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Main field and convex covariant density for quasi-linear hyperbolic systems : relativistic fluid dynamics

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Section A :
Physique théorique.

Main field and convex covariant density for quasi-linear hyperbolic systems

Relativistic Fluid Dynamics

by

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SUMMARY. — A quasi-linear hyperbolic system of the first order, in conservative form, is considered and a supplementary conservation law is supposed to exist, as a consequence of the field equations. Starting from a paper of K. O. Friedrichs [*1*], the definition of convex covariant density is introduced and it is proven through an explicitely covariant formalism that: *a*) a « main field » $\mathbf{U}'$ exists depending only on the field equations and the supplementary conservation law, but invariant through field variable mapping; *b*) the system assumes a symmetric conservative form if $\mathbf{U}'$ is chosen as field variable and the symmetric system is « generated » by the knowledge of only one four-vector; *c*) it is possible to define a covariant scalar function on a shock manifold which provides « entropy growth » (in the sense of P. D. Lax); *d*) the previous function « generates » the shock and the shock manifold are not space-like if the characteristic ones are not space-like. Finally the system of relativistic fluid dynamics is shown to possess a convex covariant density and consequences of the results *a*)-*d*) are discussed in detail.

1. GENERAL REMARKS

Let V^4 be a C^∞, 4-dimensional manifold and $\boldsymbol{x}$ a point of V^4, x^α ($\alpha = 0, i$; $i = 1, 2, 3$) being local coordinates of $\boldsymbol{x}$. The manifold is supposed to be

(*) Lavoro eseguito durante il godimento di una borsa di studio C. N. R. nell'ambito del G. N. F. M.

endowed with a pseudo-Riemannian metric. In the local coordinates x^α, $g_{\alpha\beta}$ represent the components of the metric tensor of signature $(+ - - -)$.

On V^4 we consider a quasi-linear system of N first order partial differential equations for the unknown N-vector $\mathbf{U}(x^\alpha) \in \mathbb{R}^N$:

$$A^\alpha(\mathbf{U})\mathbf{U}_\alpha = \boldsymbol{f}(\mathbf{U}), \tag{1.1}$$

the components of $\mathbf{U}$ are contravariant tensors and $\mathbf{U}_\alpha = \partial_\alpha \mathbf{U}$ is a vector the components of which are the covariant derivatives of the components of $\mathbf{U}$, A^α are $N \times N$ matrices.

DEFINITION I. — The system (1.1) is said to be hyperbolic if a timelike covector $\{ \xi_\alpha \}$ exists such that the following two statements hold:

i)
$$\det (A^\alpha \xi_\alpha) \neq 0 \tag{1.2}$$

ii) for any covector $\{ \zeta_\alpha \}$ of space type, the following eigenvalue problem:

$$A^\alpha(\zeta_\alpha - \mu\xi_\alpha)\boldsymbol{d} = 0 \tag{1.3}$$

has only real proper values μ and N linearly independent eigenvectors $\boldsymbol{d}$, i. e. forming a basis of $\mathbb{R}^N$.

The covectors $\{ \zeta_\alpha - \mu\xi_\alpha \}$ built with any proper value μ are called « characteristic », while the $\{ \xi_\alpha \}$ fulfilling *i*), *ii*) are said « subcharacteristic ».

DEFINITION II. — An hyperbolic system is said to be strictly hyperbolic if the roots μ are all distinct.

DEFINITION III. — An hyperbolic system is said to be conservative if

$$A^\alpha = \boldsymbol{\nabla}\mathbf{F}^\alpha(\mathbf{U}), \qquad (\boldsymbol{\nabla} = \partial/\partial\mathbf{U}), \tag{1.4}$$

i. e. it exhibits the form:

$$\partial_\alpha \mathbf{F}^\alpha(\mathbf{U}) = \boldsymbol{f}(\mathbf{U}). \tag{1.5}$$

DEFINITION IV. — A system of the type (1.1) is said to be symmetric hyperbolic if:

a)
$$A^\alpha = A^{\alpha T} \tag{1.6}$$

b) a covector $\{ \xi_\alpha \}$ exists such that the matrix $A^\alpha\xi_\alpha$ is positive definite $\forall \mathbf{U} \in \mathcal{D}$, $\mathcal{D}$ being a convex open subset of $\mathbb{R}^N$.

Any symmetric system, the initial data of which, belonging to a manifold having ξ_α as normal, are of class $H^s(\mathbb{R}^N)$, with $s \geq 4$, has unique solution $\in H^s(\mathbb{R}^N)$, in the neighbourhood of the initial manifold, even if the system is not stricly hyperbolic [*1*] [*2*].

2. SUPPLEMENTARY CONSERVATION LAW

Let us take an hyperbolic system in the conservative form (1.5) and let us suppose, as it usually happens for physical systems that, when differentiability conditions hold, a supplementary conservation law exists, as a consequence of the field equations:

$$\partial_\alpha h^\alpha(\mathbf{U}) = g(\mathbf{U}) \tag{2.1}$$

h^α being a contravariant vector and g a covariant scalar.

Hyperbolic systems possessing even more than one supplementary conservation law have been studied by K. O. Friedrichs in [*1*], where the existence of a symmetric hyperbolic form for the system (1.5) is proven if some compatibility condition holds and a suitable quadratic form is positive.

Here we examine the case of only one supplementary law and starting from Friedrichs conditions suitably written, and the results of refs. [*3*]-[*6*], we introduce the definition of convex covariant density and show in a covariant formulation the following results:

1) A « main field » $\mathbf{U}'$ exists so that such systems exhibit symmetric conservative form; 2) it is possible to find a scalar covariant function η, defined on a shock manifold, providing the « entropy growth » condition; 3) the knowledge of the function η is enough to determine the shock; 4) the shock manifolds are never space-like, consequently the shock speed never exceeds the velocity of light in vacuo; 5) the relativistic fluid dynamics system has a convex covariant density. Properties 1)-4) are analyzed for the fluid.

3. FRIEDRICHS CONDITIONS

In [*1*] K. O. Friedrichs analyzes a conservative system of the type:

$$\partial_\alpha \boldsymbol{q}^\alpha = \boldsymbol{p} \tag{3.1}$$

$\boldsymbol{q}^\alpha$ being a r-component column vector depending on the field $\mathbf{U} \in \mathbb{R}^{\mathrm{N}}$, $(\mathrm{N} < r)$ which is a function of the coordinates x^α.

To the system (3.1) belong N independent equations and $r - \mathrm{N}$ supplementary conservation laws. Therefore compatibility conditions are required. In particular if $r = \mathrm{N} + 1$, compatibility is ensured by the existence of an r-vector $\boldsymbol{y}(\mathbf{U})$ such that:

$$\boldsymbol{y} . \partial_\alpha \boldsymbol{q}^\alpha = \boldsymbol{y} . \boldsymbol{p} \qquad \forall \mathbf{U} \quad \text{and} \quad \forall \partial_\alpha \mathbf{U} .$$

On introducing the operator $\mathbf{\nabla} = \partial/\partial \mathbf{U}$, we have:

$$y . \mathbf{\nabla} q^\alpha = 0 \tag{3.2}$$

$$y . p = 0 \tag{3.3}$$

Equations (3.2) and (3.3) are the condition I given in [1].

In [1] a further condition is requested: a time-like covector $\{ \xi_\alpha \}$ independent on the field $\mathbf{U}$ exists such that the quadratic form:

$$\mathrm{Q} = y . \delta^2 q^\alpha \xi_\alpha > 0 \tag{3.4}$$

is positive $\forall \delta \mathbf{U}$, $\delta \mathbf{U}$ being an arbitrary non vanishing variation of the field $\mathbf{U}$ and

$$\delta^2 q^\alpha = \delta \mathbf{U} . \mathbf{\nabla \nabla} q^\alpha \delta \mathbf{U} . \tag{3.5}$$

Furthermore Friedrichs shows that condition (3.4) is invariant under $\mathbf{U}$ field mapping.

4. CONVEX COVARIANT DENSITY SYSTEMS

When we have only one supplementary conservation law ($r = \mathrm{N} + 1$), N equations in (3.1) identify with (1.5), while the remaining one is eq. (2.1). We have:

$$q^\alpha \equiv \begin{pmatrix} \mathbf{F}^\alpha \\ h^\alpha \end{pmatrix}; \qquad p \equiv \begin{pmatrix} f \\ g \end{pmatrix} \tag{4.1}$$

Since from (3.2), (3.3) y is defined except for an arbitrary scalar factor, we may write:

$$y \equiv \begin{pmatrix} -\mathbf{U}' \\ 1 \end{pmatrix} \tag{4.2}$$

Then Friedrichs conditions I and II look like:

$$\mathbf{U}' \mathrm{A}^\alpha = \mathbf{\nabla} h^\alpha \tag{4.3}$$

$$\mathbf{U}' . f = g \tag{4.4}$$

$$\mathrm{Q} = \xi_\alpha \{ - \mathbf{U}' . \delta^2 \mathbf{F}^\alpha + \delta^2 h^\alpha \} > 0 , \qquad \forall \delta \mathbf{U} \neq 0 . \tag{4.5}$$

We observe that eq. (4.3) can be written equivalently:

$$\mathbf{U}' . \delta \mathbf{F}^\alpha \equiv \delta h^\alpha \qquad (\forall \delta \mathbf{U}) \tag{4.6}$$

which shows the invariance of $\mathbf{U}'$ through $\mathbf{U}$ field mapping, $\mathbf{U}'$ depending only on $\mathbf{F}^\alpha$ and h^α.

On applying the operator δ to (4.6)

$$\delta \mathbf{U}' . \delta \mathbf{F}^\alpha + \mathbf{U}' . \delta^2 \mathbf{F}^\alpha = \delta^2 h^\alpha$$

and then (4.5) is equivalent to say:

$$Q = \delta U' . \delta F^\alpha \xi_\alpha > 0 \tag{4.7}$$

The choice of $\mathbf{U}$ is free, then we may choose:

$$\mathbf{U} = \mathbf{F}^\alpha \xi_\alpha \tag{4.8}$$

We put also:

$$h(\mathbf{U}) = h^\alpha(\mathbf{U})\xi_\alpha \tag{4.9}$$

From (4.8) we have soon

$$A^\alpha \xi_\alpha = I \tag{4.10}$$

where I is the unit matrix. Taking account of (4.3) or (4.6) we gain:

$$\mathbf{U}' = \nabla h(\mathbf{U}) \leftrightarrow \mathbf{U}' . \delta \mathbf{U} = \delta h \tag{4.11}$$

and condition (4.7) for our field becomes:

$$Q = \delta \mathbf{U}' . \delta \mathbf{U} = \delta^2 h > 0 \tag{4.12}$$

which is equivalent to the convexity of $h(\mathbf{U})$.

In order to have a compact formulation for the case of one supplementary conservation law, with the field choice (4.8), we state the following

DEFINITION OF CONVEX COVARIANT DENSITY SYSTEM. — *We say that a conservative hyperbolic system* (1.5), *endowed with a supplementary conservation law* (2.1), *is a convex covariant density system if the following conditions hold:*

(C.A) *a column* N*-vector* $\mathbf{U}'$ *exists such that:*

$$\mathbf{U}' A^\alpha = \nabla h^\alpha , \qquad \mathbf{U}' . f = g$$

(C.B) *a time-like covector* $\{ \xi_\alpha \}$ *independent of* $\mathbf{U}$ *exists and if the field* $\mathbf{U} = \mathbf{F}^\alpha \xi_\alpha$ *is chosen, the covariant density* $h = h^\alpha \xi_\alpha$ *is a convex function of* $\mathbf{U}$ *in a convex domain* $\mathscr{D} \subseteq \mathbb{R}^N : \delta^2 h > 0$.

5. MAIN FIELD AND SYMMETRIC FORM OF A CONVEX COVARIANT DENSITY SYSTEM

Let us begin the section recalling an important theorem that will be employed several times in the following.

Global invertibility Theorem (see [7]).

« *Let* f *a continuously differentiable mapping of a convex domain* $\mathscr{D}$ *in* $\mathbb{R}^N$, *into* $\mathbb{R}^N$. *If the symmetric part of the Jacobian matrix of* f *is definite (positive or negative), then* f *is globally univalent in* $\mathscr{D}$, *i. e.:*

$$\forall x, y \in \mathscr{D} : x \neq y \leftrightarrow f(x) \neq f(y) . \text{ »}$$

As a consequence of this theorem we have

LEMMA. — « *For a convex covariant density system the mapping* $\mathbf{U}'(\mathbf{U})$ *is globally invertible* ».

In fact on choosing $\mathbf{U}$ according to (4.8) it follows from (4.11) that the Jacobian matrix $\nabla\mathbf{U}'$ identifies with the Hessian matrix of $h(\mathbf{U})$ which is symmetric and results positive definite thanks to the supposed convexity of h.

Then it is possible to take $\mathbf{U}'$ as a field vector and show the:

STATEMENT I. — « *A convex covariant density system is a conservative symmetric system in the field* $\mathbf{U}'$ ».

Proof. — Let

$$h'^{\alpha} = \mathbf{U}' . \mathbf{F}^{\alpha} - h^{\alpha} \tag{5.1}$$

on taking the gradient respect to $\mathbf{U}'$, we have through (C.A):

$$\mathbf{F}^{\alpha} = \nabla' h'^{\alpha} \qquad (\nabla' = \partial/\partial\mathbf{U}') . \tag{5.2}$$

Introducing (5.2) into (1.5) we reach:

$$\mathrm{A}'^{\alpha}\mathbf{U}'_{\alpha} = f \qquad (\mathbf{U}'_{\alpha} = \partial_{\alpha}\mathbf{U}') \tag{5.3}$$

where:

$$\mathrm{A}'^{\alpha} = \nabla'\mathbf{F}^{\alpha} = \nabla'\nabla' h'^{\alpha} \tag{5.4}$$

are symmetric matrices.

Moreover from (5.1) and (4.9) we have:

$$h' = h'^{\alpha}\xi_{\alpha} = \mathbf{U}' . \mathbf{U} - h , \tag{5.5}$$

$h'(\mathbf{U}')$ is the Legendre conjugate function of $h(\mathbf{U})$ and then a convex function of $\mathbf{U}'$. It follows from (5.4) that the matrix:

$$\mathrm{A}'^{\alpha}\xi_{\alpha} = \nabla'\nabla' h' \tag{5.6}$$

is positive definite. Then, conditions *a*) and *b*) of def. IV being fulfiled, it follows that the system (5.3) is a symmetric hyperbolic conservative system in the field $\mathbf{U}'$.

The previous proof repeats through an explicitely covariant formalism the one performed in [*5*] and differs from that proposed by Friedrichs in [*1*]. (In [*1*] the existence of a symmetric form is proven, while here it is shown that the matrices A'^{α} are Hessian matrices of the known quantities h'^{α}, the field $\mathbf{U}'$ for which the system is symmetric is found and the conservative form is preserved through the mapping $\mathbf{U} \leftrightarrow \mathbf{U}'$).

We have seen that any convex covariant density system is endowed with a vector $\mathbf{U}'$ that may be expressed as a function of the field variables $\mathbf{U}$, but is not affected by transformation of $\mathbf{U}$, being determined completely only by the conservative system (1.5) and the supplementary law. Moreover we pointed out that when $\mathbf{U}'$ is chosen as field the system assumes a symmetric form, with the consequence that the local Cauchy problem is well posed. Such remarkable properties suggest us to call $\mathbf{U}'$ the « *main field* » of the system.

The possibility of finding special field variables in order to symmetrization of the field equations had been observed for the first time by G. Godunov [8] for special physical systems. Later G. Boillat [9], employing a non covariant formalism, introduced $\mathbf{U}'$ as $\boldsymbol{\nabla} h^0$, with $\mathbf{U} = \mathbf{F}^0$ pointing out the symmetrization of the field equations [5]; but this way prevents to observe the important privilege of the independence of $\mathbf{U}'$ on the choice of particular $\mathbf{U}$.

Moreover it is remarkable that the field $\mathbf{U} = \mathbf{F}^\alpha \xi_\alpha$ that we have introduced in the definition of convex covariant density systems and shall employ later to symplify calculations, is the conjugate field of $\mathbf{U}'$ in the Legendre transformation (5.5):

$$\mathbf{U}' = \boldsymbol{\nabla} h(\mathbf{U}), \qquad \mathbf{U} = \boldsymbol{\nabla}' h'(\mathbf{U}'). \tag{5.7}$$

The symmetrized field system:

$$\boldsymbol{\nabla}'\boldsymbol{\nabla}' h'^\alpha \mathbf{U}'_\alpha = f$$

is characterized by a differential operator that is known when the four-vector $h'^\alpha(\mathbf{U}')$ is assigned. Therefore we shall call $h'^\alpha(\mathbf{U}')$ the « *four-vector generating function* » of the symmetric system.

In conclusion the « main field » $\mathbf{U}'$ *the components of which are privilegiate variables and the « 4-vector generating function »* h'^α*, that generates the differential field equations, are enough to characterize the physical systems possessing a covariant convex density.*

We observe that not only on the mathematical stand-point $\mathbf{U}'$ and h'^α possess a special privilege respect to other quantities, but also physically they play an important role as we shall see later dealing with relativistic fluid, since they are related to the observables of the physical systems.

The classical quantities corresponding to $\mathbf{U}'$ and h'^α have already been evaluated in [10] for non linear adiabatic continuum mechanics; $\mathbf{U}'$ is determined also in [8] and [11] for non relativistic perfect fluid (for h'^α, in this case, see Appendix).

We finally observe that our approach (in particular the C.A) is quite similar to the way followed by I-Shih Liu [12] for the thermodynamics based on the entropy principle proposed by I. Müller [13]. The components of $\mathbf{U}'$ play, in this case, the same role of the Lagrange multipliers introduced in [12].

6. ENTROPY GROWTH ACROSS A SHOCK WAVE

Let Ω a connected open set of V^4 and Γ an hypersurface cutting Ω into two open subsets Ω_1, Ω_2. Let $\phi(x^\alpha) = 0$, $\phi \in C^m$ $(m \geq 2)$, the equation of Γ referred to any coordinate frame: we shall identify Γ with a shock hypersurface for the field $\mathbf{U}$.

Then it is known that the Rankine-Hugoniot conditions must hold:

$$[\mathbf{F}^\alpha]\phi_\alpha = 0 \qquad \text{on } \Gamma \tag{6.1}$$

brackets denoting the jump across Γ of the included function and $\phi_\alpha = \partial_\alpha \phi$.

Formally the Rankine-Hugoniot equations are obtained from the field equations (1.5) through the correspondence rule

$$\partial_\alpha \rightarrow \phi_\alpha[\ \ ]\,. \tag{6.2}$$

But the previous rule does not hold when applied on supplementary equation (2.1), in fact

$$\eta = [h^\alpha]\phi_\alpha \qquad \text{on } \Gamma \tag{6.3}$$

is generally non vanishing. Furthermore it is possible to show that η is non negative.

This result was proven through a non covariant formalism by P. D. Lax [*4*] with the introduction of an artificial viscosity into the field equations (Lax examined only one space variable system; see also on this subject the works by Kruskov [*14*] and Hopf [*15*]). A different proof was given in [*5*].

It is known that the positive signature of η for the non relativistic perfect fluid brings to the growth of thermodynamic entropy across the shock. That is why condition $\eta > 0$ is often referred in literature as « entropy growth condition » and is assumed as a criterion to pick up physical shocks among the solutions of the Rankine-Hugoniot equations. Recently G. Boillat and T. Ruggeri [*10*] pointed out entropy growth across a shock in the mechanics of hyperelastic continuous media submitted to finite strain.

We remark that the circumstance η non vanishing on Γ, roughly speaking, means that while the law (2.1) follows from the field equations when differentiability conditions hold, it does not follow for the weak solutions (as shock waves are).

In this section we shall exhibt an explicitely covariant proof of the fact that η is non negative on Γ.

Let $\{\xi_\alpha\}$ a subcharacteristic covector such that $\xi_\alpha\xi^\alpha = 1$ and σ a covariant scalar defined as:

$$\sigma = -\,\xi^\alpha\phi_\alpha\,, \tag{6.4}$$

then a space-like covector $\{\zeta_\alpha\}$ exists for which it results:

$$\phi_\alpha = -\,\sigma\xi_\alpha + \zeta_\alpha\,, \qquad \xi_\alpha\zeta^\alpha = 0 \tag{6.5}$$

Let $\Phi(x^\alpha) = 0$ the equation of a characteristic hypersurface that has locally the same « direction of propagation » ζ_α as the shock hypersurface, i. e.:

$$\det(A^\alpha\Phi_\alpha) = \det\{(-\mu\xi_\alpha + \zeta_\alpha)A^\alpha\} = 0 \qquad (6.6)$$

$$\Phi_\alpha^{(k)} = -\mu^{(k)}\xi_\alpha + \zeta_\alpha \qquad (6.7)$$

where $\mu^{(k)}$ ($k = 1, 2, \ldots, N$) are the eigenvalues of (6.6), that result real $\forall k$ for the hyperbolicity condition (1.3).

Now we consider a solution of the Rankine-Hugoniot equation (6.1) (shock):

$$\mathbf{U} \equiv \mathbf{U}(\mathbf{U}^*, \phi_\alpha), \qquad \mathbf{U} \neq \mathbf{U}^* \qquad (6.8)$$

$\mathbf{U}$, $\mathbf{U}^*$ being the perturbed and respectively the unperturbed fields evaluated as limit values on Γ. (In the following * will denote the values of any function of the field computed for $\mathbf{U} = \mathbf{U}^*$.)

Here we take only k-shocks according to the following

DEFINITION OF K-SHOCK. — We shall say that a shock is a k-shock if an integral number $k = 1, 2, \ldots, N$ exists such that:

$$\lim_{\sigma \to \mu_*^{(k)}} \mathbf{U} = \mathbf{U}^* . \qquad (6.9)$$

In words a k-shocks is a kind of shock that vanishes when the shock speed approaches to a characteristic velocity (of course these shocks become weak shocks when σ is near to $\mu_*^{(k)}$).

Now we suppose to know the explicit solution (6.8) for a k-shock and to introduce it into (6.3): we have then η as a function of $\mathbf{U}^*$ and ϕ_α:

$$\eta(\mathbf{U}^*, \phi_\alpha) = h^\alpha(\mathbf{U}(\mathbf{U}^*, \phi_\beta))\phi_\alpha - h^\alpha(\mathbf{U}^*)\phi_\alpha . \qquad (6.10)$$

We prove the

STATEMENT II. — « *For a convex covariant density system and a k-shock one has:*

$$\eta(\mathbf{U}^*, \phi_\alpha) \geq 0 \qquad \textit{when} \qquad \sigma \geq \mu_*^{(k)} \qquad (\textit{on } \Gamma) \text{ ».}$$

Proof. — On differentiating (6.10) respect to ϕ_β with constant $\mathbf{U}^*$ we reach:

$$\partial\eta/\partial\phi_\beta = h^\beta(\mathbf{U}) - h^\beta(\mathbf{U}^*) + \nabla h^\alpha\phi_\alpha . \partial\mathbf{U}/\partial\phi_\beta \qquad (6.11)$$

Now we differentiate (6.1) respect to ϕ_α:

$$A^\beta\phi_\beta\partial\mathbf{U}/\partial\phi_\alpha + \mathbf{F}^\alpha(\mathbf{U}) - \mathbf{F}^\alpha(\mathbf{U}^*) \equiv 0 \qquad (6.12)$$

and take the dot product of (6.12) with $\mathbf{U}'$, with (C.A):

$$\nabla h^\beta\phi_\beta . \partial\mathbf{U}/\partial\phi_\alpha = -\mathbf{U}' . [\mathbf{F}^\alpha] = -\nabla h . [\mathbf{F}^\alpha]$$

result that we introduce into (6.11) arriving at:

$$\partial\eta/\partial\phi_\alpha = [h^\alpha] - \nabla h . [\mathbf{F}^\alpha] ,$$

then:

$$\xi_\alpha\partial\eta/\partial\phi_\alpha = [h] - \nabla h . [\mathbf{U}] . \qquad (6.13)$$

Since h is a convex function of $\mathbf{U}$, in the convex domain $\mathscr{D}$, we have:

$$w(\mathbf{U}, \mathbf{U}^*) = -h(\mathbf{U}) + h(\mathbf{U}^*) + \nabla h.(\mathbf{U} - \mathbf{U}^*) > 0\,, \qquad \forall \mathbf{U} \neq \mathbf{U}^* \in \mathscr{D}$$

then the r. h. s. in (6.13) is equal $-w$ evaluated on Γ. Consequently:

$$\xi_\alpha \partial\eta/\partial\phi_\alpha < 0 \tag{6.14}$$

Furthermore in the frame $\mathscr{S}$ in which $\xi^0 = 1$, $\xi^i = 0$ locally, condition (6.14) writes:

$$\partial\eta/\partial\sigma > 0 \tag{6.15}$$

Now $\partial\eta/\partial\sigma$ being a scalar the inequality (6.15) is independent of the frame: η is a strictly increasing function of σ in any frame.

Since our shock is supposed to be a k-shock:

$$\lim_{\sigma \to \mu_*^{(k)}} \eta = 0$$

then it is proven that: $\eta \geq 0$ for $\sigma \geq \mu_*^{(k)}$.

7. η AS GENERATING FUNCTION OF THE SHOCK

STATEMENT III. — « *If η is known as a function of $\mathbf{U}^*$ and ϕ_α, then the following relationship holds on Γ:*

$$\nabla^*\eta = [\mathbf{U}']\mathbf{A}_*^\alpha\phi_\alpha \tag{7.1}$$

where

$$\nabla^* = \partial/\partial\mathbf{U}^*\,, \qquad \mathbf{A}_*^\alpha = \mathbf{A}^\alpha(\mathbf{U}^*)\text{ ».}$$

Eq. (7.1) means that knowing the *only* scalar function η as a function of $\mathbf{U}^*$ and ϕ_α (with ϕ_α non-characteristic) we may find the jump of $\mathbf{U}'$ and therefore of $\mathbf{U}$; η behaves as a sort of « potential » for the shock.

Of course, in practice, the evaluation of $\eta(\mathbf{U}^*, \phi_\alpha)$ follows the knowledge of the shock as a solution to the Rankine-Hugoniot equations, but it is interesting the fact that were it possible to determine η through experimental tests we should be able to have all the information of the shock.

The proof of (7.1), in covariant formalism, does not differ from that performed in [5]. Taking the gradient of (6.10) respect to $\mathbf{U}^*$ with constant ϕ_α we find:

$$\nabla^*\eta = \nabla h^\alpha \nabla^*\mathbf{U}\phi_\alpha - \nabla^* h_*^\alpha\phi_\alpha \qquad (h_*^\alpha = h^\alpha(\mathbf{U}^*)) \tag{7.2}$$

Operating with ∇^* on the Rankine-Hugoniot equations (6.1) we obtain:

$$\mathbf{A}^\alpha\nabla^*\mathbf{U}\phi_\alpha = \mathbf{A}_*^\alpha\phi_\alpha$$

and taking the dot product with $\mathbf{U}'$ we reach through (C.A):

$$\nabla h^\alpha\nabla^*\mathbf{U}\phi_\alpha = \mathbf{U}'\mathbf{A}_*^\alpha\phi_\alpha$$

introducing this in (7.2) we arrive soon to (7.1).

8. RELATIVISTIC BOUND OF THE SHOCK SPEED

The Rankine-Hugoniot equations:

$$\mathbf{F}^{\alpha}(\mathbf{U})\phi_{\alpha} = \mathbf{F}^{\alpha}(\mathbf{U}^{*})\phi_{\alpha} \tag{8.1}$$

provide N equations for the perturbed field $\mathbf{U}$ if $\mathbf{U}^{*}$, ϕ_{α} are known.

On the mathematical stand-point for any ϕ_{α} may exist non vanishing shocks ($\mathbf{U} \neq \mathbf{U}^{*}$), solution to (8.1), but physically it is necessary that $g^{\alpha\beta}\phi_{\alpha}\phi_{\beta} \leq 0$ so that the speed of the shocks does not exceed that of the light, according to relativistic principle.

Eq. (8.1) belong to the class of equations of the type:

$$f(\mathbf{U}, \phi_{\alpha}) = f(\mathbf{U}^{*}, \phi_{\alpha}) \tag{8.2}$$

which always possess the trivial solution $\mathbf{U} = \mathbf{U}^{*}$ for any ϕ_{α} and may have also non vanishing solutions $\mathbf{U} \neq \mathbf{U}^{*}$ (byfurcated solutions of the trivial solution).

We put now the following question: does it exist a set a values of ϕ_{α} such that the function f is globally invertible respect to $\mathbf{U}$ for a fixed ϕ_{α}? If the answer is affirmative, then only the trivial solution $\mathbf{U} = \mathbf{U}^{*}$ is allowed.

The problem has been examined by G. Boillat and T. Ruggeri, who proved [*6*] that non vanishing solutions (shocks) take place only if their speed is greather than the smallest characteristic speed and smaller than the greatest one.

Here we provide an explicitely covariant formulation of the proof given in [*6*].

In order to employ the global invertibility theorem enounced in sect. 5, we evaluate the Jacobian matrix of f in (8.2) respect to $\mathbf{U}'$:

$$\nabla' f = \mathrm{A}'^{\alpha}\phi_{\alpha} = \mathrm{A}'^{\alpha}(-\sigma\xi_{\alpha} + \zeta_{\alpha})\,. \tag{8.3}$$

Since from (5.6) one gets:

$$\xi_{\alpha}\mathrm{A}'^{\alpha} = \nabla'\nabla' h' = \mathrm{H}' \tag{8.4}$$

it follows:

$$\nabla' f = \zeta_{\alpha}\mathrm{A}'^{\alpha} - \sigma\mathrm{H}' \tag{8.5}$$

then the Jacobian matrix is symmetric.

Moreover being:

$$\det(\Phi_{\alpha}\mathrm{A}^{\alpha}) = \det\{(-\mu\xi_{\alpha} + \zeta_{\alpha})\mathrm{A}^{\alpha}\} = \det(-\mu\mathrm{I} + \zeta_{\alpha}\mathrm{A}^{\alpha}) = 0$$

and from (5.7)

$$\mathrm{A}'^{\alpha} = \nabla'\mathbf{F}^{\alpha} = \nabla\mathbf{F}^{\alpha}\nabla'\mathrm{U} = \mathrm{A}^{\alpha}\nabla'\nabla' h' = \mathrm{A}^{\alpha}\mathrm{H}'\,,$$

we reach:

$$\det(\zeta_{\alpha}\mathrm{A}'^{\alpha} - \mu\mathrm{H}') = 0 \tag{8.6}$$

i. e.: the $\mu^{(k)}$ are the eigenvalues of $\zeta_\alpha A'^\alpha$ respect to H′. From a well known theorem of linear algebra the matrix $\zeta_\alpha A'^\alpha - \sigma H'$ is positive definite or respectively negative definite if:

$$\sigma > \sup_{U\in\mathcal{D}} \operatorname*{Max}_k \{ \mu^{(k)} \} = M$$

or

$$\sigma < \inf_{U\in\mathcal{D}} \min_k \{ \mu^{(k)} \} = m$$

H′ being positive definite.

If σ fulfils one of the previous inequalities, for the global invertibility theorem, the mapping f in (8.2) is globally invertible and the unique solution of the Rankine-Hugoniot equations is $\mathbf{U}' = \mathbf{U}'_*$, and then $\mathbf{U} = \mathbf{U}^*$ since also the mapping $\mathbf{U}' \leftrightarrow \mathbf{U}$ is globally univalent.

Therefore non vanishing shocks happen only if:

$$m \leq \sigma \leq M \tag{8.7}$$

If we suppose that the characteristic manifolds $\Phi(x^\alpha)$ are time or light-like, so that Φ_α are space or light-like (no sum over k):

$$g^{\alpha\beta}\Phi_\alpha^{(k)}\Phi_\beta^{(k)} = \{ \mu^{(k)} \}^2 + \zeta_\alpha\zeta^\alpha \leq 0\,, \qquad \forall k, \quad \forall\zeta_\alpha, \quad \forall \mathbf{U}\in\mathcal{D},$$

it follows from (8.7):

$$g^{\alpha\beta}\phi_\alpha\phi_\beta \leq \sup_{U\in\mathcal{D}} \operatorname*{Max}_k \{ g^{\alpha\beta}\Phi_\alpha^{(k)}\Phi_\beta^{(k)} \} \leq 0$$

i. e. ϕ_α is space or light-like and $\phi(x^\alpha) = 0$ is a time or light-like manifold.

Summarizing: it holds the following:

STATEMENT IV. — *For the hyperbolic convex covariant density system the non vanishing shocks fulfil condition* (8.7) *and the shock manifolds are time or light-like if the characteristic ones are such.*

9. RELATIVISTIC HYDRODYNAMICS. EXISTENCE OF A CONVEX COVARIANT DENSITY

The equations of relativistic hydrodynamics are (see: e. g. [*16*]):

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{9.1}$$

$$\partial_\alpha(ru^\alpha) = 0 \tag{9.2}$$

∂_α denoting the covariant derivative operator and the energy-momentum tensor being:

$$T^{\alpha\beta} = rfu^\alpha u^\beta - pg^{\alpha\beta} \tag{9.3}$$

where r is the matter density, f the index of the fluid, u^α the 4-velocity ($u^\alpha u_\alpha = 1$) and p the pressure. The speed of light is taken equal unity.

From (9.1), (9.2) and taking into account the thermodynamic relations:

$$r\theta dS = rdf - dp \tag{9.4}$$

$$rf = \rho + p \tag{9.5}$$

one is able to show the existence of the supplementary conservation law [17]:

$$\partial_\alpha(rSu^\alpha) = 0 \tag{9.6}$$

S being the specific entropy (entropy of the mass unit), θ the thermodynamic absolute temperature and ρ the proper energy density of the fluid. Free entalpy:

$$G = f - \theta S - 1 \tag{9.7}$$

and its differential

$$dG = -Sd\theta + dp/r \tag{9.8}$$

will be useful in the following.

The system (9.1), (9.2) may be put in the compact form (1.5) on choosing:

$$\mathbf{F}^\alpha \equiv \begin{pmatrix} rfu^\alpha u^\beta - pg^{\alpha\beta} \\ ru^\alpha \end{pmatrix}, \qquad (\beta = 0, 1, 2, 3) \tag{9.9}$$

$$f \equiv 0\,. \tag{9.10}$$

The supplementary law (9.6) identifies with (2.1) when:

$$h^\alpha = -rSu^\alpha\,, \qquad g = 0 \tag{9.11}$$

Now we show that the system of relativistic fluid dynamics possesses a convex covariant density.

To evaluate the main field $\mathbf{U}'$ from (4.6), we put:

$$\mathbf{U}' \equiv \begin{pmatrix} w_\beta \\ \psi \end{pmatrix}, \tag{9.12}$$

where w_β and the scalar ψ must be determined. Eq. (9.12) introduced into (4.6) yields:

$$r(\theta w_\beta v^\beta + r)dS + \{ rfw_\beta + (S + \psi)v_\beta \} dv^\beta \equiv 0 \tag{9.13}$$

in which

$$v^\beta = ru^\beta \tag{9.14}$$

and dv^β must be intended as a covariant differential.

Since dS and dv^β are linearly independent it follows:

$$\begin{cases} \theta w_\beta v^\beta + r = 0 \\ rfw_\beta + (S + \psi)v_\beta = 0 \end{cases}$$

from which finally:

$$\mathbf{U}' \equiv \frac{1}{\theta}\begin{pmatrix} -u_\beta \\ f - \theta S \end{pmatrix} = \frac{1}{\theta}\begin{pmatrix} -u_\beta \\ G + 1 \end{pmatrix} \tag{9.15}$$

It is remarkable that the components of the main field (all independent) are substantially the velocity, the absolute temperature (because $u^\alpha u_\alpha = 1$) and the free-enthalpy G, i. e.: observables of the system.

So condition (C.A) is verified and it remains to be shown that:

$$h = h^\alpha \xi_\alpha = - r S u^\alpha \xi_\alpha \tag{9.16}$$

is a convex function of the field:

$$\mathbf{U} = \mathbf{F}^\alpha \xi_\alpha \equiv \begin{pmatrix} (rfu^\alpha u^\beta - pg^{\alpha\beta})\xi_\alpha \\ ru^\alpha \xi_\alpha \end{pmatrix} \tag{9.17}$$

for at least one subcharacteristic covector $\{ \xi_\alpha \}$.

We start pointing out that the free enthalpy G defined by (9.7), taken as a function of p and θ has negative definite Hessian matrix at the thermodynamic equilibrium.

In fact (see e. g. [*14*]):

$$G_{\theta\theta} < 0\,, \qquad D(G_\theta, G_p)/D(\theta, p) = G_{\theta\theta}G_{pp} - \{ G_{\theta p} \}^2 > 0 \tag{9.18}$$

$(G_\theta = \partial G/\partial\theta,\ G_p = \partial G/\partial p)$.

Convexity of $-$ G respect to the variables p and θ is equivalent to say that the quadratic form:

$$K^2 = - r^2 \{ \delta(\partial G/\partial p)\delta p + \delta(\partial G/\partial\theta)\delta\theta \} > 0 \qquad \forall(\delta p, \delta\theta) \neq 0 \tag{9.19}$$

and from (9.8) it follows

$$K^2 = \delta r \delta p + r^2 \delta\theta\delta S > 0 \qquad \forall(\delta p, \delta\theta) \neq 0 \tag{9.20}$$

Now we introduce:

$$v = u^\beta \xi_\beta\,, \qquad z^\beta = (rfu^\alpha u^\beta - pg^{\alpha\beta})\xi_\alpha \tag{9.21}$$

from (4.12), (9.15) and (9.17) we find:

$$\theta^2 Q = (u_\beta \delta\theta - \theta\delta u_\beta)\delta z^\beta + \{ \theta\delta(f - \theta S) + (\theta S - f)\delta\theta \}\, \delta(rv) \tag{9.22}$$

Taking account that:

$$u_\beta \delta u^\beta = 0\,, \qquad u_\beta \delta z^\beta = \delta(u_\beta z^\beta) - z^\beta \delta u_\beta = \delta(\rho v) + p\delta v\,,$$

$$\delta u_\beta \delta z^\beta = rfv\delta u_\beta \delta u^\beta - \delta p \delta v\,, \qquad \delta(f - \theta S) = \delta G = - S\delta\theta + \delta p/r\,,$$

one arrives after some calculations at

$$\theta^2 Q = - rfv\theta\delta u_\beta \delta u^\beta + 2\theta\delta p\delta v + v(\delta\rho - f\delta r)\delta\theta + v\theta\delta p\delta r/r \tag{9.23}$$

but (9.4), (9.5) imply:

$$\delta\rho - f\delta r = r\theta\delta S \tag{9.24}$$

Introducing (9.24) into (9.23) and taking account of (9.20) we have:

$$\theta Q = - rfv\delta u_\beta \delta u^\beta + 2\delta p\delta v + vK^2/r\,. \tag{9.25}$$

Since we are looking for the signature of Q, which is independent of the frame, because Q is a covariant scalar, we may write (9.25) in the Minkowskian rest frame of the fluid $\mathscr{S}$.

In $\mathscr{S}$ we have $u_\alpha \equiv (1, 0)$ and then:

$$\delta u^0 = 0 \quad (\text{because } u_\alpha \delta u^\alpha = 0), \qquad v = u_\alpha \xi^\alpha = \xi^0, \qquad \delta u^\alpha \delta u_\alpha = -(\delta \vec{u})^2,$$
$$\delta v = \delta u^\alpha \xi_\alpha = \delta u^i \xi_i = -\vec{\xi} \cdot \delta \vec{u} \tag{9.26}$$

Then (9.25) times v looks like:

$$\theta v \mathrm{Q} = rfv^2(\delta \vec{u})^2 - 2v\vec{\xi} \cdot \delta \vec{u} \delta p + v^2 \mathrm{K}^2/r$$

which may be written equivalently:

$$\theta v \mathrm{Q} = rf \{ v\delta \vec{u} - \vec{\xi} \delta p/(rf) \}^2 + v^2 \mathrm{K}^2/r - \vec{\xi}^2 (\delta p)^2/(rf) \tag{9.27}$$

Since

$$\xi^\alpha \xi_\alpha = \xi_0^2 - \vec{\xi}^2 = 1 \rightarrow \vec{\xi}^2 = v^2 - 1$$

and (9.27) becomes:

$$\theta v \mathrm{Q} = rf \{ v\delta \vec{u} - \vec{\xi} \delta p/(rf) \}^2 + \mathrm{K}^2/r + (v^2 - 1) \{ \mathrm{K}^2 - (\delta p)^2/f \}/r \tag{9.28}$$

Since $v > 1$ (being $v = u^\alpha \xi_\alpha$ and u^α, ξ^α unit time-like 4-vectors oriented towards the future) it is enough, for Q to be positive, to show that $f\mathrm{K}^2 - (\delta p)^2$ is non negative.

From (9.19) we have:

$$\{ (\delta p)^2 - f\mathrm{K}^2 \}/(fr^2) = \mathrm{G}_{\theta\theta}(\delta\theta)^2 + 2\mathrm{G}_{p\theta}\delta p \delta\theta + \{ \mathrm{G}_{pp} + 1/(fr^2) \} (\delta p)^2 \tag{9.29}$$

Now we look for the conditions for which the matrix of the coefficients of the quadratic form in the r. h. s. of (9.29) is negative semi-definite.

Since $\mathrm{G}_{\theta\theta} < 0$ for (9.18) it is enough to require

$$\mathrm{G}_{\theta\theta} \{ \mathrm{G}_{pp} + 1/(fr^2) \} - \{ \mathrm{G}_{p\theta} \}^2 \geq 0$$

which is equivalent to

$$\frac{\mathrm{D}(\mathrm{G}_\theta, \mathrm{G}_p)}{\mathrm{D}(\theta, p)} + \frac{1}{fr^2} \frac{\mathrm{D}(\mathrm{G}_\theta, p)}{\mathrm{D}(\theta, p)} \geq 0 \tag{9.30}$$

But the first term is positive for (9.18) and then:

$$1 + \frac{1}{fr^2} \frac{\mathrm{D}(\mathrm{G}_\theta, p)}{\mathrm{D}(\mathrm{G}_\theta, \mathrm{G}_p)} = 1 + \frac{1}{fr^2} \left(\frac{\partial p}{\partial \frac{1}{r}} \right)_{\mathrm{S}} \geq 0$$

Taking account of (9.24) the last inequality writes:

$$1 - (\partial p/\partial \rho)_{\mathrm{S}} \geq 0 \tag{9.31}$$

i. e. the sound speed lower than that of light, condition that of course is supposed to hold.

Therefore it is proven that $Q > 0$ in any frame and then the convexity of $h(\mathrm{U})$ for any unit time-like covector $\{ \xi_\alpha \}$ oriented towards the future.

Therefore the proposition stated in the general theory holds, and in particular:

1) The system of relativistic hydrodynamic equations is a symmetric system in the field U′ given by (9.15).

The system assumes the form (5.3), (5.4) with the four-vector generating function (see (5.1)) that has in this case the very simple expression:

$$h'^\alpha = \frac{p}{\theta} u^\alpha$$

2) $[\mathrm{S}] > 0$ on Γ when $u_*^\alpha \phi_\alpha < 0$.

In fact for the fluid:

$$\eta = [h^\alpha \phi_\alpha] = - [r\mathrm{S}u^\alpha \phi_\alpha] \tag{9.32}$$

and taking account of the Rankine-Hugoniot equation related to the mass conservation (9.2):

$$[ru^\alpha \phi_\alpha] = 0$$

it follows:

$$\eta = - r^* u_*^\alpha \phi_\alpha [\mathrm{S}] . \tag{9.33}$$

By employing the decomposition (6.5) we gain:

$$u_*^\alpha \phi_\alpha = - \sigma u_*^\alpha \xi_\alpha + u_*^\alpha \zeta_\alpha . \tag{9.34}$$

Recalling that Φ_α with $u_*^\alpha \Phi_a = 0$ is a characteristic covector (matter or contact wave) we have that the corresponding eigenvalue (see (6.7)) is:

$$\mu_* = (u_*^\alpha \zeta_\alpha)/(u_*^\beta \xi_\beta) \tag{9.35}$$

Introducing (9.35) into (9.34) we find:

$$u_*^\alpha \phi_\alpha = (\mu_* - \sigma) u_*^\alpha \xi_\alpha \tag{9.36}$$

Then

$$\eta = r^*(\sigma - \mu_*) u_*^\alpha \xi_\alpha [\mathrm{S}] . \tag{9.37}$$

From (9.37) one realizes that our shock is a k-shock η vanishing for $\sigma = \mu_*$ and then statement II holds. But $u_*^\alpha \xi_\alpha > 1$, u_*^α, ξ_α being unit time-like vectors oriented towards the future, then from (9.37) we have:

$$[\mathrm{S}] > 0 \quad \text{for} \quad \sigma > \mu_* \quad \text{i. e.} \quad [\mathrm{S}] > 0 \quad \text{for} \quad u_*^\alpha \phi_\alpha < 0$$

according to (9.36).

3) The knowledge of [S] as a function of U* and ϕ_α determines the shock.

This is a consequence of statement III and of the expression of the function η of the fluid given by (9.37). Therefore were it possible to measure only the jump of S across a shock wave, we would be able to calculate the jump of each field variable.

4) The velocity of propagation of the relativistic hydrodynamic shocks never exceed the speed of light.

A paper on the consequence of the general theory of the convex covariant densitv systems for the Magneto fluid dynamics is in preparation.

Finally, we point out that result 4) had already been proven for the fluid and M. H. D. through a different way by A. Lichnerowicz [*16*].

APPENDIX

I. ANOTHER CONVEX COVARIANT DENSITY FOR THE FLUID

We have seen before that the main field $\mathbf{U}'$ of a convex covariant density system is invariant under mapping of the field $\mathbf{U}$, and it may be expressed as the gradient of the convex covariant density $h = h^\alpha \xi_\alpha$, when the field choice: $\mathbf{U} = \mathbf{F}^\alpha \xi_\alpha$ is employed, h^α being the current density of quantity conserved thanks to the supplementary law. Then h represents the proper density of the conserved quantity relative to the congruence defined by the time-like covector $\{ \xi_\alpha \}$.

It is remarkable that in the fluid case it is possible to define another convex density $\hat{h}$, relative to the field dependent congruence $\{ u_\alpha \}$ with the same properties of h:

$$\hat{h} = h^\alpha u_\alpha = - r\mathrm{S} \tag{I.1}$$

the gradient of which respect to the field:

$$\hat{\mathbf{U}} = \mathbf{F}^\alpha u_\alpha \equiv \begin{pmatrix} \rho u^\beta \\ r \end{pmatrix} \tag{I.2}$$

is still equal to the same main field $\mathbf{U}'$ (9.15):

$$\mathbf{U}' = \hat{\mathbf{\nabla}} \hat{h} , \qquad (\hat{\mathbf{\nabla}} = \partial/\partial \hat{\mathbf{U}}) \tag{I.3}$$

i. e.:

$$d\hat{h} = - d(r\mathrm{S}) = - (u_\beta/\theta) d(\rho u^\beta) + (f/\theta - \mathrm{S}) dr$$

as it is immediate to verify taking account of (9.24).

Moreover the convexity of $\hat{h}(\hat{\mathbf{U}})$ is easier to be proven than that of $h(\mathbf{U})$. We exploit the proof showing that:

$$\hat{w}(\hat{\mathbf{U}}, \hat{\mathbf{U}}_*) = - \hat{h}(\hat{\mathbf{U}}) + \hat{h}(\hat{\mathbf{U}}_*) + \hat{\mathbf{\nabla}} \hat{h}(\hat{\mathbf{U}}) \cdot (\hat{\mathbf{U}} - \hat{\mathbf{U}}_*) > 0 , \qquad \forall \hat{\mathbf{U}} \neq \hat{\mathbf{U}}_* \tag{I.4}$$

We have easily:

$$\hat{w} = \tilde{w} r^*/\theta - \rho^*(1 - u^*_\alpha u^\alpha)/\theta \tag{I.5}$$

where

$$\tilde{w} = \theta(\mathrm{S} - \mathrm{S}^*) + p/r^* - f + \rho^*/r^* = \mathrm{G}^* - \mathrm{G} + \mathrm{G}^*_p(p - p^*) + \mathrm{G}^*_\theta(\theta - \theta^*) > 0$$

for the convexity of $- \mathrm{G}(p, \theta)$.

Now one verifies that:

$$1 - u^*_\alpha u^\alpha = \frac{1}{2}(u_\alpha - u^*_\alpha)(u^\alpha - u^\alpha_*) = - \frac{1}{2} q^2 < 0 \tag{I.6}$$

u_α, u^*_α being unit time-like 4-vectors oriented towards the future.

Then:

$$\hat{w} = \tilde{w} r^*/\theta + \frac{1}{2} \rho^* q^2/\theta > 0 \tag{I.7}$$

and the convexity of $- r\mathrm{S}$ is proven. We observe that the auxiliary condition $(\partial p/\partial \rho)_\mathrm{S} - 1 \leq 0$ here is not required.

As a consequence of the convexity of $\hat{h}(\hat{\mathbf{U}})$ and (I.3) we have also that the mapping $\mathrm{U}' \leftrightarrow \hat{\mathrm{U}}$ is globally univalent.

II. CLASSICAL APPROXIMATION OF THE RELATIVISTIC FLUID

Here we give the non relativistic limits for the main field and the 4-vector generating function h'^{α} of the fluid.

It is easy to verify that the main field $\mathbf{U}'$ is:

$$\mathbf{U}' \equiv \frac{1}{\theta}\begin{pmatrix} -1 \\ \vec{u} \\ G - \frac{1}{2}\vec{u}^2 \end{pmatrix}$$

where: $-1/\theta$ is the multiplier related to the energy conservation equation, $\vec{u}/\theta$ is that of the momentum equation and $\left(G - \frac{1}{2}\vec{u}^2\right)/\theta$ the multiplier of the matter conservation law.

The components of $\mathbf{U}'$ coincide with the ones gives by Godunov [8] and deduced in [11].

While $h'^{\alpha} \equiv (h'^0, h'^i)$, $h'^0 = h'$ is given by:

$$h'^0 = p/\theta\,, \qquad h'^i = pu^i/\theta\,.$$

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