_{n+1}^{(1)}(\bar{k}_1) \|_2 , \qquad n \geqslant 2, \ \| d_1^{(1)}(\bar{k}_1) \|_2 \equiv 0 \quad (4.B.3)$$

$$\alpha_n = f.v(l) . \{ [\operatorname*{Max}_{\bar{k}_1 \ldots \bar{k}_n} | e_n(E_+, \bar{0}; \bar{k}_1 \ldots \bar{k}_n) |^{1/2}] . [\operatorname*{Max}_{\bar{k}_1 \ldots \bar{k}_n} | e_n(E(-\bar{l}), -\bar{l}; \bar{k}_1 \ldots \bar{k}_n) |^{1/2}] \}^{-1} \qquad n \geqslant 2 \quad (4.B.4)$$

where the actual τ_n and σ_n are still given by Eqs. (2.B.6) and (2.B.7), but with E replaced by E_+. The recurrences (4.B.2) and (4.B.3) have structures similar to (2.B.10) and (2.B.11) respectively. Then, by applying the same arguments which led from (2.B.10), (2.B.11) to (2.B.12) and (2.B.13) respectively, one arrives at the following explicit recurrences of bounds for $\| d_n^{(1)} \|_2$ and $\| d_n^{(1)}(\bar{k}_1) \|_2$ (which satisfy the recurrences (4.B.2) and (4.B.3), respectively):

$$\| d_n^{(1)} \|_2 \leqslant Z_n \Big\{ \alpha_n . \| b_n \|_2 + \tau_n . \| d_{n-1}^{(1)} \|_2 + \sum_{l=1}^{+\infty} [(\tau_{n+1} . Z_{n+1}) . (\tau_{n+2} . Z_{n+2}) \ldots (\tau_{n+l} . Z_{n+l})] . (\alpha_{n+l} \| b_{n+l} \|_2) \Big\}, \qquad n \geqslant 2, \ \| d_1^{(1)} \|_2 = 0 \quad (4.B.5)$$

$$\| d_n^{(1)}(\bar{k}_1) \|_2 \leqslant Z'_n \Big\{ [\alpha_n . \| b_n(\bar{k}_1) \|_2 + \sigma_n(k_1) . \| d_{n-1}^{(1)} \|_2] + \frac{n-1}{n} \tau_n . \| d_{n-1}^{(1)}(\bar{k}_1) \|_2 + \sum_{l=1}^{+\infty} [(\tau_{n+1} . Z'_{n+1}) . (\tau_{n+2} . Z'_{n+2}) \ldots (\tau_{n+l} . Z'_{n+l})] . [\alpha_{n+l} . \| b_{n+l}(\bar{k}_1) \|_2 + \sigma_{n+l}(k_1) \| d_{n+l-1}^{(1)} \|_2] \Big\}, \qquad n \geqslant 2, \ \| d_1^{(1)}(\bar{k}_1) \|_2 = 0 \quad (4.B.6)$$

Let f be so small that all Z_n and Z'_n, $n \geqslant 2$, are finite and strictly positive (this agrees with and makes explicit assumption *b*) in subsection 2.A). Then, by using the properties of $\| b_n \|_2$ and $\| b_n(\bar{k}_1) \|_2$ established in subsection 2.B, and using similar techniques, it is easy to prove that

a) both series on the right-hand-sides of (4.B.5) and (4.B.6) converge

b) $\| d_n^{(1)} \|_2 \to 0$ as $n \to \infty$ and $\sum_{n=2}^{+\infty} [\| d_n^{(1)} \|_2]^2 < +\infty$

c) there exist continuous and positive functions $\sigma_n^{(2)}(k_1)$ such that
*c*1) $\| d_n^{(1)}(\bar{k}_1) \|_2 \leqslant [\, | v(k_1) | \cdot \sigma_n^{(2)}(k_1)]/\omega(k_1)^{1/2}$
*c*2) $\sigma_n^{(2)}(k_1)$ is bounded for any k_1
*c*3) $\sigma_n^{(2)}(k_1) \to 0$ if $n \to \infty$, for given k_1.

On the other hand, by expanding $(\mathbb{1} - \mathrm{W})^{-1}$, one gets

$$d^{(1)} = \left[\mathbb{1} + \sum_{n=1}^{+\infty} \mathrm{W}^n \right] . \mathrm{Y}_1^{(0)} \tag{4.B.7}$$

It is obvious that the series for each $d_n^{(1)}(\bar{k}_1 \ldots \bar{k}_n)$, $n \geqslant 2$, which results from (4.B.7) coincides with the one obtained by successive iterations of (4.B.1). By majorizing directly such a series for $d_n^{(1)}$ (via Minkowski and Schwartz inequalities, etc.), one finds two majorizing series for $\| d_n^{(1)} \|_2$ and $\| d_n^{(1)}(\bar{k}_1) \|_2$ which, in turn, can be summed up into the right-hand-sides of (4.B.5) and (4.B.6), respectively. The detailed check, term by term, of the last statement is not difficult, but rather cumbersome, and will be omitted (compare with section 4 of [*14*]).

Notice that the recurrences (4.B.2) and (4.B.3) provide a particularly convenient method for majorizing all $d_n^{(1)}$'s, which avoids the detailed study of the individual terms of the series (4.B.7). We stress the fact that the bounds (4.B.5) and (4.B.6) establish the convergence of the series (4.B.7).

We shall assume that the boson energy $\omega(k)$ is such that

e) if $e_n(\mathrm{E}, \bar{\pi}; \bar{k}_1 \ldots \bar{k}_n)$, $n \geqslant 2$, is non-vanishing for any $\bar{k}_1 \ldots \bar{k}_n$ and small or vanishing $|\bar{\pi}|$ at the given $\mathrm{E}(\mathrm{E} \leqslant \mathrm{E}_+)$, then

$$\left| \nabla_{k_1} \frac{1}{e_n(\mathrm{E}, \bar{\pi}; \bar{k}_1 \ldots \bar{k}_n)^{1/2}} \right| \leqslant \sigma^{(3)} \cdot \frac{1}{| e_n(\mathrm{E}, \bar{\pi}; \bar{k}_1 \ldots \bar{k}_n) |^{1/2}}$$

for any $\bar{k}_1 \ldots \bar{k}_n$, $n \geqslant 2$, $\sigma^{(3)}$ being a positive and finite constant.

f) for the given E_+, $e_1(\mathrm{E}_+, \bar{0}; \bar{k}_1) = [k_1 - k_1^{(0)}(\mathrm{E}_+)] \cdot e_1^{(0)}(k_1)$ where $\mathrm{k}_1^{(0)}(\mathrm{E}_+) > 0$ and $e_1^{(0)}(k_1) \neq 0$ for any $k_1 \geqslant 0$. These assumptions are automatically fulfilled by the typical non-relativistic boson energy

$$\omega(k) = \omega_0 + \omega_1 k^2 \,, \qquad \omega_1 \geqslant 0 \,.$$

For a relativistic meson energy, namely,

$$\omega(k) = (\omega_0^2 + \omega_1' k^2)^{1/2} \,, \qquad \omega_1' \geqslant 0 \,,$$

the corresponding $e_1^{(0)}(k_1)$ may have its own zeroes, generically denoted by x_1, which would violate assumption *f*). One can still allow for the relativistic $\omega(k)$, provided that *f*) be replaced by

f') $v(k)$ vanishes identically for $k > x_0$, x_0 being smaller than *all* x_1, so that the only effective zero of $e_1(\mathrm{E}_+, \bar{0}; \bar{k}_1)$ below x_0 is $k_1^{(0)}(\mathrm{E}_+)$. It is always possible to choose $v(k)$ so that assumptions *b*), *f'*), and *g*) (to be formulated in subsection 5.B) hold simultaneously and, hence, the main results of this work remain valid.

By applying $\nabla_{\bar{k}_1}$ to the whole recurrence (2.B.3), one generates a new recurrence for $\nabla_{\bar{k}_1} b_n(\bar{k}_1 \dots \bar{k}_n)$, $n \geqslant 1$, which also contains b_n, $\nabla_{\bar{k}_1} e_n$ and $\nabla_{\bar{k}_1} v(k_1)$. Upon majorizing this new recurrence by using assumption *e)* and techniques similar to those leading to (2.B.11), one gets a three-term recurrence of inequalities of the type

$$| \nabla_{\bar{k}_1} b_1(\bar{k}_1) | \leqslant \alpha_1^{(1)}(\bar{k}_1) + \tau_2 . \| \nabla_{\bar{k}_1} b_2(\bar{k}_1) \|_2 \tag{4.B.8}$$

$$\| \nabla_{\bar{k}_1} b_2(\bar{k}_1) \|_2 \leqslant \alpha_2^{(1)}(\bar{k}_1) + \frac{1}{2} . \tau_2 . | \nabla_{\bar{k}_1} b_1(\bar{k}_1) | + \tau_3 . \| \nabla_{\bar{k}_1} b_3(\bar{k}_1) \|_2 \tag{4.B.9}$$

$$\| \nabla_{\bar{k}_1} b_n(\bar{k}_1) \|_2 \leqslant \alpha_n^{(1)}(\bar{k}_1) + \frac{n-1}{n} . \tau_n \| \nabla_{\bar{k}_1} b_{n-1}(\bar{k}_1) \|_2 + \tau_{n+1} . \| \nabla_{\bar{k}_1} b_{n+1}(\bar{k}_1) \|_2 , \qquad n \geqslant 3 \tag{4.B.10}$$

Here,

$$\| \nabla_{\bar{k}_1} b_n(\bar{k}_1) \|_2 \equiv \left[\int d^3\bar{k}_2 \dots d^3\bar{k}_n \, | \nabla_{\bar{k}_1} b_n(\bar{k}_1 \bar{k}_2 \dots \bar{k}_n) |^2 \right]^{1/2} , \qquad n \geqslant 2$$

and $\alpha_l^{(1)}(\bar{k}_1)$, $l \geqslant 1$, whose expressions are omitted for brevity, depend on v, $\nabla_{\bar{k}_1} v(k_1)$, $\| b_n \|_2$ and $\| b_n(\bar{k}_1) \|_2$. In turn, the latter two are regarded as known, by virtue of (2.B.12-13). By applying to the recurrence (4.B.8-10) methods analogous to those yielding (2.B.13) and the results *a)*, *b)* and *c)* at the end of subsection 2.B, one arrives at

$$| \nabla_{\bar{k}_1} b_1(\bar{k}_1) | \leqslant \sigma^{(4)}(\bar{k}_1) \, | v(k_1) | + \sigma^{(5)}(\bar{k}_1) . \, | \nabla_{\bar{k}_1} v(k_1) | \tag{4.B.11}$$

where $\sigma^{(i)}(\bar{k}_1)$, $i = 4, 5$, are continuous, positive and bounded for any $\bar{k}_1$. Similarly, by taking $\nabla_{\bar{k}_1}$ in the recurrence (4.B.1) and majorizing (using again assumption *e)*), one derives the analogue of (4.B.8-10) for

$$\| \nabla_{\bar{k}_1} d_n^{(1)}(\bar{k}_1) \|_2 \left(\equiv \left[\int d^3\bar{k}_2 \dots d^3\bar{k}_n \, | \nabla_{\bar{k}_1} d_n^{(1)}(\bar{k}_1 \bar{k}_2 \dots \bar{k}_n) |^2 \right]^{1/2} \right), \qquad n \geqslant 2 .$$

Finally, by extending the methods which led from (4.B.8-10) to (4.B.11) and using (4.B.8-11), one derives

$$\| \nabla_{\bar{k}_1} d_2^{(1)}(\bar{k}_1) \|_2 \leqslant \sigma^{(6)}(\bar{k}_1) . \, | v(k_1) | + \sigma^{(7)}(\bar{k}_1) . \, | \nabla_{\bar{k}_1} v(k_1) | \tag{4.B.12}$$

$\sigma^{(i)}(\bar{k}_1)$, $i = 6, 7$ being also continuous, positive and bounded for any $\bar{k}_1$. The bounds (4.B.11-12) will be useful in section 5.

4.C. An expression for $d_2^{(2)}$ in terms of y_1.

Our next task is to construct mathematically the first component $d_2^{(2)}(\bar{k}_1 \bar{k}_2)$ of $(\mathbb{1} - \mathrm{W})^{-1} \mathrm{Y}_2^{(0)}$. Unfortunately, it is difficult since $d_2^{(2)}$ depends on y_1, which contains elastic scattering singularities, so that the majoration techniques used in subsection 4.B for $d_n^{(1)}$, $n \geqslant 2$, cannot be extended to $d_2^{(2)}$ and other methods have to be applied. Let us consider all possible contributions to $d_2^{(2)}(\bar{k}_1\bar{k}_2)$ which arise from the formal expansion

$$\left[\mathbb{1} + \sum_{n=1}^{+\infty} W^n\right] Y_2^{(0)}$$

(through a detailed analysis for $n = 1, 2, 3, 4$ and a suitable induction for larger n). Then, one shows without difficulty that

a) there exist, in a formal sense at least, four functions s_i, $i = 0, 1, 2, 3$ such that

$$\begin{aligned} d_2^{(2)}(\bar{k}_1\bar{k}_2) = s_0(\bar{k}_1\bar{k}_2).d^{(0)}(\bar{k}_1\bar{k}_2) &+ \int d^3\bar{k}'_1 . s_1(\bar{k}_1\bar{k}_2;\bar{k}'_1)d^{(0)}(\bar{k}'_1\bar{k}_2) \\ &+ \int d^3\bar{k}'_2 s_2(\bar{k}_1\bar{k}_2;\bar{k}'_2).d^{(0)}(\bar{k}_1\bar{k}'_2) \\ &+ \int d^3\bar{k}'_1 d^3\bar{k}'_2 s_3(\bar{k}_1\bar{k}_2;\bar{k}'_1\bar{k}'_2)d^{(0)}(\bar{k}'_1\bar{k}'_2) \quad (4.C.1) \end{aligned}$$

b) all s_i, $i = 0, 1, 2, 3$, are free of δ-functions of threemomenta.

*c*1) $s_0(\bar{k}_1\bar{k}_2) = s_0(\bar{k}_2\bar{k}_1)$

*c*2) $s_1(\bar{k}_2\bar{k}_1;\bar{k}'_2) = s_2(\bar{k}_1\bar{k}_2;\bar{k}'_2)$

*c*3) $s_3(\bar{k}_1\bar{k}_2;\bar{k}'_1\bar{k}'_2) = s_3(\bar{k}_2\bar{k}_1;\bar{k}'_1\bar{k}'_2)$

Since $d^{(0)}(\bar{k}_1\bar{k}_2) = d^{(0)}(\bar{k}_2\bar{k}_1)$, the properties *c*1), *c*2) and *c*3) imply

$$d_2^{(2)}(\bar{k}_1\bar{k}_2) = d_2^{(2)}(\bar{k}_2\bar{k}_1).$$

The structure of (5.C.1) suggests that the s_i can be regarded as a kind of Green's functions.

Let f be sufficiently small (recall assumption *b*) in subsection 2.A). Then, one can establish rigorously the existence of the four functions s_i, $i = 0, 1, 2, 3$ satisfying the above properties *a*), *b*) and *c*) and the following ones

*d*1) $|s_0(\bar{k}_1\bar{k}_2)|$ is continuous and bounded for any $\bar{k}_1, \bar{k}_2$

*d*2) $$|s_1(\bar{k}_1\bar{k}_2;\bar{k}'_1)| \leqslant s_{1,M}(\bar{k}_1\bar{k}_2;\bar{k}'_1).\left[\frac{f^2.|v(k_1)|.|v(k'_1)|}{|e_2(E_+,\bar{0};\bar{k}_1\bar{k}_2)|^{1/2}.|e_2(E_+,\bar{0};\bar{k}'_1\bar{k}_2)|^{1/2}}\right]$$

$$|s_2(\bar{k}_1\bar{k}_2;\bar{k}'_2)| \leqslant s_{2,M}(\bar{k}_1\bar{k}_2;\bar{k}'_2).\left[\frac{f^2.|v(k_2)|.|v(k'_2)|}{|e_2(E_+,\bar{0};\bar{k}_1\bar{k}_2)|^{1/2}.|e_2(E_+,\bar{0};\bar{k}_1\bar{k}'_2)|^{1/2}}\right]$$

$$s_{1,M}(\bar{k}_1\bar{k}_2;\bar{k}'_1) = s_{2,M}(\bar{k}_2\bar{k}_1;\bar{k}'_1)$$

*d*3) $$|s_3(\bar{k}_1\bar{k}_2;\bar{k}'_1\bar{k}'_2)| \leqslant s_{3,M}(\bar{k}_1\bar{k}_2;\bar{k}'_1\bar{k}'_2).\left[\frac{f^4.|v(k_1)|.|v(k_2)|.|v(k'_1)|.|v(k'_2)|}{|e_2(E_+,\bar{0};\bar{k}_1\bar{k}_2)|^{1/2}.|e_2(E_+,\bar{0};\bar{k}'_1\bar{k}'_2)|^{1/2}}\right]$$

$d4)$ $s_{1,M}$, $s_{2,M}$ and $s_{3,M}$ are positive, continuous and bounded for any $\bar{k}_1, \bar{k}_2, \bar{k}'_1, \bar{k}'_2$.

$d5)$ $\left| \frac{\partial^{n_1}}{\partial k^{n_1}_{1,\alpha}} \frac{\partial^{n_2}}{\partial k^{n_2}_{2,\beta}} s_0(\bar{k}_1 \bar{k}_2) \right|$ is bounded for any $\bar{k}_1, \bar{k}_2$, if $n_1 + n_2 < +\infty$, $(\alpha, \beta = 1, 2, 3)$

$$d6) \left| \frac{\partial^{n_1}}{\partial k^{n_1}_{2,\alpha}} \cdot \frac{\partial^{n_2}}{\partial k'^{n_2}_{2,\beta}} \cdot \frac{\partial^{n_3}}{\partial k^{n_3}_{1,\gamma}} s_2(\bar{k}_1 \bar{k}_2 ; \bar{k}'_2) \right|$$
$$\leqslant \left[\sum_{h_1=0}^{n_1} \sum_{h_2=0}^{n_2} s^{(h_1,h_2)}_{2,M;\alpha\beta\gamma}(\bar{k}_1 \bar{k}_2 ; \bar{k}'_2 ; n_1 n_2 n_3) . \left| \frac{\partial^{h_1} v(k_2)^*}{\partial k^{h_1}_{2,\alpha}} \right| . \left| \frac{\partial^{h_2} v(k'_2)}{\partial k'^{h_2}_{2,\beta}} \right| \right]$$
$$\cdot \frac{1}{| e_2(E_+, \bar{0}; \bar{k}_1 \bar{k}_2) |^{1/2} . | e_2(E_+, \bar{0}; \bar{k}_1 \bar{k}'_2) |^{1/2}}$$

where $\sum_{i=1}^{3} n_i < +\infty$, all $s^{(h_1,h_2)}_{2,M;\alpha\beta\gamma}$ are positive, continuous and bounded for any $\bar{k}_1, \bar{k}_2, \bar{k}'_2$ and so on for s_1

$$d7) \left| \frac{\partial^{n_1}}{\partial k^{n_1}_{1,\alpha}} \cdot \frac{\partial^{n_2}}{\partial k'^{n_2}_{1,\beta}} \cdot \frac{\partial^{n_3}}{\partial k^{n_3}_{2,\gamma}} \cdot \frac{\partial^{n_4}}{\partial k'^{n_4}_{2,\delta}} s_3(\bar{k}_1 \bar{k}_2 ; \bar{k}'_1 \bar{k}'_2) \right|$$
$$\leqslant \left\{ \sum_{h_1=0}^{n_1} \sum_{h_2=0}^{n_2} \sum_{h_3=0}^{n_3} \sum_{h_4=0}^{n_4} \left| \frac{\partial^{h_1} v(k_1)^*}{\partial k^{h_1}_{1,\alpha}} \right| . \left| \frac{\partial^{h_2} v(k_2)^*}{\partial k^{h_2}_{2,\beta}} \right| . \left| \frac{\partial^{h_3} v(k'_1)}{\partial k'^{h_3}_{1,\gamma}} \right| . \left| \frac{\partial^{h_4} v(k'_2)}{\partial k'^{h_4}_{2,\delta}} \right| \right.$$
$$\left. . \, s^{(h_1,h_2,h_3,h_4)}_{3,M;\alpha\beta\gamma\delta}(\bar{k}_1 \bar{k}_2 ; \bar{k}'_1 \bar{k}'_2 ; n_1 n_2 n_3 n_4) \right\}$$
$$\cdot \frac{1}{| e_2(E_+, \bar{0}; \bar{k}_1 \bar{k}_2) |^{1/2} . | e_2(E_+, \bar{0}; \bar{k}'_1 \bar{k}'_2) |^{1/2}}$$

where $\sum_{i=1}^{3} n_i < +\infty$ and all $s^{(h_1,h_2,h_3,h_4)}_{3,M;\alpha\beta\gamma\delta}$ are positive, continuous and bounded for any $\bar{k}_1, \bar{k}_2, \bar{k}'_1, \bar{k}'_2$.

$d8)$ each s_i, $i = 0, 1, 2, 3$, is really a function only of the scalar products of the vectors which appear as its arguments (no privileged directions exist). Thus, s_0 depends only on $\bar{k}_1^2$, $\bar{k}_2^2$, $\bar{k}_1 . \bar{k}_2$ and so on for the others.

Notice that all s_i, $i = 0, 1, 2, 3$, depend on E_+ and, through it, also on l, and that they do not have an additional and explicit $\bar{l}$-dependence. In fact, *i*) they are uniquely determined by the kernel W, which depends on E_+ and which does not have an explicit $\bar{l}$-dependence, *ii*) E_+ depends on $\omega(l)$ and $E(-\bar{l})$ both of which only depend on l.

The proof of the above results, which are essential for the rigorous construction of y_1, requires a careful study and majoration of all Feynman diagrams which, arising from $\left[\mathbb{1} + \sum_{n=1}^{+\infty} W^n\right].Y_2^{(0)}$, contribute to the functions s_i, $i = 0, 1, 2, 3$. Such a proof, which is rather lengthy, and explicit estimates will be given in a forthcoming paper. Our methods also provide the basis for an effective construction of all s_i, $i = 0, 1, 2, 3$, in the form $s_i = s_{i,F} + s_{i,R}$, where $s_{i,F}$ is the sum of a finite number of Feynman diagrams (say, all perturbative contributions to s_i up to some order f^{2N}) and $s_{i,R}$ is the remainder. In fact, the techniques to be presented in such a forthcoming paper will allow to majorize all $|s_{i,R}|$.

5. THE ELASTIC SCATTERING INTEGRAL EQUATION FOR y_1 (STEP 2)

5.A. Derivation of the elastic scattering integral equation.

Let us consider Eq. (3.3) for $n = 1$ and replace in it $y_2(\bar{k}_0\bar{k}_1)$ by

$$|e_2(E_+, \bar{0}; \bar{k}\bar{k}_1)|^{-1/2}.\left[\sum_{i=1}^{2} d_2^{(i)}(\bar{k}\bar{k}_1)\right],$$

where, in turn, $d_2^{(2)}(\bar{k}\bar{k}_1')$ is to be substituted by the right-hand-side of Eq. (4.C.1) and $d_2^{(1)}(\bar{k}_0\bar{k}_1)$ is a known function (as it is given by the convergent series which results from (4.A.5) for $n = 2$ and (4.B.7) and it depends only on completely known functions). Moreover, let us express $d^{(0)}$ in terms of y_1 via the second Eq. (4.A.3), use Eq. (3.3) for $n = 0$ in order to eliminate y_0 in terms of y_1 and rearrange terms. Then, Eq. (3.3) for $n = 1$ yields finally the following inhomogeneous, linear and singular integral equation for y_1:

$$y_1(\bar{k}_1) = D(k_1)^{-1}.y_{in}(\bar{k}_1) + \int d^3\bar{k}_1'.D(k_1)^{-1}.\left[A(\bar{k}_1, \bar{k}_1') + \frac{f^2 v(k_1)^* v(k_1')}{e_0(E_+, \bar{0})}\right] y(\bar{k}_1') \quad (5.A.1)$$

$$y_{in}(\bar{k}_1) = D_1(\bar{k}_1) + \frac{f^2 v(k_1)^* v(l)}{e_0(E_+, \bar{0})} \quad (5.A.2)$$

$$D(k_1) = e_1(E_+, \bar{0}; \bar{k}_1) + i\varepsilon - D_2(k_1) \quad (5.A.3)$$

$$D_1(\bar{k}_1) = \frac{f.v(l).b_1(-\bar{l}\,; \bar{k}_1)}{|e(E(-\bar{l}), -\bar{l}; \bar{k}_1)|^{1/2}} + 2^{1/2}.f.\int d^3\bar{k}' \frac{v(k').d_2^{(1)}(\bar{k}'\bar{k}_1)}{|e_2(E_+, \bar{0}; \bar{k}'\bar{k}_1)|^{1/2}} \quad (5.A.4)$$

$$\mathrm{D}_2(k_1) = f^2\left[\int d^3\bar{k}'\frac{|v(k')|^2 s_0(\bar{k}'\bar{k}_1)}{e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}_1)} + \int d^3\bar{k}'d^3\bar{k}''\frac{v(k')v(k'')^* s_1(\bar{k}'\bar{k}_1\,;\,\bar{k}'')}{|e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}_1)|^{1/2}.e_2(\mathrm{E}_+, \bar{0}; \bar{k}_1\bar{k}'')^{1/2}}\right] \quad (5.\mathrm{A}.5)$$

$$\begin{aligned}\mathrm{A}(\bar{k}_1, \bar{k}'_1) = {} & \frac{f^2 v(k_1)^* v(k'_1) s_0(\bar{k}'_1\bar{k}_1)}{e_2(\mathrm{E}_+, \bar{0}; \bar{k}'_1\bar{k}_1)} \\ & + f^2\int d^3\bar{k}'\frac{v(k_1)^* v(k') s_1(\bar{k}'\bar{k}_1\,;\,\bar{k}'_1)}{|e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}_1)|^{1/2}.e_2(\mathrm{E}_+, \bar{0}; \bar{k}_1\bar{k}'_1)^{1/2}} \\ & + f^2\int d^3\bar{k}'\frac{|v(k')|^2.s_2(\bar{k}'\bar{k}_1\,;\,\bar{k}'_1)}{|e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}_1)|^{1/2}.e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}'_1)^{1/2}} \\ & + f^2\int d^3\bar{k}'\frac{v(k')^* v(k'_1).s_2(\bar{k}'_1\bar{k}_1\,;\,\bar{k}')}{|e_2(\mathrm{E}_+, \bar{0}; \bar{k}'_1\bar{k}_1)|^{1/2}.e_2(\mathrm{E}_+, \bar{0}; \bar{k}'_1\bar{k}')^{1/2}} \\ & + f^2\int d^3\bar{k}'d^3\bar{k}''\frac{v(k'')^* v(k')}{|e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}_1)|^{1/2}} \\ & \cdot\left[\frac{s_3(\bar{k}'\bar{k}_1\,;\,\bar{k}''\bar{k}'_1)}{e_2(\mathrm{E}_+, \bar{0}; \bar{k}''\bar{k}'_1)^{1/2}} + \frac{s_3(\bar{k}'\bar{k}_1\,;\,\bar{k}'_1\bar{k}'')}{e_2(\mathrm{E}_+, \bar{0}; \bar{k}'_1\bar{k}'')^{1/2}}\right] \quad (5.\mathrm{A}.6)\end{aligned}$$

Remarks. — 1) Since $e_1(\mathrm{E}_+, \bar{0}; \bar{k}_1)$ vanishes for some $\bar{k}_1$ (recall assumption *d*) in section 3)) and in order to ensure the correct elastic-scattering singularities and outgoing wave behavior for y_1, we have replaced $e_1(\mathrm{E}_+, \bar{0}; \bar{k}_1)$ by $e_1(\mathrm{E}_+, \bar{0}; \bar{k}_1) + i\varepsilon$ ($\varepsilon \to 0^+$), according to the general prescriptions of scattering theory [*1*].

2) By virtue of properties *d*1), *d*2) and *d*4) in subsection 4.C and assumption *a*) in subsection 2.A, $\mathrm{D}_2(k_1)$ is bounded for any $\bar{k}_1$, vanishes if $k_1 \to \infty$ and, for any $\bar{k}_1$, becomes as small as desired if f is suitably small.

3) The results *d*1)-*d*4) in subsection 4.C imply

$$|\mathrm{A}(\bar{k}_1, \bar{k}'_1)| \leqslant \mathrm{A_M}(\bar{k}_1, \bar{k}'_1).\,|v(k_1)|\,.\,|v(k'_1)|\,.\,[\omega(k_1).\omega(k'_1)]^{-1/2},$$

where $\mathrm{A_M}(\bar{k}_1, \bar{k}'_1)$ is bounded for any $\bar{k}_1, \bar{k}'_1$ and vanishes if k_1 or k'_1 approach infinity. Moreover, A becomes as small as one likes, provided that f be adequately small.

4) By using

$$\left|\int\frac{d^3\bar{k}'v(k')d_2^{(1)}(\bar{k}'\bar{k}_1)}{|e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}_1)|^{1/2}}\right| \leqslant \left[\int d^3\bar{k}'\frac{|v(k')|^2}{|e_2(\mathrm{E}_+, \bar{0}; \bar{k}'\bar{k}_1)|}\right]^{1/2}.\,\|d_2^{(1)}(\bar{k}_1)\|_2$$

and the results *a*) in subsection 2.B (between Eqs. (2.B.13) and (2.B.14)) and *c*1) in subsection 4.B (between (4.B.6) and (4.B.7)), one gets

$$|\mathrm{D}_1(\bar{k}_1)| \leqslant \mathrm{D}_{1,\mathrm{M}}(\bar{k}_1).\,|v(k_1)|\,.\omega(k_1)^{-1/2}$$

where $\mathrm{D}_{1,\mathrm{M}}(\bar{k}_1)$ is bounded for any $\bar{k}_1$ and vanishes if $k_1 \to \infty$.

5) By using assumption *a*), the results *d*1)-*d*4) in subsection 4.C, the results obtained for b_1 (subsection 2.B) and $d_2^{(1)}$ (subsection 4.B) and Eqs. (5.A.4), (5.A.6), it is easy to prove that $\int d^3\bar{k}_1 \,|\, D_1(\bar{k}_1)\,|^2 < +\infty$ and $\int d^3\bar{k}_1 d^3\bar{k}'_1 \,|\, A(\bar{k}_1, \bar{k}'_1)\,|^2 < +\infty$.

6) From Eqs. (5.A.3), (5.A.5) and (2.B.2) for $\bar{\pi} = \bar{0}$, the result *d*8) and the comment just after it in subsection 4.C, it follows that $D_2(k_1)$ and, hence, $D(k_1)$ only depend on k_1 and E_+ (but not on the direction of $\bar{k}_1$).

Let f be suitably small (recall assumption *b*) in subsection 2.A). Then, by using assumption *f*) in subsection 4.A, the above remark 6), and noticing that $D_2(k_1)$ becomes as small as desired for suitably small f, one has

$$D(k_1) = [k_1 - (k'_1(E_+) + i\varepsilon)]D'(k_1) \qquad (5.A.7)$$

where $k'_1(E_+) > 0$ and $D'(k_1) \neq 0$ for $k_1 \geqslant 0$. Notice that:

i) $k'_1(E_+) - k_1^{(0)}(E_+)$ and $D'(k_1) - e_1^{(0)}(k_1)(k_1^{(0)}(E_+), e_1^{(0)}(k_1)$ being the same as in assumption *f*)) approach zero as f^2, if $f \to 0$,

ii) $D'(k_1)/e_1^{(0)}(k_1) \to 1$ as $k_1 \to \infty$, since $D_2(k_1) \to 0$ (recall remark 2) above).

We shall introduce

$$\begin{aligned} y'_1(\bar{k}_1) &= D(\bar{k}_1).y_1(\bar{k}_1) \\ A_1(\bar{k}_1, \bar{k}'_1) &= [D'(k'_1)]^{-1}.\left[A(\bar{k}_1, \bar{k}'_1) + \frac{f^2 v(k_1)^* v(k'_1)}{e_0(E_+, \bar{0})}\right] \end{aligned} \qquad (5.A.8)$$

so that Eq. (5.A.1) becomes the desired elastic scattering integral equation

$$y'_1(\bar{k}_1) = y_{in}(\bar{k}_1) + \int d^3\bar{k}'_1 \frac{A_1(\bar{k}_1, \bar{k}'_1)}{k'_1 - [k'_1(E_+) + i\varepsilon]} . y'_1(\bar{k}'_1) \qquad (5.A.9)$$

Using standard abstract notation and manipulating, Eq. (5.A.9) reads

$$y'_1 = y_{in} + \hat{A}.y'_1, \quad y'_1(\bar{k}_1) \equiv \langle \bar{k}_1 \,|\, y'_1 \rangle, \quad y_{in}(\bar{k}_1) \equiv \langle \bar{k}_1 \,|\, y_{in} \rangle \qquad (5.A.10)$$

$$\begin{aligned} \frac{A_1(\bar{k}_1, \bar{k}'_1)}{k_1 - [k'_1(E_+) + i\varepsilon]} &\equiv \langle \bar{k}_1 \,|\, \hat{A} \,|\, \bar{k}'_1 \rangle \\ &= \int_0^{+\infty} dk''_1 \frac{\langle \bar{k}_1 \,|\, \hat{\Delta}(k''_1) \,|\, \bar{k}'_1 \rangle}{k''_1 - [k'_1(E_+) + i\varepsilon]} \end{aligned} \qquad (5.A.11)$$

$$\begin{aligned} \langle \bar{k}_1 \,|\, \hat{\Delta}(k''_1) \,|\, \bar{k}'_1 \rangle = \int_{-1}^{+1} d(\cos\theta''_1) \int_0^{2\pi} d\varphi''_1 . \delta\,[\cos\theta''_1 - \cos\theta'_1\,] \\ \delta[\varphi''_1 - \varphi'_1\,].\delta[k''_1 - k'_1\,].A_1(\bar{k}_1, \bar{k}''_1) \end{aligned} \qquad (5.A.12)$$

where (θ'_1, φ'_1) and $(\theta''_1, \varphi''_1)$ are the polar angles determining $\bar{k}'_1$ and $\bar{k}''_1$.

5.B. Rigorous solution of the elastic scattering integral equation.

Some time ago, Lovelace presented a compactness proof for the Lippmann-Schwinger equation in two-particle potential scattering [*23*]. As we shall see, it is possible to extend such a proof to Eq. (5.A.9). We shall give the essential arguments and bounds for that purpose, and omit certain details which can be found in [*23*]. Let us consider the Banach space C_1 of all complex functions $\Phi(\bar{k}_1)$ such that both $\Phi(\bar{k}_1)$ and $\nabla_{\bar{k}_1}\Phi(\bar{k}_1)$ are continuous and bounded in magnitude for any $\bar{k}_1$. The norm in C_1, which makes the latter a Banach space, is

$$||| \Phi ||| = \operatorname*{Max}_{\bar{k}_1} | \Phi(\bar{k}_1) | + \sigma^{(8)} . \operatorname*{Max}_{\bar{k}_1} | \nabla_{\bar{k}_1}\Phi(\bar{k}_1) | \tag{5.B.1}$$

$\sigma^{(8)}$ being a strictly positive constant such that both terms in Eq. (5.B.1) have the same dimensions (for instance, $\sigma^{(8)} = \omega_0 > 0$).

We shall make the following additional assumption: *g*) $v(k)$ is such that

*g*1)
$$| k_1'^2 . A_1(\bar{k}_1, \bar{k}_1') | \leqslant \frac{\sigma^{(10)}}{1 + \sigma^{(9)} [\ln (k_1'/\omega_0)]^2},$$
$$| k_1'^2 \nabla_{\bar{k}_1} A_1(\bar{k}_1, \bar{k}_1') | \leqslant \frac{\sigma^{(11)}}{1 + \sigma^{(9)} [\ln (k_1'/\omega_0)]^2}$$

are true uniformly for any $\bar{k}_1$ and $\bar{k}_1'$, $\sigma^{(i)}$, $i = 9, 10$ and 11 being certain non-negative constants, with $\sigma^{(9)} > 0$ strictly, and

*g*2) both

$$k_1'^{1/2}\left[\left(2k_1' + \frac{k_1'^2}{\sigma^{(8)}}\right) | A_1(\bar{k}_1, \bar{k}_1') | + k_1'^2 \left| \frac{\partial A_1(\bar{k}_1, \bar{k}_1')}{\partial k_1'} \right| \right]$$
$$k_1'^{1/2}\left[\left(2k_1' + \frac{k_1'^2}{\sigma^{(8)}}\right) | \nabla_{\bar{k}_1} A(\bar{k}_1, \bar{k}_1') | + k_1'^2 \left| \nabla_{\bar{k}_1} \frac{\partial A_1(\bar{k}_1, \bar{k}_1')}{\partial k_1'} \right| \right]$$

are uniformly bounded for any $\bar{k}_1$ and $\bar{k}_1'$.

By recalling the second Eq. (5.A.8), majorizing it and Eq. (5.A.6), recalling Eq. (5.A.7) and the comments below it and using the results *d*1) to *d*7) in order to make v and $\nabla_{\bar{k}} v$ appear, one concludes that there always exist cut-off functions v such that both *g*1) and *g*2) hold. We shall not write down the resulting conditions on v and $\nabla_{\bar{k}} v$, which are straightforward but rather cumbersome. Notice that *g*1) and *g*2) generalize, respectively, the conditions named (2.11) and (2.14) in Lovelace's paper [*23*]. Moreover, both $\sigma^{(10)}$ and $\sigma^{(11)}$ become arbitrarily small, if f is suitably small.

The main steps necessary for solving Eq. (5.A.9) in C_1 are the following.

1) Assumption *c*) and our previous results in subsections 2.B and 4.A (recall the result *a*) in subsection 2.B, the result *c*1) in subsection 4.B,

the bounds (4.B.11-12) and Eqs. (5.A.2) and (5.A.4)) imply that $b_1(-\bar{l};\bar{k}_1)$,

$$\int d^3\bar{k}'\,\frac{v(k')d_2^{(1)}(\bar{k}'\bar{k}_1)}{|\,e_2(\mathrm{E}_+,\bar{0};\bar{k}'\bar{k}_1)\,|^{1/2}},$$

$\mathrm{D}_1(\bar{k}_1)$ and $y_{\mathrm{in}}(\bar{k}_1)$ all belong to C_1.

2) If Φ belongs to C_1, so does $\hat{\Delta}(k_1'')\Phi$, and one has

$$|||\,\hat{\Delta}(k_1'')\Phi\,||| \leqslant \sigma^{(12)}(k_1'').\,|||\,\Phi\,|||$$

uniformly in k_1'', where

$$\sigma^{(12)}(k_1'') = \operatorname*{Max}_{\bar{k}_1}\left[k_1''^2\int_{-1}^{+1}d(\cos\theta_1'')\int_0^{2\pi}d\varphi_1''\,|\,\mathrm{A}_1(\bar{k}_1,\bar{k}_1'')\,|\right] + \sigma^{(8)}.\operatorname*{Max}_{\bar{k}_1}\left[k_1''^2\int_{-1}^{+1}d(\cos\theta_1'')\int_0^{2\pi}d\varphi_1''\,|\,\nabla_{\bar{k}_1}\mathrm{A}_1(\bar{k}_1,\bar{k}_1'')\,|\right] \quad (5.\mathrm{B}.2)$$

This bound implies that $\hat{\Delta}(k_1''=0)=0$.

3) Step 2) and assumption $g1$) imply

$$|||\,\hat{\Delta}(k_1'')\Phi\,||| \leqslant \frac{\sigma^{(13)}}{1+\sigma^{(9)}.\,[\ln(k_1''/\omega_0)]^2}\,.\,|||\,\Phi\,||| \quad (5.\mathrm{B}.3)$$

uniformly in k_1'', $\sigma^{(13)}$ being a positive constant (compare with (2.10) in [*23*]).

4) If Φ belongs to C_1, so does $\dfrac{d\hat{\Delta}(k_1'')}{d(k_1''^{1/2})}\Phi$. Moreover, by virtue of assumption $g2$), one has $\left|\left|\left|\dfrac{d\hat{\Delta}(k_1'')}{d(k_1''^{1/2})}\Phi\right|\right|\right| \leqslant \sigma^{(14)}(k_1'').\,|||\,\Phi\,|||$, uniformly in k_1'', where

$$\sigma^{(14)}(k_1'') = 2.k_1''^{3/2}\left\{\operatorname*{Max}_{\bar{k}_1}\int_{-1}^{+1}d(\cos\theta_1'')\int_0^{2\pi}d\varphi_1''\left[\left(2+\frac{k_1''}{\sigma^{(8)}}\right)|\,\mathrm{A}_1(\bar{k}_1,\bar{k}_1'')\,| + k_1''\left|\frac{\partial\mathrm{A}_1(\bar{k}_1,\bar{k}_1'')}{\partial k_1''}\right|\right]\right.$$
$$\left. + \sigma^{(8)}.\operatorname*{Max}_{\bar{k}_1}\int_{-1}^{+1}d(\cos\theta_1'')\int_0^{2\pi}d\varphi_1''\left[\left(2+\frac{k_1''}{\sigma^{(8)}}\right)|\,\nabla_{\bar{k}_1}\mathrm{A}_1(\bar{k}_1,\bar{k}_1'')\,| + k_1''\left|\nabla_{\bar{k}_1}\frac{\partial\mathrm{A}_1(\bar{k}_1,\bar{k}_1'')}{\partial k_1''}\right|\right]\right\} \quad (5.\mathrm{B}.4)$$

5) Steps 2) and 3) imply the following Hölder conditions:

$$\begin{aligned}&|||\,(\hat{\Delta}(k_1'')-\hat{\Delta}(k_2''))\Phi\,||| \leqslant \sigma^{(15)}.\,|\,k_1''-k_2''\,|\,.\,|||\,\Phi\,|||,\quad k_2''>0\\ &|||\,\hat{\Delta}(k_1'')\Phi\,||| \leqslant \sigma^{(16)}.k_1''^{1/2}.\,|||\,\Phi\,|||\end{aligned} \quad (5.\mathrm{B}.5)$$

$\sigma^{(15)}$ and $\sigma^{(16)}$ being positive constants (compare with [*23*]).

6) If Φ belongs to C_1, $\hat{A}\Phi$ also belongs to C_1. Moreover, Eqs. (5.A.10-11) and steps 2), 3), 4) and 5) yield:

i) if $k_1'(E_+) > 0$

$$|||\hat{A}\Phi||| \leqslant \left\{ \frac{\sigma^{(13)}}{1 + [\ln(k_1'(E_+)/\omega_0)]^2} \left[\pi + \left| \ln \frac{k_0 - k_1'(E_+)}{k_1'(E_+)} \right| \right] + \sigma^{(15)}.k_0 + \sigma^{(13)} \int_{k_0}^{+\infty} \frac{dk_1''}{[k_1'' - k_1'(E_+)][1 + \sigma^{(9)}[\ln(k_1''/\omega_0)]^2]} \right\} . |||\Phi||| \quad (5.B.6)$$

k_0 being an arbitrary finite momentum, with $k_0 > k_1'(E_+)$.

ii) if E_+ is such that $k_1'(E_+) = 0$ and k_0' is arbitrary but strictly positive

$$|||\hat{A}\Phi||| \leqslant \left[2\sigma^{(16)}.k_0'^{1/2} + \sigma^{(13)} \int_{k_0}^{+\infty} \frac{dk_1''}{k_1''.[1 + [\ln(k_1''/\omega_0)]^2]} \right] . |||\Phi||| \quad (5.B.7)$$

Notice that the two integrals appearing in (5.B.6) and (5.B.7) converge and that $\sigma^{(i)}$, $i = 13, 14, 15$ and 16 become as small as desired as $f \to 0$.

From the above properties, and extending Lovelace's arguments [*23*], it follows that: *a*) $\hat{A}$ is a compact (and, hence, a continuous and bounded) operator in C_1, *b*) $\hat{A}$ can be arbitrarily closely approximated by an operator of finite rank and, hence, Eq. (5.A.9) can be approximated by a finite matrix equation as accurately as desired, *c*) for suitably small *f*, the series for $y_1'(\bar{k}_1)$ formed by all successive iterations of Eq. (5.A.9) converges in C_1 (as the two constants multiplying $|||\Phi|||$ on the right-hand-sides of (5.B.6) and (5.B.7) are less than unity). Further results can be obtained by applying to the kernel $\hat{A}$ and Eq. (5.A.9) other standard properties of compact operators. For brevity, we shall omit them.

6. CONSTRUCTION OF y_0 AND y_n, $n \geqslant 2$ (STEP 3)

Using Eqs. (5.A.7) and the first Eq. (5.A.8), Eq. (3.3) for $n = 0$ becomes

$$y_0 = \frac{1}{e_0(E_+, \bar{0})} \left\{ f.v(l) + f.\int_0^{+\infty} \frac{dk}{k - [k_1'(E_+) + i\varepsilon]} \cdot \left[\frac{k^2}{D_1'(k)} \cdot \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi \,.\, v(k) \,.\, y_1'(\bar{k}) \right] \right\} \quad (6.1)$$

Here, (θ, φ) are the polar angles of $\bar{k}$ and $y_1'(\bar{k})$ is the solution of Eq. (5.A.9), which exists and belongs to C_1 (at least, for small *f*). By extending the techniques used to establish step 5) above, one proves easily that the singular integral in Eq. (6.1) converges, which implies the finiteness of y_0.

Similar methods allow to establish the convergence of all integrals in Eq. (5.C.1), when $d^{(0)}$ is expressed in terms of y'_1 via the second Eq. (4.A.3). This implies that $D(k_1).D(k_2).d_2^{(2)}(\bar{k}_1\bar{k}_2)$ and

$$\nabla_{\bar{k}_i}[D(k_1)D(k_2)d_2^{(2)}(\bar{k}_1\bar{k}_2)], \qquad i = 1, 2$$

are continuous and bounded for any $\bar{k}_1, \bar{k}_2$. This, by virtue of the results obtained in subsection 4.B and Eq. (4.A.5), completes the characterization of y_2.

Once y_0, y_1 and y_2 are known, the construction of y_n for $n \geqslant 3$ poses no problem of principle, although it is rather cumbersome. We shall sketch the determination of y_3 only. Let us consider all Eqs. (3.3) for $n \geqslant 3$ and, using Eq. (4.A.1), cast them as

$$Y' = Y_1'^{(0)} + Y_2'^{(0)} + W'Y',$$

$$Y_2'^{(0)} = \begin{pmatrix} \dfrac{f}{e_3(E_+, \bar{0}; \bar{k}_1\bar{k}_2\bar{k}_3)^{1/2}} \cdot \sum_{i=1}^{3} v(k_i)^* y_2(\bar{k}_j . \bar{k}_h) \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \tag{6.2}$$

where $(j, h) = (2, 3), (1, 3), (1, 2)$ for $i = 1, 2, 3$. Y' and $Y_1'^{(0)}$ are given by the right-hand-sides of Eqs. (4.A.2) respectively, with the first component omitted and W' is the corresponding new kernel. The construction of $(\mathbb{1} - W')^{-1}.Y_1'^{(0)}$ proceeds by generalizing directly that of $d^{(1)}$ in subsection 4.B. Clearly, y'_3 equals the first component of $(\mathbb{1} - W')^{-1}.Y_1'^{(0)}$, plus that of $(\mathbb{1} - W')^{-1}.Y_2'^{(0)}$. In turn, the latter is given by the generalization of Eq. (5.C.1), with: *i*) new functions s'_j, $j = 0, 1, \ldots, 8$, instead of s_i, which can be constructed by generalizing the techniques to be presented in a separate paper, *ii*) $d^{(0)}$ replaced by the non-vanishing component of $Y_2'^{(0)}$, which is already known. The elastic scattering amplitude could be determined, in principle, in terms of $\Psi_{sc}(\bar{l}; -\bar{l})$, by generalizing directly the developments given in Schweber [*6*].

We stress that the techniques used in this and the following paper provide basis for: *i*) effective reductions of the actual elastic scattering problem in a field-theoretic model to n-body problems (as they control the contributions of all possible Feynman diagrams), *ii*) non-perturbative studies, for increasing values in f, (following the spirit of subsection 5.C of [*14*]). In order not to make this work longer, we shall omit them and the discussion of some open problems.

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