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O. BRANDER
K. CHADAN

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Section A :
Physique théorique.

A Tauberian theorem in quantum mechanical inverse scattering theory

by

O. BRANDER (*)

and

K. CHADAN

Laboratoire de Physique Théorique et Hautes Énergies (**),
Université de Paris-Sud, 91405 Orsay, France

ABSTRACT. — Let $V(r)$ be a spherically symmetric potential, which is locally L^1, except at $r = 0$. Assume further, that

$$\int_\varepsilon^\infty |V(r)|\,dr < \infty \qquad \varepsilon > 0$$

and that

$$W(r) \in L^1(0, \infty), \qquad \text{where} \qquad W(r) \equiv -\int_r^\infty V(s)ds\,.$$

Denote by W_0 the class of functions $W(r)$, fulfilling these conditions, that is being $L^1(0, \infty)$ and absolutely continuous except at $r = 0$. We prove the following theorem: the necessary and sufficient condition for $W(r)$ (with the potential $V = W'$) to belong to the class W_0 is that the Fourier sine transform of the phase shift $\delta(k)$ (or of certain other, equivalent scattering data) belongs to the class W_0. Generalization is also given to potentials that fail to be absolutely integrable at infinity. This Symmetrical Tauberian theorem shows the intimate connection between the Fourier sine transform of the phase shifts and the integral of the potential, and gives a precise meaning to the heuristical argument according to which the phase shifts and the potential are related by a kind of Fourier transform.

(*) Permanent address: Institute of Theoretical Physics S-41296, Göteborg, Sweden.
(**) Laboratoire associé au Centre National de la Recherche Scientifique.

1. INTRODUCTION

In this paper we consider the inverse scattering problem for the non-relativistic Schrödinger equation with a spherically symmetric potential V(r). As usual, we put $h = 2M = 1$, so that the energy E equals k^2, where k is the wave number.

As sufficient conditions on the potential for the usual theory to apply, it is customary to take local integrability of $r\,|\,V(r)\,|$ and integrability of $|\,V(r)\,|$ at infinity. Usually, one also assumes the integrability of $r\,|\,V(r)\,|$ up to infinity, in order to ensure a finite number of bound states. This amounts to

$$\int_0^\infty r\,|\,V(r)\,|\,dr < \infty \tag{1.1}$$

and it was for the class of potentials fulfilling this condition that the Gel'fand-Levitan-Marchenko inverse scattering theory was initially developed [*1*].

However, it has become clear during recent years that the above conditions are not necessary, and that one can have perfectly respectable theories without any modification of the usual formalism with a much larger class of potentials [*2-7*]. One class, which was studied in references 2, 3 and 4, consists in modifying condition (1.1) at the origin, replacing it by

$$W(r) \in L^1(0, 1)\,, \qquad \lim_{r\to 0} rW(r) = 0\,, \tag{1.2}$$

where

$$W(r) = -\int_r^\infty V(r)dr\,. \tag{1.3}$$

In references 5 and 6 similar modifications were done at infinity.

As an example not satisfying eq. (1.1) at the origin, we take

$$W(r) = r^{-1/2} \sin\left(\exp\frac{1}{r}\right)e^{-\mu r}\,. \tag{1.4}$$

Indeed, in ref. 2 all the familiar properties in scattering theory were proved under this modified condition, including the existence and asymptotic completeness of the wave operators. Moreover, it was shown that the Gel'fand-Levitan-Marchenko formalism for the inverse scattering problem is perfectly valid without modifications. In ref. 4 the solution of the inverse problem within this class of potentials was discussed, and the close connection was shown between oscillations of the potential on the one hand and the phase shift on the other.

We intend here to generalize the following theorem, due to Agranovich and Marchenko [*8*] [*1*].

Let $S(k)$ be the scattering matrix, satisfying the usual conditions of unitarity, etc., and let

$$S(k) = 1 + \int_{-\infty}^{\infty} \Sigma(t) \exp(-ikt)dt . \tag{1.5}$$

Then the following two statements are equivalent:

$$t\Sigma'(t) \in L^1(-\infty, \infty) \Leftrightarrow rV(r) \in L^1(0, \infty) . \tag{1.6}$$

It is especially the close and symmetrical connection between $V(r)$ and $\Sigma'(t)$ exhibited in this equation that we shall examine more fully.

Beside eq. (1.5), let us introduce the following Fourier integral representations of the Jost function $F(k)$ and the phase shift $\delta(k)$:

$$F(k) = 1 + \int_0^{\infty} \Gamma(t) \exp(ikt)dt \tag{1.7}$$

$$| F(k) |^{-2} = 1 + 2\int_0^{\infty} H(t) \cos(kt)dt \tag{1.8}$$

$$\delta(k) = -\int_0^{\infty} \gamma(t) \sin(kt)dt \tag{1.9}$$

Another motivation for the present study is the following, heuristic argument. Consider the Born approximation for the phase shift

$$\delta_B(k) = \int_0^{\infty} \frac{\sin^2 kr}{k} V(r)dr \tag{1.10}$$

and rewrite it in the form

$$\delta_B(k) = \int_0^{\infty} drV(r)\int_0^{r} ds \sin 2ks$$
$$= -\int_0^{\infty} dsW(s) \sin 2ks .$$

Comparing this with eq. (1.9), we see that

$$\gamma_B(t) = \frac{1}{2} W\left(\frac{1}{2} t\right). \tag{1.11}$$

This suggests a close connection between γ and W in general, which we are going to examine below.

For convenience, let us now introduce the following class of functions:

DEFINITION. — A function f on R^+ [or R] is said to belong to the class W_0 if

(A) $f(x) \in L^1(0, \infty)$ [and $L^1(-\infty, 0)$]

(B) $f'(x) \in L^1(\varepsilon, \infty)$ [and $L^1(-\infty, -\varepsilon)$]

for any $\varepsilon > 0$.

Note that for a function in the class W_0, the second of the conditions (1.2) is automatic, lemma A2.1.

We can now formulate the theorem we are going to prove as follows:

THEOREM 1.1. — Let Ω be any one of the functions Σ, Γ, γ or H. Then the necessary and sufficient condition that the corresponding W-function belongs to the class W_0 is that Ω belongs to the class W_0.

This theorem will be proved in several steps below. The first step is expressed by the following lemma, which is proved in appendix 1:

LEMMA 1.1. — If any one of the functions Σ, Γ, γ and H belongs to the class W_0, so do all the others.

The second, and central part of the proof is concerned with the behaviour of the interaction at the origin. Without any *a priori* assumptions on the potential we show in section 2 that if H is a W_0-function, the Gel'fand-Levitan equation has a unique solution, at least near the origin, and we derive new integral representations for V and W. In appendix 2 it is proved directly from this integral representation that W is a W_0 function, at least near the origin.

In section 3 we derive a similar integral representation for the potential at large distances, and add up the different parts of the proof of theorem 1.1. Finally, in section 4 we discuss some possible generalizations.

For simplicity of notation, we shall throughout this paper consider the S-wave only, and no bound states, since it is a standard procedure to generalize on these points.

2. DERIVATION OF THE PROPERTIES OF W(*r*) FROM THE GEL'FAND-LEVITAN EQUATION

It is well known [*1*] [7] [*8*] that when the potential satisfies eq. (1.1) then the Gel'fand-Levitan equation

$$K(r, t) + G(r, t) + \int_0^r ds K(r, s) G(s, t) = 0, \qquad 0 \leqslant t \leqslant r \tag{2.1}$$

where

$$G(r, t) = H(r - t) - H(r + t) \tag{2.2}$$

and H is the function defined by eq. (1.8), is a Fredholm equation, with a square integrable kernel. It possesses a unique solution for any $r \in (0, \infty)$. Moreover, the potential is given by

$$V(r) = 2 \frac{d}{dr} K(r, r). \tag{2.3}$$

It is also known [7], that if the condition (1.1) is violated at the origin, then K(*r*, *t*) develops a singularity at $t = r$, so that eq. (2.3) is not well defined. It was shown in ref. 4 that this singularity of $K(r, t)$ is connected with a similar singularity of $H(x)$ at $x = 0$. There, the solution of the Gel'fand-Levitan equation was discussed for this singular case, and in several examples the direct connection was proved between the properties of γ, H and W.

Although appropriate in many special cases, the method used in ref. 4 for solving the Gel'fand-Levitan equation fails for the most singular cases, where eq. (1.2) is just satisfied, e. g. when

$$W(r) = 0\left(\frac{1}{r\,|\log r\,|^{1+\varepsilon}}\right), \qquad \varepsilon > 0, \quad r \to 0. \tag{2.4}$$

In order to treat the general case, and to prove our theorem 1.1, we have to study the Gel'fand-Levitan equation (2.1) under the assumption that $H(x)$ is a W_0-function, that is satisfies conditions (A) and (B) above. By definition, H is also an even function.

Let us now define the resolvent kernel R of G on the interval $(0, r)$ by the equation

$$\begin{cases} R(r\,;x, y) = -G(r-x, r-y) - \displaystyle\int_0^r dz G(r-x, r-z)R(r\,;z, y) \\ R(r\,;y, x) = R(r\,;x, y), \end{cases} \tag{2.5}$$

where, for later convenience, we have used the variables $x = r - s$ and $y = r - t$. The solution of the Gel'fand-Levitan equation is given now simply by

$$K(r, t) = R(r\,;r-t, 0) = R(r\,;0, r-t). \tag{2.6}$$

The problem we now encounter is the lack of a general theory for Fredholm equations with L^1-kernels. (Except for ref. 9, on a half-line with difference-type kernels, which is not enough for our purpose.) However, the problem is at $x = 0$, the lower end of the interval, so it would be a good beginning to solve eq. (2.5) for very small r. But this can be done for any L^1-kernel, and the following lemma be proved:

LEMMA 2.1. — If $H(x) \in W_0$ and $b > 0$ is chosen such that

$$\sup_{x\in(-\infty,\infty)} \int_x^{x+b} |H(y)|\,dy \leqslant \frac{1}{2}\delta, \qquad \delta < 1, \tag{2.7}$$

then eq. (2.5) has a unique solution $R(r\,;x, y)$ for $r \leqslant b$. The solution is $L^1(0, b)$ in the x- and y-variables, and absolutely continuous in r on $0 \leqslant r \leqslant b$ for fixed x, y, $x \neq y$.

In passing we would like to mention about this lemma, that it can, with appropriate changes in notation, be proved also for similar equations on other intervals than $(0, b)$. The important point is that the L^1-norm of the kernel is $\leqslant \delta$ on the interval in question. In particular, it is valid for the Marchenko equation on (c, ∞), if c is large enough.

Proof of lemma 2.1. — Iterate eq. (2.5) to obtain the infinite series

$$R(r\,;\,r-s,\,r-t) = -\,G(s,t) + \int_0^r G(s,s_1)G(s_1,t)ds_1$$
$$+ \sum_{n=3}^{\infty}(-1)^n\int_0^r ds_1 \ldots \int_0^r ds_{n-1}G(s,s_{n-1})G(s_{n-1},s_{n-2}) \ldots G(s_1,t) \quad (2.8)$$

Here

$$\int_0^r |\,G(s,t)\,|\,ds \leqslant \int_{-t}^{r-t} |\,H(x)\,|\,dx + \int_t^{r+t} |\,H(x)\,|\,dx \leqslant \delta \qquad (2.9)$$

for any $r \leqslant b$ by the assumption (2.7), and for the n: th term $T_n(r\,;\,s,t)$

$$\int_0^r |\,T_n(r\,;\,s,t)\,|\,ds \leqslant \int_0^r ds_{n-1}\int_0^r ds\,|\,G(s,s_{n-1})T_{n-1}(r\,;\,s_{n-1},t)\,|$$
$$\leqslant \int_0^r ds_{n-1}\left[\int_{-s_{n-1}}^{r-s_{n-1}} |\,H(x)\,|\,dx + \int_{s_{n-1}}^{r+s_{n-1}} |\,H(x)\,|\,dx\right]|\,T_{n-1}(r\,;\,s_{n-1},t)\,|$$
$$\leqslant \delta\int_0^r |\,T_{n-1}(r\,;\,s_{n-1},t)\,|\,ds_{n-1}\,. \qquad (2.10)$$

By induction we thus have that

$$\int_0^r |\,T_n(r\,;\,s,t)\,|\,ds \leqslant \delta^n\,, \qquad (2.11)$$

uniformly in t and $r \leqslant b$, and consequently

$$\int_0^r |\,R(r\,;\,x,y)\,|\,dx \leqslant \sum_{n=1}^{\infty}\delta^n = \frac{\delta}{1-\delta} < \infty\,. \qquad (2.12)$$

Thus each term in the series (2.8) is $L^1(0,b)$ in the x-variable for any y and $r \leqslant b$, and the series converges almost everywhere (Beppo Levi's theorem) to an $L^1(0,b)$-function $R(r\,;\,x,y)$ for any fixed y and $r \leqslant b$.

This function obviously satisfies the integral equation (2.5). That this solution is unique follows from the fact that if $\bar{R}$ is another L^1-solution, then

$$\int_0^r |\,R(r\,;\,x,y) - \bar{R}(r\,;\,x,y)\,|\,dx \leqslant \int_0^r dx\int_0^r dz\,|\,G(r-x,\,r-z)\,|$$
$$|\,R(r\,;\,z,y) - \bar{R}(r\,;\,z,y)\,|$$
$$\leqslant \delta\int_0^r dz\,|\,R(r\,;\,z,y) - \bar{R}(r\,;\,z,y)\,|\,, \qquad (2.13)$$

implying

$$\int_0^r dx\,|\,R(r\,;\,x,y) - \bar{R}(r\,;\,x,y)\,| = 0 \Rightarrow R = \bar{R} \quad \text{a. e.} \qquad (2.14)$$

Let us finally consider the derivative with respect to r of the n: th term of eq. (2.8). It is given by

$$\frac{d}{dr}\mathrm{T}_n(r; x, t) = (-1)^n \sum_{m=1}^{n-1} \left[\int_0^r ds_1 \ldots \int_0^r ds_{m-1} \mathrm{G}(r, s_{m-1}) \ldots \mathrm{G}(s_1, t) \right] \times \left[\int_0^r ds_{m+1} \ldots \int_0^r ds_{n-1} \mathrm{G}(s, s_{n-1}) \ldots \mathrm{G}(s_{m+1}, r) \right]. \quad (2.15)$$

Here, each expression in square brackets is an $\mathrm{L}^1(0, b)$-function in r by the estimates above, eqs. (2.9) and (2.10) for $r = b$. Moreover, if we also assume $\mathrm{H} \in \mathrm{W}_0$, that is that the only singular point of $\mathrm{H}(x)$ is at $x = 0$, then the only singular points of these L^1-functions are at $r = t$ and $r = s$, respectively. If $s \neq t$ it follows that the products, and thus the sum in eq. (2.15) is $\mathrm{L}^1(0, b)$ and, therefore, that each term in eq. (2.8) is absolutely continuous as a function of r on $0 \leqslant r \leqslant b$, provided $x \neq y$. Since the series is absolutely and uniformly convergent, the same result follows for the sum. Q. E. D.

Combining eqs. (2.3) and (2.6) above, we obtain

$$\mathrm{V}(r) = 2\frac{d}{dr}\mathrm{R}(r; 0, 0). \quad (2.16)$$

Unfortunately, the lemma contains no direct information on the derivative with respect to r for $x = y = 0$. However, consider the quantity

$$\mathrm{M}(r; x) = \frac{d}{dr}\mathrm{R}(r; x, 0), \quad (2.17)$$

which, according to the lemma, is L^1 in r for fixed $x > 0$. It satisfies the integral equation

$$\mathrm{M}(r; x) = \mathrm{J}(r; x) - \int_0^r dz \mathrm{G}(r - x, r - z)\mathrm{M}(r; z), \quad (2.18)$$

where

$$\mathrm{J}(r; x) = 2\mathrm{H}'(2r - x) + 2\int_0^r dy \mathrm{H}'(2r - x - y)\mathrm{R}(r; y, 0), \quad (2.19)$$

obtained from eq. (2.5) by differentiation.

Here, the inhomogeneous term $\mathrm{J}(r; x)$ is $\mathrm{L}^1(0, r)$ in x in spite of the strong singularity of H' in the integrand for $x = y = r$. This is connected to the fact that $\mathrm{R}(r; r, 0) = 0$, and can be proved as follows. The only problem being with the integral in eq. (2.19) near the upper end of the integration interval, let us consider

$$\int_{r/2}^r dy \mathrm{H}'(2r - x - y)\mathrm{R}(r; y, 0) = -\mathrm{H}\left(\frac{3}{2}r - x\right)\mathrm{R}\left(r; \frac{1}{2}r, 0\right) - \int_{r/2}^r dy \mathrm{H}(2r - x - y)\frac{\partial \mathrm{R}(r; y, 0)}{\partial y} \quad (2.20)$$

obtained by partial integration. From eq. (2.8) we obtain for this partial derivative

$$\begin{aligned}\frac{\partial R(r\,;y,0)}{\partial y} &= -H'(y) - H'(2r-y)\\ &+ \sum_{n=2}^{\infty}(-1)^n \int_0^r ds_1 \ldots \int_0^r ds_{n-1}\,[H'(y-r+s_{n-1}) + H'(r-y+s_{n-1})]\\ &\times G(s_{n-1}, s_{n-2}) \ldots G(s_1, r)\,, \end{aligned} \tag{2.21}$$

showing that it is strongly singular at $y = 0$ like $H'(y)$, but is otherwise L^1 on the interval needed, that is up to $y = r$. Thus the right-hand side of eq. (2.20) is $L^1(0, r)$ in x, and so is $J(r\,; x)$. Note that J is also $L^1(\varepsilon, b)$ in r, for any positive $\varepsilon < b$ and $x \leqslant r$.

Next we note that eq. (2.18) has the same kernel as eqs. (2.1) and (2.5). Thus, for $r \in (0, b)$, it has a unique solution, which can be expressed in terms of the resolvent kernel R by

$$M(r\,; x) = J(r\,; x) + \int_0^r dz R(r\,; x, z) J(r\,; z)\,. \tag{2.22}$$

From the properties of R and J, obtained above, it follows directly that M is $L^1(0, r)$ in x and $L^1(\varepsilon, b)$, $\varepsilon > 0$, in r for $x \leqslant r$.

Now, according to eqs. (2.16) and (2.17),

$$V(r) = 2M(r\,; 0)\,. \tag{2.23}$$

Employing also eqs. (2.22) and (2.6), we obtain

$$V(r) = 2J(r\,; 0) + 2\int_{\to 0}^r dz K(r\,; r - z) J(r\,; z)\,, \tag{2.24}$$

where we have assumed that the limit $x \to 0$ exists, and can be taken before the integration. On the interval $\varepsilon \leqslant r \leqslant b$, $0 < \varepsilon < b$, where b is chosen according to lemma 2.1, this can be expected to be true almost everywhere, since the functions involved are L^1, and, therefore, the right-hand side of eq. (2.24) is $L^1(\varepsilon, b)$. This is proved in a more direct way in appendix 2.

Using also eq. (2.19), eq. (2.24) can be rewritten in the form

$$V(r) = 4H'(2r) + 8\int_0^r ds K(r, s) H'(s + r) + 4\int_{\to 0}^r ds \int_0^r dt K(r, s) K(r, t) H'(s + t) \tag{2.25}$$

which, to our knowledge, is a new integral representation for the potential as a solution of the inverse problem.

Since eq. (2.25) is deduced from eq. (2.3), the two equations are equivalent when both are well defined, which is the case when the potential (or, equivalently, H′) is absolutely integrable at the origin. However, eq. (2.25) has a larger region of validity, as seen above for $r > \varepsilon$.

In fact, if we construct W from eq. (2.25) by integration, we obtain

$$W(r) = 2H(2r) - 8\int_r^\infty dv \int_0^v ds K(v, s)H'(s + v) - 4\int_r^\infty dv \int_{\to 0}^v ds \int_0^v dt K(v, s)K(v, t)H'(s + t), \quad (2.26)$$

which can be proved to define an $L^1(0, b)$-function for any function H of the class W_0. This is done in appendix 2, and concludes the second and most difficult part of the proof of theorem 1.1, the part concerned with the behaviour of the interaction at the origin.

The restriction to the interval $(0, b)$, where b is defined in lemma 2.1, and can be very small, is unimportant. The interval can easily be extended to $(0, c)$, any $c < \infty$. This follows from dividing H into two parts

$$H(x) = H_1(x) + H_2(x), \quad (2.27)$$

where

$$\begin{cases} H_1(x) = 0 & \text{for} \quad x > b \\ H_2(x) = 0 & \text{for} \quad x < b, \end{cases} \quad (2.28)$$

and applying the transformation operator techniques. Since H_2 is non-singular on any finite interval, only standard steps are involved, so we leave them out and conclude that the only remaining difficulty is at infinity.

3. THE MARCHENKO EQUATION AND THE BEHAVIOUR AT INFINITY. CONCLUSIONS

To study the behaviour of the potential at infinity, we first assume that eq. (1.1) is fulfilled and use the Marchenko equation

$$A(r, t) = \Sigma(r + t) + \int_r^\infty A(r, s)\Sigma(s + t)ds, \qquad t \geqslant r \quad (3.1)$$

where Σ is defined by eq. (1.5). Under this assumption, this equation is known to have a unique L^2 solution for any r, from which the potential can be obtained as

$$V(r) = -2\frac{d}{dr}A(r, r). \quad (3.2)$$

Therefore, define the quantity

$$B(r; x) = \frac{d}{dr}A(r, r + x) \quad (3.3)$$

and differentiate eq. (3.1) to obtain the integral equation

$$B(r\,;x) = I(r\,;x) + \int_0^\infty dz B(r\,;z)\Sigma(2r + x + z)\,, \tag{3.4}$$

where

$$I(r\,;x) = 2\Sigma'(2r + x) + 2\int_0^\infty dy A(r, r + y)\Sigma'(2r + y + x)\,. \tag{3.5}$$

Note in particular, that to obtain $V(r)$ for, say, $r \geq b > 0$, only $\Sigma(t)$ for $t \geq 2b$ is involved. Thus as long as $\Sigma(t)$ is well defined for $t > 0$, its possible singularity at $t = 0$ should not result in any singularity of $V(r)$ for $r > 0$.

In analogy with the previous section, we can define the resolvent kernel of Σ, write the solution of eq. (3.4) in terms of this, and obtain an integral representation for the potential. The result of this procedure is

$$V(r) = -4\Sigma'(2r) - 8\int_r^\infty ds A(r, s)\Sigma'(s + r) - 4\int_r^\infty ds\int_r^\infty dt A(r, s)A(r, t)\Sigma'(s + t)\,, \tag{3.6}$$

with a striking similarity to eq. (2.25).

Like that expression, eq. (3.6) can be used to prove the properties of V and W from the properties of Σ. We do not wish to be involved in any detailed argument again on this point, since the analogy is very close to the previous case. Let us just note, in passing, that according to the arguments at the end of section 2, what remains is to prove that for any $\Sigma \in W_0$, V is $L^1(c, \infty)$ for some $c < \infty$. But this is almost immediate, since for large enough c, it follows from the same kind of arguments as in lemma 2.1 that the iteration series converges and thus that $A(r, s)$ is absolutely continuous in both variables like $\Sigma(r + s)$ on the interval in question, that is $c \leq r \leq s < \infty$. Eq. (3.6) then trivially implies that V is L^1 when Σ' is. Also, W is $L^1(c, \infty)$ when Σ is.

This concludes the proof of the sufficiency of the conditions of theorem 1.1. We repeat the essential steps of this argument.

Assuming that $\Sigma \in W_0$, it follows from lemma 1.1 that also $H \in W_0$. By the argument just given, it follows from eq. (3.6) that $W, W' \in L^1(c, \infty)$, $c < \infty$. In section 2 and appendix 2 it is proved that $W \in L^1(0, b)$, $b > 0$, and $W' \in L^1(\varepsilon, b)$, $\varepsilon > 0$. Since according to the argument at the end of section 2 the finite interval (b, c) presents no difficulty, we conclude that W satisfies conditions (A) and (B), thus is a W_0-function.

The necessity of the conditions is, more or less, already known from earlier work on the direct problem. It can be seen in the following way.

Assuming that $W \in W_0$, it follows from the estimates of ref. 2 that the wave function $\varphi(k, r)$ is at least C^1 in r on any finite interval, say $0 \leq r \leq b$.

Therefore, the function [7]

$$\tilde{\psi}(r, t) = \frac{1}{\pi} \int_0^\infty \left[\varphi(k, r) - \frac{\sin kr}{k} \right] \cos kt dk \tag{3.7}$$

is absolutely continuous in r and t, since the integral is absolutely and uniformly convergent by the same estimates. It then follows that

$$K(r, t) = \frac{\partial \tilde{\psi}(r, t)}{\partial t} \tag{3.8}$$

is $L^1(0, b)$ in both variables, and solving the Gel'fand-Levitan equation for $G(r, t)$ gives, according to lemma 2.1, the same property for this function. Therefore, also H and Σ are $L^1(0, b)$. This is all we need to prove for r near the origin.

Similarly, for r large we need to prove that Σ and Σ' belong to $L^1(c, \infty)$, $c < \infty$, as soon as W and W′ do so. Since eq. (1.1) is not necessarily satisfied, the bounds for $A(r, t)$ of ref. 1 are not finite. Therefore, consider the integral representation [1]

$$A(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk [f(k, r) - e^{ikr}] e^{-ikt}, \tag{3.9}$$

where $f(k, r)$ is the Jost solution. Using here the integral equation for the Jost solution [1], we obtain

$$\begin{aligned} A(r, t) = & -\frac{1}{2} W\left(\frac{r+t}{2}\right) \theta(t - r) \\ & + \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_r^\infty ds_1 \int_{s_1}^\infty ds_2 \frac{\sin k(s_1 - r)}{k} V(s_1) \frac{\sin k(s_2 - s_1)}{k} V(s_2) \\ & \times f(k, s_2) e^{-ikt} . \end{aligned} \tag{3.10}$$

Here, using the bounds for f similar to those for φ, the integral can be proved to be absolutely and uniformly convergent, and thus to represent an absolutely continuous function of r and t. It also goes to zero at infinity in both variables ($t > r$) faster than the first term on the right-hand side of eq. (3.10), so that $A(r, t)$ is both absolutely continuous and integrable for $c \leq r < t \leqslant \infty$. Solving eq. (3.1) for $\Sigma(r + t)$ then gives the same properties for this function, by lemma 2.1, provided c is large enough. This concludes the proof of the necessity of the conditions of theorem 1.1.

4. GENERALIZATIONS

In this paper, we have succeeded in obtaining a precise connection between the properties of the S-matrix and the potential for a class of potentials larger than the class for which a corresponding connection was known before [1] [8]. The generalization obtained concerns the behaviour of the potential at the origin, and is given by the W_0-class, as compared to the old class satisfying eq. (1.1). However, from what has been known for a few years about the generalization of scattering theory to potentials which are not absolutely integrable at infinity [5-6], one would expect that a theorem similar to ours should be true also in that case.

In fact, it seems clear from the present work, that our theorem 1.1 can be translated literally to the case where W is defined by

$$W(r) = - \int_r^{\to\infty} V(s)ds \,, \tag{4.1}$$

and assumed to be absolutely continuous on every finite interval, and to belong to $L^1(0, \infty)$. Indeed, from the integral of eq. (3.6)

$$\begin{aligned} W(r) = & - 2\Sigma(2r) + 8\int_r^{\to\infty} dv \int_v^{\to\infty} ds A(v, s)\Sigma'(s + v) \\ & + 4\int_r^{\to\infty} dv \int_v^{\to\infty} ds \int_v^{\to\infty} dt A(v, s)A(v, t)\Sigma'(s + t) \,, \end{aligned} \tag{4.2}$$

and following the same line of reasoning as in section 3 and appendix 2, these properties of W could be proved from the same properties of Σ, providing us with the crucial part of the proof of the new theorem.

Of course, one can combine the two theorems into a single one, dealing with singularities both at the origin and at infinity.

Similarly, related singular behaviour is allowed for Σ' and W' also at finite points. However, the generalization to this case is not completely trivial. This is so because convolutions tend to move the singularities, so that in order for the $\mathscr{W}$ of the Wiener-Lévy theorem to be a ring, one has to allow singularities at infinitely many points, as soon as one has one singular point other than zero or infinity. We intend to return to this question in a future publication.

APPENDIX 1

In this appendix we study the relation between the Fourier transforms of the functions $S(k)$, $F(k)$, $|F(k)|^{-2}$ and $\delta(k)$ in order to prove lemma 1.1.

We shall consider the following functions:

$$\Gamma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[F(k) - 1]\exp(-ikt) \tag{A1.1}$$

$$\Pi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left[\frac{1}{F(k)} - 1\right]\exp(-ikt) \tag{A1.2}$$

$$H(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\left[\frac{1}{|F(k)|^2} - 1\right]\exp(-ikt) \tag{A1.3}$$

$$\Sigma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk[S(k) - 1]\exp(ikt) \tag{A1.4}$$

$$\gamma(t) = -\frac{2}{\pi}\int_{0}^{\infty} dk\delta(k)\sin(kt). \tag{A1.5}$$

As explained in the introduction, we assume no bound states. This implies among other things that $F(k)$ and $1/F(k)$ are regular in the upper half plane, so that $\Gamma(t) = \Pi(t) = 0$ for $t < 0$.

Lemma 1.1 tells that if any one of the functions Γ, H, Σ and γ belongs to W_0, so do all the others. Our strategy to prove this can schematically be expressed in the following way:

$$\Gamma \Leftrightarrow \Pi \Rightarrow \Sigma \Rightarrow \gamma \Rightarrow \Gamma, \qquad \Pi \Rightarrow H \Rightarrow \Gamma, \tag{A1.6}$$

using, respectively, lemmas A1.1, A1.2, A1.3, A1.4, A1.5 and A1.6, given below.

Our main tool to prove these lemmas is a slightly modified Wiener-Lévy theorem [*10*]. To formulate it, we define $\mathscr{L}$ to be the ring of all functions $\Phi(k)$, representable in the form [*9*]

$$\Phi(k) = C + \int_{-\infty}^{\infty} dt\varphi(t)\exp(ikt) \tag{A1.7}$$

with $\varphi \in L^1(-\infty, \infty)$, and $\mathscr{W}_0$ to be the subring with $\varphi \in W_0$. That $\mathscr{W}_0$ is a subring of $\mathscr{L}$ follows from the fact that the product $\Phi\Psi$ between two elements of $\mathscr{W}_0$ also belongs to $\mathscr{W}_0$ since it is represented by the convolution

$$(\varphi * \psi)(t) = \int_{-\infty}^{\infty} dx\varphi(x)\psi(t-x) = \int_{-\infty}^{t/2} dx\varphi(x)\psi(t-x) + \int_{-\infty}^{t/2} dx\varphi(t-x)\psi(x). \tag{A1.8}$$

From the last form of the convolution it follows immediately that $(\psi * \psi)' \in L^1(\varepsilon, \infty)$, and from a similar form that $(\varphi * \psi)' \in L^1(-\infty, -\varepsilon)$ and thus that $\varphi * \psi \in W_0$ or $\Phi\Psi \in \mathscr{W}_0$.

THEOREM (Wiener-Lévy). — Let $G(z)$ be analytic in a domain D of the complex plane and let $\Phi(k)$ be such that the curve $z = \Phi(k)$, $k \in (-\infty, \infty)$, lies inside D. Then if $\Phi(k)$ belongs to the ring $\mathscr{W}_0$, so does $G(\Phi(k))$.

The usual formulation of the theorem is with the ring $\mathscr{L}$ (see ref. 9, also 1 or 7). From the argument above, that the property of belonging to $\mathscr{W}_0$ is preserved under multiplication, it can be easily seen that the proof given by Akhiezer [*11*] goes through equally well with $\mathscr{W}_0$.

LEMMA A1.1. — $\Gamma \in W_0 \Leftrightarrow \Pi \in W_0$.

Proof. — Follows immediately from the Wiener-Lévy theorem.

LEMMA A1.2. — $\Gamma \in W_0 \Rightarrow \Sigma \in W_0$.

Proof. — From the relation between S and F the following relation follows for their Fourier transforms:

$$\Sigma(t) = \int_0^\infty dx \Pi(x)\Gamma(x+t) + \Pi(-t) + \Gamma(t). \tag{A1.9}$$

Therefore

$$\Sigma'(t) = \begin{cases} \int_0^\infty dx \Pi(x)\Gamma'(x+t) + \Gamma'(t) & \text{for} \quad t > 0 \\ -\int_0^\infty dx \Pi'(x-t)\Gamma(x) - \Pi'(-t) & \text{for} \quad t < 0, \end{cases} \tag{A1.10}$$

and it follows that $\Sigma \in W_0$.

LEMMA A1.3. — $\Pi \in W_0 \Rightarrow H \in W_0$.

Proof. — In analogy with the preceding proof we use the relation

$$H(t) = \int_0^\infty dx \Pi(x)\Pi(x+t) + \Pi(-t) + \Pi(t), \tag{A1.11}$$

and the result follows.

LEMMA A1.4. — $\Sigma \in W_0 \Rightarrow \gamma \in W_0$.

Proof. — The phase shift is given by

$$\delta(k) = \frac{1}{2} i \log S(k) = \frac{1}{2} i \log \frac{1}{S(-k)}. \tag{A1.12}$$

Since we have assumed no bound states, the Levinson theorem tells us that

$$\delta(-\infty) = \delta(\infty) = \delta(0) = 0,$$

and an application of the Wiener-Lévy theorem gives $\gamma \in W_0$.

LEMMA A1.5. — $\gamma \in W_0 \Rightarrow \Gamma \in W_0$.

Proof. — Define the function

$$\Delta(k) = \int_0^\infty \gamma(t) \exp(ikt)dt. \tag{A1.13}$$

Then [1] [7]

$$F(k) = \exp\{\Delta(k)\} \tag{A1.14}$$

and the result follows directly from the Wiener-Lévy theorem.

LEMMA A1.6. — $H \in W_0 \Rightarrow \Gamma \in W_0$.

Proof. — From the knowledge of H we can, for k real, construct

$$\text{Re}\, \Delta(k) = \log |F(k)| = -\frac{1}{2} \log \left[1 + 2\int_0^\infty H(t) \cos(kt)dt \right]. \tag{A1.15}$$

From eq. (A1.13) it follows that $\Delta(k)$ is analytic in the upper half k-plane, continuous in $\text{Im}\, k \geq 0$ and goes to zero at infinity. Therefore, using Hilbert transforms

$$\text{Im}\, \Delta(k) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\text{Re}\, \Delta(k')}{k'-k} dk', \qquad \text{Im}\, k \geqslant 0, \tag{A1.16}$$

the whole analytic function can be constructed. But then we have the functional relationship between F and H, needed to apply the Wiener-Lévy theorem, and the result follows.

APPENDIX 2

In this appendix we demonstrate some properties of functions of the class W_0, and we prove that the W of eq. (2.26) belongs to this class of functions. Let us restate:

DEFINITION. — A function f is said to belong to the class W_0 if

$$\begin{aligned} &(A) \qquad f(x) \in L^1(0, \infty) \\ &(B) \qquad f'(x) \in L^1(\varepsilon, \infty) \quad \text{for any} \quad \varepsilon > 0 . \end{aligned} \tag{A2.1}$$

If $f(-x)$, $x > 0$ is defined, it should also satisfy the same conditions.

LEMMA A2.1. — For any function $f \in W_0$.

$$\lim_{x \to 0} x f(x) = 0 . \tag{A2.2}$$

Proof. — Condition (B) implies that $f(x)$ is absolutely continuous on any interval not containing $x = 0$, and thus $xf(x)$ is continuous, except possibly at $x = 0$. But then we must have

$$| xf(x) | < | \log | x | |^{-1} \qquad \text{on some interval} \qquad | x | \leqslant \varepsilon_1 > 0 \tag{A2.3}$$

since otherwise we would have

$$| xf(x) | \geq | \log | x | |^{-1}$$

on some part Δ of measure > 0 of this interval. If Δ does not contain $x = 0$, we make ε_1 smaller, such that (A2.3) becomes true. If Δ contains the point zero we would have

$$\int_\Delta | f(x) | dx \geq \int_\Delta \frac{dx}{| x \log | x | |} = \infty ,$$

which contradicts condition (A). Thus (A2.3) is true, and the lemma follows.

LEMMA A2.2. — For any two functions $g, h \in W_0$ the integral

$$F_1(x) = \int_x^\infty dz \int_0^z dy g(z - y) h'(z + y) \tag{A2.4}$$

defines an $L^1(0, \infty)$ function.

Proof. — Consider the function

$$G_1(z) = \int_0^z dy g(y) h(2z - y) . \tag{A2.5}$$

Its derivative is given by

$$\begin{aligned} G_1'(z) &= g(z)h(z) + 2\int_0^z dy g(y) h'(2z - y) \\ &= g(z)h(z) - 2F_1'(z) . \end{aligned}$$

Thus

$$F_1(x) = -\frac{1}{2} G_1(x) - \frac{1}{2} \int_x^\infty dz g(z) h(z) , \tag{A2.6}$$

since all the functions go to zero at infinity.

Here, G_1 is $L^1(0, \infty)$ since it is a convolution of two L^1-functions and

$$\int_0^\infty dx \int_x^\infty dz \, | g(z)h(z) | = \int_0^\infty dz \, | zg(z)h(z) | \leq C \int_0^\infty dz \, | h(z) | < \infty ,$$

since $zg(z)$ is continuous and bounded (lemma A2.1) and $h(z)$ is L^1. Thus F_1 is $L^1(0, \infty)$.

Q. E. D.

LEMMA A2.3. — For any two functions g, $h \in W_0$ the integral

$$F_2(x) = \int_x^\infty dz \int_{\to 0}^z dy_1 \int_0^z dy_2 g(z - y_1) g(z - y_2) h'(y_1 + y_2) \tag{A2.7}$$

defines an $L^1(0, \infty)$ function.

Proof. — Consider the function

$$G_2(x) = \int_0^z dy_1 \int_0^z dy_2 g(y_1) g(y_2) h(2z - y_1 - y_2). \tag{A2.8}$$

Its derivative is given by

$$\begin{aligned} G_2'(z) &= 2g(z)\int_0^z dy g(y) h(z - y) + 2\int_0^{\to z} dy_1 \int_0^z dy_2 g(y_1) g(y_2) h'(2z - y_1 - y_2) \\ &= 2g(z)\int_0^z dy g(y) h(z - y) - 2F_2'(z). \end{aligned}$$

Thus

$$F_2(x) = -\frac{1}{2} G_2(x) - \int_x^\infty dz g(z) \int_0^z dy g(y) h(z - y) \tag{A2.9}$$

and it follows that $F_2(x) \in L^1(0, \infty)$, in complete analogy to the preceding proof. Q. E. D.

LEMMA A2.4. — The functions F_1 and F_2 of the preceding two lemmas belong to the class W_0.

Proof. — It remains to prove condition (B). For $z \geq 2\varepsilon$ we can write

$$\begin{aligned} F_1'(z) = &- \int_0^{z/2} dy g(y) h'(2z - y) + g(z)h(z) - g\left(\frac{1}{2}z\right) h\left(\frac{3}{2}z\right) \\ &- \int_{z/2}^z dy g'(y) h(2z - y), \end{aligned}$$

where the integrals are convolutions of L^1-functions and the other terms are absolutely continuous. Thus $F_1'(z) \in L^1(2\varepsilon, \infty)$. For F_2 we write

$$\begin{aligned} F_2'(z) = &- \int_0^z dy_1 g(y_1) \left[\int_0^{z/2} dy_2 g(y_2) h'(2z - y_1 - y_2) - g(z) h(z - y_1) \right. \\ &\left. + g\left(\frac{1}{2}z\right) h\left(\frac{3}{2}z - y_1\right) + \int_{z/2}^z dy_2 g'(y_2) h(2z - y_1 - y_2) \right], \end{aligned}$$

and it follows in the same way that $F_2'(z) \in L^1(2\varepsilon, \infty)$. Q. E. D.

LEMMA A2.5. — For any $H \in W_0$ the solution of the Gel'fand-Levitan equation has the structure

$$K(r, t) = g_1(r - t) + g_2(r, t) \tag{A2.10}$$

where (at least for $0 \leq t \leq r \leq b$, where b is defined in lemma 2.1), $g_1 \in W_0$ and g_2 is absolutely continuous.

Proof. — It is well known that K has the structure

$$K(r, t) = N_{2r}(r - t) - N_{2r}(r + t), \tag{A2.11}$$

where N_{2r} is the solution of Krein's equation [*1*] [*7*]:

$$N_{2r}(x) + H(x) + \int_0^{2r} H(x - y) N_{2r}(y) dy = 0, \qquad 0 \leq x \leq 2r. \tag{A2.12}$$

From lemma 2.1, slighly modified to apply to this equation instead of eq. (2.5), it follows that $N_{2r}(x)$ is $L^1(0, 2b)$ in x and absolutely continuous in r. Furthermore, the only singularity is at $x = 0$, so that $N_{2r}(x)$, like $H(x)$, is absolutely continuous for $x \geq \varepsilon$. Thus we can define the W_0 function

$$g_1(x) = N_{2b}(x), \tag{A2.13}$$

such that

$$\Delta(r\,;\,x) \equiv N_{2r}(x) - g_1(x) \tag{A2.14}$$

satisfies the equation

$$\Delta(r\,;\,x) + \int_0^{2r} H(x-y)\Delta(r\,;\,y)dy = \int_{2r}^{2b} H(x-y)g_1(y)dy\,. \tag{A2.15}$$

Since the right-hand side of this equation is absolutely continuous also in the x-variable, so is $\Delta(r\,;\,x)$ and the lemma follows with

$$g_2(r, t) = \Delta(r\,;\,r-t) - N_{2r}(r+t)\,. \tag{A2.16}$$

LEMMA A2.6. — For any $H \in W_0$, the function $W(r)$, defined by eq. (2.26) is $L^1(0, b)$ and absolutely continuous on any interval (ε, b), $0 < \varepsilon < b$.

Proof. — Because of the structure of K according to lemma A2.5, the most singular part of W is the term $2H(2r)$ and two integrals of the form of lemmas A2.2 and A2.3, respectively, which according to lemma A2.4 belong to the class W_0. The rest of W is expressible in $g_2(r, t)$ of the preceding lemma, and easily shown to be absolutely continuous.

Q. E. D.

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Note added in proof. — To show that $K(r, t)$ defined by (3.8) is $L^1(0, b)$ in both variables, one can also proceed as follows. We define

$$\Psi(r, t) = \frac{2}{\pi}\int_0^{\infty} dk \cos(kt) \int_t^r dr' \left[\varphi(k, r') - \frac{\sin kr'}{k}\right]$$

From the integral equation and the bounds for φ obtained in [2], it follows that the above integral is absolutely and uniformly convergent, and, therefore, that Ψ is an absolutely continuous function in both variables. It then follows that [7]

$$K(r, t) = -\frac{\partial}{\partial t}\tilde{\psi} = -\frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\Psi(r, t)\right]$$

is $L^1(0, b)$ in both variable.

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