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Section A :
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Predictive relativistic mechanics of systems of N particles with spin

by

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ABSTRACT. — We develop the Predictive Relativistic Mechanics of isolated systems of particles with spin, using both the Manifestly Predictive Formalism and the Manifestly Covariant one. The paper starts by generalizing the concept of Poincaré Invariant Predictive System to include the spins of the particles and ends with the construction of the Canonical Hamiltonian Formalism for separable systems.

1. INTRODUCTION

The concept of Free Elementary Particle is one of the basic physical concepts of Classical [*1*] Relativistic Mechanics. At present this concept is unambiguously defined by identifying it with an Elementary Dynamical System (E. D. S.) [*2-4*] which is a mathematical concept. Shortly speaking an E. D. S. is a dynamical system which is invariant under the Poincaré group and which has the following two properties: *i*) There exists an Hamiltonian Formalism compatible with the Poincaré invariance, and *ii*) The realisation of the Poincaré group on the co-phase space (the space of initial conditions) acts transitively.

According to the preceding definition each E. D. S. can be characterized [*2-4*] by two non negative parameters (m, s) which are interpreted as the mass and the modulus of the spin of the corresponding free particle. The fact that the spin, which was thought to be a pure quantum mechanical concept, has actually a classical analog is not yet a well divulgated result among physicists.

The relativistic theory of isolated systems of interacting particles with no spin ($m > 0$, $s = 0$) has been satisfactorily developped in recent years. This theory, usually called Predictive Relativistic Mechanics (P. R. M.) involves two fundamental concepts: the concept of a Poincaré Invariant Predictive System (P. I. P. S.) [*5-7*] and the concept of a Compatible Hamiltonian Formalism (C. H. F.) [*5*] [*6*]. The first concept restricts the class of admissible interactions which satisfy the Principle of Predictivity and the Principle of Relativity. A C. H. F. leads to unambiguous definitions of conserved quantities as the total Energy, Linear Momentum, or Angular Momentum.

Up to now only a few papers [*8*] have been published dealing with the relativistic theory of isolated systems of interacting particles with spin ($m > 0$, $s > 0$). This paper is a systematic contribution to this problem which parallels the theory of systems of particles with no spin. In Sect. 2 we generalize the concept of P. I. P. S. to include the spin of the particles. We use here the Manifestly Predictive Formalism which makes explicit the fact the initial positions, velocities and orientations of the spin determine the future evolution of the system. We obtain the necessary and sufficient conditions which a dynamical system has to satisfy to be a P. I. P. S., conditions which are a generalisation of Currie-Hill's equations.

In Section 3 we use the Manifestly Covariant Formalism and we generalize the Arens-Droz-Vincent's equations. We proof also the equivalence of this formalism with the preceding one and we establish the correspondence between them.

Finally in Section 4 we generalize the concept of a C. H. F. of a P. I. P. S. We remind in it some of Souriau [*2*] and Arens [*3*] results related to this problem for systems of free particles and we establish the connection between them. We consider then the case of interacting particles and we establish the recurrent algorithm which for separable systems permits the calculation in perturbation theory of the Hamiltonian Form and related quantities such as the Energy, Momentum or Angular Momentum of the system.

Some of the material contained in this paper deals with quite straightforward generalizations of known proofs or results of the theory of systems of particles without spin. It has been included here either because we felt that it was necessary to emphasize the spin dependence or because the corresponding material for particles with no spin had been published under the form of interior reports only.

2. INVARIANT SYSTEMS : THE MANIFESTLY PREDICTIVE FORMALISM

Let us consider an isolated system of N interacting particles with spin. We shall assume that the Principle of Predictivity is satisfied which means that the knowledge of the initial positions, the initial velocities and the initial orientations of the spin with respect to any galilean reference system determines the evolution of the system. In other words, we assume that this evolution is governed by a first order system of ordinary differential equations, on the space $[T(R^3) \times S^2]^N$, with the following structure:

$$\left\{\begin{aligned} \frac{dx_a^i}{dt} &= v_a^i \\ \frac{dv_a^i}{dt} &= \mu_a^i(t\,;\, x_b^j,\, v_c^k,\, \alpha_d^l) \\ \frac{d\alpha_a^i}{dt} &= \rho_a^i(t\,;\, x_b^j,\, v_c^k,\, \alpha_d^l)\,, \end{aligned}\right. \tag{2.1}$$

where x_a^i $(a, b, c, d, \ldots = 1, 2, \ldots, N\,;\; i, j, k, l, \ldots = 1, 2, 3)$ are the position coordinates of particle a at the instant t, and where v_a^i are the corresponding components of the velocities; the quantities α_a^i are the components of the unit vector defining the orientation of the spin of particle a at the instant t; therefore [9]:

$$\alpha_a^i \alpha_{ai} = 1\,, \qquad \forall a, \quad \forall t \tag{2.2}$$

and thus the functions ρ_a^i must be such that:

$$\alpha_a^i \rho_{ai} = 0\,. \tag{2.3}$$

This condition implies that $\alpha_a^i \alpha_{ai}$ are first integrals and therefore equations (2.2) are consistent constraints.

Let us notice that the functions μ_a^i and ρ_a^i, which characterize the interaction which is considered, will depend also on the masses m_a and the modulus s_a of the particles of the system; they might also depend on other physical parameters as electric charges or similar ones.

The concept of the spin of a particle has been initially defined with precision for a free particle, i. e., for a particle which is at rest or which moves along a straight line with constant velocity. Therefore we need to make clear the meaning of the quantities α_a^i involved in the evolution equations (2.1). We shall do that by establishing the connection between α_a^i and the intrinsic orientation of the spin of the particle, i. e., the orientation with respect to the rest frame of reference of the corresponding particle. By definition α_a^i will be the 3-vector which under a Lorentz boost corresponding to the velocity v_a^i has as image the intrinsic orientation of the spin at the corresponding instant of time. More precisely, let u_a^λ

($\lambda, \mu, \ldots = 0, 1, 2, 3$) be the unit 4-velocity [10] of particle a at the instant t, i. e.:

$$\begin{cases} u_a^0 \equiv \left(1 - v_a^2/c^2\right)^{-1/2} \\ u_a^i \equiv \dfrac{v_a^i}{c}\left(1 - v_a^2/c^2\right)^{-1/2} \qquad (v_a^2 \equiv v_a^i v_{ai}) \end{cases} \tag{2.4}$$

and let $B_{\overrightarrow{v_a}}$ be the Lorentz boost corresponding to v_a^i; by definition the 4-vector with components $(0, \alpha_a^i)$ is such that:

$$(0, \alpha_a^i) \overset{B_{\overrightarrow{v_a}}}{\rightarrow} \gamma_a^\lambda : \begin{cases} \gamma_a^0 = u_{ai}\alpha_a^i \\ \gamma_a^j = \left(\delta_i^j + \dfrac{u_a^j u_{ai}}{1 + u_a^0}\right)\alpha_a^i , \end{cases} \tag{2.5}$$

where γ_a^λ is the space-like 4-vector which represents the intrinsic orientation of the spin at the instant t. Let us notice that from equations (2.4) and (2.5) it follows that:

$$\gamma_a^\lambda u_{a\lambda} = 0 \tag{2.6 a}$$

$$\gamma_a^\lambda \gamma_{a\lambda} = \alpha_a^i \alpha_{ai} , \tag{2.6 b}$$

identities which were necessary to insure the correctness of the definition above. Reciprocally we have of course that under a boost corresponding to the velocity $-v_a^i$, which we shall note $\mathfrak{B}_{\overrightarrow{v_a}}$, the image of γ_a^λ is the 4-vector with components $(0, \alpha_a^i)$, i. e.:

$$\gamma_a^\lambda \overset{\mathfrak{B}_{\overrightarrow{v_a}}}{\rightarrow} \alpha_a^i \equiv \gamma_a^i - \frac{\gamma_a^0}{1 + u_a^0} u_a^i . \tag{2.7}$$

Let us consider the general solution of the system of differential equation (2.1), which we shall write as follows:

$$\begin{cases} x_a^i = \varphi_a^i(x_0, v_0, \alpha_0 ; t) & (2.8\ a) \\ \alpha_a^i = \psi_a^i(x_0, v_0, \alpha_0 ; t) , & (2.8\ b) \end{cases}$$

where $x_0 = (x_{a0}^i)$, $v_0 = (v_{a0}^i)$ and $\alpha_0 = (\alpha_{a0}^i)$ are the initial conditions for $t = 0$, and therefore we have:

$$\begin{cases} \varphi_a^i(x_0, v_0, \alpha_0 ; 0) \equiv x_{a0}^i & (2.9\ a) \\ \dot{\varphi}_a^i(x_0, v_0, \alpha_0 ; 0) \equiv v_{a0}^i , \quad \left(\dot{} \equiv \dfrac{d}{dt}\right) & (2.9\ b) \\ \psi_a^i(x_0, v_0, \alpha_0 ; 0) \equiv \alpha_{a0}^i . & (2.9\ c) \end{cases}$$

For each set of initial conditions (x_0, v_0, α_0) equation (2.8 a) determine a set of N *geometrical* curves in Minkowski *affine* space-time $\mathfrak{M}_4$; at the same time equation (2.8 b) in conjunction with equation (2.5) associate to each point of these curves a space-time vector γ_a^λ orthogonal to the corresponding curve.

We shall say that a geometrical curve of $\mathfrak{M}_4$ has a spin if this curve is time-like and we have defined a smooth field of space-like unit and orthogonal vectors along it. According to this terminology we can say that the general solution (2.8) of the system of differential equation (2.1) is a family (parametrised by the initial conditions) of sets of N curves with spin.

The Poincaré group $\mathfrak{P}$ acts as follows on the space of curves with spin. If $\Lambda = (\mathrm{L}.\mathrm{A})$ is an element of $\mathfrak{P}$, $\mathrm{L} = (\mathrm{L}^{\mu}_{\nu})$ being a matrix of the Lorentz group and $\mathrm{A} = (\mathrm{A}^{\mu})$ a vector of Minkowski *vector* space-time M_4 and if $\{ (t, x^i), \gamma^{\lambda} \}$ is a generic point with its spin of a curve with spin, then by definition the image of this curve by Λ is the curve with spin defined by the mappings:

$$(t, x^i) \overset{\Lambda}{\rightarrow} \Lambda(t, x^i) \tag{2.10 a}$$

$$\gamma^{\lambda} \overset{\mathrm{L}}{\rightarrow} \mathrm{L}(\gamma^{\lambda}) . \tag{2.10 b}$$

Let us notice that from (2.10 *b*), (2.5) and (2.7) it follows that the image β^i of the instantaneous orientation of the spin α^i is given by the following set of mappings:

$$\alpha^i \overset{\mathrm{B}_{\vec{v}}}{\rightarrow} \gamma^{\lambda} \overset{\mathrm{L}}{\rightarrow} \mathrm{L}(\gamma^{\lambda}) \overset{\mathfrak{A}_{\vec{w}}}{\rightarrow} \beta^i , \tag{2.11}$$

where $\vec{w}$ is the 3-velocity of the image of the corresponding curve at the corresponding point.

We shall say that the system of differential equation (2.1) is a Poincaré Invariant Predictive System (P. I. P. S.) if the family $\mathscr{F}$ of sets of N curves with spin defined by the general solution (2.8) is stable under the action of $\mathfrak{P}$. Taking into account (2.5), (2.7) and (2.11) we can more explicitly state the following definition:

DEFINITION 2.1. — « A system of differential equations of the type (2.1) is a P. I. P. S. if there exist functions:

$$\left\{ \begin{aligned} y^i_{a0} &= f^i_a(x_0, v_0, \alpha_0; \Lambda) \\ w^i_{a0} &= g^i_a(x_0, v_0, \alpha_0 ; \Lambda) , \qquad \Lambda \in \mathfrak{P} \\ \beta^i_{a0} &= h^i_a(x_0, v_0, \alpha_0 ; \Lambda) \end{aligned} \right. \tag{2.12}$$

such that for each t, (x_0, v_0, α_0) and Λ we have:

$$\mathrm{L}^j_i[\varphi^i_a(x_0, v_0, \alpha_0 ; t) - \mathrm{A}^i] + \mathrm{L}^j_0.(ct - \mathrm{A}^0) = \varphi^j_a(y_0, w_0, \beta_0 ; \tau_a) \tag{2.13}$$

$$\begin{aligned} &\mathrm{L}^j_i \left\{ \delta^i_l + \frac{u^i_a(x_0, v_0, \alpha_0 ; t)\cdot u_{al}(x_0, v_0, \alpha_0 ; t)}{1+u^0_a(x_0, v_0, \alpha_0 ; t)} \right\} \cdot \psi^l_a(x_0, v_0, \alpha_0 ; t) \\ &\quad + \mathrm{L}^j_0 u_{al}(x_0, v_0, \alpha_0 ; t)\cdot \psi^l_a(x_0, v_0, \alpha_0 ; t) \\ &= \left\{ \delta^j_l + \frac{u^j_a(y_0, w_0, \beta_0 ; \tau_a)\cdot u_{al}(y_0, w_0, \beta_0 ; \tau_a)}{1+u^0_a(y_0, w_0, \beta_0 ; \tau_a)} \right\} \psi^l_a(y_0, w_0, \beta_0 ; \tau_a) \end{aligned} \tag{2.14 a}$$

$$\begin{aligned} &\mathrm{L}^0_i \left\{ \delta^i_l + \frac{u^i_a(x_0, v_0, \alpha_0 ; t)\cdot u_{al}(x_0, v_0, \alpha_0 ; t)}{1+u^0_a(x_0, v_0, \alpha_0 ; t)} \right\} \cdot \psi^l_a(x_0, v_0, \alpha_0 ; t) \\ &\quad + \mathrm{L}^0_0 \cdot u_{al}(x_0, v_0, \alpha_0 ; t)\cdot \psi^l_a(x_0, v_0, \alpha_0 ; t) \\ &= u_{al}(y_0, w_0, \beta_0 ; \tau_a)\cdot \psi^l_a(y_0, w_0, \beta_0 ; \tau_a) , \end{aligned} \tag{2.14 b}$$

where we have used the following notations:

$$c \cdot \tau_a \equiv L_l^0 \{ \varphi_a^l(x_0, v_0, \alpha_0 ; t) - A^l \} + L_0^0(ct - A^0) \qquad (2.15)$$

$$u_a^0(x_0, v_0, \alpha_0 ; t) \equiv + \left\{ 1 - \frac{1}{c^2} \dot\varphi_{aj}(x_0, v_0, \alpha_0 ; t) \cdot \dot\varphi_a^j(x_0, v_0, \alpha_0 ; t) \right\}^{-1/2} \qquad (2.16\,a)$$

$$u_a^i(x_0, v_0, \alpha_0 ; t) \equiv \frac{1}{c} u_a^0(x_0, v_0, \alpha_0 ; t) \cdot \varphi_a^j(x_0, v_0, \alpha_0 ; t), \qquad (2.16\,b)$$

c being the speed of light in vacuum ».

We shall assume that the system of differential equation (2.1) is such that inequalities $v_{0a}^2 < c^2$, which we shall always impose, imply $v_a^2 < c^2$ in the interval of t for which the solution exists. In what follows this assumption will be often used to guarantee the existence of functions defined implicitly.

It is important to notice that if the system 2.1 is a P. I. P. S. in the sense of the preceding definition, then the functions (2.12) are completely determined by the general solution (2.8). In fact, taking the derivative of eq. (2.13) with respect to t and using eq. (2.15) we have:

$$L_i^j \cdot \dot\varphi_a^i(x_0, v_0, \alpha_0 ; t) + L_0^j c = \dot\varphi_a^j(y_0, w_0, \beta_0 ; \tau_a) \left[\frac{1}{c} L_l^0 \dot\varphi_a^l(x_0, v_0, \alpha_0 ; t) + L_0^0 \right] \qquad (2.17)$$

considering now equations (2.13), (2.17) and (2.14) for $\tau_a = 0$ and using equation (2.9) and the definition (2.16) we get:

$$y_{a0}^j = L_i^j[\varphi_a^i(x_0, v_0, \alpha_0 ; T_a) - A^i] + L_0^j(cT_a - A^0) \qquad (2.18\,a)$$

$$w_{a0}^j = \left[\frac{1}{c} L_l^0 \dot\varphi_a^l(x_0, v_0, \alpha_0 ; T_a) + L_0^0 \right]^{-1} \cdot \{ L_i^j \dot\varphi_a^i(x_0, v_0, \alpha_0 ; T_a) + L_0^j c \} \qquad (2.18\,b)$$

$$\begin{aligned} \beta_{a0}^j = {} & L_i^j \left\{ \delta_l^i + \frac{u_a^i(x_0, v_0, \alpha_0 ; T_a) \cdot u_{al}(x_0, v_0, \alpha_0 ; T_a)}{1 + u_a^0(x_0, v_0, \alpha_0 ; T_a)} \right\} \psi_a^l(x_0, v_0, \alpha_0 ; T_a) \\ & + L_0^j u_{al}(x_0, v_0, \alpha_0 ; T_a) \cdot \psi_a^l(x_0, v_0, \alpha_0 ; T_a) \\ & - \frac{v_{a0}^j}{1 + v_{a0}^0} \left\{ L_i^0 \left[\delta_l^i + \frac{u_a^i(x_0, v_0, \alpha_0 ; T_a) \cdot u_{al}(x_0, v_0, \alpha_0 ; T_a)}{1 + u_a^0(x_0, v_0, \alpha_0 ; T_a)} \right] \psi_a^l(x_0, v_0, \alpha_0 ; T_a) \right. \\ & \left. + L_0^0 u_{al}(x_0, v_0, \alpha_0 ; T_a) \cdot \psi_a^l(x_0, v_0, \alpha_0 ; T_a) \right\}, \end{aligned} \qquad (2.18\,c)$$

where, according to (2.15), the quantities T_a are implicitly defined by the equations:

$$L_l^0 \{ \varphi_a^l(x_0, v_0, \alpha_0 ; T_a) - A^l \} + L_0^0(cT_a - A^0) = 0 \qquad (2.19)$$

and where we have used the following notations:

$$v_{a0}^{0} \equiv + \left\{ 1 - \frac{w_{a0}^{l} w_{al0}}{c^2} \right\}^{-1/2}$$
$$v_{a0}^{j} \equiv \frac{1}{c} v_{a0}^{0} w_{a0}^{j} . \tag{2.20}$$

Equations (2.18) determine the functions (2.12) we were looking for. Moreover, for each element $\Lambda \in \mathscr{P}$, these equations define a transformation on the space of initial conditions which we shall call the *induced transformation.* More precisely, by a rather long but straightforward calculation, it is possible to prove the following theorem which is a generalisation of the corresponding theorem for particles without spin [*13*].

THEOREM 2.1. — « For each P. I. P. S. the family (2.18) of induced transformations in the cophase space $[T(R^3) \times S^2]^N$ is a realisation of the Poincaré group ». We shall call this realisation the *Induced Realisation.*

Let us examine now what conditions the functions μ_a^i and ρ_a^i, which characterize the interactions satisfy when the system (2.1) is a P. I. P. S. Let us write equations (2.13), (2.17) and (2.14) for $A^i = 0$ and $L_\nu^\mu = \delta_\nu^\mu$. The equations which we obtain together with eqs. (2.15) and (2.16) lead to the following relations:

$$\varphi_a^j(y_0, w_0, \beta_0 ; t - A^0/c) = \varphi_a^j(x_0, v_0, \alpha_0 ; t) \tag{2.21 a}$$
$$\dot{\varphi}_a^j(y_0, w_0, \beta_0 ; t - A^0/c) = \dot{\varphi}_a^j(x_0, v_0, \alpha_0 ; t) \tag{2.21 b}$$
$$\psi_a^j(y_0, w_0, \beta_0 ; t - A^0/c) = \psi_a^j(x_0, v_0, \alpha_0 ; t) ; \tag{2.21 c}$$

now, taking here $t = A^0/c$ and remembering (2.9), it follows:

$$y_{a0}^j = \varphi_a^j(x_0, v_0, \alpha_0 ; A^0/c) \tag{2.22 a}$$
$$w_{a0}^j = \dot{\varphi}_a^j(x_0, v_0, \alpha_0 ; A^0/c) \tag{2.22 b}$$
$$\beta_{a0}^j = \psi_a^j(x_0, v_0, \alpha_0 ; A^0/c) . \tag{2.22 c}$$

Equations (2.21) and (2.22) express that to the general solution of the system of differential equation (2.1) there corresponds a one parameter group of transformations on co-phase space; as it is well know this is equivalent to saying that the system is autonomous, i. e.:

$$\frac{\partial \mu_a^i}{\partial t} = 0 , \qquad \frac{\partial \rho_a^i}{\partial t} = 0 . \tag{2.23}$$

Let us calculate now the generators of the Induced Realisation (2.18). We remind that if a group G_r with r parameters acts on a manifold V_n of dimension n as a group of transformations:

$$y^A = f^A(x^B ; g^L) , \tag{2.24}$$

where $\{ x^A \}$ (A, B, ... = 1, 2, ..., n) is a system of coordinates of V_n

and where $\{ g^L \}$ (L, M, ... = 1, 2, ..., r) is a parametrisation of G_r, then by definition the infinitesimal generators associated with this parametrisation are the r vector fields:

$$\xi_L^A \equiv \left\{ \frac{\partial f^A}{\partial g^L} \right\}_{(g=e)}, \tag{2.25}$$

e being the neutral element of the group. According to this definition and using the parametrisation A^0, A^i, ω^j, V^k of the Poincaré group which is defined in the Appendix, the application of the formula (2.25) to the Induced Realisation (2.18) yields by a direct calculation and in the corresponding order the following ten vector fields of $[T(R^3) \times S^2]^N$:

$$\vec{H} \equiv v_a^j \frac{\partial}{\partial x_a^j} + \mu_a^j(x, v, \alpha) \frac{\partial}{\partial v_a^j} + \rho_a^j(x, v, \alpha) \frac{\partial}{\partial \alpha_a^j} \tag{2.26 a}$$

$$\vec{P}_i \equiv - \varepsilon_a \frac{\partial}{\partial x_a^i}, \qquad (\varepsilon_a = 1 \quad \forall a) \tag{2.26 b}$$

$$\vec{J}_i \equiv \eta_i{}^j{}_l \left(x_a^l \frac{\partial}{\partial x_a^j} + v_a^l \frac{\partial}{\partial v_a^j} + \alpha_a^l \frac{\partial}{\partial \alpha_a^j} \right) \qquad [11] \tag{2.26 c}$$

$$\vec{K}_i \equiv \frac{1}{c^2} x_{ai} v_a^j \frac{\partial}{\partial x_a^j} - \left\{ \varepsilon_a \delta_i^j - \frac{1}{c^2} v_{ai} v_a^j - \frac{1}{c^2} x_{ai} \mu_a^j(x, v, \alpha) \right\} \frac{\partial}{\partial v_a^j}$$
$$- \frac{1}{c^2} \left\{ \varkappa_a \left(\alpha_{al} v_a^l \delta_i^j - \alpha_{ai} v_a^j \right) - x_{ai} \rho_a^j(x, v, \alpha) \right\} \frac{\partial}{\partial \alpha_a^j}, \tag{2.26 d}$$

where we have suppressed the sub-indices zero and we have used the notation:

$$\varkappa_a \equiv \{ 1 + (1 - v_a^2/c^2)^{1/2} \}^{-1}. \tag{2.27}$$

Let us notice that the condition (2.3) guarantees that these vector fields are tangent to $[T(R^3) \times S^2]^N$: $\alpha_a^i \alpha_{ai}$ remains constant along the trajectories of each vector field (2.26). Therefore the derivatives with respect to α_a^j can be considered as independent notwithstanding the constraints (2.2). The vector fields (2.26) are respectively the generators of time evolution, space translations, spatial rotations and pure Lorentz transformations.

Since the vector fields (2.26) are the infinitesimal generators of a realisation of the Poincaré group, they must satisfy the appropriate commutation relations corresponding to its Lie algebra; the commutation relations associated with the parametrisation which we have used are:

$$[\vec{P}_i, \vec{P}_j] = 0, \qquad [\vec{J}_i, \vec{P}_j] = \eta_{ij}{}^l \vec{P}_l, \qquad [\vec{J}_i, \vec{J}_j] = \eta_{ij}{}^l \vec{J}_l \tag{2.28 a}$$

$$[\vec{P}_i, \vec{H}] = 0, \qquad [\vec{J}_i, \vec{H}] = 0, \qquad [\vec{K}_i, \vec{H}] = \vec{P}_i \tag{2.28 b}$$

$$[\vec{K}_i, \vec{P}_j] = \frac{1}{c^2} \delta_{ij} \vec{H}, \qquad [\vec{K}_i, \vec{J}_j] = \eta_{ij}{}^l \vec{K}_l, \qquad [\vec{K}_i, \vec{K}_j] = - \frac{1}{c^2} \eta_{ij}{}^l \vec{J}_l \tag{2.28 c}$$

where [,] means the Lie bracket of two vector fields.

The commutation relations (2.28 *a*) involve only the generators (2.26 *b*) and (2.26 *c*) of the euclidean group which do not contain the functions μ_a^i and ρ_a^i and therefore they do not bring in any restrictions to the dynamical system (2.1). On the other hand a simple calculation shows that the commutation relations (2.28 *b*) and (2.28 *c*) are equivalent to the following conditions on the functions μ_a^i and ρ_a^i:

$$\mathscr{L}(\vec{\mathrm{P}}_j)\mu_a^i = 0\,, \qquad \mathscr{L}(\vec{\mathrm{P}}_j)\rho_a^i = 0 \tag{2.29 a}$$

$$\mathscr{L}(\vec{\mathrm{J}}_j)\mu_a^i = \eta_j{}^i{}_l\mu_a^l\,, \qquad \mathscr{L}(\vec{\mathrm{J}}_j)\rho_a^i = \eta_j{}^i{}_l\rho_a^l \tag{2.29 b}$$

$$\begin{aligned} &\mathscr{L}(\vec{\mathrm{K}}_j)\mu_a^i - \frac{1}{c^2}x_{aj}\mathscr{L}(\vec{\mathrm{H}})\mu_a^i = \frac{1}{c^2}(2v_{aj}\mu_a^i + v_a^i\mu_{aj}) \\ &\mathscr{L}(\vec{\mathrm{K}}_j)\rho_a^i - \frac{1}{c^2}x_{aj}\mathscr{L}(\vec{\mathrm{H}})\rho_a^i = -\frac{1}{c^2}\varkappa_a\left\{(v_{al}\rho_a^l + \alpha_{al}\mu_a^l)\delta_j^i - v_a^i\rho_{aj} - \alpha_{aj}\mu_a^i\right\} \\ &\qquad + \frac{1}{c^2}v_{aj}\rho_a^i - \frac{1}{c^2}\varkappa_a^2\left(1 - \frac{v_a^2}{c^2}\right)^{-1/2} v_{ak}\mu_a^k(\alpha_{al}v_a^l\delta_j^i - \alpha_{aj}v_a^i)\,, \end{aligned} \tag{2.29 c}$$

where $\mathscr{L}(\ )$ is the Lie derivative operator. These equations generalize the Currie-Hill [7] equations to which they reduce when the particles have no spin. Their interpretation is the following:

i) The equations (2.29 *a*) express the invariance of the functions μ_a^i and ρ_a^i under the sub-group of spatial translations, and therefore these functions will depend on v_a^i, α_a^i and the relative positions.

ii) Eq. (2.29 *b*) tell us that μ_a^i and ρ_a^i behave like vectors under the rotation sub-group; therefore these functions have to be vector functions of vector arguments.

iii) Eq. (2.29 *c*) do not have a simple interpretation. They are related to the subset of pure Lorentz transformations (This subset is not a subgroup as it can be seen from the consideration of the third bracket in eq. (2.28 *c*)) and to the time evolution sub-group generated by $\vec{\mathrm{H}}$. It must be realized that taking into account the explicit expressions of $\vec{\mathrm{H}}$ and $\vec{\mathrm{K}}_i$ these eq. (2.29 *c*) are a system of *non linear* partial differential equations.

The preceding conclusions can be summarized by stating the following theorem:

THEOREM 2.2. — « If the dynamical system (2.1) is a P. I. P. S. then the functions μ_a^i and ρ_a^i are a solution of the system of differential equations (2.23) and (2.29) ».

Using the theory of groups of transformations it is possible to prove that reciprocally if μ_a^i and ρ_a^i are solutions of eqs. (2.23) and (2.29), then the corresponding dynamical system (2.1) is a P. I. P. S. The proof of this statement is rather long but almost identical to the proof of the corresponding statement for systems of particles without spin [*13*].

To finish this section we make the following remarks:

Remark 2.1. — If we consider a system with a single particle (N = 1) then eqs. (2.23) and (2.29) lead immediately to the conclusion that $\mu^i = \rho^i = 0$. This proves the « Principle of Inertia » for particles with spin.

Remark 2.2. — If in eq. (2.29) we take the limit $c \to \infty$ we obtain the conditions which express the invariance under the Galileo group. At this limit the functions μ^i_a and ρ^i_a are uncoupled.

3. COVARIANT FORMALISM

As it is well know from the study of Isolated Systems of point particles without spin, the Manifestly Predictive Formalism which we have used in the preceding section in dealing with systems of particles with spin, it is not always the more appropriate from a technical point of view. In this section we extend the *Covariant Formalism* which has already been extensively used in Predictive Relativistic Mechanics to our more general problem. It will be divided in three sub-sections. The first one contains an analysis of the Principle of Predictivity in this formalism. In the second one we shall study the implication of the invariance under the Poincaré group. The third sub-section will be devoted to establish the connections between the Manifestly Predictive Formalism and the Manifestly Invariant one.

A. — Let us consider a family of predictive systems of type (2.1) satisfying the constraints (2.3), parametrized by the N masses m_a of the particles:

$$\left\{\begin{aligned} &\frac{dx^i_a}{dt} = v^i_a \\ &\frac{dv^i_a}{dt} = \mu^i_a(m_e\,;\,t\,;\,x^j_b,\,v^k_c,\,\alpha^l_d) \\ &\frac{d\alpha^i_a}{dt} = \rho^i_a(m_e\,;\,t\,;\,x^j_b,\,v^k_c,\,\alpha^l_d)\,, \qquad \alpha^i_a\rho_{ai} = 0 \quad (\alpha^i_a\alpha_{ai} = 1) \end{aligned}\right. \tag{3.1}$$

and let us write its general solution as follows:

$$x^i_a = \varphi^i_a(m_e\,;\,x_0,\,v_0,\,\alpha_0,\,t_0\,;\,t) \tag{3.2 a}$$

$$\alpha^i_a = \psi^i_a(m_e\,;\,x_0,\,v_0,\,\alpha_0,\,t_0\,;\,t)\,, \tag{3.2 b}$$

where $x_0 = (x^i_{a0})$, $v_0 = (v^i_{a0})$ and $\alpha_0 = (\alpha^i_{a0})$ are the initial conditions for $t = t_0$, and therefore [14]:

$$\varphi^i_a(m_e\,;\,x_0,\,v_0,\,\alpha_0,\,t_0\,;\,t_0) \equiv x^i_{a0} \tag{3.3 a}$$

$$\dot\varphi^i_a(m_e\,;\,x_0,\,v_0,\,\alpha_0,\,t_0\,;\,t_0) \equiv v^i_{a0}\,, \qquad \left(\dot\varphi \equiv \frac{d\varphi}{dt}\right) \tag{3.3 b}$$

$$\psi^i_a(m_e\,;\,x_0,\,v_0,\,\alpha_0,\,t_0\,;\,t_0) \equiv \alpha^i_{a0}\,, \qquad (\alpha^i_{a0}\alpha_{ai0} = 1) \tag{3.3 c}$$

We shall keep explicit the dependence on the masses because they will play an important role in what follows.

Let us consider a system of first order ordinary differential equations defined on $[T(\mathfrak{M}_4) \times M_4]^N$ and having the following structure:

$$\begin{cases} \dfrac{dx_a^\lambda}{d\tau} = \pi_a^\lambda \\ \dfrac{d\pi_a^\lambda}{d\tau} = \theta_a^\lambda(x_b^\mu, \pi_c^\nu, \gamma_d^\rho) \\ \dfrac{d\gamma_a^\lambda}{d\tau} = \Delta_a^\lambda(x_b^\mu, \pi_c^\nu, \gamma_d^\rho), \end{cases} \tag{3.4}$$

where (x_a^λ) represent N points of Minkowski affine space-time $\mathfrak{M}_4$ and where $(\pi_a^\lambda, \gamma_c^\nu)$ represent 2N tangent vectors at the corresponding point. We shall assume that the functions θ_a^λ and Δ_a^λ satisfy the following relations:

$$\pi_{a\rho}\Delta_a^\rho + \gamma_{a\rho}\theta_a^\rho = 0 \tag{3.5 a}$$

$$\gamma_{a\rho}\Delta_a^\rho = 0, \tag{3.5 b}$$

which mean that the 2N quantities $(\pi_a\gamma_a) \equiv \pi_a^\rho\gamma_{a\rho}$ and $\gamma_a^2 = \gamma_a^\rho\gamma_{a\rho}$ are first integrals of the system of equation (3.4), to which we shall always assign the values:

$$(\pi_a\gamma_a) = 0, \qquad \gamma_a^2 = 1. \tag{3.6}$$

This is equivalent to saying that we shall restrict the co-phase space, i. e., the space of initial conditions, of system (3.4) to be the hypersurface $\Sigma \subset [T(\mathfrak{M}_4) \times M_4]^N$ defined by equation (3.6). Let us notice that the dimension of Σ are to 10N.

Let us write now the general solution of the differential system (3.4) taking into account the constraints (3.5) and (3.6):

$$\begin{cases} x_a^\lambda = \Phi_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; \tau) & (3.7\ a) \\ \gamma_a^\lambda = \Psi_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; \tau), & (3.7\ b) \end{cases}$$

where $\tilde{x} = (\tilde{x}_a^\lambda)$, $\tilde{\pi} = (\tilde{\pi}_a^\lambda)$ and $\tilde{\gamma} = (\tilde{\gamma}_a^\lambda)$ are the initial conditions for $\tau = 0$, and therefore:

$$\begin{cases} \Phi_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; 0) \equiv \tilde{x}_a^\lambda & (3.8\ a) \\ \dot{\Phi}_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; 0) \equiv \tilde{\pi}_a^\lambda, \qquad \left(\dot{\Phi} \equiv \dfrac{d\Phi}{d\tau}\right) & (3.8\ b) \\ \Psi_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; 0) \equiv \tilde{\gamma}_a^\lambda & (3.8\ c) \end{cases}$$

Moreover:

$$(\tilde{\pi}_a\tilde{\gamma}_a) = 0, \qquad \tilde{\gamma}_a^2 = 1. \tag{3.9}$$

In the sense of the terminology of section 2 we can say that the general solution (3.2) associated with the family of predictive systems (3.1) defines a family $\mathscr{F}^{\flat}$ (parametrized by the masses and the initial conditions) of sets of N *curves with spin* on $\mathfrak{M}_4$. Moreover, the general solution (3.7) of eqs. (3.4)-(3.6) represent a family $\mathscr{F}^{\sharp}$ (parametrised by the initial conditions) of sets of N *parametric* curves on which there is defined an orthogonal field of unit vectors along them.

The purpose of this sub-section is to develop a formalism into which the set (3.1) of predictive systems with N parameters would be described by a system of equations of the type (3.4)-(3.6). As it is obvious, this program requires that the geometrical support of the family $\mathscr{F}^{\sharp}$ coincides with the family $\mathscr{F}^{\flat}$. Let us consider the following definition.

DÉFINITION 3.1. — « We shall say that the system of equations (3.4)-3.6) is a *Lift* of the family of predictive systems (3.1) if $\forall(\tilde{x}, \tilde{\pi}, \tilde{\gamma}) \in \Sigma$ we have:

$$\begin{cases} \Phi_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\,;\, \tau) = \varphi_a^i(m_e\,;\, x_0, v_0, \alpha_0, t_0\,;\, t_a) & (3.10\ a) \\ \Phi_a^0(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\,;\, \tau) = c.t_a & (3.10\ b) \end{cases}$$

$$\Psi_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\,;\, \tau) - \frac{\Psi_a^0(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\,;\, \tau)}{m_a c + \dot{\Phi}_a^0(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\,;\, \tau)} \dot{\Phi}_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\,;\, \tau) = \psi_a^i(m_e\,;\, x_0, v_0, \alpha_0, t_0\,;\, t_a)\,, \quad (3.10\ c)$$

where:

i) the parameters m_a are defined as follows [*15*]:

$$m_a \equiv \frac{1}{c}\{ -\tilde{\pi}_a^{\lambda}\tilde{\pi}_{a\lambda} \}^{1/2} \quad (3.11)$$

ii) the initial conditions $(x_{a0}^i, v_{b0}^j, \alpha_{c0}^k)$ are implicitly defined by the equations:

$$\tilde{x}_a^i = \varphi_a^i\left(m_e\,;\, x_0, v_0, \alpha_0, t_0\,;\, \frac{1}{c}\tilde{x}_a^0\right) \quad (3.12\ a)$$

$$(\tilde{\pi}_a^0)^{-1}\tilde{\pi}_a^i = \frac{1}{c}\dot{\varphi}_a^i\left(m_e\,;\, x_0, v_0, \alpha_0, t_0\,;\, \frac{1}{c}\tilde{x}_a^0\right) \quad (3.12\ b)$$

$$\tilde{\gamma}_a^i - \frac{\tilde{\gamma}_a^0}{m_a c + \tilde{\pi}_a^0}\tilde{\pi}_a^i = \psi_a^i\left(m_e\,;\, x_0, v_0, \alpha_0, t_0\,;\, \frac{1}{c}\tilde{x}_a^0\right) \quad (3.12\ c)$$

which are a consequence of eq. (3.10).

iii) finally, t_a are N functions of τ implicitly defined for each a by the equation:

$$\tau = m_a^{-1}\int_{\frac{1}{c}\tilde{x}_a^0}^{t_a} \left\{ 1 - \frac{1}{c^2}\dot{\varphi}_a^i \cdot \dot{\varphi}_{ai}(m_e\,;\, x_0, v_0, \alpha_0, t_0\,;\, t) \right\}^{1/2} dt\,. \text{ »} \quad (3.13)$$

Let us notice first that the relations (3.10 *c*) and (3.12 *c*) are consistent with the constraints $\alpha_a^i\alpha_{ai}$. In fact, the consistency of (3.12 *c*) follows trivially from eqs. (3.9) and from the identity (3.11). The consistency of (3.10 *c*) is a direct consequence of eq. (3.6) and of the following lemma:

LEMMA 3.1. — « If the system of equations (3.4)-(3.6) is a Lift of (3.1), then:

$$\dot{\Phi}_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau)\cdot\dot{\Phi}_{a\lambda}(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau) = \tilde{\pi}_a^\lambda\tilde{\pi}_{a\lambda} \equiv -m_a^2c^2 , \qquad (3.14)$$

i. e., the N quantities $\pi_a^2 = -\pi_a^\lambda\pi_{a\lambda}$ are first integrals of the systems ».

Proof. — From eq. (3.13) it follows that:

$$\frac{dt_a}{d\tau} = m_a\left\{1 - \frac{1}{c^2}\dot{\varphi}_a^i\cdot\dot{\varphi}_{ai}(m_e ; x_0, v_0, \alpha_0, t_0 ; t_a)\right\}^{-1/2} \qquad (3.15)$$

and therefore considering the derivatives with respect to τ of both members of eqs. (3.10 *a*) and (3.10 *b*) we have:

$$\dot{\Phi}_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau) = m_a\cdot\left(1 - \frac{1}{c^2}\dot{\varphi}_a^j\dot{\varphi}_{aj}\right)^{-1/2}\cdot\dot{\varphi}_a^i(m_e ; x_0, v_0, \alpha_0, t_0 ; t_a) \quad (3.16\ a)$$

$$\dot{\Phi}_a^0(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau) = m_ac\left\{1 - \frac{1}{c^2}\dot{\varphi}_a^j\cdot\dot{\varphi}_{aj}(m_e ; x_0, v_0, \alpha_0, t_0 ; t_a)\right\}^{-1/2}, \quad (3.16\ b)$$

where from we can derive immediately eq. (3.14).

Let us notice also that taking into account the positivity of the integrand of eq. (3.13), the condition $\tau = 0$ implies $ct_a = \tilde{x}_a^0$. Therefore eqs. (3.10) and (3.16), together with the constraints (3.6), are consistent with the identities (3.8).

Let us prove now a lemma which will play an important role.

LEMMA 3.2. — « If the differential system (3.4)-(3.6) is a Lift of the family of predictive systems (3.1), then we have:

$$\Phi_a^\lambda[\Phi_b^\mu(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_b), \dot{\Phi}_c^\nu(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_c), \Psi_d^\rho(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_d) ; \tau] = \Phi_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a+\tau) \quad (3.17\ a)$$

$$\dot{\Phi}_a^\lambda[\Phi_b^\mu(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_b), \dot{\Phi}_c^\nu(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_c), \Psi_d^\rho(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_d) ; \tau] = \dot{\Phi}_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a+\tau) \quad (3.17\ b)$$

$$\Psi_a^\lambda[\Phi_b^\mu(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_b), \dot{\Phi}_c^\nu(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_c), \Psi_d^\rho(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_d) ; \tau] = \Psi_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a+\tau), \quad (3.17\ c)$$

i. e., its general solution can be interpreted as an N-parameter abelian group of transformations of the co-phase space ».

Proof. — Calling *projection* of $(\tilde{x}, \tilde{\pi}, \tilde{\gamma})$ to the functions (x_0, v_0, α_0) defined by equation (3.12) we can prove that the projection of

$$[\Phi_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a), \Phi_b(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_b), \Psi_c(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_c)]$$

is independent of the values of τ_d, and therefore this projection is (x_0, v_0, α_0).

In fact, using the notation $(\hat{x}_0, \hat{v}_0, \hat{\alpha}_0)$ for the projection and taking into account the lemma 3.1, we have by definition:

$$\Phi_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a) = \varphi_a^i\left(m_e ; \hat{x}_0, \hat{v}_0, \hat{\alpha}_0, t_0 ; \frac{1}{c}\Phi_a^0\right) \quad (3.18\,a)$$

$$(\dot{\Phi}_a^0)^{-1} \cdot \dot{\Phi}_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a) = \frac{1}{c}\dot{\varphi}_a^i\left(m_e ; \hat{x}_0, \hat{v}_0, \hat{\alpha}_0, t_0 ; \frac{1}{c}\Phi_a^0\right) \quad (3.18\,b)$$

$$\Psi_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a) - \frac{\Psi_a^0(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a)}{m_a c + \Phi_a^0(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a)}\dot{\Phi}_a^i(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_a) = \psi_a^i\left(m_e ; \hat{x}, \hat{v}_0, \hat{\alpha}_0, t_0 ; \frac{1}{c}\Phi_a^0\right); \quad (3.18\,c)$$

considering now eqs. (3.10) and (3.16) we see that the 1-h-t's of these equations are equal to the r-h-t's after substituting $(\hat{x}_0, \hat{v}_0, \hat{\alpha}_0)$ by (x_0, v_0, α_0); therefore $(\hat{x}_0, \hat{v}_0, \hat{\alpha}_0) = (x_0, v_0, \alpha_0)$.

Using this result and the lemma (3.1) eqs. (3.10 *a*), (3.10 *b*) and (3.13) lead to the following relations:

$$\Phi_a^i[\Phi_b(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_b), \dot{\Phi}_c(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_c), \Psi_d(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_d) ; \tau] = \varphi_a^i(m_e ; x_0, v_0, \alpha_0, t_0 ; l_a) \quad (3.19\,a)$$

$$\Phi_a^0[\Phi_b(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_b), \dot{\Phi}_c(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_c), \Psi_d(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau_d) ; \tau] = c \cdot l_a, \quad (3.19\,b)$$

where the functions $l_a(\tau)$ are implicitly defined for each a by the equation:

$$\tau = m_a^{-1}\int_{\frac{1}{c}\Phi_a^0(\tilde{x},\tilde{\pi},\tilde{\gamma};\,\tau_a)}^{l_a}\left\{1 - \frac{1}{c^2}\dot{\varphi}_a^i \cdot \dot{\varphi}_{ai}(m_e ; x_0, v_0, \alpha_0, t_0 ; t)\right\}^{1/2} dt. \quad (3.20)$$

Now, from (3.13) and (3.10 *b*) we have:

$$\tau_a = m_a^{-1}\int_{\frac{1}{c}\tilde{x}_a^0}^{\frac{1}{c}\Phi_a^0(\tilde{x},\tilde{\pi},\tilde{\gamma};\,\tau_a)}\left\{1 - \frac{1}{c^2}\dot{\varphi}_a^i \cdot \dot{\varphi}_{ai}(m_e ; x_0, v_0, \alpha_0, t_0 ; t)\right\}^{1/2} dt, \quad (3.21)$$

and this expression added to (3.20) leads to:

$$\tau_a + \tau = m_a^{-1}\int_{\frac{1}{c}\tilde{x}_a^0}^{l_a}\left\{1 - \frac{1}{c^2}\dot{\varphi}_a^i \cdot \dot{\varphi}_{ai}(m_e ; x_0, v_0, \alpha_0, t_0 ; t)\right\}^{1/2} dt, \quad (3.22)$$

and therefore the r-h-t's of equation (3.19) are equal to the 1-h-t's of equations (3.10 *a*) and (3.10 *b*) after substituting τ for $\tau_a + \tau$. This proves the eq. (3.17 *a*). To prove eq. (3.17 *b*) it is sufficient to derive preceding equations with respect to τ. Using eq. (3.10 *c*) and taking into account the constraints (3.6) and (3.9) we can prove eq. (3.17 *c*) by a similar calculation to that which we used to prove eq. (3.17 *a*).

An important conclusion which can be derived from the preceding lemma is the following theorem:

THEOREM 3.1. — « Each family of predictive systems (3.1) possesses a Lift and this Lift is unique ».

Proof. — Taking $\tau_a = \tau$, $\forall a$, eq. (3.17) tell us that the functions $\Phi_a^\lambda(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; \tau)$, $\Phi_b^\mu(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; \tau)$ and $\Psi_c^\nu(\tilde{x}, \tilde{\pi}, \tilde{\gamma}; \tau)$, defined by eqs. (3.10) and (3.6), can be interpreted as defining a one parameter group of transformations of the hypersurface Σ which we defined above. Therefore these functions can be identified as the general solution of an autonomous system of differential equations of the type (3.4)-(3.6); this system is associated with the infinitesimal generator of the group.

The lemmas 3.1 and 3.2 state two properties of the Lift of any family of predictive systems depending on N parameters. As we shall see these two properties characterize a Lift. This suggests the following definition:

DEFINITION 3.2. — « We shall say that an autonomous system of differential equations of the type (3.4)-(3.6) is a *Projectable System* if his general solution satisfies the properties (3.14) and (3.17) ».

This definition is useful in connection with the following theorem:

THEOREM 3.2. — « The necessary and sufficient conditions that the functions θ_a^λ and Δ_a^λ of a system of differential equations of the type (3.4)-(3,6) must satisfy to be a Projectable System are the following:

$$\pi_{a\rho}\theta_a^\rho = 0 \tag{3.23}$$

$$\left\{\begin{aligned} &\pi_{a'}^\rho \frac{\partial \theta_a^\lambda}{\partial x^{a'\rho}} + \theta_a^{\rho'} \frac{\partial \theta_a^\lambda}{\partial \pi^{a'\rho}} + \Delta_{a'}^\lambda \frac{\partial \theta_a^\lambda}{\partial \gamma^{a'\rho}} = 0 &\quad (3.24\,a)\\ &\pi_{a'}^\rho \frac{\partial \Delta_a^\lambda}{\partial x^{a'\rho}} + \theta_{a'}^\rho \frac{\partial \Delta_a^\lambda}{\partial \pi^{a'\rho}} + \Delta_{a'}^\rho \frac{\partial \Delta_a^\lambda}{\partial \gamma^{a'\rho}} = 0\,, &\quad (3.24\,b)\end{aligned}\right.$$

where, as their position indicates, there is *no* sommation on the indices a or $a' \neq a$ ».

Proof. — Let us prove first that the conditions are necessary. Considering the derivatives with respect to τ of eqs. (3.14), (3.17 *b*) and (3.17 *c*) and taking $\tau = 0$ we get the following results (dropping the twiddles):

$$\pi_{a\rho} \cdot \theta_a^\rho(x, \pi, \gamma) = 0 \tag{3.25}$$

$$\theta_a^\lambda[\Phi_b^\mu(x, \pi, \gamma; \tau_b), \dot{\Phi}_c^\nu(x, \pi, \gamma; \tau_c), \Psi_d^\rho(x, \pi, \gamma; \tau_d)] = \ddot{\Phi}_a^\lambda(x, \pi, \gamma; \tau_a) \tag{3.26 a}$$

$$\Delta_a^\lambda[\Phi_b^\mu(x, \pi, \gamma; \tau_b), \dot{\Phi}_c^\nu(x, \pi, \gamma; \tau_c), \Psi_d^\rho(x, \pi, \gamma; \tau_d)] = \dot{\Psi}_a^\lambda(x, \pi, \gamma; \tau_a)\,. \tag{3.26 b}$$

Eq. (3.25) is identical to eq. (3.23). Moreover, considering the derivatives of eqs. (3.26) with respect to $\tau_{a'}$ $(a' \neq a)$ and taking the values $\tau_c = 0$, we obtain immediately eqs. (3.24).

Let us prove now that the conditions are sufficient. Eq. (3.14) follow immediately from eq. (3.23) because these equations can be written:

$$\frac{d}{d\tau}(\pi_{a\rho}\pi_a^\rho) = 0\,. \tag{3.27}$$

To prove the sufficiency of eq. (3.24), let us consider the following system of partial differential equations:

$$\frac{\partial x_a^\lambda}{\partial \tau^b} = \delta_{ab}\pi_a^\lambda \tag{3.28 a}$$

$$\frac{\partial \pi_a^\lambda}{\partial \tau^b} = \delta_{ab}\cdot\theta_a^\lambda(x, \pi, \gamma) \tag{3.28 b}$$

$$\frac{\partial \gamma_a^\lambda}{\partial \tau^b} = \delta_{ab}\cdot\Delta_a^\lambda(x, \pi, \gamma)\,, \tag{3.28 c}$$

where we assume that the functions θ_a^λ and Δ_a^λ satisfy the constraints (3.5). As a consequence of eq. (3.24) this system of equations is completely integrable. It follows then from its particular structure and the theory of groups of transformations *[12]* that its general solution associated with initial conditions $(\tilde{x}, \tilde{\pi}, \tilde{\gamma})$ satisfying the constraints (3.9) at $\tau^c = 0$ defines an abelian group of transformations of Σ depending on N parameters. Moreover it is evident that the one parameter sub-group defined by $\tau^a = \tau$ can be identified with the general solution of the system of equations (3.4)-(3.6). Therefore eq. (3.17) will be satisfied and this completes the proof of the theorem.

We have seen up to now that to each family of predictive systems depending on N parameters can be unambiguously associated with a Projectable System, e. g., its Lift. We shall see now that reciprocally every Projectable System can be associated with a family of predictive systems depending on N parameters which Lift coincides with it. This result will complete the proof of the equivalence of both concepts.

DEFINITION 3.3. — « We shall call *Projection* of a Projectable System the family of Predictive Systems depending on N parameters which general solution associated with initial conditions (x_0, v_0, α_0) for each $t = t_0$ is the following:

$$\varphi_a^i(m_e\,; x_0, v_0, \alpha_0, t_0\,; t) \equiv \Phi_a^i(\bar{x}, \bar{\pi}, \bar{\gamma}\,; \tau_a) \tag{3.29 a}$$

$$\psi_a^i(m_e\,; x_0, v_0, \alpha_0, t_0\,; t) \equiv \Psi_a^i(\bar{x}, \bar{\pi}, \bar{\gamma}\,; \tau_a) - \frac{\Psi_a^0(\bar{x}, \bar{\pi}, \bar{\gamma}\,; \tau_a)}{m_a c + \dot{\Phi}_a^0(\bar{x}, \bar{\pi}, \bar{\gamma}\,; \tau_a)}\dot{\Phi}_a^i(\bar{x}, \bar{\pi}, \bar{\gamma}\,; \tau_a) \tag{3.29 b}$$

where :

i) the functions Φ_a^λ and Ψ_a^λ are the general solution of the Projectable System which we are considering, *ii*) the initial conditions $(\bar{x}, \bar{\pi}, \bar{\gamma})$ take the values:

$$\bar{x}_b^0 = ct_0 , \qquad \bar{x}_b^i = x_{bo}^i \tag{3.30 a}$$

$$\bar{\pi}_b^0 = m_b c(1 - v_{bo}^2/c^2)^{-1/2} , \qquad \bar{\pi}_b^i = m_b v_{bo}^i(1 - v_{bo}^2/c^2)^{-1/2} \tag{3.30 b}$$

$$\bar{\gamma}_b^0 = \frac{1}{m_b c}\bar{\pi}_{bi}\alpha_{bo}^i , \qquad \bar{\gamma}_b^i = \left\{ \delta_j^i + \frac{\bar{\pi}_b^i \bar{\pi}_{bj}}{m_b c(m_b c + \bar{\pi}_b^0)} \right\} \alpha_{bo}^j \tag{3.30 c}$$

iii) finally, τ_a are defined for each value of a by the equation:

$$ct = \Phi_a^0(\bar{x}, \bar{\pi}, \bar{\gamma} ; \tau_a) \text{ ».} \tag{3.31}$$

Let us notice that, effectively, for $t = t_0$ the r-h-t's of eq. (3.29) and the r-h-t of the derivative with respect to t of eq. (3.29 *a*) take the values (x_0, v_0, α_0). Let us derive now the expressions of the corresponding functions μ_a^i and ρ_a^i in terms of the functions θ_a^λ and Δ_a^λ which define the Projectable System which we are considering. Deriving twice with respect to the variable time eq. (3.29 *a*) and once eq. (3.29 *b*) and considering the value of the resulting expressions for $t = t_0$ we obtain taking into account eqs. (3.30) and (3.31) (We drop the sub-indices zero):

$$\mu_a^i(m_e ; t ; x, v, \alpha) = m_a^{-2}\left(1 - \frac{v_a^2}{c^2}\right)\left(\bar{\theta}_a^i - \frac{v_a^i}{c}\bar{\theta}_a^0\right) \tag{3.32 a}$$

$$\rho_a^i(m_e ; t ; x, v, \alpha) = m_a^{-1}h_a^{-1}\left\{ \bar{\Delta}_a^i - \frac{1}{c}m_a^{-1}(1 + h_a)^{-1}\bar{\gamma}_a^0\bar{\theta}_a^i - \frac{1}{c}h_a(1 + h_a)^{-1}v_a^i\left[\bar{\Delta}_a^0 - \frac{1}{c}m_a^{-1}(1 + h_a)^{-1}\bar{\gamma}_a^0\bar{\theta}_a^0\right]\right\}, \tag{3.32 b}$$

where:

$$h_a \equiv +\left(1 - v_a^2/c^2\right)^{-1/2} \tag{3.33}$$

and where $\bar{\theta}_a^\lambda$ and $\bar{\Delta}_a^\lambda$ mean the values of the functions θ_a^λ and Δ_a^λ when their arguments take the values (3.30).

Let us prove now a theorem which guarantees the equivalence which we mention before:

THEOREM 3.3. — « The Lift of the Projection of a Projectable System coincides with it ».

Proof. — Let us consider a Projectable System. The general solution of its Projection is defined in eqs. (3.29)-(3.31). We have to prove that the general solution of the initial system satisfies the conditions of definition 3.1, i. e., satisfies eqs. (3.10)-(3.13).

Let us choose admissible values of $(\tilde{x}, \tilde{\pi}, \tilde{\gamma})$, i. e., such that $\tilde{\pi}_a^0 > 0$, $\tilde{\pi}_a^\rho \tilde{\pi}_{a\rho} < 0$ and satisfying the constraints (3.9). Let us consider now the values $\hat{\tau}_a$, which for each a, are defined by the equation:

$$\bar{x}_a^0 \equiv \Phi_a^0(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \hat{\tau}_a) = ct_0 \tag{3.34}$$

and let us also define $(\bar{x}^i_a, \bar{\pi}^\lambda_b, \bar{\gamma}^\mu_c)$ as follows:

$$\left\{\begin{array}{l} \bar{x}^i_a \equiv \Phi^i_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \hat{\tau}_a) \\ \bar{\pi}^\lambda_a \equiv \dot{\Phi}^\lambda_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \hat{\tau}_a) \\ \bar{\gamma}^\lambda_a \equiv \Psi^\lambda_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \hat{\tau}_a)\,, \end{array}\right. \tag{3.35}$$

wherefrom according to the lemma 3.2 we shall have:

$$\Phi^\lambda_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau) = \Phi^\lambda_a(\bar{x}, \bar{\pi}, \bar{\gamma}\ ;\ -\ \hat{\tau}_a + \tau) \tag{3.36 a}$$

$$\Psi^\lambda_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau) = \Psi^\lambda_a(\bar{x}, \bar{\pi}, \bar{\gamma}\ ;\ -\ \hat{\tau}_a + \tau)\,. \tag{3.36 b}$$

Therefore, taking into account eqs. (3.29)-(3.31) which define the Projection of a System, we obtain:

$$\left\{\begin{array}{rl} \varphi^i_a(m_e\ ;\ x_0, v_0, \alpha_0, t_0\ ;\ t_a) = \Phi^i_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau) & (3.37\ a) \\ ct_a = \Phi^0_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau) & (3.37\ b) \end{array}\right.$$

$$\psi^i_a(m_e\ ;\ x_0, v_0, \alpha_0, t_0\ ;\ t_a) = \Psi^i_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau) - \frac{\Psi^0_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau)}{m_ac + \dot{\Phi}^0_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau)}\dot{\Phi}^i_a(\tilde{x}, \tilde{\pi}, \tilde{\gamma}\ ;\ \tau)\,, \tag{3.37 c}$$

and also:

$$\{-\ \bar{\pi}^\rho_a\bar{\pi}_{a\rho}\}^{1/2} = m_ac \tag{3.38}$$

$$x^i_{a0} = \bar{x}^i_a\,, \qquad v^i_{a0} = c\left(\bar{\pi}^0_a\right)^{-1}\bar{\pi}^i_a\,, \qquad \alpha^i_{a0} = \bar{\gamma}^i_a - \frac{\bar{\gamma}^0_a}{m_ac + \bar{\pi}^0_a}\bar{\pi}^i_a \tag{3.39}$$

$$ct_a = \Phi^0_a(\bar{x}, \bar{\pi}, \bar{\gamma}\ ;\ -\ \hat{\tau}_a + \tau) \tag{3.40}$$

The lemma 3.1 tells us that eqs. (3.38) and (3.11) are the same equation. Remembering the definition 3.1 we see that it remains only to be proved that τ and t_a are connected by eq. (3.13). To this end let us consider the following integral:

$$\mathrm{I}_a \equiv m_a^{-1}\int_{\frac{1}{c}\bar{x}^0_a}^{\Phi^0_a(\tilde{x},\tilde{\pi},\tilde{\gamma};\tau)}\left\{1 - \frac{1}{c^2}\dot{\varphi}^i_a\cdot\dot{\varphi}_{ai}(m_e\ ;\ x_0, v_0, \alpha_0, t_0\ ;\ t)\right\}^{1/2}dt\,, \tag{3.41}$$

which, taking into account (3.29 a) and (3.31), can be re-written as follows:

$$\mathrm{I}_a = m_a^{-1}\int_{\frac{1}{c}\bar{x}^0_a}^{\Phi^0_a(\bar{x},\bar{\pi},\bar{\gamma};\tau)}\{1 - (\dot{\Phi}^0_a)^{-2}\dot{\Phi}^i_a\cdot\dot{\Phi}_{ai}(\bar{x}, \bar{\pi}, \bar{\gamma}\ ;\ \tau')\}^{1/2}dt\,, \tag{3.42}$$

τ' being the function of t implicitly defined by:

$$ct = \Phi^0_a(\bar{x}, \bar{\pi}, \bar{\gamma}\ ;\ \tau')\,. \tag{3.43}$$

Using now τ' as variable of integration instead of t in eq. (3.42) we obtain immediately $\mathrm{I}_a = \tau$ because the integrand is m_a. This completes the proof of the theorem.

B. — Let us assume now that the Predictive Systems of the N parameter family (3.1) are Poincaré invariant in the sense of Section 2. Because of the geometrical meaning of this invariance, it is clear that the general solution of its Lift will satisfy the following equations:

$$\Phi_a^\lambda[L_\nu^\mu(\tilde{x}_b^\nu - A^\nu), L_\sigma^\rho\tilde{\pi}_c^\sigma, L_\beta^\alpha\tilde{\gamma}_d^\beta ; \tau] = L_\delta^\lambda\cdot\{\Phi_a^\delta(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau) - A^\delta\} \quad (3.44\ a)$$

$$\Psi_a^\lambda[L_\nu^\mu(\tilde{x}_b^\nu - A^\nu), L_\sigma^\rho\tilde{\pi}_c^\sigma, L_\beta^\alpha\tilde{\gamma}_d^\beta ; \tau] = L_\delta^\lambda\cdot\Psi_a^\delta(\tilde{x}, \tilde{\pi}, \tilde{\gamma} ; \tau), \quad (3.44\ b)$$

for each element (L_ν^μ, A^ρ) of the Poincaré group. This suggests the following definition:

DEFINITION 3.4. — « We shall say that a Projectable System is *Invariant* if its general solution satisfies eq. (3.44) ».

Let us prove now a theorem which connects this definition with a set of differential equations satisfied by the functions θ_a^λ and Δ_a^λ:

THEOREM 3.4. — « The necessary and sufficient conditions that the functions θ_a^λ and Δ_a^λ of a Projectable System satisfy when this system is Invariant are the following equations:

$$\varepsilon_b\frac{\partial\theta_a^\lambda}{\partial x_b^\rho} = 0, \qquad \varepsilon_b\frac{\partial\Delta_a^\lambda}{\partial x_b^\rho} = 0, \qquad (\varepsilon_b = 1 \quad \forall b) \quad (3.45\ a)$$

$$\left\{\begin{aligned}(\delta_\mu^\rho\eta_{\nu\sigma} - \delta_\nu^\rho\eta_{\mu\sigma})\left(x_b^\sigma\frac{\partial\theta_a^\lambda}{\partial x_b^\rho} + \pi_b^\sigma\frac{\partial\theta_a^\lambda}{\partial\pi_b^\rho} + \gamma_b^\sigma\frac{\partial\theta_a^\lambda}{\partial\gamma_b^\rho}\right) &= \delta_\mu^\lambda\theta_{a\nu} - \delta_\nu^\lambda\theta_{a\mu}\\ (\delta_\mu^\rho\eta_{\nu\sigma} - \delta_\nu^\rho\eta_{\mu\sigma})\left(x_b^\sigma\frac{\partial\Delta_a^\lambda}{\partial x_b^\rho} + \pi_b^\sigma\frac{\partial\Delta_a^\lambda}{\partial\pi_b^\rho} + \gamma_b^\sigma\frac{\partial\Delta_a^\lambda}{\partial\gamma_b^\rho}\right) &= \delta_\mu^\lambda\Delta_{a\nu} - \delta_\nu^\lambda\Delta_{a\mu}\,.\text{ »}\end{aligned}\right. \quad (3.45\ b)$$

Proof. — Let us see that these equations are necessary conditions. Considering twice the derivative with respect to τ of eq. (3.44 *a*) and once the derivative of eq. (3.34 *b*) and letting then $\tau = 0$, we get immediately (dropping the twiddles):

$$\theta_a^\lambda[L_\nu^\mu(x_b^\nu - A^\nu), L_\sigma^\rho\pi_c^\sigma, L_\beta^\alpha\gamma_d^\beta] = L_\delta^\lambda\cdot\theta_a^\lambda(x, \pi, \gamma) \quad (3.46\ a)$$

$$\Delta_a^\lambda[L_\nu^\mu(x_b^\nu - A^\nu), L_\sigma^\rho\pi_c^\sigma, L_\beta^\alpha\gamma_d^\beta] = L_\delta^\lambda\cdot\Delta(x, \pi, \gamma)\,. \quad (3.46\ b)$$

Taking now the derivatives with respect to A^ρ of these equations and putting then $L_\nu^\mu = \delta_\nu^\mu$ and $A^\rho = 0$, we get eq. (3.45 *a*). Let us consider moreover eq. (3.46) with $A^\mu = 0$ and let us write the Lorentz matrices in infinitesimal form:

$$L_\nu^\mu = \delta_\nu^\mu + \varepsilon_\nu^\mu, \qquad \varepsilon^{\lambda\mu} + \varepsilon^{\mu\lambda} = 0, \qquad \varepsilon^{\lambda\mu} \equiv \eta^{\lambda\nu}\varepsilon_\nu^\mu; \quad (3.47)$$

developping both members of eq. (3.46) in the neighbourhood of $\varepsilon_\nu^\mu = 0$ and keeping first order terms we easily obtain eq. (3.45 *b*).

The proof of the sufficiency of eq. (3.45) is harder to obtain but does not involve essential difficulties. Let us mention only that the first step of the proof consists in using eq. (3.45) to establish eq. (3.46). From them we obtain eq. (3.44) by using the theorem of existence and unicity of solutions of differential equations.

Let us consider now the Projection of an Invariant Projectable System. Considering the definition 3.3 and the lemma 3.2 it is clear that such a Projection is an N parameter family of P. I. P. S. This result, together with the theorem 3.3, allows the identification of an N-parameter family of P. I. P. S. with an Invariant Projectable System. We consider that the Invariant Projectable Systems are N-parameters families of P. I. P. S. described with a covariant formalism.

It must be noticed once again that, as it occurred in the case where the particles had no spin [*16*], the covariant formalism we just discussed is not the only possible one. For this reason it must be emphasized that from the physical point of view the important concept is the concept of P. I. P. S. and not that of Invariant Projectable System. Moreover it must be emphasized also that the concept of a single P. I. P. S. is a self-consistent one wether or not this system can be imbedded into a N-parameter family of P. I. P. S.

C. — The purpose of this third sub-section is to summarize the properties of the Invariant Projectable Systems in the language of differential geometry emphasizing the symmetries of the formalism.

Let us start by writting the infinitesimal generators of the N parameter abelian groups which is defined by the general solution of any Invariant Projectable System (lemma 3.2). Remembering the definition (2.25), these generators are:

$$\vec{\mathrm{H}}_a \equiv \delta_{ab}\left(\pi^\rho_b \frac{\partial}{\partial x^\rho_b} + \theta^\rho_b \frac{\partial}{\partial \pi^\rho_b} + \Delta^\rho_b \frac{\partial}{\partial \gamma^\rho_b}\right), \tag{3.48}$$

which, as we see, depend directly on the corresponding dynamical system and therefore they can be used to define it. The condition saying that the group is abelian can be written in terms of the vector field (3.48) as follows:

$$[\vec{\mathrm{H}}_a, \vec{\mathrm{H}}_b] = 0\,, \tag{3.49}$$

equations, which as it can be easily checked, are strictly equivalent to eq. (3.24).

Let us consider now the natural action of the Poincaré group on the space $[\mathrm{T}(\mathfrak{M}_4) \times \mathrm{M}_4]^{\mathrm{N}}$, i. e.:

$$\begin{cases} x^\lambda_a \rightarrow \mathrm{L}^\lambda_\mu(x^\mu_a - \mathrm{A}^\mu) \\ \pi^\lambda_a \rightarrow \mathrm{L}^\lambda_\mu \pi^\mu_a \\ \gamma^\lambda_a \rightarrow \mathrm{L}^\lambda_\mu \gamma^\mu_a\,, \end{cases} \tag{3.50}$$

which infinitesimal generators are the ten vector fields:

$$\vec{P}_\mu \equiv -\varepsilon_b \frac{\partial}{\partial x_b^\mu} \tag{3.51 a}$$

$$\vec{J}_{\lambda\mu} \equiv (\delta_\lambda^\rho \eta_{\mu\sigma} - \delta_\mu^\rho \eta_{\lambda\sigma})\left(x_b^\sigma \frac{\partial}{\partial x_b^\rho} + \pi_b^\sigma \frac{\partial}{\partial \pi_b^\rho} + \gamma_b^\sigma \frac{\partial}{\partial \gamma_b^\rho} \right), \tag{3.51 b}$$

where $\vec{P}_\mu$ are the generators of space-time translations and where $\vec{J}_{\lambda\mu} = -\vec{J}_{\mu\lambda}$ are the generators of the Lorentz group. Using these vectors as a basis of the Lie algebra of the Poincaré group this Lie algebra is characterized by the following Lie Brackets:

$$[\vec{P}_\lambda, \vec{P}_\mu] = 0 \tag{3.52 a}$$

$$[\vec{P}_\nu, \vec{J}_{\lambda\mu}] = \eta_{\nu\mu}\vec{P}_\lambda - \eta_{\nu\lambda}\vec{P}_\mu \tag{3.52 b}$$

$$[\vec{J}_{\lambda\mu}, \vec{J}_{\nu\sigma}] = \eta_{\lambda\nu}\vec{J}_{\mu\sigma} + \eta_{\mu\sigma}\vec{J}_{\lambda\nu} - \eta_{\lambda\sigma}\vec{J}_{\mu\nu} - \eta_{\mu\nu}\vec{J}_{\lambda\sigma}\,. \tag{3.52 c}$$

From the definitions (3.51) it is easy to see that the eq. (3.45), which expressed the invariance of the dynamical system with respect to the Poincaré group, can be written as follows:

$$\mathscr{L}(\vec{P}_\mu)\theta_a^\lambda = 0\,, \qquad \mathscr{L}(\vec{P}_\mu)\Delta_a^\lambda = 0 \tag{3.53 a}$$

$$\begin{cases} \mathscr{L}(\vec{J}_{\lambda\mu})\theta_a^\nu = (\delta_\lambda^\nu \eta_{\mu\rho} - \delta_\mu^\nu \eta_{\lambda\rho})\theta_a^\rho \\ \mathscr{L}(\vec{J}_{\lambda\mu})\Delta_a^\nu = (\delta_\lambda^\nu \eta_{\mu\rho} - \delta_\mu^\nu \eta_{\lambda\rho})\Delta_a^\rho \end{cases} \tag{3.53 b}$$

or also, taking into account the definitions (3.48), in the more compact form:

$$[\vec{P}_\mu, \vec{H}_a] = 0\,, \qquad [\vec{J}_{\lambda\mu}, \vec{H}_a] = 0\,. \tag{3.54}$$

Eqs. (3.49), (3.52) and (3.54) express that the vector fields H_a, $\vec{P}_\mu$ and $\vec{J}_{\lambda\mu}$ generate an abelian extension of the Lie algebra of the Poincaré group. We shall say that the group generated by this extended Lie algebra is the *Complete Group* of *Symmetries* of the dynamical System.

Moreover it is clear that eqs. (3.5) and (3.23) are equivalent to the following ones:

$$\mathscr{L}(\vec{H}_a)(\pi_{a\rho}\gamma_a^\rho) = 0 \tag{3.55 a}$$

$$\mathscr{L}(\vec{H}_a)(\gamma_{a\rho}\gamma_a^\rho) = 0\,, \tag{3.55 b}$$

$$\mathscr{L}(\vec{H}_a)(\pi_{a\rho}\pi_a^\rho) = 0\,, \tag{3.55 c}$$

which, as we knew already, express in a trivial manner that the 3N quantities $(\pi_a\gamma_a)$, γ_a^2 and $m_a^2c^2 = -\pi_a^\rho\pi_{a\rho}$ are first integrals. As we said already we assign to them the values (3.6) and we shall interpret m_a as being the masses of the particles. Let us notice that from the structure of the constraints (3.6) and from eqs. (3.55 *a*) and (3.55 *b*) it follows that the vector fields H_a, $\vec{P}_\mu$ and $\vec{J}_{\lambda\mu}$ are effectively tangent, as it should, to the co-phase space Σ.

4. HAMILTONIAN FORMULATION

A. — Let W_{2n} be an even dimensional manifold; a symplectic form Ω on W_{2n} is an exterior 2-form such that:

$$\begin{cases} \text{rang}\,(\Omega) = \dim\,(W_{2n}) \equiv 2n & (4.1\,a) \\ d\Omega = 0\,, \quad (d : \text{exterior differential}) & (4.1\,b) \end{cases}$$

We list below some of the basic results and additional definitions which will be useful to us:

i) there always exist [17] local coordinates $\{ q^A, p_B \}$ (A, B, ... = 1, 2, ... n) of W_{2n} such that:

$$\Omega = dq^A \wedge dp_A\,, \qquad (\wedge : \text{exterior product})\,, \tag{4.2}$$

these coordinates are called canonical coordinates. The pairs (q^A, p_A) are called canonical mates.

ii) A transformation of W_{2n} is said to be canonical if it leaves Ω invariant. Or also, when we interpret it as a change of coordinates, if it transforms canonical coordinates into canonical coordinates.

iii) Given two functions F and G defined on W_{2n}, its Poisson bracket in the sense of Ω is by definition the function which using canonical coordinates has the following expression:

$$[F, G] \equiv \frac{\partial F}{\partial q^A}\frac{\partial G}{\partial p_A} - \frac{\partial G}{\partial q^A}\frac{\partial F}{\partial p_A}\,. \tag{4.3}$$

iv) Let $\vec{\Lambda}$ be a vector field of W_{2n} which leaves invariant the symplectic form Ω, i. e.:

$$\mathscr{L}(\vec{\Lambda})\Omega = 0\,. \tag{4.4}$$

Then to $\vec{\Lambda}$ we can associate a function Λ, defined up to an additive constant, by the formula:

$$i(\vec{\Lambda})\Omega = -\,d\Lambda\,, \tag{4.5}$$

where $i(\ )$ is the interior product operator.

v) If $\vec{\Lambda}_1$, and $\vec{\Lambda}_2$ are two vector fields which satisfy eq. (4.4), then:

$$i([\vec{\Lambda}_1, \vec{\Lambda}_2])\Omega = -\,d[\Lambda_1, \Lambda_2]\,, \tag{4.6}$$

i. e., the function associated to the Lie bracket $[\vec{\Lambda}_1, \vec{\Lambda}_2]$ is, up to an additive constant, the Poisson bracket of the corresponding functions Λ_1 and Λ_2.

vi) Finally, if F is an arbitrary function of W_{2n} and $\vec{\Lambda}$ is a vector field which satisfies eq. (4.4), then it follows that:

$$\mathscr{L}(\vec{\Lambda})F = [\Lambda, F]\,. \tag{4.7}$$

Let us consider now an Invariant Projectable System, in the sense of Section 3, defined by the vector fields (3.48). As we did [5] when we considered the systems of particles with no spin, we shall say that our System admits a Compatible Hamiltonian Formulation (C. H. F.), if there exists a symplectic form Ω defined on co-phase space Σ which is left invariant by the Complete Group of Symmetries: or in other words if this group acts as a group of canonical transformations, i. e.:

$$\mathscr{L}(\vec{H}_a)\Omega = 0 \tag{4.8 a}$$

$$\mathscr{L}(\vec{P}_\mu)\Omega = 0 \tag{4.8 b}$$

$$\mathscr{L}(\vec{J}_{\lambda\mu})\Omega = 0 \,. \tag{4.8 c}$$

Assuming the existence of such Ω and using eq. (4.5) it is possible to associate to the N + 10 vector fields $\vec{H}_a$, $\vec{P}_\mu$ and $\vec{J}_{\lambda\mu}$, N + 10 corresponding functions H_a, P_μ, $J_{\lambda\mu}$ ($J_{\lambda\mu} = - J_{\mu\lambda}$) defined up to additive constants. Moreover, taking into account eq. (4.6) and eqs. (3.49), (3.52) and (3.54) the Poisson brackets of these functions will satisfy the following relations:

$$[H_a, H_b] = C_{ab}\,, \qquad [P_\mu, H_a] = C_{a\mu}\,, \qquad [J_{\lambda\mu}, H_a] = C_{a\lambda\mu} \tag{4.9 a}$$

$$\begin{aligned} [P_\lambda, P_\mu] = C_{\lambda\mu}\,, \qquad [P_\rho, J_{\lambda\mu}] &= \eta_{\rho\mu}P_\lambda - \eta_{\rho\lambda}P_\mu + C_{\rho,\lambda\mu} \\ [J_{\lambda\mu}, J_{\rho\sigma}] = \eta_{\lambda\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\lambda\rho} &- \eta_{\lambda\sigma}J_{\mu\rho} - \eta_{\mu\rho}J_{\lambda\sigma} + C_{\lambda\mu,\rho\sigma}\,, \end{aligned} \tag{4.9 b}$$

where C_{ab}, $C_{a\mu}$, $C_{a\lambda\mu}$, $C_{\rho\sigma}$, $C_{\rho,\lambda\mu}$ and $C_{\rho\sigma,\lambda\mu}$ are constants.

Using the properties of the Poisson brackets it is possible to prove [18] easily that there always exists a possible choice of the arbitrary constants involved in the expressions of the functions P_λ and $J_{\lambda\mu}$ such that for this choice the preceding constants of (4.9 *b*) be zero:

$$C_{\lambda\mu} = 0\,, \qquad C_{\rho,\lambda\mu} = 0\,, \qquad C_{\lambda\mu,\rho\sigma} = 0\,. \tag{4.10}$$

Therefore these functions generate a Poisson algebra with the structure of the Lie algebra of the Poincaré group. In what follows we shall use eq. (4.10) as supplementary conditions to define unambiguously the functions P_α and $J_{\lambda\mu}$, which will depend then only on the dynamical system and the symplectic form which we consider.

Moreover it is easy to prove [5] that necessarily:

$$C_{a\mu} = 0\,, \qquad C_{a\lambda\mu} = 0\,. \tag{4.11}$$

And finally if include among our assumptions the invariance of the dynamical system and Ω under the permutation group S_N, i.e., the invariance under an exchange of the numeration of the particles, then we can conclude [5] also that:

$$C_{ab} = 0\,. \tag{4.12}$$

Summarizing: To a C. H. F. of the dynamical system which we consider correspond N + 10 functions H_a, P_μ, $J_{\lambda\mu}$ which generate a Poisson algebra with identical structure to that of the Lie algebra of the Complete Symmetry Group. P_μ and $J_{\lambda\mu}$ which are unambiguously defined, are of course interpreted as the Energy-Momentum and the generalized Angular Momentum of the system. The functions H_a which remain undefined up to an additive arbitrary constant will be called the Covariant Hamiltonians. Its physical interpretation will be made more precise latter on.

It must be emphasized that according to eq. (4.7) and eqs. (4.9)-(4.12) these functions H_a, P_μ, $J_{\lambda\mu}$ are solutions of the following completely integrable differential systems:

$$\mathscr{L}(\vec{H}_b)H_a = 0\,, \qquad \mathscr{L}(\vec{P}_\rho)H_a = 0\,, \qquad \mathscr{L}(\vec{J}_{\rho\sigma})H_a = 0 \tag{4.13}$$

$$\begin{cases} \mathscr{L}(\vec{H}_a P_\mu = 0\,, \qquad \mathscr{L}(\vec{P}_\rho)P_\mu = 0 \\ \mathscr{L}(\vec{J}_{\rho\sigma})P_\mu = \eta_{\rho\mu}P_\sigma - \eta_{\sigma\mu}P_\rho \end{cases} \tag{4.14 a}$$

$$\begin{cases} \mathscr{L}(\vec{H}_a)J_{\lambda\mu} = 0\,, \qquad \mathscr{L}(\vec{P}_\rho)J_{\lambda\mu} = \eta_{\rho\mu}P_\lambda - \eta_{\rho\lambda}P_\mu \\ \mathscr{L}(\vec{J}_{\rho\sigma})J_{\lambda\mu} = \eta_{\rho\lambda}J_{\sigma\mu} - \eta_{\sigma\lambda}J_{\rho\mu} + \eta_{\rho\mu}J_{\lambda\sigma} - \eta_{\sigma\mu}J_{\lambda\rho}\,. \end{cases} \tag{4.14 b}$$

Eq. (4.13) express that the Covariant Hamiltonians H_a are first integrals of the system and that they behave as scalars under the Poincaré group. Eq. (4.14 *a*) tell us that the four functions P_α are first integrals of the system which are invariant under the space-time translations sub-group and that they behave as the components of a four vector under the Lorentz group. Eq. (4.14 *b*) tell us that the six functions $J_{\lambda\mu}$ are also first integrals; they tell us now they transform under space-time translations and they express that they behave as the components of a skew-symmetric tensor under Lorentz transformations.

We see from the preceding considerations how important is for a given dynamical system to admit a C. H. F.: the concepts of Energy-Momentum and of Generalized Angular Momentum depend on this formulation. On the other hand a C. H. F is really useful only if a sufficient set of supplementary conditions are given such that they make the symplectic form Ω unique. If we had for a given dynamical system two or more inequivalent admissible C. H. F. we would have an essential ambiguity coming for example from the existence of two or more sets of admissible functions H_a, P_μ or $J_{\lambda\mu}$ and such ambiguity would jeopardize the physical interpretation of these quantities. The next two sections deal with this unicity problem.

B. — Let us consider a system of no interacting, or free, particles. We mean obviously by this that:

$$\theta_a^\lambda = \Delta_a^\lambda \equiv 0 \Leftrightarrow \vec{H}_a = \vec{H}_a^F \equiv \delta_{ab}\pi_b^\rho \frac{\partial}{\partial x_b^\rho}\,. \tag{4.15}$$

It is evident that this system is an Invariant Projectable System.

If N = 1 Souriau has proved [*2*] that for the system (4.15) there exists a unique C. H. F. under the important supplementary condition that the action of the Poincaré group on co-phase space is transitive. We shall assume, N being arbitrary, that for a system of free particles the symplectic form Ω^F is the sum over the particle indices of Souriau's symplectic form Ω_a, i. e.:

$$\Omega^F \equiv dx_a^\rho \wedge d\pi_\rho^a + \frac{1}{2}\sum_{a=1}^{N} s_a \pi_a^{-3} \Gamma_{a\lambda\mu} d\Gamma_a^{\lambda\rho} \wedge d\Gamma_{a\ \rho}^{\mu}\,, \tag{4.16}$$

where s_a means the modulus of the spin [20] of particle a and where we have used the following notation:

$$\Gamma_{a\lambda\mu} \equiv \eta_{\lambda\mu\nu\sigma}\pi_a^\nu \gamma_a^\sigma\,. \tag{4.17}$$

It must be emphasized that the 2-form (4.16) must be considered on Σ, i. e., with the constraints:

$$(\pi_a \gamma_a) = 0\,, \qquad \gamma_a^2 = 1\,. \tag{4.18}$$

Taking them into account it is easy to see that Ω^F is indeed a symplectic form. Moreover a straightforward calculation proves that:

$$i(\vec{H}_a^F)\Omega^F = -\,d\left\{\frac{1}{2}\pi_a^2\right\} \tag{4.19 a}$$

$$i(\vec{P}_\mu)\Omega^F = -\,d\left\{\varepsilon_a \pi_\mu^a\right\} \tag{4.19 b}$$

$$i(\vec{J}_{\lambda\mu})\Omega^F = -\,d\left\{x_{a\lambda}\pi_\mu^a - x_{a\mu}\pi_\lambda^a + \sum_{a=1}^{N} s_a \pi_a^{-1}\Gamma_{a\lambda\mu}\right\}, \tag{4.19 c}$$

and therefore the functions associated with the generators of the Complete Group of Symmetry are:

$$H_a^F \equiv \frac{1}{2}\pi_a^2 \tag{4.20 a}$$

$$P_\mu^F \equiv \varepsilon_a \pi_\mu^a \tag{4.20 b}$$

$$J_{\lambda\mu}^F \equiv x_{a\lambda}\pi_\mu^a - x_{a\mu}\pi_\lambda^a + \sum_{a=1}^{N} s_a \pi_a^{-1}\Gamma_{a\lambda\mu}\,, \tag{4.20 c}$$

where we have already made the appropriate choice of arbitrary constants to verify eqs. (4.10)-(4.12).

Since $\frac{1}{c^2}\pi_a^2$ must be interpreted as the square m_a^2 of the mass of particle a (see Section 3), eq. (4.20 a) gives, for free particles, the interpretation of the Covariant Hamiltonians H_a^F. The Energy-Momentum and the Generalized Angular Momentum given by eqs. (4.20 b) and (4.20 c) are just the usual and well known expressions including the spin-dependent part of $J_{\lambda\mu}$.

Let us consider now the same system (4.15) of free particles from the point of view of the Manifestly Predictive Formalism. The projection of (4.15) is obtained using the formulas (3.32), which give the trivial result:

$$\mu_a^i = \rho_a^i = 0 . \tag{4.21}$$

Therefore the infinitesimal generators (2.26) of the corresponding Induced Realisation can be written as follows:

$$\vec{H}^F \equiv v_a^j \frac{\partial}{\partial x_a^j} \tag{4.22 a}$$

$$\vec{P}_i \equiv - \varepsilon_a \frac{\partial}{\partial x_a^i} \tag{4.22 b}$$

$$\vec{J}_i \equiv \eta_i{}^j{}_l \left(x_a^l \frac{\partial}{\partial x_a^j} + v_a^l \frac{\partial}{\partial v_a^j} + \alpha_a^l \frac{\partial}{\partial \alpha_a^j} \right) \tag{4.22 c}$$

$$\vec{K}_i^F \equiv \frac{1}{c^2} x_{ai} v_a^j \frac{\partial}{\partial x_a^j} - \left\{ \varepsilon_a \delta_i^j - \frac{1}{c^2} v_{ai} v_a^j \right\} \frac{\partial}{\partial v_a^j} - \frac{\varkappa_a}{c^2} \left(\alpha_{al} v_a^l \delta_i^j - \alpha_{ai} v_a^j \right) \frac{\partial}{\partial \alpha_a^j} . \tag{4.22 d}$$

The new problem which we have to face now is to determine a symplectic form σ^F defined on the co-phase space $[T(R^3) \times S^2]^N$ and invariant under the Induced Realisation, i. e., such that the Lie derivative with respect to the vector fields (4.22) be zero. In this connection it is easy to prove that the restriction, in the sense of eq. (3.30), of Souriau's 2-form (4.16) gives a symplectic form σ^F with the desired properties. A straightforward but lengthy calculation yields:

$$\sigma^F = dq_a^i \wedge dp_i^a + \frac{1}{2} \sum_{a=1}^{N} s_a \eta_{lij} \alpha_a^l d\alpha_a^i \wedge d\alpha_a^j \tag{4.23}$$

where we have used the following definitions:

$$\vec{p}_a \equiv \frac{m_a \vec{v}_a}{\sqrt{1 - \frac{v_a^2}{c^2}}} \tag{4.24 a}$$

$$\vec{q}_a \equiv \vec{x}_a + \frac{1}{c^2} m_a^{-2} s_a (1 + h_a)^{-1} \vec{p}_a \wedge \vec{\alpha}_a , \tag{4.24 b}$$

where we have used usual vector notations and where $\wedge$ means in (4.24 *b*) the usual vector product.

To write the symplectic form σ^F in terms of a canonical system of coordinates let us use a « polar » representation of the orientations α_a^i of the spins, i. e.:

$$\begin{cases} \alpha_a^1 = \sin\theta_a \cdot \cos\phi_a \\ \alpha_a^2 = \sin\theta_a \cdot \sin\phi_a \\ \alpha_a^3 = \cos\theta_a . \end{cases} \tag{4.25}$$

Substituting (4.25) in eq. (4.21) we get by a simple calculation:

$$\sigma^F = dq_a^i \wedge dp_i^a + \sum_{a=1}^{N} s_a d(-\cos\theta_a) \wedge d\phi_a , \tag{4.26}$$

expression which shows that $\{ q_a^i, -s_b \cos\theta_b; p_c^i, \phi_d \}$ is a canonical coordinate system for σ^F. Moreover from eq. (4.26) it follows very easily that the position coordinates x_a^i can *not* be part of a system of canonical coordinates. This result is important because it departs from the corresponding one for systems of free particles with *no* spin. On the other hand this result is the analog of the corresponding one for systems of interacting particles where again the position coordinates can not be part of a system of canonical coordinates [*6*] [*21*].

Let us notice that the symplectic form (4.23) formally coincides (for $N = 1$) with one of the 2-forms proposed by Arens [*3*] to deal with a free particle, except for an important difference. In Arens paper the variables q_a^i, which they have here an unambiguous meaning given by eqs. (4.24 *b*) and (4.24 *a*), are apparently identified with the position coordinates.

To finish this sub-section let us write the functions H^F, P_i^F, J_i^F and K_i^F which the symplectic form σ^F associates to the generators (4.22). Using the formula (4.5) we get very easily:

$$H^F \equiv \varepsilon^a m_a c^2 h_a , \qquad h_a \equiv +\left(1 - v_a^2/c^2\right)^{-1/2} \equiv +\left(1 + \vec{p}_a^{\,2}\right)^{1/2} \tag{4.27 a}$$

$$P_i^F \equiv \varepsilon_a p_i^a \tag{4.27 b}$$

$$J_i^F \equiv \varepsilon^a\left(\eta_{ijl} q_a^j p_a^l + s_a \alpha_{ai}\right) \tag{4.27 c}$$

$$K_i^F \equiv \varepsilon^a \left\{ m_a c^2 h_a q_{ai} + m_a^{-1} s_a (1 + h_a)^{-1} \eta_{ijl} p_a^j \alpha_a^l \right\} , \tag{4.27 d}$$

where, as in eq. (4.20), we have already made the appropriate choice of the arbitrary constants to guarantee that the Poisson brackets are formally identical with eq. (2.28).

As we see, the function H^F (Hamiltonian) is the total energy of the system and P_i^F are the components of the total linear momentum. The functions J_i^F are the components of the angular momentum, which is the sum of an « orbital » part, where the q_a^i play the role of positions, and a spin part. K_i^F are the functions which generalize the center of mass formula.

C. — Let us consider now a system of interacting particles with spin, i. e., a dynamical system for which the functions θ_a^λ and Δ_a^λ are not identically zero. The purpose of this final sub-section is to prove that it is possible to impose supplementary conditions to a symplectic form Ω on Σ satisfying eq. (4.8) such that they make it unique, for a large class of dynamical systems.

The ideas, the assumptions and the technique used in this section are essentially the same which were used in references 5 and 23 for the case of particles with no spin. The technique though will be slightly different and more appropriate to avoid the complications introduced by the spins of the particles. At the same time, the formulas being more compact some of them might appear to be less transparent that in reference 23. A comparison with the latter may then be useful.

The first assumption that we make is to assume that the dynamical system which we consider is separable and has an index of separability sufficiently great. This means that when the space distance between the particles tend to infinity, the functions θ_a^λ and Δ_a^λ tend to zero quickly enough. More precisely we shall assume that a value of h, sufficiently great, exists such that:

$$\lim_{\tau \to \pm\infty} \tau^h \varphi_\tau^* \theta_a^\lambda = \lim_{\tau \to \pm\infty} \tau^h \varphi_\tau^* \Delta_a^\lambda = 0 \,, \tag{4.28}$$

where φ_τ^* means the « reciprocal image » transformation [*24*] of the transformation φ_τ defined by the general solution, associated with the dynamical system (4.15), interpreted as a one parameter family of transformations of Σ, i. e.:

$$\varphi_\tau : \begin{cases} x_a^\alpha \to x_a^\alpha + \pi_a^\alpha \tau \\ \pi_a^\alpha \to \pi_a^\alpha \\ \gamma_a^\alpha \to \gamma_a^\alpha \,. \end{cases} \tag{4.29}$$

Coherently with (4.28), we shall assume also that the symplectic form Ω becomes the form Ω^F of free particles in the « infinite past », i. e., before the interaction between the particles started. We express this condition as follows:

$$\lim_{\tau \to -\infty} \varphi_\tau^* \Omega = \Omega^F \,. \tag{4.30}$$

Let us write now eq. (4.8 *a*) in the following form:

$$\mathscr{L}(\vec{H}_a^F)\Omega = -\mathscr{L}(\vec{H}_a^I)\Omega \,, \tag{4.31}$$

where $\vec{H}_a^F$ are given by eq. (4.15) and where according to eq. (3.48), the vector fields $\vec{H}_a^I$ are defined as follows:

$$\vec{H}_a^I \equiv \delta_{ab}\left(\theta_b^\rho \frac{\partial}{\partial \pi_b^\rho} + \Delta_b^\rho \frac{\partial}{\partial \gamma_b^\rho} \right). \tag{4.32}$$

Since the vector fields H_a^F commute, it is clear that if Ω is a solution of (4.31) it will be a solution also of the integrability conditions:

$$\mathscr{L}(\vec{H}_a^F)\mathscr{L}(\vec{H}_b^I)\Omega = \mathscr{L}(\vec{H}_b^F)\mathscr{L}(\vec{H}_a^I)\Omega \,. \tag{4.33}$$

Moreover, taking into account eqs. (4.29) and (4.15), it is not difficult to prove the following identity which is valid $\forall\tau$ and $\forall\Omega$:

$$\frac{d}{d\tau}\varphi_\tau^*\Omega \equiv \varepsilon^a\varphi_\tau^* \cdot \mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{F}})\Omega\,, \qquad \varepsilon^a \equiv 1 \tag{4.34}$$

hence if Ω is a solution of eq. (4.31) we shall have:

$$\frac{d}{d\tau}\varphi_\tau^*\Omega = -\,\varepsilon^a\varphi_\tau^*\mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}})\Omega\,. \tag{4.35}$$

Integrating this equation in the interval $(0, -\infty)$ and remembering (4.29) and the condition (4.30) we obtain finally the following integral equation for the symplectic form Ω:

$$\Omega = \Omega^{\mathrm{F}} + \varepsilon^a\int_0^{-\infty} d\tau \cdot \varphi_\tau^*\mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}})\Omega\,. \tag{4.36}$$

Therefore we may state the following result:

LEMMA 4.1. — « If the symplectic form Ω is a solution of eqs. (4.31) and satisfies the asymptotic condition (4.30), then Ω is also a solution of the integrability conditions (4.33) and of the integral equation (4.36) ».

Let us prove now the reciprocal result. Namely:

LEMMA 4.2. — « If Ω is a solution of eqs. (4.33) and (4.36), then it is also a solution of eqs. (4.31) and satisfies the asymptotic condition (4.30) ».

Proof. — From eq. (4.36) and taking into account eqs. (4.16) and (4.29) we get:

$$\varphi_{\tau'}^*\Omega = \Omega^{\mathrm{F}} + \varepsilon^a\int_0^{-\infty} d\tau \cdot \varphi_{\tau+\tau'}^*\mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}})\Omega\,, \tag{4.37}$$

where from changing the variable of integration from τ to $\tau + \tau' = \tau''$ and letting then $\tau' \to -\infty$ we obtain the asymptotic condition (4.30). Moreover, from eqs. (4.33) and (4.36) and taking into account eq. (4.34) we have:

$$\mathscr{L}(\vec{\mathrm{H}}_b^{\mathrm{F}})\Omega = \int_0^{-\infty} d\tau \cdot \frac{d}{d\tau}\varphi_\tau^*\mathscr{L}(\vec{\mathrm{H}}_b^{\mathrm{I}})\Omega\,, \tag{4.38}$$

expression which leads to eqs. (4.31) if the integral of (4.36) is zero in the infinite past [*25*]. This completes the proof of the lemma.

These two lemmas are interesting in connection with perturbation theory. We shall assume that the functions θ_a^λ and Δ_a^λ can be developped as power series of N parameters e_b which are supposed to be responsible for the interaction. Thus we shall have:

$$\theta_a^\lambda = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} e_1^{n_1} \dots e_N^{n_N} \theta_a^{\lambda(n_1,\dots,n_N)}$$
$$\Delta_a^\lambda = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} e_1^{n_1} \dots e_N^{n_N} \Delta_a^{\lambda(n_1,\dots,n_N)}, \quad (4.39)$$

with:

$$\theta_a^{\lambda(n_1,\dots,n_{a-1},0,n_{a+1},\dots,n_N)} = \Delta_a^{\lambda(\ )} \equiv 0, \qquad \forall a, \quad \forall n_b \geqslant 0$$
$$\theta_a^{\lambda(0,\dots,0,n_a,0,\dots,0)} = \Delta_a^{\lambda(\ )} \equiv 0, \qquad \forall a, \quad \forall n_a \geqslant 1. \quad (4.40)$$

Coherently with this assumption we shall assume that the symplectic form Ω can be developped as follows:

$$\Omega = \sum_{n_1=0}^{\infty} \dots \sum_{n_N=0}^{\infty} e_1^{n_1} \dots e_N^{n_N} \Omega^{(n_1,\dots,n_N)} \quad (4.41)$$

where:

$$\Omega^{(0,0,\dots,0)} \equiv \Omega^F \quad (4.42\ a)$$
$$\Omega^{(0,\dots,0,n_b,0,\dots,0)} \equiv 0, \qquad \forall b, \quad \forall n_b \geqslant 1. \quad (4.42\ b)$$

Substituting now the developments (4.39) and (4.41) into eqs. (4.33) and (4.36) we obtain the following eqs. for $\Omega^{(n_1,n_2,\dots,n_N)}$:

$$\mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{F}}) \sum_{(r)+(s)=(n)} \mathscr{L}(\vec{\mathrm{H}}_b^{\mathrm{I}(r)})\Omega^{(s)} = \mathscr{L}(\vec{\mathrm{H}}_b^{\mathrm{F}}) \sum_{(r)+(s)=(n)} \mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}(r)})\Omega^{(s)} \quad (4.43)$$

$$\Omega^{(n)} = \varepsilon^a \int_0^{-\infty} d\tau \cdot \varphi_\tau^* \sum_{(r)+(s)=(n)} \mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}(r)})\Omega^{(s)}, \qquad (n) \geqslant 1 \quad (4.44)$$

where (n), (r) and (s) represent respectively the one row matrices $(n_1, \dots, n_N)$, $(r_1, \dots, r_N)$ and $(s_1, \dots, s_N)$, and where $(n) \geqslant 1$ means that one of the elements at least of this matrix is greater or equal to 1; we have also used the following notation:

$$\vec{\mathrm{H}}_a^{\mathrm{I}(r)} \equiv \delta_{ab}\left(\theta_b^{\rho(r)} \frac{\partial}{\partial \pi_b^\rho} + \Delta_b^{\rho(r)} \frac{\partial}{\partial \gamma_b^\rho}\right). \quad (4.45)$$

Let us notice that eq. (4.44) do not include (0) order term of eq. (4.36). This (0) order term is just (4.42 *a*) as it follows immediately from (4.40). Let us notice also the coherence of eqs. (4.44) with assumptions (4.42 *b*) and (4.40).

Obviously, eq. (4.44) make possible a recurrent calculation of $\Omega^{(n)}$ starting from Ω^{F}. The 2-form thus obtained is symplectic because Ω^{F} is symplectic and because the exterior differential commutes with the Lie derivative. Moreover it can be easily proved that the integrability conditions (4.33) are automatically satisfied as a consequence of eq. (3.49). Finally let us remark that the symplectic 2-form Ω calculated by this recurrent algorithm satisfies also eqs. (4.8 *b*) and (4.8 *c*) as a consequence of eq. (3.54) and the fact that Ω^{F} satisfies them also.

« We conclude then that any dynamical system that satisfies the assumptions that we stated (the integrals (4.44) are convergent when the separability index is great enough) admits *one and only one* C. H. F. satisfying the asymptotic condition (4.30) ».

To end this paper, let us mention that starting from the first equations (4.13) and (4.14) it is possible to calculate directly the developments, similar to (4.41), of the conserved quantities H_a, P_μ and $J_{\lambda\mu}$. Thus, for instance, the formulas which are obtained for P_μ are the following:

$$P_\mu^{(0)} = P_\mu^F \equiv \varepsilon_a \pi_\mu^a \tag{4.46 a}$$

$$P_\mu^{(n)} = \varepsilon^a \int_0^{-\infty} d\tau \cdot \varphi_\tau^* \sum_{(r)+(s)=(n)} \mathscr{L}(\vec{H}_a^{I(r)}) P_\mu^{(s)}, \qquad (n) \geqslant 1 . \tag{4.46 b}$$

For the Covariant Hamiltonians H_a, taking into account (3.55 *c*), we obtain:

$$H_a = H_a^F \equiv \frac{1}{2} \pi_a^2 \tag{4.47}$$

result which holds at any order of perturbation theory independently of the dynamical system which is being considered [*26*].

APPENDIX

Let us consider Minkowski's affine space $\mathfrak{M}_4$ referred to a galilean coordinate system $\{ x^\mu \}$, where $x^0 = ct$ represents the time coordinate and $\{ x^i \}$ represent the space coordinates; and let us consider the group of Poincaré acting on $\mathfrak{M}_4$:

$$x^\lambda \to L^\lambda_\mu(x^\mu - A^\mu), \tag{A.1}$$

where L^λ_μ is a matrix of the Lorentz group:

$$L^\lambda_\mu L^\nu_\rho \eta_{\lambda\nu} = \eta_{\mu\rho}, \tag{A.2}$$

and where A^μ is an arbitrary vector of R^4 representing a space-time translation.

A possible parametrization of the Lorentz matrices L^λ_μ is the following:

$$\begin{aligned} L^0_0 &= \Gamma \equiv + \left\{ 1 - \frac{V^2}{c^2} \right\}^{-1/2}, \qquad L^0_j = -\frac{1}{c}\Gamma V_k \\ L^i_j &= R^i_l \left\{ \delta^l_j + \frac{\Gamma^2}{1+\Gamma} \cdot \frac{V^l V_j}{c^2} \right\}, \qquad L^j_0 = -\frac{1}{c}\Gamma R^j_l V^l, \end{aligned} \tag{A.3}$$

where V^i can be interpreted as the relative velocity of two galilean systems of reference and R^i_j as the rotation relating the corresponding cartesian axis. The matrices R^i_j of the rotation group can be parametrised as follows:

$$R^i_l = \left(1 - \frac{1}{2}\omega^2\right)\delta^i_l + \frac{1}{2}\omega^i\omega_l + \left(1 - \frac{1}{4}\omega^2\right)^{1/2} \eta^i{}_{lj}\omega^j \tag{A.4}$$

where:

$$\omega^2 \equiv \omega^i\omega_i \equiv 4\sin^2\frac{\theta}{2}$$

δ^i_l : Kronecker's symbol

η_{ilj} : Levi-Civita's symbol

ω^i being the components of the vector defining the axis of rotation and θ being the angle of rotation.

Using the parametrisation $\{ A^0, A^i, \omega^j, V^k \}$, the infinitesimal generators of the Poincaré group (A.1) are respectively the following ten vector fields of $\mathfrak{M}_4$:

$$\vec{P}_0 = -\frac{1}{c}\frac{\partial}{\partial t} \tag{A.5 a}$$

$$\vec{P}_i = -\frac{\partial}{\partial x^i} \tag{A.5 b}$$

$$\vec{J}_i = \eta_i{}^j{}_l x^l \frac{\partial}{\partial x^j} \tag{A.5 c}$$

$$\vec{K}_i = -\frac{x_i}{c^2}\frac{\partial}{\partial t} - t\frac{\partial}{\partial x^i}. \tag{A.5 d}$$

With respect to this basis the Lie algebra of the Poincaré group is characterized by the Lie brackets (2.28) or (3.52). In the first case we have to make the identification:

$$\vec{H} \equiv c\vec{P}_0 \tag{A.6}$$

and in the second one:

$$\vec{J}_{jl} = \eta_{jl}{}^i \vec{J}_i \tag{A.7 a}$$

$$\vec{J}_{0i} = -c\vec{K}_i. \tag{A.7 b}$$

REFERENCES

[1] We use the expression « Classical Mechanics » as opposed to « Quantum Mechanics ».

[2] J. M. SOURIAU, *Structure des Systèmes Dynamiques*, Dunod Université (Paris, 1970).

[3] R. ARENS, *Comm. Math. Phys.*, t. **21**, 1971, p. 139.

[4] L. MARTINEZ ALONSO, *J. Math. Phys.*, t. **20**, 1979, p. 219.

[5] L. BEL and J. MARTIN, *Ann. Inst. H. Poincaré*, t. **22 A**, 1975, p. 173.

[6] J. MARTIN and J. L. SANZ, *J. Math. Phys.*, t. **19**, 1978, p. 780.

[7] See for instance : D. G. CURRIE, *Phys. Rev.*, t. **142**, 1966, p. 817; R. N. HILL, *J. Math. Phys.*, t. **8**, 1967, p. 201; L. BEL, *Ann. Inst. H. Poincaré*, t. **12**, 1970, p. 307; Ph. DROZ-VINCENT, *Physica Scripta*, t. **2**, 1970, p. 129; L. BEL, *Ann. Inst. Poincaré*, t. **14**, 1970, p. 189; R. ARENS, *Arch. for Rat. Mech.* and *Analysis*, t. **47**, 1972, p. 255; L. BEL and X. FUSTERO, *Ann. Inst. H. Poincaré*, t. **24**, 1979, p. 411.

[8] See for instance : B. M. BARKER and R. F. O'CONNELL, *Phys. Rev. D*, t. **12**, 1975, p. 329; X. FUSTERO and E. VERDAGUER, « Interaction among systems of finite size in Predictive Relativistic Mechanics » Preprint Universidad Autonoma de Barcelona. Spain, 1979; J. M. GRACIA-BONDIA, *Physics Letters*, **75 A**, t. **4**, 1980, p. 262.

[9] We use the sommation convention for all kind of indices: repeated indices, one in a covariant and the other one in a contravariant position, are summed over their range.

[10] We shall us the signature + 2. Therefore $u_a^\lambda u_{a\lambda} = -1$.

[11] η_{ijl} represents the Levi-Civita symbol in three dimensions with $\eta_{123} = +1$. In M_4 we shall assume that $\eta_{0123} = +1$.

[12] L. P. EISENHART, *Continuous groups of transformations*. Dover Publications Inc. (New York, 1961).

[13] For systems of particles with no spin the proof of this result can be found in L. BEL, ref. 7, **12** and in L. BEL, *Lecciones sobre Mecànica Relativista Predictiva*. Departamento de Fìsica Teorica, Universidad Autònoma de Barcelona. Spain, 1976.

[14] In this sub-section we consider an arbitrary initial time. This is more convenient because we shall forget momentarily the Poincaré invariance.

[15] We assume that the vectors $\tilde{\pi}_a^\lambda$ are time-like and future oriented ($\tilde{\pi}_a^0 > 0$).

[16] See an alternative formulation in : H. P. KUNZLE, *J. Math. Phys.*, t. **15**, 1974, p. 1033.

[17] See for instance : J. M. SOURIAU, ref. 2; R. ABRAHAM, *Foundation of Mechanics*, W. A. Benjamin, 1967; C. GODBILLON, « Géométrie différentielle et Mécanique Analytique », Hermann, 1969.

[18] See for instance the proof of ref. 21.

[19] This assumption is the most natural one which is consistent with the fact that the co-phase space for a system of N particles is the Nth cartesian power of the co-phase space for one particle.

[20] We introduce here the modulus of the spin to make eq. (4.15) dimensionally homogeneous. s_a has the dimensions of an action.

[21] D. G. CURRIE, T. F. JORDAN and E. C. G. SUDARSHAN, *Rev. of Mod. Phys.*, t. **35**, 1963, p. 530.

[*22*] This is due to the fact that we use « instantaneous spins » instead of « intrinsic spins » (See Section 2).

[*23*] L. BEL and X. FUSTERO, *Ann. Inst. H. Poincaré*, t. **24**, 1979, p. 411.

[*24*] See for instance : Y. CHOQUET-BRUHAT, *Géométrie différentielle et Systèmes extérieurs*, Dunod. Paris, 1968.

[*25*] For more details see ref. 23.

[*26*] Let us remark that this construction defines the quantities H_a unambiguously (there are no more arbitrary additive constants).

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