ANNALES DE L'I. H. P., SECTION A

C. DUVAL

On the polarizers of compact semi-simple Lie groups. Applications

Annales de l'I. H. P., section A, tome 34, n° 1 (1981), p. 95-115

<http://www.numdam.org/item?id=AIHPA_1981__34_1_95_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXIV, n° 1, 1981, p. 95-115. *Physique théorique.*

On the polarizers of compact semi-simple Lie groups. Applications

by

C. DUVAL
Faculté de Sciences de Luminy et Centre de Physique Théorique, C. N. R. S.,
70, route Léon-Lachamp, Case 907, F-13288 Marseille Cedex 2

ABSTRACT. — The new notion of polarizer of a prequantizable symplectic manifold is introduced. A specific expression for the invariant polarizers of the coadjoint orbits of a compact semi-simple Lie group is proposed. It is shown that a severe selection on the associated representations must be necessarily performed. The explicit construction of the prequantization of these orbits is carried out in full details. Applications to hadron spectroscopy are discussed.

INTRODUCTION

The quest for the correspondence between irreducible unitary representations of a Lie group and some of its coadjoint orbits still constitutes a major endeavour in representation theory since the pioneer work of Kirillov [*1*] and Kostant [*2*]. Much has already been achieved in this domain and the celebrated Borel-Weil-Bott theorem establishes the requested one to one correspondence between all (finite dimensional) irreducible unitary representations of a compact semi-simple simply connected Lie group and the so called prequantizable orbits of its coadjoint action [*10*]. An illustration of this theorem will be given in the sequel.

It has been emphasized for a long time that geometric quantization (known as the Kostant-Souriau theory) provides a well suited framework to attack that problem on purely geometrical grounds. The notions of prequantization and polarization of a symplectic manifold have been devised in order to work out a geometrical setting for a quantization procedure adapted to the dynamical systems physics has to deal with [*2*] [*3*]. Polarizations generalize real Lagrangian distributions and have the advantage of making it possible to quantize a fair number of dynamical systems (e. g. symplectic models of spinning particles, possibly endowed with an internal structure [*3*] [*5*] [*12*]). The usual quantum mechanical treatment is in fact recovered by introducing the notion of polarized functions (wave functions in the sequel) [*2*] [*3*]. When homogeneous dynamical systems are taken into account, the space of wave functions associated with an invariant polarization can be given (in some « good » cases) a Hilbertian structure which serves to represent the Lie group under consideration. Whence the relationship with the latter point of view.

Now compact semi-simple Lie groups are widely accepted as good candidates for gauge groups in particle physics [*6*]. Also the fact that much remains to investigate within the proposed models of elementary particles with internal degrees of freedom (see [*5*] for an account on the phase spaces associated with classical isospin, isospin-hypercharge,...) prompted us to revisit the quantization of the coadjoint orbits of compact semi-simple Lie groups.

At this stage it is worth noticing that not all irreducible representations (multiplets) of a given internal symmetry group (e. g. SU(2), SU(3) are actually taken into account for the classification of elementary particles [*6*] [*13*]. A novel selection—in addition to the prequantization selection—among the coadjoint orbits of a gauge group, has to be carried out by geometrical means.

The purpose of this article is to show that this programme can be undertaken by introducing the new notion of *polarizer* of a prequantizable symplectic manifold (originally due to Souriau). Roughly speaking, polarizers are complex differential forms which give rise to polarizations. The subtlety is that the existence of invariant polarizers of homogeneous dynamical systems turns out, in some cases, to be quite a strong requirement which leads to a drastic selection of coadjoint orbits. Equation (6.4) illustrates this fact in the case of a compact semi-simple simply connected Lie group. When specialized to SU(2) and SU(3), the selection seems to be in fairly good accordance with physical data of hadron spectroscopy. This will be discussed below. Our main original contribution is presented in chapters 6, 8 and 9 where several other results concerning twistors and SU(4) polarizers are analysed in order to strengthen our point of view.

We believe that Souriau's definition of elementary systems [*3*], introduced as prequantizable coadjoint orbits of a Lie group (e. g. the restricted Poincaré group ($\times$ a gauge group), SU(2, 2), etc.) should be improved by requiring the existence of invariant polarizers as a must. Since our results agree with the commonly accepted concept of elementarity, that additional assumption should be taken as a reasonable one.

Chapters 1 and 5 are devoted to the detailed construction of the coadjoint orbits of a compact semi-simple simply connected Lie group together with the explicit prequantization of the prequantizable ones. The general philosophy is that either the group itself and its left-invariant Maurer-Cartan 1-form must be treated basically in many respects. They turn out to provide us with both symplectic structure and prequantization in a fairly straightforward manner. We have found it necessary to present a constructive description of these geometrical structures in a somewhat different guise than customary [*2*] [*8*].

As for chapters 2, 3, 7, they are mainly intended to introduce some new aspects of geometric quantization. The basic facts concerning semi-simple Lie algebras are recalled in chapter 4. The physical motivations which pervade this article are discussed in the last chapter.

1. CANONICAL HAMILTONIAN STRUCTURE ON A LIE GROUP

Let G be a real connected Lie group with Lie algebra $\mathcal{G}$ (1), and q be a fixed element of $\mathcal{G}^*$, we can now define on G the following 1-form associated to q

$$\boxed{\varpi := q \, . \, \theta} \tag{1.1}$$

where θ is the Maurer-Cartan structure on G [*7*] (the unique left-invariant 1-form on G with values in $\mathcal{G}$ such that $\forall Z \in \mathcal{G}$, $\theta(Z) = Z$).

The Maurer-Cartan equations yield $d\varpi = - q \, . \, [\theta, \theta]$. Now the infinitesimal left action of G on itself is given by

$$\tilde{Z}(g) = (r_g)_*(Z)(g) = (\mathrm{ad}\,(g^{-1})Z)(g)\,, \qquad g \in G \quad \text{and} \quad Z \in \mathcal{G}.$$

Next $d\varpi(\tilde{Z}(g), \tilde{Z}'(g)) = - q \, . \, [\theta(\tilde{Z}(g)), \theta(\tilde{Z}'(g))] = - q_g \, . \, [Z, Z']$ where $q_g = \mathrm{ad}^*(g)q = \pi(g)$. But the right-hand side of the previous equation represents the pull-back by π of the canonical symplectic structure σ of the coadjoint orbit $\mathcal{O}_q = \pi(G) = G/G_q$ (2) [*3*] (G_q is the isotropy subgroup of q for ad*). Hence

$$\boxed{d\varpi = \pi^*\sigma} \tag{1.2}$$

(1) The space of left-invariant (differentiable) vector fields on G.

(2) $g \sim g.g'$ iff $g' \in G_q$.

G inherits in this way a structure of presymplectic manifold; the leaves of the involutive distribution ker ($d\varpi$) are diffeomorphic to $G_q(L^*_{\tilde{G}}\varpi = \varpi)$. The moment [3] of G is obviously q_g, as can be checked by

$$\varpi(\tilde{Z}(g)) = q.\theta\,(\mathrm{ad}\,(g^{-1})Z) = q_g.Z\,.$$

(1.3) If G is compact, connected and simply connected, the same is true for $\mathcal{O}_q$, for G_q turns out to be connected (and compact) under these circumstances [1].

2. PREQUANTIZATION

Let (M, σ) be a symplectic manifold (σ is a 2-form on M such that $d\sigma = 0$ and ker (σ) = $\{ 0 \}$). The couple (Y, ω) is called a *prequantization* of (M, σ) if Y is a principal U(1)-fibre bundle over M, with projection $p : Y \to M$, equipped with a connection form $i\omega$ such that $d\omega = p^*\sigma$ (see [3] for an equivalent setting). A symplectic manifold is said to be *prequantizable* if there exists a prequantization of it. Two prequantizations (Y_1, ω_1) and (Y_2, ω_2) are equivalent iff there exists a diffeomorphism $f : Y_1 \to Y_2$ satisfying $\omega_1 = f^*\omega_2$, $p_1 = p_2 \circ f$. In fact, all prequantizations of a prequantizable simply connected symplectic manifold turn out to be equivalent [2] [3]. In view of (1.3)

(2.1) *All prequantizable coadjoint orbits of a compact simply connected Lie group have a unique prequantization.* We will find it convenient to build it up from the group itself.

Let us recall that (M, σ) is prequantizable if $1/2\pi\int_{S^2}\sigma$ is integral for for any 2-cycle S^2 in M [2].

(2.2) A cotangent bundle $(M, \sigma) = \left(T^*Q, \sum_j dp_j \wedge dq^j\right)$ is trivially prequantized by $(Y, \omega) = \left(M \times U(1), \sum_j p_j dq^j + dz/iz\right)$.

(2.3) An example of non trivially prequantizable symplectic manifolds is given by the minimal orbits of the coadjoint representation of SU(n + 1) together with their canonical symplectic structure. It is well known that they are symplectomorphic to $\mathbb{P}_n(\mathbb{C})$ and that the Hopf fibration $\pi : S^{2n+1} \to \mathbb{P}_n(\mathbb{C})$, $\pi(Z) = m/i(Z.\bar{Z} - \mathbb{1}/n)$, $Z \in \mathbb{C}^{n+1}$, $\bar{Z}.Z = 1$ ([3]), provides us with the prequantum bundle $(S^{2n+1}/\mathbb{Z}_{|m|}, m\bar{Z}.dZ/i)$ where

([3]) The bar « − » means transposition plus conjugation $\left(\bar{Z}.Z = \sum_{\alpha=1}^{n+1} \bar{Z}_\alpha Z^\alpha ;\text{ here } \bar{Z}_\alpha = \overline{Z^\alpha}\right)$.

$m \in \mathbb{Z} - \{0\}$ (see (5.7) for a hint) and $Z \sim Ze^{2i\pi p/m}$, $p = 1, 2, \ldots, |m|$ (the method of fusion [3], also (5.8)). These results specialise those of chapter 5 to the simplest case of $\mathbb{P}_n(\mathbb{C})$ if $G = SU(n+1)$.

3. POLARIZATIONS AND POLARIZERS

A polarization of a symplectic manifold (M, σ) is defined as an involutive distribution F of maximal isotropic subspaces : $F_x \subset T_x^{\mathbb{C}}M$,

$$\mathbb{C} - \dim(F_x) = \dim(M)/2, \qquad \sigma(F_x, F_x) = 0.$$

F is called admissible if $F \cap \bar{F}$ has constant dimension and $F + \bar{F}$ is involutive [2]. If (Y, ω) prequantizes (M, σ) and if F can be lifted horizontally, one gets a *Planck polarisation* $F^*(p_*F^* = F, \omega(F^*) = 0)$ [3].

(3.1) Let (Y, ω) prequantize (M, σ), $\dim(M) = 2n$, we will call *polarizer* of (Y, ω) *any nowhere vanishing complex n-form ϕ on* Y *such that*

i)	$\operatorname{rank}(\phi) = n$
ii)	$d\phi = c\omega \wedge \phi, \qquad c = \text{const.} \in \mathbb{C}^*$

ϕ is thus of minimal rank (locally $\phi = \phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n$ where the ϕ's are complex 1-forms of Y).

Differentiating (3.1, *ii*) yields $d\omega \wedge \phi = 0$, and $\forall X, X' \in \ker(\phi)$ $d\omega(X, X')\phi = 0$, whence $d\omega \restriction \ker(\phi) = 0$. Ker (ϕ) is thus isotropic, in fact maximal isotropic in view of (3.1, *i*). Let ξ denote the $U(^1)$ infinitesimal generator on $Y(\omega(\xi) = 1, \xi \lrcorner d\omega = 0)$. Since in particular $\xi \in \ker(\phi)^{\perp}$ (4), $\xi \in \ker(\phi)$ because $\ker(\phi)$ is self orthogonal. Furthermore

$$L_\xi\phi = d(\xi \lrcorner \phi) + \xi \lrcorner d\phi = c\phi.$$

Integration along the fibres of Y yields that necessarily $c = ik$, $k \in \mathbb{Z} - \{0\}$. Hence (3.1, *ii*) reads now

(3.2)

$d\phi = ik\omega \wedge \phi$	$k \in \mathbb{Z} - \{0\}$

ϕ turns out to be semi-basic. It is easily shown that every 1-form ρ which anihilates $\ker(\phi)$, satisfies $\rho \wedge \phi = 0$. Actually $\ker(\phi)^0 = \operatorname{coker}(\phi)$, but $\omega(\xi) = 1$ and $\xi \lrcorner \phi = 0$, so it readily follows that $\omega \notin \ker(\phi)^0$. Thus $\omega \wedge \phi \neq 0$, at last $d\phi$ never vahishes. Also $\ker(d\phi) = \ker(\omega) \cap \ker(\phi) \subset \ker(\phi)$ is n-dimensional. If $X \in F^* := \ker(d\phi)$, $L_X\phi = 0$ and if $X, X' \in F^*$,

$$[X, X'] \lrcorner d\phi = L_{[X,X']}\phi - d([X, X'] \lrcorner \phi) = 0 - d(L_X(X' \lrcorner \phi) - X' \lrcorner L_X\phi) = 0.$$

(4) « $\perp$ » means: orthogonal with respect to $d\omega$.

The distribution F^* is involutive and

$$\ker (d\phi) \textit{ is a Planck polarization of } (Y, \omega)$$

No information on the admissibility of such a polarization is available from its polarizer.

(3.3) A polarizer ϕ will be called invariant (hence its associated polarization) with respect to a subgroup G of Quant (Y) if $a^*\phi = \phi\ \forall a \in G$. Quant (Y) is the group of all automorphisms of the prequantum structure of Y [*3*].

(3.4) F is called a *real polarization if* $F = \bar{F}$. For example, in the case of $T^*\mathbb{R}^n$ (2.2) $F = \sum_{j=1}^{n} \mathbb{R}\partial/\partial p_j$. It can be easily checked that

$$\phi = z^k dq^1 \wedge dq^2 \wedge \ldots \wedge dq^n$$

is a k-polarizer giving rise to F. Such a ϕ is certainly not unique since $f\phi$, $f \in C^\infty(\mathbb{R}^n)$, $f \neq 0$ is another k-polarizer for the same F.

(3.5) A k-polarizer ϕ *will be said to be Kählerian if* $\bar{\phi} \wedge \phi \neq 0$, Since $L_\xi(\bar{\phi} \wedge \phi) = 0$ (3.2), $\bar{\phi} \wedge \phi = p^*(i^n f \lambda)$ for some $f \in C^\infty(M, \mathbb{R})$ with constant sign on M; λ denotes the Liouville measure of M. Thus

$$\ker (\bar{\phi}) \cap \ker (\phi) = \ker (d\omega)$$

and

$$\ker (d\bar{\phi}) \cap \ker (d\phi) = \ker(\omega) \cap \ker (d\omega) = \{ 0 \} .$$

Whence the justification of the terminology [*2*] [*8*].

The material of this chapter has been borrowed to unpublished notes of Souriau's (to appear in S. D. S. [*3*]).

4. A COMPENDIUM ON SEMI-SIMPLE LIE ALGEBRAS

Let $\mathscr{G}$ be a (real) semi-simple Lie algebra and $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{G}$. One has the following decomposition:

$$\mathscr{G}^{\mathbb{C}} = \mathscr{H}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta} \mathscr{G}_\alpha$$

where Δ is a finite subset of $\mathscr{H}^{\mathbb{C}*}$ (the root system of $\mathscr{H}^{\mathbb{C}}$) and $\mathscr{G}_\alpha$ a complex 1-dimensional eigenspace of Ad ($\mathscr{H}^{\mathbb{C}}$), i. e. $e_\alpha \in \mathscr{G}_\alpha$ if

(4.1) $$[h, e_\alpha] = \alpha(h)e_\alpha \qquad \forall h \in \mathscr{H}^{\mathbb{C}}$$

The Killing form K of $\mathscr{G}$ extends naturally to $\mathscr{G}^{\mathbb{C}}$, when restricted to $\mathscr{H}^{\mathbb{C}}$K is still non denegerate and

(4.2) $$K(h, e_\alpha) = 0 \qquad \forall h \in \mathscr{H}^{\mathbb{C}}, \quad \forall \alpha \in \Delta$$

(4.3) $$K(e_\alpha, e_\beta) = 0 \qquad \forall \alpha, \beta \in \Delta, \qquad \alpha + \beta \neq 0$$

Also $\Delta = \Delta^+ \cup \Delta^-$ ($\Delta^- = -\Delta^+, \Delta^+ \cap \Delta^- = \emptyset$), $\Delta^+ \supset B$ where B is a basis for $\mathcal{H}^{\mathbb{C}*}$ (simple roots) i. e. $\forall \alpha \in \Delta^+ \exists m_\beta \in \mathbb{N}$ ($\beta \in B$) such that

$$\alpha = \sum_{\beta \in B} m_\beta \beta .$$

The Cartan matrix entries

$$n(\beta, \alpha) := 2K^{-1}(\alpha, \beta)/K^{-1}(\alpha, \alpha) \qquad (\alpha, \beta \in \Delta)$$

take their values in $\{-3, -2, -1, 0, 1, 2, 3\}$.

Furthermore, it is always possible to choose e_α and $e_{-\alpha}$ ($\alpha \in \Delta^+$) in order to define a unique $h_\alpha \in \mathcal{H}^{\mathbb{C}}$ such that

(4.4) $$[e_\alpha, e_{-\alpha}] = h_\alpha , \qquad [h_\alpha, e_\alpha] = 2e_\alpha , \qquad [h_\alpha, e_{-\alpha}] = -2e_{-\alpha} .$$

(4.5) $$K(e_\alpha, e_{-\alpha}) = 1 .$$

The Lie algebra spanned by $(h_\alpha, e_\alpha, e_{-\alpha})$ is isomorphic to sl (2, $\mathbb{C}$). It is clear that $\alpha = 2K(h_\alpha, .)/K(h_\alpha, h_\alpha)$ because $\alpha(h_\alpha) = 2$ and again

(4.6) $$\begin{aligned} n(\beta, \alpha) &= \beta(h_\alpha) = 2K^{-1}(\alpha, \beta)/K^{-1}(\alpha, \alpha) \\ &= 2K(h_\alpha, h_\beta)/K(h_\beta, h_\beta) \end{aligned}$$

We need to recall the usefull decomposition

$$\mathcal{G}^{\mathbb{C}} = \sum_{\beta \in B} \mathbb{C}h_\beta \oplus \sum_{\alpha \in \Delta^+} \mathbb{C}e_\alpha \oplus \mathbb{C}e_{-\alpha}$$

Let us now assume that $\mathcal{G}$ is compact (K is negative definite), then, as a classical result

$$\mathcal{G} = \sum_{\beta \in B} i\mathbb{R}h_\beta \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}(e_\alpha - e_{-\alpha}) \oplus i\mathbb{R}(e_\alpha + e_{-\alpha})$$

Finally $\mathcal{H} = \mathcal{H}^{\mathbb{C}} \cap \mathcal{G}$ is r-dimensional (r = card (B) = rank ($\mathcal{G}$)) and $K \restriction \mathcal{H}$ is negative definite.

For all the material of this section see [*10*] [*11*].

5. PREQUANTIZATION OF THE COADJOINT ORBITS OF A COMPACT SEMI-SIMPLE LIE GROUP

Let G be a (real) compact semi-simple Lie group. G is assumed to be connected and simply connected as well. We denote by $\mathcal{G}$ its Lie algebra.

Let H be a maximal torus in G with Lie algebra $\mathscr{H}$, a Cartan subalgebra of $\mathscr{G}$. Since ad* $(G)\mathscr{H}^* = \mathscr{G}^*$ where ad* represents the coadjoint action on $\mathscr{G}^*$, each orbit $\mathcal{O}_q = \text{ad}^*(G)q$ $(q \in \mathscr{G}^*)$ intersects $\mathscr{H}^*$. We thus may choose q in $\mathscr{H}^*$ and define h by $q := K(h, .)/i$. It is well known that all (co)-adjoint orbits are uniquely parametrized by those h which satisfy

$$\alpha(h) \geqslant 0 \qquad \forall \alpha \in \Delta^+ \tag{5.1}$$

once a set of positive roots Δ^+ has been chosen [9].

From now on we will find it convenient to express the canonical 1-form ϖ (1.1) in terms of the following decomposition of the Maurer-Cartan structure

$$\theta =: \sum_{\beta \in B} \theta_\beta h_\beta + \sum_{\alpha \in \Delta^+} \theta_\alpha^+ e_\alpha + \theta_\alpha^- e_{-\alpha} \tag{5.2}$$

Put $h = \sum_{\beta \in B} \eta_\beta h_\beta$ $(\eta_\beta \in \mathbb{R}\ \forall \beta \in B)$, then $\alpha(h) = \sum_{\beta \in B} \eta_\beta n(\alpha, \beta) \geqslant 0\ \ \forall \alpha \in \Delta^+$

(see (4.6)). Now ϖ reads

$$\varpi = K(h, \theta)/i = \sum_{\alpha, \beta \in B} \eta_\beta K(h_\alpha, h_\beta)/i\theta_\alpha = \sum_{\alpha \in B} K(h_\alpha, h_\alpha)\Big(\sum_{\beta \in B} \eta_\beta n(\alpha, \beta)\Big)/2i\theta_\alpha$$

(see (4.2)).

Since $K \restriction \mathscr{H}$ is negative definite, $K(h_\alpha, h_\alpha) > 0\ \forall \alpha \in \Delta$ and

$$\boxed{\varpi = \sum_{\beta \in B} m_\beta \theta_\beta / i \quad \Big| \quad m_\beta \geqslant 0 \qquad \forall \beta \in B} \tag{5.3}$$

The m's are the invariants of $\mathcal{O}_q$. It is then straigtforward to show that the Lie algebra $\mathscr{G}_q$ of the isotropy subgroup G_q is

$$\mathscr{G}_q = \mathscr{H} \oplus \sum_{\alpha \in \Delta_h^+} \mathbb{R}(e_\alpha - e_{-\alpha}) \oplus i\mathbb{R}(e_\alpha + e_{-\alpha}) \tag{5.4}$$

where

$$\Delta_h^+ := \{ \alpha \in \Delta^+ / \alpha(h) = 0 \} \tag{5.5}$$

If we set

$$\overline{\Delta_h^+} := \Delta^+ - \Delta_h^+ \tag{5.5 bis}$$

we get $\dim(\mathcal{O}_q) = 2 \operatorname{card}(\overline{\Delta_h^+})$.

Having in mind (1.3) we shall write $G_q = \exp(\mathscr{G}_q)$ since exp is onto in this case. Let us recall that $\mathcal{O}_q = G/\sim$ where $g \sim g.g'$ iff $g' \in G_q$. Each tangent vector to the leaf passing through g is of the form $Z(g)$, $Z \in \mathcal{G}_q$. In order to introduce the prequantum bundle Y of $\mathcal{O}_q$, let us set

(5.6) $$\chi(\exp(X)) := e^{i\varpi(X)} \qquad X \in \mathcal{G}_q$$

Using the Campbell-Haussdorf formula

$$\exp(X).\exp(X') = \exp(X + X' - [X, X']/2 + [X - X', [X, X']]/12 + \dots)$$

we get

$$\chi(\exp(X).\exp(X')) = \chi(\exp(X)).\chi(\exp(X')) \qquad \forall X, X' \in \mathcal{G}_q$$

(remember that $q.\mathrm{Ad}(X) = q.\mathrm{Ad}(X') = 0$). Moreover $2\pi i h_\beta \in \ker(\exp)$ $\forall \beta \in B$, and χ is a well defined character of G_q if

(5.7) $$\boxed{m_\beta \in \mathbb{N} \quad \forall \beta \in B.}$$

These are in fact the *prequantization conditions* for the coadjoint orbits $\mathcal{O}_q$:

$$\varpi(X) \in 2\pi\mathbb{Z} \qquad \forall X \in \ker(\exp) \subset \mathcal{G}_q$$

(cf. (5.6) and remember that $\theta_\beta(h_\alpha) = \delta_{\alpha\beta}$, $\theta_\beta(e_{\pm\alpha}) = 0$).
Define then $\tilde{G}_q := \ker(\chi)$ and compute the derivative of (5.6) (5)

$$D(\chi)(g).Z(g) = i\varpi(Z)\chi(g)$$

where $Z \in \mathcal{G}_q$ and $g \in G_q$. Every tangent vector to $\tilde{\pi}^{-1}(\tilde{\pi}(g))$ at $g \in G$ ($\tilde{\pi}$ denotes the projection: $G \to G/\tilde{G}_q$) is of the form $Z(g)$, $\varpi(Z) = 0$. As a consequence $Z \in \ker(\varpi) \cap \ker(d\varpi)$, hence ϖ passes to the quotient $Y := G/\tilde{G}_q$, i. e. there exists a 1-form ω on Y such that $\varpi = \tilde{\pi}^*\omega$. Moreover the following formula

$$\chi(g) \circ \tilde{\pi} = \tilde{\pi} \circ r_g \qquad (g \in G_q)$$

defines a free action of $U(^1)$ on Y. Now $r_g^*\theta = \mathrm{ad}(g^{-1})\theta$ and thus $r_g^*\varpi = \varpi$ $\forall g \in G_q$ (1.1). This enables us to conclude that

$$\chi(g)^*\omega = \omega \qquad \forall g \in G_q$$

Putting

$$\pi = p \circ \tilde{\pi}$$

we obtain, as a consequence of (1.2)

$$d\omega = p^*\sigma.$$

We have just proved that

(5.8) $(Y := G/\tilde{G}_q, \omega)$ *is the prequantum bundle* (see (2.1)) *of* $\mathcal{O}_q$ *provided conditions* (5.7) *hold.*

---

(5) Use the variation formula $\delta g = Z(g)$, $Z = \int_0^1 \mathrm{ad}(g_t^{-1})\delta X dt$, $g_t = \exp(tX)$, $g = g_1$.

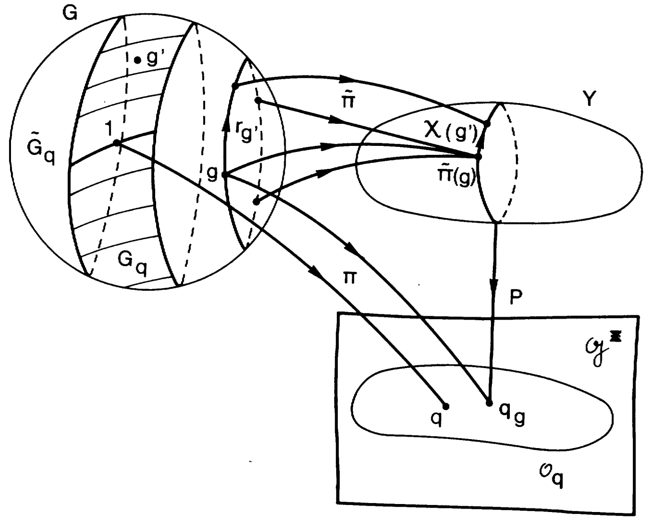

FIG. 1.

The prequantization conditions $[\sigma] \in H^2(\mathcal{O}_q, \mathbb{Z})$ [2] [12] can thus be alternatively expressed in our case as $[\tau^*\varpi] \in H^1(G_q, \mathbb{Z})$ where τ denotes the inclusion : $G_q \to G$ [1].

6. POLARIZERS ON COMPACT SEMI-SIMPLE LIE GROUPS

We shall retain the notations of the preceding chapters. Let us now introduce the following complex form of G

$$\Psi := \bigwedge_{\alpha \in \overline{\Delta_h^+}} K(e_\alpha, \theta) \tag{6.1}$$

which is obviously a non vanishing monomial n-form

$$\left(n = \text{card}\left(\overline{\Delta_h^+}\right) = \dim(\mathcal{O}_q)/2\right)$$

which does not vanish on G. We can write it, according to (5.2) and (5.5 *bis*) as

$$\Psi = \bigwedge_{\alpha(h) > 0} \theta_\alpha^-$$

We are now seeking conditions for the polarizer equation (3.2) to be satisfied by Ψ on G.

We have $d\theta_\alpha^- = -\mathrm{K}(e_\alpha, [\theta, \theta])$ and

$$[\theta, \theta] = \sum_{\substack{\beta\in\mathbf{B}\\ \gamma\in\Delta^+}} \left([h_\beta, e_\gamma]\theta_\beta \wedge \theta_\gamma^+ + [h_\beta, e_{-\gamma}]\theta_\beta \wedge \theta_\gamma^-\right)$$

$$+ \sum_{\gamma,\gamma'\in\Delta^+} \left([e_\gamma, e_{-\gamma'}]\theta_\gamma^+ \wedge \theta_{\gamma'}^- + 1/2[e_\gamma, e_{\gamma'}]\theta_\gamma^+ \wedge \theta_{\gamma'}^+ + 1/2[e_{-\gamma}, e_{-\gamma'}]\theta_\gamma^- \wedge \theta_{\gamma'}^-\right)$$

With the help of (4.1-3.5) we readily obtain

$$\mathrm{K}(e_\alpha, [h_\beta, e_\gamma]) = \gamma(h_\beta)\mathrm{K}(e_\alpha, e_\gamma) = 0 \qquad (\alpha + \gamma \neq 0),$$

$$\sum_{\substack{\beta\in\mathbf{B}\\ \gamma\in\Delta^+}} \mathrm{K}(e_\alpha, [h_\beta, e_{-\gamma}])\theta_\beta \wedge \theta_\gamma^- = -\sum_{\beta\in\mathbf{B}} \alpha(h_\beta)\theta_\beta \wedge \theta_\alpha^-$$

We need, at this stage, some extra results on semi-simple Lie algebras:

$$(6.2) \qquad \begin{aligned}[e_\alpha, e_\beta] &= \mathrm{N}_{\alpha,\beta}e_{\alpha+\beta} && \text{if} \quad \alpha + \beta \in \Delta \\ &= h_\alpha && \text{if} \quad \beta = -\alpha \\ &= 0 && \text{otherwise}\end{aligned}$$

The coefficients $\mathrm{N}_{\alpha,\beta}$ satisfy amongst other ones the following identity

$$\mathrm{N}_{-\alpha,\alpha+\beta} = \mathrm{N}_{-\beta,-\alpha}$$

which is used together with (6.2) to obtain

$$\sum_{\gamma,\gamma'\in\Delta^+} \mathrm{K}(e_\alpha, [e_\gamma, e_{-\gamma'}])\theta_\gamma^+ \wedge \theta_{\gamma'}^- = \sum_{\gamma\in\Delta^+} \mathrm{N}_{\alpha,\gamma}\theta_\gamma^+ \wedge \theta_{\alpha+\gamma}^-$$

$$\sum_{\gamma,\gamma'\in\Delta^+} \mathrm{K}(e_\alpha, [e_\gamma, e_{\gamma'}])\theta_\gamma^+ \wedge \theta_{\gamma'}^+ = 0 \qquad (\alpha + \gamma + \gamma' \neq 0)$$

$$\sum_{\gamma,\gamma'\in\Delta^+} \mathrm{K}(e_\alpha, [e_{-\gamma}, e_{-\gamma'}])\theta_\gamma^- \wedge \theta_{\gamma'}^- = \sum_{\gamma\in\Delta^+} \mathrm{N}_{\alpha,-\gamma}\theta_\gamma^- \wedge \theta_{\alpha-\gamma}^-$$

At last

$$(6.3)\, d\theta_\alpha^- = \sum_{\beta\in\mathbf{B}} \alpha(h_\beta)\theta_\beta \wedge \theta_\alpha^- - \sum_{\gamma\in\Delta^+} (\mathrm{N}_{\alpha,\gamma}\theta_\gamma^+ \wedge \theta_{\alpha+\gamma}^- + \mathrm{N}_{\alpha,-\gamma}\theta_\gamma^- \wedge \theta_{\alpha-\gamma}^-)$$

The last two terms do have a zero contribution in the final expression of $d\Psi$, for $\Delta^+ = \Delta_h^+ \cup \overline{\Delta_h^+}$ (disjoint union) and thus

— if $\gamma \in \Delta_h^+$ then $\alpha + \gamma \in \overline{\Delta_h^+}$ (and $\alpha + \gamma \neq \alpha$). Clearly $\forall \gamma \in \Delta_h^+$

$$\theta_\gamma^+ \wedge \theta_{\alpha+\gamma}^- \bigwedge_{\alpha' \in \overline{\Delta_h^+} - \{\alpha\}} \theta_{\alpha'}^- = 0 .$$

The same is true for the third term in (6.3),

— if $\gamma \in \overline{\Delta_h^+}$ then necessarily $\alpha + \gamma \in \overline{\Delta_h^+}$ (and $\alpha + \gamma \neq \alpha$). If not, $\alpha(h) = -\gamma(h) > 0$ (see (5.1, 5.5)), contradiction since $\gamma \in \Delta^+$. This case is easier to investigate for the last term $\theta_\gamma^- \wedge \theta_{\alpha-\gamma}^-$, because necessarily $\gamma \neq \alpha (\alpha - \gamma \in \Delta^+)$. Again $\forall \gamma \in \overline{\Delta_h^+}$

$$\theta_\gamma^+ \wedge \theta_{\alpha+\gamma}^- \bigwedge_{\alpha' \in \overline{\Delta_h^+} - \{\alpha\}} \theta_{\alpha'}^- = \theta_\gamma^- \wedge \theta_{\alpha-\gamma}^- \bigwedge_{\alpha' \in \overline{\Delta_h^+} - \{\alpha\}} \theta_{\alpha'}^- = 0$$

We can now settle that

$$d\Psi = \sum_{\substack{\beta \in \mathrm{B} \\ \alpha \in \overline{\Delta_h^+}}} \alpha(h_\beta)\theta_\beta \wedge \Psi$$

The polarizer equation (3.2) may be lifted to G and (5.3) used to finally get the supplementary conditions (see (4.6))

$$(6.4) \qquad \boxed{km_\beta = \sum_{\alpha(h)>0} n(\alpha, \beta) \quad \forall \beta \in \mathrm{B}, \quad \text{for some} \quad k \in \mathbb{Z} - \{0\}}$$

A prequantizable symplectic manifold will be called polarizable if a polarizer can live on a prequantization of it. *We have just proved that not all prequantizable coadjoint orbits of a compact semi-simple and simply connected Lie group are polarized by* (6.1), *but rather the only ones satisfying* (6.4). The existence of a polarizer for a prequantizable symplectic manifold seems to be quite a strong additional assumption leading to more drastic selection of homogeneous symplectic manifolds than the sole prequantization.

We wish to show that Ψ (6.1) is actually a polarizer of Y (5.8) i. e. that it passes to the quotient. Let $Z \in \mathcal{G}_q$, $\varpi(Z) = 0$, be a tangent vector to the foliation by $\tilde{G}_q$, so, (5.4) and (4.2) 4.3) yield $Z \lrcorner \Psi = 0$. Next, we have assumed that $d\Psi = ik\varpi \wedge \Psi$ and thus $Z \lrcorner d\Psi = 0$, hence

$$Z \in \ker(\Psi) \cap \ker(d\Psi),$$

which was needed to have $\Psi := \tilde{\pi}^*\phi$ (ϕ is the sought polarizer).

Due to the fact that $l_G^*\theta = \theta$, *ϕ turns out to be naturally a* G-*invariant polarizer* (3.3).

As for the associated Planck polarization F*

$$F^* := \tilde{\pi}_* \tilde{F} \qquad \text{(see chapter 3)} \quad (^6)$$

$$\tilde{F}^{\cdot} := \ker(d\Psi) = \tilde{\mathcal{G}}_q^{\mathbb{C}} \oplus \sum_{\alpha(h)>0} \mathbb{C} e_\alpha$$

it appears as a *positive* one [*12*] since $id\varpi(e_{-\alpha}, e_\alpha) = \alpha(h) > 0 \; \forall \alpha \in \overline{\Delta_h^+}$ (4.4).

At last ϕ *is a Kähler polarizer* (3.5) since $\bar{\Psi} \wedge \Psi \neq 0$ (clear) and the orbit turns out to be endowed in this manner with a structure of (compact) Kähler manifold [*2*] [*7*] [*9*].

7. WAVE FUNCTIONS

Let us start from a prequantization (Y, ω) of a given symplectic manifold (M, σ). Suppose then that a Planck polarization F* has been singled out on Y. Wave functions associated to F* can be possibly introduced as those complex valued functions f of Y which are constant along F*

(7.1) $$X(f) = 0 \qquad \forall X \in F^*$$

and which satisfy the so called circulation condition

$$z^*(f) = z.f \qquad \forall z \in U(1)$$

Infinitesimally

(7.2) $$\xi(f) = if$$

(see [*2*] [*3*], also [*12*] for a general account on geometric quantization).

Things can be best formulated in terms of polarizers (if any). Let ϕ be a k-polarizer of Y (3.1, 3.2), *we shall call wave function associated to* ϕ *any function* $f : Y \to \mathbb{C}$ *such that* $f\phi$ *satisfies the equation of a* $k+1$ *polarizer* (see (3.2)), in other words

(7.3) $$\boxed{(df - if\omega) \wedge \phi = 0}$$

which is equivalent to (7.1, 7.2) because if $X \in F^*$,

$$X \lrcorner ((df - if\omega) \wedge \phi) = X(f)\phi = 0 \,,$$

hence $X(f) = 0$; next $\xi \lrcorner ((df - if\omega) \wedge \phi) = (\xi(f) - if)\phi = 0$ and $\xi(f) = if$.

(6) The polarizations $F = p_* F^*$ are thus in one to one correspondence with the parabolic subalgebras $\mathcal{G}_q^{\mathbb{C}} \oplus \sum_{\alpha(h)>0} \mathbb{C} e_\alpha$ [*10*], and this seems to indicate that all invariant polarizers should have the expression (6.1).

Conversely if $X(f) = X \lrcorner\, df = 0$ $\forall X \in F^* = \ker(d\phi)$, $df \in \ker(d\phi)^0$ i. e. $df \wedge \phi = g\omega \wedge \phi$ for some complex function g of Y, and

$$\xi \lrcorner\, (df \wedge \phi) = \xi(f)\phi = if\phi = g\phi\,,$$

hence $g = if$.

In the case (2.2), (3.4) $f(p, q, z) = zF(q)$ where $F \in C^\infty(\mathbb{R}^n)$ (configuration space picture).

If f (resp. g) is a wave function associated to the k-polarizer ϕ (resp. ψ) and if ϕ and ψ are transversal, i. e. $\bar{\phi} \wedge \psi \neq 0$ we can define the *pairing* of f and g according to

$$(7.4) \qquad \langle f, g \rangle := \int_M \bar{f} g \,|\, \bar{\phi} \wedge \psi/\lambda \,|^{1/k} \lambda$$

provided this expression makes sense (see (3.5)).

This expression can certainly be improved in order to take into account the points of non transversality (caustics) by means of some generalization of the Maslov index [*3*]. These considerations are however beyond the scope of our paper, and will not be needed for our purpose. In fact (7.4) yields a natural (finite dimensional) Hilbert space structure on the space of wave functions of the polarized orbits of semi-simple compact Lie groups.

8. THE EXAMPLES OF SU(2) AND SU(3)

A) SU(2)

We can choose the set of diagonal anti-hermitian and tracefree matrices as a Cartan subalgebra. In the case of SU(2) we set (7)

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\varpi = m_\alpha \theta_\alpha / i \qquad (5.3)$$

since $\Delta^+ = \{\alpha\}$. Now any element of SU(2) will be written as (Z_1, Z_2) with $Z_1, Z_2 \in \mathbb{C}^2$, $\bar{Z}_1 . Z_1 = \bar{Z}_2 . Z_2 = 1$, $\bar{Z}_1 . Z_2 = 0$, vol $(Z_1, Z_2) = 1$. Vol stands for the volume element of $\mathbb{C}^2$.

In canonical coordinates $\theta_\alpha = \theta_1^1$ and if we put

$$s := m_\alpha / 2$$

we can write

$$\boxed{\varpi = 2s\theta_1^1 / i = 2s\bar{Z}_1 . dZ_1 / i}$$

We have $\mathcal{O}_q \cong \mathbb{P}_1(C)$ where

$$q = s/i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad s \in \mathbb{N}/2 \quad (5.7)$$

(7) We have chosen the Killing structure to be rather $K(X, Y) := \mathrm{Tr}\,(X . Y)$ $\forall X, Y \in su(n)$.

The invariant s is interpreted as the *spin (or isospin)* for physical purposes [3] [5].

Because $\Delta_h^+ = \emptyset$ ($h = sh_\alpha$), $\phi = K(e_\alpha, \theta) = \theta_2^1$ is a polarizer of those orbits which satisfy

$$ks = 1$$

(see (6.4) and remember that $n(\alpha, \alpha) = 2$). Hence the only SU(2) prequantizable coadjoint orbits on which polarizers of the form (6.7) can be defined are the orbits of spin 1/2 ($k = 2$) and spin 1 ($k = 1$).

As for the wave functions associated to these orbits, they are looked after as all functions $f : \mathrm{SU}(2) \to \mathbb{C}$ such that

(8.1) $$(df - 2s\theta_1^1) \wedge \theta_1^2 = 0 . \qquad (7.3)$$

A usefull technical device is necessary, namely

(8.2) $$df = \sum_{j,k=1}^{2} (f_j^k \theta_k^j + \tilde{f}_j^k \bar{\theta}_k^j)$$

where

(8.3) $$f_j^k := \frac{\partial f}{\partial Z_k} . Z_j ; \qquad \tilde{f}_j^k := \bar{Z}_j . \frac{\partial f}{\partial \bar{Z}_k}$$

We obtain from (8.1)

(8.4) $\tilde{f}_j^k = 0 \quad \forall j, k = 1, 2 ; \qquad f_j^2 = 0 \quad \forall j = 1, 2 ; \qquad f_1^1 = 2sf .$

which means that f is holomorphic with respect to (Z_1, Z_2), actually independant of Z_2; the last condition of (8.4) tells us that f is homogeneous of degree $2s$.

In the case $s = 1/2 f(Z_1, Z_2) = \psi(Z_1)$ where ψ is a complex linear form of $\mathbb{C}^2$.

In the case $s = 1 f(Z_1, Z_2) = \psi(Z_1, Z_1)$ where ψ is a symmetric complex bilinear application from $\mathbb{C}^2 \times \mathbb{C}^2$ to $\mathbb{C}$.

We do recover the familiar 2 (resp. 3) dimensional Hilbert space description of spin 1/2 (resp. 1) as irreducible and unitary representation carrier spaces for SU(2).

Fortunately enough these representations correspond to the ones particle physics has to deal with: nucleons, pions ..., are respectively classified according to isospin 1/2, and 1 [6].

B) SU(3)

Let us now turn to the case of SU(3), and set

$$\alpha = h_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \beta = h_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$e_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad e_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $B = \{ \alpha, \beta \}$ and $\Delta^+ = \{ \alpha, \beta, \alpha + \beta \}$. SU(3) is of type A_2 [*11*].
According to (5.3) we put

$$\varpi = (m_\alpha \theta_\alpha + m_\beta \theta_\beta)/i \,. \tag{8.5}$$

Unlike the last case the positive integers m_α, m_β have no direct physical interpretation [*6*].

The expression (8.5) can alternatively by written in terms of the triple $(Z_1, Z_2, Z_3) \in SU(3)$ $(\bar{Z}_j . Z_k = \delta_{jk}$, $j, k = 1, 2, 3$ and $\text{vol}(Z_1, Z_2, Z_3) = 1)$

$$\boxed{\varpi = ((m_\alpha + m_\beta)\bar{Z}_1 . dZ_1 + m_\beta \bar{Z}_2 . dZ_2)/i}$$

The reference point q of the orbits $\mathcal{O}_q$ is given by

$$iq = \frac{1}{3}\begin{pmatrix} 2m_\alpha + m_\beta & 0 & 0 \\ 0 & m_\beta - m_\alpha & 0 \\ 0 & 0 & -m_\alpha - 2m_\beta \end{pmatrix}$$

Since $\alpha(h) = m_\alpha$, $\beta(h) = m_\beta$, it is clear that the orbits split into two non trivial strata, namely

i) $m_\alpha, m_\beta \neq 0 . \Delta_h^+ \neq \emptyset$, $\overline{\Delta_h^+} = \Delta^+$, these are the maximal orbits (of complex dimension 3) with topology $\mathbb{P}T(\mathbb{P}_2(\mathbb{C}))$ (flag manifolds),

ii) $m_\alpha \neq 0$, $m_\beta = 0$; or $m_\alpha = 0$, $m_\beta \neq 0 . \Delta_h^+ = \{ \beta \}$ or $\{ \alpha \}$, these are the minimal orbits with topology $\mathbb{P}_2(\mathbb{C})$.

The maximal orbits which posses a polarizer of the form

$$\phi = K(e_\alpha, \theta) \wedge K(e_{\alpha+\beta}, \theta) \wedge K(e_\beta, \theta) = \theta_1^2 \wedge \theta_1^3 \wedge \theta_2^3$$

must obey

$$k m_\alpha = k m_\beta = 2 \qquad \text{(see (6.4))}$$

that is to say: $m_\alpha = m_\beta = 1$ $(k = 2)$ and: $m_\alpha = m_\beta = 2$ $(k = 1)$.

As for the minimal ones

a) $\underline{m_\beta = 0}$,

$$\phi = K(e_\alpha, \theta) \wedge K(e_{\alpha+\beta}, \theta) = \theta_1^2 \wedge \theta_1^3$$

is a polarizer whenever

$$k m_\alpha = 3$$

i. e. $m_\alpha = 1$ $(k = 3)$, $m_\alpha = 3$ $(k = 1)$.

b) $\underline{m_\alpha = 0}$,

Again (6.1) might provide us with a polarizer, but we find it convenient to write rather

$$i\varpi = m_\beta(\bar{Z}_1 . dZ_1 + \bar{Z}_2 . dZ_2) = - m_\beta \bar{Z}_3 . dZ_3 = m_\beta d\bar{Z}_3 . Z_3$$

in order to have a well suited form for the polarizer in this case

$$(8.6)\qquad \phi = K(e_\alpha, \theta) \wedge K(e_{\alpha+\beta}, \theta) = \theta_2^3 \wedge \theta_1^3 = \overline{\theta_3^2} \wedge \overline{\theta_3^1}$$

Conditions (6.4) hold for the orbits $m_\beta = 1$, $m_\beta = 3$.

Let us finally investigate the wave functions spaces corresponding to this finite set of orbits.

i)
$$(df - m_\alpha(2\theta_1^1 + \theta_2^2)) \wedge \theta_1^2 \wedge \theta_1^3 \wedge \theta_2^3 = 0$$

Using the decomposition (8.2, 8.3), we find $\tilde{f}_j^k = 0 \quad \forall j, k = 1, 2, 3$; $f_j^3 = 0 \quad \forall j = 1, 2, 3$; $f_1^2 = 0$; $f_1^1 = 2f_2^2 = 2m_\alpha f$.

In other words

$m_\alpha = m_\beta = 1$	$f(Z_1, Z_2, Z_3) = \psi(Z_1, Z_1, Z_2)$ with $\psi(Z_1, Z_1, Z_1) = 0$. Representation $\{ 8 \}$.

From now on ψ will represent a complex multilinear application with arguments in $\mathbb{C}^3$ and values in $\mathbb{C}$.

$m_\alpha = m_\beta = 2$	$f(Z_1, Z_2, Z_3) = \psi(Z_1, Z_1, Z_1, Z_1, Z_2, Z_2)$ with $\psi(Z_1, Z_1, Z_1, Z_1, Z_1, Z_2) = 0$. Representation $\{ 27 \}$.

ii) The wave functions equation (7.3) reads in the case

a)
$$(df - m\theta_1^1) \wedge \theta_1^2 \wedge \theta_1^3 = 0$$

and yields $\tilde{f}_j^k = 0 \quad \forall j, k = 1, 2, 3$; $f_j^k = 0 \quad \forall j = 1, 2, 3 \quad \forall k = 2, 3$; $f_1^1 = m_\alpha f$.

$m_\alpha=1$, $m_\beta=0$	$f(Z_1, Z_2, Z_3)=\psi(Z_1)$. Representation $\{ 3 \}$.
$m_\alpha=3$, $m_\beta=0$	$f(Z_1, Z_2, Z_3)=\psi(Z_1, Z_1, Z_1)$. Representation $\{ 10 \}$.

b)
$$(df - m_\beta\overline{\theta_3^3}) \wedge \overline{\theta_3^2} \wedge \overline{\theta_3^1} = 0 \qquad \text{(see (8.6))}$$

and yields $f_j^k = 0 \quad \forall j, k = 1, 2, 3$; $\tilde{f}_j^k = 0 \quad \forall j = 1, 2, 3 \quad \forall k = 1, 2$; $\tilde{f}_3^3 = m_\beta f$.

$m_\alpha=0$, $m_\beta=1$	$f(Z_1, Z_2, Z_3)=\psi(\bar{Z}_3)$. Representation $\{ \bar{3} \}$.
$m_\alpha=0$, $m_\beta=3$	$f(Z_1, Z_2, Z_3)=\psi(\bar{Z}_3, \bar{Z}_3, \bar{Z}_3)$. Representation $\{ \overline{10} \}$.

We are now ready to conclude that the only prequantizable SU(3) co-adjoint orbits polarized by the polarizers (6.1) are labelled by the invariants (1, 0), (0, 1), (3, 0), (0, 3), (1, 1), (2, 2) and correspond respectively to the unitary irreducible representations $\{ 3 \}$, $\{ \bar{3} \}$, $\{ 10 \}$, $\{ \overline{10} \}$ ($\mathbb{P}_2(\mathbb{C})$), $\{ 8 \}$, $\{ 27 \}$ ($\mathbb{P}T(\mathbb{P}_2\mathbb{C})$).

Now in the isospin-hypercharge scheme, hadrons are actually classified according the weight diagrams of the previous representations, except perhaps the { 27 } [*6*] [*13*].

9. MISCELLANEOUS EXAMPLES

The polarizers of the minimal $SU(n+1)$ coadjoint orbits $((P_n(C), m d\bar{Z} \wedge dZ/i)$ which are prequantized by $(S^{2n+1}/\mathbb{Z}_{|m|}, m\bar{Z}.dZ/i)$

$$m \in \mathbb{Z} - \{0\}$$

(see (2.3) and (6.1)) have the simple expression

$$(9.1) \qquad Z \lrcorner \, \mathrm{vol} \qquad (m > 0)$$

$$(9.1 \; bis) \qquad \overline{Z \lrcorner \, \mathrm{vol}} \qquad (m < 0)$$

when

$$(9.2) \qquad km = n + 1 \qquad k \in \mathbb{Z} - \{0\}$$

(cf. (6.1), (6.4) or note that

$$d(Z \lrcorner \, \mathrm{vol}) = (n+1)\, \mathrm{vol} \quad \text{and} \quad \bar{Z}.dZ \wedge (Z \lrcorner \, \mathrm{vol}) = \mathrm{vol}) .$$

Accordingly, the associated wave functions are readily obtained as all holomorphic (resp. anti-holomorphic) functions f or S^{2n+1} which are homogeneous of degree $|m|$ if $m > 0$ (resp. $m < 0$),

$$f(Z) = \psi(\underbrace{Z, Z, \ldots, Z}_{m}) \qquad (m > 0)$$

$$f(Z) = \psi(\underbrace{\bar{Z}, \bar{Z}, \ldots, \bar{Z}}_{|m|}) \qquad (m < 0)$$

The dimension of these irreducible, unitary representations is $C^{|m|}_{n+|m|}$.

It is clear that the expressions (9.1, 9.1 *bis*) still work in the case of the minimal semi-simple coadjoint orbits of $SU(p, q)$.

The condition (9.2) holds as well and provides us with a bonus of the theory: *if we insist on the existence of polarizers on the prequantum bundle of the space of motions of relativistic massless spinning particles, the only helicities which are allowed turn out to be* $\pm 1/2$, ± 1, ± 2. This attractive result can be easily established in terms of *twistors* [*12*] since

$$(\{ Z \in \mathbb{C}^{2,2}, \bar{Z}.Z = 1 \} / \mathbb{Z}_{|2s|}, 2s\bar{Z}.dZ/i) \quad (^8)$$

prequantizes the classical model of a massless spinning particle of helicity $s \in \mathbb{Z}/2 - \{0\}$ (which is at the same time a coadjoint orbit of the restricted

(8) $\bar{Z}.Z = \sum_{\alpha=1}^{4} \bar{Z}_\alpha Z^\alpha$ where $\bar{Z}_1 = \overline{Z^1}$, $\bar{Z}_2 = \overline{Z^2}$, $\bar{Z}_3 = -\overline{Z^3}$, $\bar{Z}_4 = -\overline{Z^4}$ for example.

Poincaré group and SU(2, 2)). Gravitons (?) seem to be a specific outcome of the conformal invariance of the model.

Let us end up with the case of SU(4) which might be of some interest for the (still provisory) charmed particles spectroscopy [6]. The representations associated to polarized prequantizable coadjoint orbits are respectively: { 4 }, { 10 }, { 35 }, and their conjugate ($\mathbb{P}_3(\mathbb{C}) \cong SU(4)/U(3)$); { 6 }, { 20 }, { 105 } (SU(4)/S(U(2) × U(2))); { 280 } and its conjugate (SU(4)/U(1) × U(2)); { 64 }, { 729 } (SU(4)/U(1) × U(1) × U(1)).

DISCUSSION

We wish to discuss now some features of the physical ideas which have motivated this article.

First of all, our investigations rely on the fact that classical dynamical systems must play a central role in any attempt to describe quantum mechanical systems. This constitutes the principle of correspondence which is considered as the cornerstone of quantum mechanics. It has been emphasized by Souriau that a purely classical treatment of spinning particles is possible in terms of coadjoint orbits of (the universal covering of) the restricted Poincaré group [3]. Also particles endowed with an internal structure can be given a classical status in the same manner, once the Poincaré group has been enlarged by a gauge group [4] [5].

The phase spaces associated with internal degrees of freedom are precisely the prequantizable (co)adjoint orbits of compact semi-simple Lie groups. Although they are intimately related to the group, these symplectic manifolds gain an autonomous existence likely to describe classical systems such as multiplets of particle physics. The rich geometrical structures of the prequantum bundles above these orbits serve for almost all physical purposes (equations of motion, minimal coupling, symmetry breaking, etc.).

On the other hand, multiplets of elementary particles are usually introduced via unitary irreducible representations of an internal symmetry group (e. g. SU(2), SU(3)) [6]. We have shown that this point of view is related to the quantization of the previous orbits.

Moreover, a finite number of representations are actually taken into account for the classification of all known elementary particles in multiplets. These representations must not be « too big » (heavy nuclei such as U^{238} are obviously disregarded) [13]. The question arises: is it possible to devise a selection rule to discard the redundant representations? We have tried to answer this by going back to the orbit framework and by looking for invariant polarizers.

The resulting selection of orbits seems to be in fairly good agreement with physical standards:

— the spin of massive spinning particles turns out to have only two allowed values namely 1/2 and 1 (spinless massive particles obviously fit into our scheme). These are in fact the spins of most observed elementary particles. The case of spin 3/2 poses a problem, but since spin 3/2 particles are actually resonances, their description in terms of elementary dynamical systems (which take into account a minimal space-time elongation) might be unadapted,

— the multiplets of elementary particles which are selected within the SU(2) picture are again associated with the representations $\{2\}$ and $\{3\}$,

— isospin 3/2 shows up in the representation $\{10\}$ (and $\{\overline{10}\}$) polarizers have selected together with the representations $\{3\}$, $\{\overline{3}\}$, $\{8\}$, $\{27\}$ in the SU(3) scheme. The requested orbits associated with quarks, baryon decuplets, baryon and meson octets, are fortunately supplied by our procedure. As for the $\{27\}$ multiplet, no experimental evidence is up to now available, although its occurence in particle physics has long been foreseen [*13*]. The intermediate representations $\{6\}$, $\{15\}$, $\{24\}$ are discarded by polarizers while they, in fact, have no physical relevance. This might be considered as a clue of the physical content of our theory,

— the observed charmed particles should fit into a $\{20\}$ representation of SU(4) which is also retained.

In all previous cases the trivial representation corresponds to the trivial orbits:

— one of the most exciting properties of twistors polarizers is to allow (in addition to neutrinos and photons) gravitons, and to forbid spin 3/2 gravitinos. Applications to the global gauge groups of quantum chromodynamics remain to be undertaken.

From a mathematical point of view, several open questions deserve special attention:

— the unicity of the invariant polarizers (6.1) of a compact semi-simple Lie group is still conjectural,

— also the geometrical interpretation of the integer k (3.2) remains to be clarified,

— it should be interesting to determine which of the previous results survive in the non compact case.

By way of conclusion, we are convinced that pursuing these lines should provide further insights into some aspects of either particle physics and representation theory.

ACKNOWLEDGMENTS

It is a pleasure to thank J. M. Souriau for several enlightened advices on geometric quantization: especially those related to polarizers. Discussions with J. Carmona, J. Elhadad, B. Kostant, P. Mery and E. de Rafael are warmly acknowledged.

REFERENCES

[1] A. KIRILLOV, *Elements de la théorie des représentations.* Éd. MIR, Moscou, 1974.
[2] B. KOSTANT, Quantization and Unitary Representations. *Lect. Notes in Math.*, t. **170**, Springer-Verlag, Berlin, 1970.
[3] J. M. SOURIAU, *Structure des systèmes dynamiques.* Dunod, Paris, 1970. *Structure of Dynamical Systems* (in preparation).
[4] S. STERNBERG, On the Role of Field Theories in our Physical Conception of Geometry. *Differential Methods in Mathematical Physics II*, Proceedings Bonn, 1977, Springer-Verlag, Berlin, 1978.
[5] C. DUVAL, *Sur les mouvements classiques dans un champ de Yang-Mills.* Preprint C. P. T., Marseille, 1978, p. 1056. On the Prequantum Description of Spinning Particles in an External Gauge Field. Méthodes de Géométrie Différentielle en Physique-Mathématique, *Colloque Int. C. N. R. S.*, Proceedings Aix-en-Provence, 1979, Springer-Verlag (to appear).
[6] D. B. LICHTENBERG, *Unitary Symmetry and Elementary Particles.* 2^{nd} Ed., Academic Press, New-York, 1978.
[7] S. KOBAYASHI, K. NOMIZU, *Foundations of Differential Geometry.* Interscience, New York, t. **1** et **2**, 1962, 1969.
[8] N. R. WALLACH, Symplectic Geometry and Fourier Analysis. *Math. Sci. Press*, 1977.
[9] J. LOEB, *Formule de Kirillov pour les groupes de Lie semi-simples compacts. Analyse harmonique sur les groupes de Lie*, Séminaire Nancy-Strasbourg, 1973-1975, Springer-Verlag, Berlin, 1975.
[10] G. WARNER, *Harmonic Analysis on Semi-Simple Lie Groups I.* Springer-Verlag, Berlin, 1972.
[11] J. P. SERRE, *Algèbres de Lie semi-simples complexes.* Benjamin, New York, 1966.
[12] D. J. SIMMS, N. M. J. WOODHOUSE, Lectures on Geometric Quantization. *Lect. notes in Physics*, t. **53**, Springer-Verlag, Berlin, 1976.
[13] M. GELL-MANN, Y. NE'EMAN, The Eightfold Way. Benjamin, 1964.

(Manuscrit reçu le 26 septembre 1980)