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Conditional probability in quantum axiomatics

by

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ABSTRACT. — Two axiom systems for quantum theory are formulated with the aim to improve and complete the « quantum logic » and the « algebraic » axiomatic framework respectively, being presently the two main alternatives for quantum axiomatics. By a careful analysis of the basic properties of the experimental procedures corresponding to quantum-mechanical propositions, two sets of postulates are formulated, connected respectively with the quantum logic and with the algebraic axiomatic scheme. It is shown that within these axiom systems we are able to overcome the old difficulties of the two « classic » axiomatic frameworks mentioned above; in particular, we are in a position to explain the physical meaning of the covering law in quantum logic and to establish the structure of the Jordan-Banach algebra in the set of bounded observables associated with the physical system under study.

INTRODUCTION

In the present paper we formulate two sets of axioms for nonrelativistic quantum mechanics with the aim to improve and complete actually existing axiom systems. Presented here, in Section 1, is a general axiomatic scheme (Axioms 1.1-1.4) which may be developed at least in two directions. The first possibility, described in details in Section 2, is very close to the well known quantum logic approach originated yet by the classic work of Birkhoff and von Neumann [*7*] and later on developed and improved by many writers (see, e. g., Mackey [*39*], [*40*], Zierler [*64*], Piron [*50*], [*51*], Mac Laren [*41*], Jauch [*34*], Ludwig [*38*], Varadarajan [*62*], Maczynski [*42*], [*43*]), while the second outcome of our general axiomatics presented in Section 3, is based on introducing the structure of a partially ordered real vector space in the set O_b of bounded observables and then establishing the Jordan-Banach algebra structure in O_b, the latter being deduced from a set of physically plausible postulates. We thus have obtained in such a way a development of our general axiomatics based on Axioms 1.1-1.4 along the lines of the algebraic axiomatic scheme initiated yet by works of Jordan, von Neumann and Wigner [*36*], [*48*] and later on modified by Segal [*57*], [*58*] (see also Emch [*18*]).

It is well known that in both the « classic » axiomatic frameworks mentioned above, which are known today under the name « quantum logic » and « algebraic approach » respectively, there is a possibility to determine partially the ordinary quantum-mechanical formalism, but it is also well known on the other hand that these axiom systems are plagued by troubles which still remain to be solved. For instance, one still needs to show that the coordinatizing division ring which appears in the representation theorem for the quantum logic is the real, complex or quaternionic number field. Some results in this direction have been obtained by Zierler [*64*], [*65*], Cirelli *et al.* [*11*], [*12*], and others, but the assumptions that have been made to obtain the desired result seem to be extremely unphysical.

The other, more important question of the quantum logic approach, concerns the complete lattice structure of the propositional logic and the other requirements which are characteristic to this axiomatic approach, like the atomicity or the validity of the so-called covering law in the logic of propositions.

The algebraic axiomatic scheme, inaugurated by the Jordan algebra approach of Jordan, Wigner and von Neumann, has perhaps more serious defects than the quantum logic. The main trouble is that the axioms are here formal rather than physical (for instance, there is no physical justification for assuming the distributivity of the Jordan product). The other serious difficulty is connected with the absence of any representation theorem for infinite-dimensional Jordan algebras, which clearly is the case of the algebra of quantum-mechanical (bounded) observables. Clearly, in order to overcome the requirement of the finite dimension it is necessary to introduce certain topological assumptions, and this observation was the starting point of Segal's axiomatic scheme [*57*], [*58*]. However, the question how to deduce from physically motivated axioms imposed on the abstract Segal algebra that the latter consists of the self-adjoint elements of some C*-algebra is still unanswered.

The two axiomatic frameworks described above have been developed and improved by both mathematicians and physicists, but the difficulties mentioned above still remain unsolved. In particular, much work has been done in order to justify the lattice assumption of the quantum logic approach, however, the general conclusion is that there is no empirical basis supporting it (see, e. g., Mac Laren [*41*], Srinivas [*59*]). Nevertheless, it can be shown that the propositional logic is a lattice under some additional assumptions, the most remarkable of which is perhaps the postulate that for any two bounded observables there exists its sum (for details see Mac Laren [*41*] or Gudder [*20*]). However, this additional assumption is in fact beyond the scope of the quantum logic approach, as it is characteristic to the algebraic axiomatic scheme in which the basic object under study is the family of bounded observables.

But, it should be noticed at this moment that there is a possibility to develop other axiom systems, closely related to the quantum logic, in which the questions of the (complete) lattice property and atomicity do not appear so problematic, and are solved by a suitable extension of the propositional logic (see Bugajska *et al.* [*10*], Guz [*23*], [*27*], [*28*]). As regards the covering law, one must say that although many attempts have been made to justify it (see, e. g., Pool [*53*], Jauch and Piron [*35*], Ochs [*49*], Bugajska and Bugajski [*9*], [*10*], Guz [*28*], [*29*], [*30*], [*31*]), the covering law is still left without a satisfactory empirical justification.

In the present paper we attempt to give a justification to the covering law by deducing the latter as a consequence of the physically clear properties of the experimental procedures (« filters ») corresponding to the quantum-mechanical propositions (see Section 2 of the paper). Moreover, both our axiom systems, described respectively in Sections 2 and 3, are based on the observation that one of the basic assumptions underlying any axiomatization of quantum mechanics is that for every observable (and, in particular, for every proposition) there exists a corresponding experimental procedure or a measurement process, which can be carried out on a single physical system (A measuring device used in the experiment corresponding to a given proposition is usually called the *filter* associated with the proposition, and we often identify propositions with its associated filters). This assumption is very close, in its spirit, to the underlying idea of the so-called « operational approach », developed mainly by Ludwig [*38*], Gunson [*21*], Pool [*52*], [*53*], [*54*], Mielnik [*44*], [*45*], Davies and Lewis [*13*], Edwards [*15*], Srinivas [*59*] and others, in which the basic concepts are the partially ordered real vector space spanned by states of a physical system and the set of the so-called operations on this space.

It is the writer's conviction that the most natural continuation of the general axiomatics based on Axioms 1.1-1.4 (see Section 1) is its development in the spirit of the operational approach, that is, in accordance with the operational treatment of propositions in terms of the associated experimental procedures (filters). This leads both to explaining the physical significance of the covering law in quantum logic (see Section 2) and to establishing the structure of the Jordan-Banach algebra in O_b (Section 3), which is obtained on the basis of a set of physically motivated postulates.

Moreover, the quantum logic approach, as modified in Section 2, is here shown to be intimately connected with the Jordan-Banach algebraic scheme, and in this point we are following the pioneering work of Gunson [21], where the connections between quantum logics and Jordan algebras were established for the first time, but, unfortunately, several unprecise statements of Section 4 of Gunson's work made some important conclusions of this paper incorrect.

On the other hand, the ideas and methods of Section 3 of the present paper were stimulated by works of Alfsen *et al.* [3], [5], [6], which are the corner-stone of the noncommutative probability and the noncommutative spectral theory.

1. GENERAL SETTING

1.1. Basic axioms and notation.

Let O be the set of all observables, and S —the set of all states of a given physical system. We do not answer the question what are the observables and states but, after Mackey [40], accept them as the primitive concepts of the theory we will develop.

Following Mackey [40] we assume (R denotes here the real line, R_+ is its nonnegative part, and B(R) stands for the σ-algebra of all Borel subsets of R) :

AXIOM 1.1. — There is a function $p : O \times S \times B(R) \to R_+$ which, for fixed $A \in O$ and $m \in S$, is a probability measure on B(R).

AXIOM 1.2. — If $p(A_1, m, E) = p(A_2, m, E)$ for all $m \in S$ and $E \in B(R)$, then $A_1 = A_2$.

AXIOM 1.3. — If $p(A, m_1, E) = p(A, m_2, E)$ for all $A \in O$ and $E \in B(R)$, then $m_1 = m_2$.

AXIOM 1.4. — For each sequence $m_1, m_2, \ldots$ of states and each sequence $t_1, t_2, \ldots$ of positive real numbers with $\sum_{i=1}^{\infty} t_i = 1$, there exists a state $m \in S$ such that

$$p(A, m, E) = \sum_{i=1}^{\infty} t_i p(A, m_i, E)$$

for all $A \in O$ and $E \in B(R)$.

The physical interpretation of the axioms introduced above is very clear. The number $p(A, m, E)$ gives us the probability that a measurement of an observable A for the system being in a state m yields to a value in a

Borel set E. The probability measure $p(A, m, \cdot)$ is called the *probability distribution of the observable A in the state m.* Axiom 1.2 says that different observables must have different probability distributions in, at least, one state. Axiom 1.3 tells us that our knowledge of the state of a physical system is complete if we know the probability distributions of all observables in this state, so that every state m can be identified with the mapping

$$p_m : A \to p(A, m, \cdot),$$

which to every observable $A \in O$ assigns its probability distribution in the state m. Finally, the state m defined in Axiom 1.4, being uniquely determined by sequences $\{ m_i \}_{i=1}^{\infty}$ and $\{ t_i \}_{i=1}^{\infty}$ (see Axiom 1.3), is interpreted physically as the *mixture* of the states m_i in the proportion $t_1 : t_2 : \ldots$, and denoted by $\sum_{i=1}^{\infty} t_i m_i$. The definition of the « purity » of a state is now standard: m is said to be *pure* if it cannot be written as a nontrivial mixture of two other states; otherwise we call m *mixed.*

An ordered pair $(A, E) \in O \times B(R)$ is customarily identified with the experimentally verifiable proposition (Maczynski [*42*], [*43*]; Mackey [*40*]) saying that « a measurement of an observable A yields to a value in a Borel set E », and the number $p(A, m, E)$ is then interpreted as the probability that the proposition (A, E) is true for the system in the state m.

In the set $O \times B(R)$ one can define two operations (Maczynski [*42*], [*43*]), called the *implication* and the *negation*, respectively :

$$(A, E) \to (B, F) \quad \text{iff} \quad p(A, m, E) \leqslant p(B, m, F) \quad \text{for all } m \in S,$$
$$\neg (A, E) = (A, R \setminus E).$$

We say that two propositions (A, E) and (B, F) are *equivalent*, and write $(A, E) \sim (B, F)$, if $(A, E) \to (B, F)$ and $(B, F) \to (A, E)$, i. e., if

$$p(A, m, E) = p(B, m, F)$$

for every $m \in S$. In other words, two propositions $(A, E), (B, F)$ are considered equivalent if they are equiprobable in any state $m \in S$.

The relation $\sim$ defined above is, clearly, an equivalence relation in $O \times B(R)$, and the set $L = (O \times B(R))/\sim$ of all the equivalence classes of this relation, called the *logic of a physical system* (or the *logic of propositions,* see Maczynski [*42*], Mackey [*40*]), is shown to be a partially ordered set with an involution, provided we define ($|(A, E)|$ stands here for the equivalence class of the proposition (A, E)) :

$$|(A, E)| \leqslant |(B, F)| \quad \text{iff} \quad (A, E) \to (B, F)$$
$$|(A, E)|' = |\neg (A, E)|.$$

Moreover, there exist in L the greatest element $1 = |(A, R)|$ and the least element $0 = |(A, \emptyset)|$ (A being an arbitrary observable) and, obviously, $1' = 0$.

Remark. — The equivalence classes $|(A, E)|$ will also be called *propositions*. We say that two propositions $a = |(A, E)|$ and $b = |(B, F)|$ are *mutually exclusive* or *orthogonal*, and write $a \perp b$, if $a \leqslant b'$. Note that this relation is obviously symmetric, and later on we shall show that it is also irreflexive.

1.2. **The partially ordered vector space spanned by states of a physical system.**

We will use the following notation :

M = the partially ordered real Banach space of bounded signed measures on B(R) (for a definition see, e. g., Yosida [*63*]),

M_+ = the positive cone of M, consisting of bounded measures on B(R),

M_p = the convex set of probability measures on B(R).

The set S of all states of a physical system, after identifying it with the family of mappings $p_m : O \to M_p$, becomes a subset of the vector space M^O (where by M^O is denoted, as usually, the set of all mappings from O to M), which becomes itself a partially ordered real vector space if we define the partial ordering in it by

$$x \leqslant y \quad \text{iff} \quad x(A) \leqslant y(A) \quad \text{for all } A \in O \qquad (x, y \in M^O).$$

The set M_+^O is, clearly, the positive cone of M^O, i. e., $M_+^O = (M^O)_+$. Moreover, M_+^O generates M^O, that is $M^O = M_+^O - M_+^O$.

Let us now consider the subspace W of M^O which consists of all bounded mappings $x : O \to M$, that is

$$W = \{ x \in M^O : \exists_{K_x \in R} \forall_{A \in O} \| x(A) \| \leqslant K_x \},$$

where $\| \cdot \|$ denotes the standard norm in M (for the definition of $\| \cdot \|$ see, e. g., Yosida [*63*]).

If for $x \in W$ we put by definition

$$\| x \| = \sup_{A \in O} \| x(A) \|$$

then W becomes a partially ordered normed vector space being positively generated, i. e., $W = W_+ - W_+$, where W_+ denotes, as usually, the positive cone of W (see Guz [*25*]). Furthermore, since $\| p_m(A) \| = 1$ for all $A \in O$, we have $\| p_m \| = 1$, and therefore $m \to p_m$ is in fact an injection of S into $W_+ \cap S^1$, S^1 being the unit sphere in W.

Consider finally the subspace $V \subseteq W$ spanned by states of a physical system, that is

$$V = \left\{ \sum_{i=1}^{n} s_i p_{m_i} : s_i \in R,\ m_i \in S,\ n = 1, 2, \ldots \right\}.$$

Obviously, $V = V_+ - V_+$, where $V_+ = R_+ . \hat{S}$, $\hat{S} = \{ p_m : m \in S \}$.

Note that the partial ordering induced in V by the proper cone V_+ via the formula

$$x \leqslant y \quad \text{iff} \quad y - x \in V_+ \qquad (x, y \in V)$$

is obviously stronger than the one introduced above.

It can also easily be verified (Guz [*25*]) that the above-defined norm $\|\cdot\|$ is additive on V_+, so $(V, V_+, \|\cdot\|)$ is a space of the type GL_0 (1). However, V_+ is not, in general, norm-closed.

Let us define a new cone in V, including V_+ as a subcone, by

$$V^+ = V \cap M_+^O .$$

It is then not difficult to prove that V^+ is a norm-closed generating proper cone in V, and as a consequence of this fact it will be shown that $(V, V^+, \|\cdot\|)$ is not only GL_0, but actually a GL-space (2).

THEOREM 1.1. — $(V, V^+, \|\cdot\|)$ is a GL-space, that is every positive linear functional on V is norm-continuous.

Proof. — The additivity of the norm $\|\cdot\|$ on V^+ follows by applying the arguments which are in fact the same as that used by Guz [*25*], pp. 153-154.

So, there remains to be proved the second part of the theorem. Since V^+ is norm-closed, it will be sufficient to show that V^+ has a nonempty interior (see, e. g., Schaefer [*56*]). Suppose to the contrary that $\text{Int } V^+ = \emptyset$. Then every point $x \in V^+$ is boundary, which means that for a given $\varepsilon > 0$ there exists $y \in V \setminus V^+$ with $\|x - y\| < \varepsilon$. But $y \notin V^+$ means that $y(A_0) \notin M_+$ for some $A_0 \in O$, so that there exists a Borel set $E_0 \in B(R)$ such that $(y(A_0))(E_0) < 0$, and we then have

$$\begin{aligned} \|x-y\| &= \sup\{\|(x-y)(A)\| : A \in O\} \geqslant \sup\{((x-y)(A))(E) : A \in O, E \in B(R)\} \\ &\geqslant \sup\{(x(A))(E) - (y(A_0))(E_0) : A \in O, E \in B(R)\} \\ &= \sup\{(x(A))(E) : A \in O, E \in B(R)\} - (y(A_0))(E_0) \\ &= \sup\{\|x(A)\| : A \in O\} - (y(A_0))(E_0) \\ &= \|x\| - (y(A_0))(E_0) > \|x\|, \end{aligned}$$

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(1) A normed real vector space X with a generating cone C is said to be a *GL_0-space* (Guz [*26*]) if its norm is additive on C.

(2) A GL_0-space $(X, C, \|\cdot\|)$ is said to be a *GL-space* (Miles [*46*]) if every positive linear functional on X is norm-continuous.

which leads to a contradiction when $x \neq 0$ and when ε is chosen, e. g., as $\| x \|$. (Note that if $x = 0$, then x is clearly boundary). Thus the theorem is proved.

Remark. — We have actually proved above that $\text{Int } V^+ = V^+ \setminus \{0\}$.

Making use of the norm $\| \cdot \|$ one can define another norm in V, being of more direct physical significance. It is defined by setting :

$$\| x \|_1 = \inf \{ \| x_1 \| + \| x_2 \| : x_1, x_2 \in V_+ \, ; \; x_1 - x_2 = x \} .$$

Notice the following properties of $\| \cdot \|_1$ (Guz [*26*]) :

(1) $\| \cdot \|_1$ coincides with $\| \cdot \|$ on V_+.

(2) $\| \cdot \|_1$ is the greatest element in the set of all norms $\| \cdot \|'$ in V for which the states $p_m \in \hat{S}$ are the elements of the unit ball $K^1 = \{ x \in V : \|x\|' \leqslant 1 \}$. In particular, $\| x \| \leqslant \| x \|_1$ for all $x \in V$.

(3) $\| \cdot \|_1$ is the base-norm (see Appendix A for a definition) associated with the strictly positive linear functional d on V defined by

$$d(x) = \| x_1 \| - \| x_2 \| ,$$

where $x_1, x_2 \in V_+$ and $x_1 - x_2 = x$.

Moreover, it is not difficult to show that the space $(V, \| \cdot \|_1)$ is complete, and this is actually implied by the fact that S, the set of states, is closed not only under the formation of finite mixture but also under the formation of countable mixtures (σ-convexity of S, see Axiom 1.4).

Indeed, it is easy to see that the σ-convexity of S implies the following property:

For every monotone increasing sequence $\{ x_n \}_{n=1}^{\infty} \subseteq V_+$ (i. e., satisfying $x_m - x_n \in V_+$, when $n \leqslant m$) such that $\| x_n \|_1$ is bounded above, there exists a unique element $x \in V_+$ such that $x_n \leqslant x$ (all n) and $\| x_n \|_1 \to \| x \|_1$. (Here $x_n \leqslant x$ means that $x - x_n \in V_+$).

However, under the condition above it can be shown (Edwards and Gerzon [*16*]) that $(V, \hat{S})$ is a complete base-norm space, and since $\| \cdot \|_1$ is the base-norm associated with $\hat{S}$, the proof is finished.

Summarizing the results that we have obtained, we can write:

THEOREM 1.2. — $(V, \| \cdot \|_1)$, the real vector space spanned by states of a physical system and endowed with the norm $\| \cdot \|_1$, is a complete base-norm space with a generating proper cone $V_+ = R_+ . \hat{S}$, and with $\hat{S}$ as its base.

Note that the norm $\| \cdot \|_1$, in addition to the advantages of purely mathematical character expressed by properties (1)-(3) and Theorem 1.2, has also a clear physical meaning, since the metric induced by $\| \cdot \|_1$ in the set S of states is easily shown (Guz [*25*]) to be equivalent to the following metric introduced by Gudder [*66*]:

$$\sigma(m_1, m_2) = \inf \{ t \in (0, 1) : (1-t)m_1 + tm'_1 = (1-t)m_2 + tm'_2 \text{ for some } m'_1, m'_2 \in S \} ,$$

whose physical significance is obvious.

For all the reasons mentioned above, the norm $\|\cdot\|_1$ will be called the *natural norm* of the space V.

Remark. — Note that V^+ is $\|\cdot\|_1$—closed, since V^+ is $\|\cdot\|$—closed and $\|\cdot\| \leqslant \|\cdot\|_1$.

1.3. The functional form of the propositional logic.

With each proposition $(A, E) \in O \times B(R)$ one can associate a bounded positive linear functional $q_{(A,E)} : V \to R$, called a *propositional functional*, defined by

$$q_{(A,E)}(x) = (x(A))(E), \qquad x \in V.$$

It can easily be seen (Guz [*25*]) that $\| q_{(A,E)} \| \leqslant 1$ for all $A \in O$ and $E \in B(R)$, and that $q_{(A,E)} = q_{(B,F)}$ if and only if $(A, E) \sim (B, F)$, so that the mapping

$$q : |(A, E)| \to q_{(A,E)} \in V'_+$$

is one-one.

Furthermore, the functional $q_{(A,R)}$ does not depend on A, and it is strictly positive, as $q_{(A,R)} = \|\cdot\|$ on V_+. We denote it by d. Obviously, $d \geqslant q_{(A,E)}$ for all $A \in O$ and $E \in B(R)$.

Moreover, the identification map q defined above preserves also the algebraic structure of the propositional logic L (Guz [*25*]) :

The mapping $q : L \to [0, d]$ is an injection of the propositional logic $(L, \leqslant, ')$ into $([0, d], \leqslant, ')$, the latter endowed with the partial ordering inherited from the order dual (3) (V^p, V^*_+) and with the involution $f \to d - f$, $f \in [0, d]$, that is

$$|(A, E)| \leqslant |(B, F)| \quad \text{iff} \quad q_{(A,E)} \leqslant q_{(B,F)},$$
$$q(|(A, E)|') = d - q_{(A,E)},$$
$$q(0) = 0, \quad q(1) = d.$$

Let us note, finally, that $\| q_{(A,E)} \|_1 \leqslant 1$ (for all $A \in O$, $E \in B(R)$), where $\|\cdot\|_1$ stands for the standard norm in the Banach dual of $(V, \|\cdot\|_1)$. Indeed, since the Banach dual $(V', \|\cdot\|_1)$ coincides with the order dual V^p endowed with the order-unit norm $\|\cdot\|_d$ induced by d, the latter being an order-unit in V^p (3), we get

$$\| q_{(A,E)} \|_1 = \| q_{(A,E)} \|_d = \inf \{ t > 0 : -td \leqslant q_{(A,E)} \leqslant td \} \leqslant 1.$$

Remark. — Let us note that the inequality $\|\cdot\| \leqslant \|\cdot\|_1$ in V implies the converse inequality $\|\cdot\| \geqslant \|\cdot\|_1$ in V'.

Let now $(X, C, \|\cdot\|)$ be an arbitrary GL_0-space. A mapping $h : B(R) \to C^*$, where $C^* = \{ f \in X^* : f \geqslant 0 \text{ on } C \}$, is said to be a *positive-valued measure over* $(X, C, \|\cdot\|)$ (shortly, *p. v. measure*, see Guz [*25*]) if for all $x \in C$ one has

(3) See Appendix A for a definition.

i) $(h(\emptyset))(x) = 0$, $(h(\mathrm{R}))(x) = \| x \|$,

ii) $\left(h\left(\bigcup_{i=1}^{\infty} \mathrm{E}_i\right)\right)(x) = \sum_{i=1}^{\infty} (h(\mathrm{E}_i))(x)$, whenever $\mathrm{E}_j \cap \mathrm{E}_k = \emptyset$ for $j \neq k$.

Note that every p. v. measure h is monotone, that is, $\mathrm{E} \subseteq \mathrm{F}$ (where $\mathrm{E}, \mathrm{F} \in \mathrm{B(R)}$) implies $h(\mathrm{E}) \leqslant h(\mathrm{F})$, i. e., $h(\mathrm{F}) - h(\mathrm{E}) \in \mathrm{C}^*$.

A family $\{ h_j \}_{j \in \mathrm{J}}$ of p. v. measures is said to be *total* (Guz [*25*]) if the set of all functionals $h_j(\mathrm{E})$, where $j \in \mathrm{J}$ and $\mathrm{E} \in \mathrm{B(R)}$, is total.

Clearly, for any fixed observable $\mathrm{A} \in \mathrm{O}$ the mapping $q_{(\mathrm{A},.)} : \mathrm{E} \to q_{(\mathrm{A},\mathrm{E})}$ is a p. v. measure over $(\mathrm{V}, \mathrm{V}_+, \| \cdot \|)$, the GL_0-space spanned by states of a physical system, hence also over $(\mathrm{V}, \mathrm{V}_+, \| \cdot \|_1)$, the latter being the base-norm space associated with $(\mathrm{V}, \mathrm{V}_+, \| \cdot \|)$. Moreover, the map $\mathrm{A} \to q_{(\mathrm{A},.)}$, which to every observable A assigns its p. v. measure, is one-one; hence the set O of all observables may be identified with the family $\{ q_{(\mathrm{A},.)} : \mathrm{A} \in \mathrm{O} \}$ of p. v. measures over V, and finally it can easily be seen that this family is total (Guz [*25*]).

So, one can write the following statement (compare Guz [*25*]) :

If $(\mathrm{O}, \mathrm{S}, p)$ is a triple satisfying Axioms 1.1-1.4, then by using the map $m \to p_m$ we may identify the set S with $\mathrm{V}_+ \cap \mathrm{S}^1$, the intersection of the positive cone $\mathrm{V}_+ = \mathrm{R}_+ . \hat{\mathrm{S}}$ and the unit sphere S^1 of the complete base-norm space $(\mathrm{V}, \mathrm{V}_+, \| \cdot \|_1)$ spanned by $\hat{\mathrm{S}}$, the set O can then be identified with a total family $\{ q_{(\mathrm{A},.)} : \mathrm{A} \in \mathrm{O} \}$ of p. v. measures over V, and finally

$$p(\mathrm{A}, m, \mathrm{E}) = (q_{(\mathrm{A},.)}(\mathrm{E}))(p_m)$$

for all $\mathrm{A} \in \mathrm{O}$, $m \in \mathrm{S}$, and $\mathrm{E} \in \mathrm{B(R)}$.

Conversely, it is not difficult to verify that having a complete base-norm space $(\mathrm{V}, \mathrm{V}_+, \| \cdot \|_1)$, which admits a total family O of p. v. measures over V, and setting $\mathrm{S} = \mathrm{V}_+ \cap \mathrm{S}^1$, $p(\mathrm{A}, m, \mathrm{E}) = (\mathrm{A}(\mathrm{E}))(m)$, where $\mathrm{A} \in \mathrm{O}$, $m \in \mathrm{S}$, $\mathrm{E} \in \mathrm{B(R)}$, we will obtain a triple $(\mathrm{O}, \mathrm{S}, p)$ satisfying all the axioms 1.1-1.4.

The two statements above can be seen as the « representation theorem » for the triple $(\mathrm{O}, \mathrm{S}, p)$ described by Axiom 1.1-1.4. It is, however, too general in order to get any physically relevant information about the physical system which we want to describe.

1.4. The bounded observables.

Let A be an arbitrary observable. The smallest closed set $\mathrm{F} \subseteq \mathrm{R}$ satisfying $p(\mathrm{A}, m, \mathrm{F}) = 1$ for all $m \in \mathrm{S}$ is called the *spectrum* of A and denoted by sp A.

An observable $\mathrm{A} \in \mathrm{O}$ is said to be *bounded* if its spectrum is a bounded set. The number $\sup \{ | t | : t \in \mathrm{sp\,A} \}$ is then called the *spectral norm of A* and denoted by $\| \mathrm{A} \|_{\mathrm{sp}}$. It is not difficult to check that

$$\mathrm{sp\,A} \subseteq [- \| \mathrm{A} \|_{\mathrm{sp}}, \| \mathrm{A} \|_{\mathrm{sp}}] .$$

The set of all bounded observables will be denoted by O_b.

Now let A be an arbitrary observable again, and let $m \in S$. If there exists a finite integral $\int_{-\infty}^{\infty} tp(A, m, dt)$, we shall call it the *expected value* (or *mean value*) of the observable A for the system in the state m, and denote it by $\langle A, m \rangle$.

If A is bounded, then obviously $\langle A, m \rangle \leqslant \| A \|_{sp}$, so that every bounded observable has a finite expected value in all states.

Note finally that with each bounded observable $A \in O_b$ one can associate a linear functional on V defined by

$$L_A(x) = \int_{-\infty}^{\infty} t(x(A))(dt), \qquad x \in V.$$

Obviously, $L_A(p_m) = \langle A, m \rangle$. The notation $\langle A, x \rangle$ will be extended to all $x \in V$, i. e. we put by definition $\langle A, x \rangle = L_A(x)$.

THEOREM 1.3. — For every bounded observable $A \in O_b$ the functional L_A is $\| \cdot \|$ – continuous, and $\| L_A \| \leqslant \| A \|_{sp}$, where $\| \cdot \|$ denotes the standard norm in the Banach dual of $(V, \| \cdot \|)$.

Proof. — Let $x \in V$. Since for each $A \in O$, $x(A)$ is a bounded signed measure on B(R), we have for every $E \in B(R)$ (see, e. g., Dynkin [*14*]) :

$$| x(A) | (E) = \sup \left| \int_E f(t)(x(A))(dt) \right| \tag{1.1}$$

where $| x(A) |$ stands for the total variation of $x(A)$, and sup is taken over all bounded Borel functions from R to R satisfying $\sup_{t \in E} | f(t) | \leqslant 1$.

One can assume without loss of generality that $\| A \|_{sp} \neq 0$, (If $\| A \|_{sp} = 0$, then of course $\| L_A \| = \| L_A \|_1 = 0$). Then for all $x \in V$ we have :

$$\| A \|_{sp}^{-1} \cdot | L_A(x) | = \| A \|_{sp}^{-1} \cdot \left| \int_{sp\,A} t(x(A))(dt) \right|$$
$$= \left| \int_{sp\,A} \| A \|_{sp}^{-1} t(x(A))(dt) \right| \leqslant | x(A) | (sp\,A),$$

where the last inequality follows by (1.1).

Hence

$$\| A \|_{sp}^{-1} \cdot | L_A(x) | \leqslant | x(A) | (R) = \| x(A) \| \leqslant \| x \|,$$

which means that

$$| L_A(x) | \leqslant \| A \|_{sp} \cdot \| x \|, \qquad \text{all } x \in V, \tag{1.2}$$

and this gives us

$$\| L_A \| \leqslant \| A \|_{sp}.$$

The theorem is therefore proved.

As a direct consequence of Theorem 1.3 and the fact that on V′ we have $\|\cdot\| \geq \|\cdot\|_1$, we obtain:

COROLLARY 1.4. — For each $A \in O_b$ the functional L_A is $\|\cdot\|_1$ — continuous.

Let $S_0 \subseteq S$. We shall say that the set S_0 is *sufficiently large* or, shortly, *sufficient* if for every non-void proposition $(A, E) \in O \times B(R)$ there is a state $m \in S_0$ such that $p(A, m, E) = 1$. (A proposition (A, E) is said to be *non-void* if $(A, E) \notin 0 = |(B, \emptyset)|$).

THEOREM 1.5. — If the set S of all states is sufficient, then for every bounded observable $A \in O_b$ we have

$$\|L_A\| = \|L_A\|_1 = \|A\|_{sp}.$$

Proof. — Assume that S is sufficient. Then, we shall prove that

$$\|A\|_{sp} = \sup \{ |L_A(p_m)| : m \in S \}. \tag{1.3}$$

Let us first note that $\|A\|_{sp}$ is an upper bound for all $|L_A(p_m)|$, where $m \in S$. (Indeed, for an arbitrary $m \in S$ we have

$$|L_A(p_m)| \leq \int_{sp\,A} |t|\, p(A, m, dt) \leq \|A\|_{sp}).$$

Since for any bounded observable A its spectrum sp A is a compact subset of R, one can choose $s \in sp\,A$ with $|s| = \sup\{ |t| : t \in sp\,A \} = \|A\|_{sp}$. We then have, for an arbitrary $\varepsilon > 0$, $p(A, m, (s-\varepsilon, s+\varepsilon)) \neq 0$ for at least one state m. This means that the proposition (A, E), where $E = (s - \varepsilon, s + \varepsilon)$, is non-void, so that there exists, by the assumption, $m_1 \in S$ with $p(A, m_1, E) = 1$.

Since $L_A(p_{m_1}) = \int_E tp(A, m_1, dt) \in [s - \varepsilon, s + \varepsilon]$, we have $|L_A(p_{m_1}) - s| \leq \varepsilon$, and therefore

$$\big| |L_A(p_{m_1})| - \|A\|_{sp} \big| = \big| |L_A(p_{m_1})| - |s| \big| \leq |L_A(p_{m_1}) - s| \leq \varepsilon.$$

Hence, in particular,

$$|L_A(p_{m_1})| \geq \|A\|_{sp} - \varepsilon,$$

which proves (1.3).

As a consequence of (1.3) one obtains

$$\|L_A\|_1 = \sup\{ |L_A(x)| : x \in V,\ \|x\|_1 = 1 \} \geq \sup\{ |L_A(p_m)| : m \in S \} = \|A\|_{sp},$$

which together with the opposite inequality $\|L_A\|_1 \leq \|L_A\| \leq \|A\|_{sp}$ proved previously gives us the required equality.

Remark. — Note that the assumption of the sufficiency of the set S of all states, although itself has no clear physical significance, can fortunately be deduced as a consequence of an obvious physical assumption, the so-called « repeatibility hypothesis », which states that the measurement of a proposition repeated immediately will always give the same result.

2. TRANSITION PROBABILITY, PURE FILTERS, AND ALL THAT

2.1. From quantum logic to the phase geometry.

One of the crucial assumptions of the quantum logic approach to the foundations of quantum mechanics is the so-called « orthogonality postulate » (see, e. g., Mackey [*40*]) which asserts the following:

AXIOM 2.1. — If $a_i = |(A_i, E_i)|$, $i = 1, 2, \ldots$, is a sequence of pairwise orthogonal propositions from L, then there is a proposition $a = |(A, E)|$ such that

$$p(A, m, E) = \sum_{i=1}^{\infty} p(A_i, m, E_i)$$

for all $m \in S$.

Having assumed Axioms 1.1, 1.2, 1.3 and 2.1 we are in a position to prove (Maczynski [*42*]) that the propositional logic $(L, \leqslant, ')$ becomes then an orthomodular σ-orthoposet, that is, an orthomodular σ-orthocomplete orthocomplemented partially ordered set with 0 and 1 (the least upper bound for an orthogonal sequence $\{ a_i \}_{i=1}^{\infty} \subseteq L$ is given by the proposition $a \in L$ defined above in Axiom 2.1).

Moreover (see Maczynski [*42*]), any state $m \in S$ can then be identified with the probability measure μ_m on L defined by $\mu_m(|(A, E)|) = p(A, m, E)$, and every observable $A \in O$ — with the L-valued measure x_A (that is, x_A is a σ-homomorphism from B(R) to L) defined by $x_A(E) = |(A, E)|$. Furthermore, we have

$$p(A, m, E) = \mu_m(x_A(E)),$$

and the family $\{ \mu_m : m \in S \}$ of all the probability measures associated with states of a physical system is easily seen to be order determining.

Thus, the propositional logic L appears now as a primary object of the theory, and the sets of states and observables become secondary, as they arise here as some constructions (namely, the probability measures on L and the L-valued measures respectively) that we have built on L.

Remark. — After we identify the states with the corresponding probability measures on L, we shall write $m(a)$ instead of $\mu_m(a)$, where $a \in L$. Moreover, since $\mu_m(|(A, E)|) = q_{(A,E)}(p_m)$, we get, after performing the identification $m \leftrightarrow \mu_m \leftrightarrow p_m$ and $|(A, E)| \leftrightarrow q_{(A,E)}$, the formula

$$m(a) = a(m),$$

where $m \in S$, $a \in L$. In this section, however, we shall prefer the notation $m(a)$.

Now, we shall introduce the concept of a pure state not by simply stating that it is an extreme point of the set S of all states, but by specifying all the essential properties that we expect to be satisfied by the set of all pure states.

We assume the following (Guz [*27*], [*31*]):

AXIOM 2.2. — There exists a subset $P \subseteq S$ whose members, called pure states, are assumed to satisfy the following requirements:

i) For every non-zero proposition $a \in L$ there is a pure state $p \in P$ such that $p(a) = 1$.

ii) If for every pure state $p \in P$ satisfying $p(a) = 1$ we also have $p(b) = 1$, where $a, b \in L$, then $a \leqslant b$.

iii) For each pure state $p \in P$ there is a proposition $a \in L$ such that $p(a) = 1$ and $q(a) < 1$ for all pure states q distinct from p.

Note that the name « pure state » for a member of the set P satisfying the conditions *i*)-*iii*) above is fully justified, since one can easily verify (Guz [*31*]) that every p from P is an extreme point of the σ-convex set of probability measures on L spanned by P.

As concerns the assumptions *i*)-*iii*) above, one can say that for the first time *i*) and *iii*) have been assumed as postulates by Mac Laren [*41*], and *ii*) by Gudder [*19*]. Their physical meaning is clear; for instance, the assumption *iii*) asserts asserts that pure states may be realized in the laboratory, since *iii*) tells us that there is a measuring device answering the experimental question (described by $a \in L$ in *iii*)): « Is the physical system in the pure state p ? ». The interpretation of the other assumptions, *i*) and *ii*), is obvious.

It has been shown (Guz [*31*]) that having assumed Axioms 1.1, 1.2, 1.3 and 2.1, we are in a position to prove the equivalence of Axiom 2.2 with the following statement:

The propositional logic L is atomistic (i. e., L is atomic and each $a \in L$ is the least upper bound of the atoms contained in it), and there is a bijection $s : P \to A(L)$ of the set P of all pure states onto the set A(L) of all atoms in L such that, for every $p \in P$,

(1) $p(s(p)) = 1$,

(2) $p(a) = 1$, where $a \in L$, implies $a \geqslant s(p)$.

The atomic proposition $s(p)$ is called the *carrier* or *support of* p (Zierler [*64*], Pool [*53*]), and it is also denoted by carr p or supp p.

Let now $m_1, m_2 \in S$. We shall say that states m_1 and m_2 are *mutually exlusive* or *orthogonal* (Gudder [*19*]), and write $m_1 \perp m_2$, if for some proposition $a \in \mathrm{L}$ one has $m_1(a) = 1$ and $m_2(a) = 0$. Note that this orthogonality relation is, obviously, symmetric.

The pair $(\mathrm{P}, \perp)$, where $\perp$ stands for the above-defined orthogonality restricted to P, plays a very essential role in quantum axiomatics. However, before seeing the significance of $(\mathrm{P}, \perp)$, one needs to introduce some definitions.

For any subset $\mathrm{M} \subseteq \mathrm{P}$ we define $\mathrm{M}^{\perp}$ to be the set of all pure states $p \in \mathrm{P}$ such that $p \perp q$ for all $q \in \mathrm{M}$, and write M^{-} instead of $\mathrm{M}^{\perp\perp}$. Clearly $\mathrm{M} \subseteq \mathrm{M}^{-}$, and if $\mathrm{M} = \mathrm{M}^{-}$, we call the set M *closed.* The family $\mathrm{C}(\mathrm{P}, \perp)$ of all closed subsets of P is called the *phase geometry* associated with a physical system (Guz [*23*]).

It can easily be shown (Guz [*27*]) that, under set inclusion, $\mathrm{C}(\mathrm{P}, \perp)$ becomes an atomistic complete lattice with joins and meets given by

$$\bigvee_j \mathrm{M}_j = \Big(\bigcup_j \mathrm{M}_j\Big)^{-}, \qquad \bigvee_j \mathrm{M}_j = \bigcap_j \mathrm{M}_j$$

($\{ \mathrm{M}_j \}$ being an arbitrary family of closed subsets of P), and that $\mathrm{C}(\mathrm{P}, \perp)$ is orthocomplemented by the correspondence $\mathrm{M} \to \mathrm{M}^{\perp}$ $(\mathrm{M} \in \mathrm{C}(\mathrm{P}, \perp))$. For the empty set $\emptyset$ we put, by definition, $\emptyset^{\perp} = \mathrm{P}$, which leads immediately to $\emptyset, \mathrm{P} \in \mathrm{C}(\mathrm{P}, \perp)$.

It has been shown the following embedding theorem (Guz [*27*]):

For every $a \in \mathrm{L}$ the set $a^1 = \{ p \in \mathrm{P} : p(a) = 1 \}$ belongs to $\mathrm{C}(\mathrm{P}, \perp)$, and the correspondence $a \to a^1$ defines an orthoinjection of the propositional logic L into the phase geometry $\mathrm{C}(\mathrm{P}, \perp)$, the latter being an atomistic complete orthocomplemented lattice.

Now, the importance of the concept of phase geometry is easily seen, since the theorem above answers some old questions connected with the quantum logic approach, like the question of the complete lattice structure of the propositional logic L or its atomisticity (see also the theorem on page 76).

Let now m_1, m_2 be two arbitrary states of a physical system. The number

$$(m_1 : m_2) = \inf \{ m_1(a) : a \in \mathrm{L},\ m_2(a) = 1 \}$$

will be called the *degree of dependence of* m_1 *on* m_2 (Guz [*24*]).

The number $(m_1 : m_2)$ was introduced independently several years ago by Mielnik [*44*] under the name « transition probability between m_1 and m_2 ». In this paper, however, we shall refer to $(m_1 : m_2)$ as to *transition probability from* m_1 *to* m_2 only when both m_1 and m_2 are pure states.

Note that when m_1, m_2 are the ordinary quantum-mechanical pure states, i. e. the rays in a complex Hilbert space H, the number $(m_1 : m_2)$ gives us the transition probability between m_1 and m_2. If m_1, m_2 are mixed states (density operators in a Hilbert space H), the number $(m_1 : m_2)$

coincides then with the so-called semi-inner product between m_1 and m_2 (see Guz [24]) defined by Kossakowski [37] as follows:

$$[m_1, m_2] = \| m_2 \| \operatorname{tr}(m_1 \operatorname{sign} m_2),$$

where tr stands for the trace, and $\| \cdot \|$ denotes the trace-norm in the Banach space of the trace-class operators in H defined by $\| m \| = \operatorname{tr}(mm^*)^{\frac{1}{2}}$.

It can easily be shown (Guz [31]) that in the general axiomatic framework described by Axioms 1.1, 1.2, 1.3, 2.1 and 2.2 the transition probability between pure states is given by the formula

$$(p : q) = p(s(q)), \qquad p, q \in \mathrm{P}.$$

The transition probability is thus evidently nonsymmetric with respect to the variables $p, q \in \mathrm{P}$.

Finally, the following properties of the transition probability can easily be verified (Guz [31]):

i) $0 \leqslant (p : q) \leqslant 1$ for all $p, q \in \mathrm{P}$,
ii) $(p : q) = 0$ iff $p \perp q$,
iii) $(p : q) = 1$ iff $p = q$.

2.2. Conditioning of states, pure filters and covering law.

Our main goal in this subsection is to describe a very important class of operations acting on the set of states, the so-called conditional probability mappings. We begin, for simplicity, with the standard Hilbert space model of quantum mechanics, and we will follow the generally accepted assertion of the quantum theory of measurements, which assumes the following (this is in fact the famous projection postulate of von Neumann):

If the initial state of a physical system is described by the density operator m, and, after a measurement performed on the system, the proposition described by the projection operator P is verified to be true, then the subsequent state of the system is described by the density operator

$$m_{\mathrm{P}} = \mathrm{P}m\mathrm{P}/\operatorname{tr}(\mathrm{P}m).$$

Thus, if we ignore the normalization of the state, we obtain a linear mapping $m \to \mathrm{P}m\mathrm{P}$, called the *conditional probability mapping* (as m_{P} is the state of the system conditioned by the fact of the occurrence of the « event » P), from the Banach space of the self-adjoint trace-class operators (acting on the Hilbert space corresponding to the quantum-mechanical system under study) into itself, which is positive, i. e. preserves the cone of positive elements.

If one wishes to define conditional probability mappings $\mathrm{E}_a : m \to m_a$ ($a \in \mathrm{L}$) in the general framework of the axiomatic scheme described by Axioms 1.1-1.4, 2.1 and 2.2, their basic properties extracted by a careful analysis of the standard Hilbert space model become then the following (Pool [52]):

(1) The domain $D(E_a)$ of the mapping E_a consists of those $m \in S$ for which $m(a) > 0$, and for every $m \in D(E_a)$ we have $(E_a m)(a) = 1$.

(2) If $m(a) = 1$, then $E_a m = m$.

(3) If m is pure and $m \in D(E_a)$, then so is $E_a m$.

The collection of the properties (1)-(3) above is not complete in the sense that it is not sufficient to prove physically significant results for these axiomatically defined conditional probability mappings, so we usually need some additional requirements for E_a, as for example (Pool [52]):

(4) If $a \leqslant b$ $(a, b \in L)$ and $m \in D(E_b)$, then $(E_b m)(a) = m(a)/m(b)$.

There is, however, another interesting way to specify all the crucial properties of the conditional probability mappings E_a, which is based upon the concept of the transition probability introduced in Subsection 2.1.

We shall assume, for simplicity, that each E_a is defined on the set of pure states only (this is the case of the pure filters, see condition (3) above), and require for E_a the properties (1), (2).

Any collection $\{ E_a \}$ of transformations of the set P of pure states into itself, indexed by non-zero propositions from L and satisfying the conditions (1), (2), will be called the *family of (pure) filters* (or *pure conditional probability mappings*) associated with the propositional logic L.

We shall come back, for the moment, to the transition probability between pure states. In order to study the properties of mappings E_a it will be convenient to extend the transition probability function $(:)$ onto the set $P_0 = P \cup \{ 0 \}$, where 0 denotes the improper « pure » state, called the *zero state*, adjoined to P and defined as the zero function on L (i. e., $0(a) = 0$ for all $a \in L$). We put, by definition, $(0 : p) = (p : 0) = 0$ for all $p \in P_0$.

Now, one can extend every E_a onto a whole set P_0 by setting (here $p \in P_0$ and $a \in L$; the latter runs over all the propositions from L):

$$\tilde{E}_a p = \begin{cases} E_a p, & \text{if } p(a) > 0\,, \\ 0, & \text{if } p(a) = 0\,. \end{cases} \tag{2.1}$$

Note that $\tilde{E}_0 = 0$. In the sequel the tilde over E_a will be omitted, as this does not lead to a confusion.

Keeping in mind the obvious interpretation of the number $(p : p_a)$ as the probability that the proposition $a \in L$ is true (or, in other words, as the probability of the occurrence of the « event » $a \in L$) for the system being in a pure state p, that is, expecting the validity of the equality $(p : p_a) = p(a)$, one can readily translate the properties (1), (2) of the conditional probability mappings into the language of the transition probability. This consists of the following:

(1′) If $(p : p_a) > 0$, then $(p_a : (p_a)_a) = 1$.

(2′) If $(p : p_a) = 1$, then $p_a = p$.

Let us note that the assumption (2′) is actually superfluous, since it follows from the axioms we have assumed, as we have shown the following general property of the transition probability: $(p:q) = 1 \Rightarrow p = q$.

Note also that the property (1′) is equivalent to the assertion that every $E_a : P_0 \to P_0$ is an idempotent, i. e., that $E_a^2 = E_a$. Indeed, having assumed (1′) one gets

$$E_a^2 p = (p_a)_a = p_a = E_a p$$

when $(p:p_a) > 0$, and by definition (2.1)

$$E_a^2 p = E_a 0 = 0 = E_a p$$

when $(p:p_a) = p(a) = 0$.

Conversely, if $E_a^2 = E_a$ and $(p:p_a) = p(a) > 0$, then the latter implies $p_a \neq 0$, and therefore $(p_a:(p_a)_a) = (p_a:p_a) = 1$.

Note, finally, that from definition (2.1) it follows that

(3′) $(p:p_a) = 0$ implies $p_a = 0$.

To be more precise, we now introduce the following definition:

A mapping E_a ($E_a : p \to p_a$) of the set $P_0 = P \cup \{0\}$ into itself is said to be a *(pure) filter associated with the proposition* $a \in L$ if it satisfies the following conditions:

i) $(p:p_a) = p(a)$ for all $p \in P_0$;
ii) E_a is an idempotent mapping;
iii) $(p:p_a) = 0$ implies $p_a = 0$.

Let us remark that any filter E_a possesses the following property:

$$(p:p_a) \geqslant (p:q_a) \tag{2.2}$$

for all $p, q \in P_0$.

Indeed, if $q_a \neq 0$, then $q_a(a) = (q_a:q_a) = 1$, so that

$$(p:q_a) = \inf\{p(b) : b \in L, q_a(b) = 1\} \leqslant p(a) = (p:p_a).$$

Note that the inequality (2.2) is physically obvious. Indeed, p_a is the final state to which the initial state p goes, after the proposition $a \in L$ has been verified to be true, so that the transition probability $(p:q_a)$ has to attain its maximum for $q_a = p_a$. Moreover, it is clear that the inequality (2.2) would be strict if $q_a \neq p_a \neq 0$, and this leads us to the following definition, and next to the Axiom 2.3 below:

A filter E_a is said to be *proper* if the inequality (2.2) becomes strict, whenever $p_a \neq 0$ and $p_a \neq q_a$.

AXIOM 2.3. — With every nonzero proposition $a \in L$ there is associated a proper pure filter $E_a : P_0 \to P_0$.

The axiom above, although physically obvious, leads to strong restrictions on the propositional logic L. In particular, having assumed this postulate we are in a position to prove (Guz [*31*]) that the logic L possesses then the so-called *covering property* (or *covering law*), which asserts the following:

For each $a \in L$ and each atom $e \in L$ there exists in L their least upper bound $a \vee e$, and if $e \nleq a$, then $a \vee e$ covers a, i. e., $a \vee e \geqslant b \geqslant a$ implies either $b = a$ or $b = a \vee e$.

Moreover, the phase geometry $C(P, \perp)$, which can be identified with the completion by cuts of the propositional logic L (Guz [*28*]), is then shown (Guz [*28*]) to be not only atomistic but also orthomodular and satisfying the covering law, so that our Axioms 1.1-1.3 and 2.1-2.3, implying the above-mentioned properties of $C(P, \perp)$, are sufficient to deduce the well-known Piron-Mac Laren's Hilbert space representation theorem for $C(P, \perp)$ (see Piron [*50*], Mac Laren [*41*]; also Varadarajan [*62*]), and thus for L also, provided of course we assume that $C(P, \perp)$ (or, equivalently, L) is irreducible and of the projective dimension not smaller than 4.

Note that the irreducibility of $C(P, \perp)$ is not a severe restriction, as if it does not hold, then any irreducible part of $C(P, \perp)$ may be taken into consideration instead of the whole $C(P, \perp)$. Moreover, the irreducibility of $C(P, \perp)$ has also a direct physical significance, as it can be closely related to the so-called « superposition principle » (for details, see Guz [*22*], [*23*]; also Pulmannova [*55*]).

Note, finally, that each proper filter E_a can be identified with the corresponding *Sasaki projection* $s_a : A(L) \cup \{0\} \to A(L) \cup \{0\}$ defined by

$$s_a(e) = a' \vee e - a' = (a' \vee e) \wedge a,$$

where $a \in L$, $e \in A(L) \cup \{0\}$ (see Guz [*32*]; here $\wedge$ stands, as usually, for the greatest lower bound in L, and A(L) denotes the set of all atoms in L).

More precisely,

$$p_a = E_a p = s^{-1}(a' \vee s(p) - a'), \tag{2.3}$$

so that the map $E_a : p \to p_a$ can be determined as the composition

$$E_a = s^{-1} s_a s$$

(with the convention that $s(0) = 0$).

As a consequence of (2.3) one easily finds that for $e \in A(L)$

$$p_e = \begin{cases} s^{-1}(e), & \text{if } p(e) \neq 0, \\ 0 & , \text{if } p(e) = 0. \end{cases}$$

Often, it is convenient to extend E_a onto the set

$$\bar{P} = R_+ . P = \{ tp : t \in R_+, p \in P \}$$

of all unnormalized pure states. The extended E_a will be denoted by P_a and defined as follows (Guz [*30*]):

$$P_a g = \begin{cases} g(a) E_a(g/g(1)), & \text{if } g \neq 0, \\ 0 & , \text{if } g = 0, \end{cases}$$

where $g \in \bar{P}$.

Note that the final state of the physical system, after the proposition $a \in \mathrm{L}$ is verified to be true, is in fact not $\mathrm{E}_a m$ but the unnormalized one $m(a)\mathrm{E}_a m$ (where m is the initial state of the system) with the intensity diminished by the factor $m(a)$, the latter being the probability of the occurrence of the « event » $a \in \mathrm{L}$ in the state m, so that the physical significance can be attached to the map P_a rather than to E_a, and we usually have in mind just P_a when we speak about the (pure) filter associated with $a \in \mathrm{L}$.

It is not difficult to prove the following properties of P_a (Guz [*30*]):

(1) $(\mathrm{P}_a g)(a) = g(a) = (\mathrm{P}_a g)(1) = \| \mathrm{P}_a g \|$
(2) $a \leqslant b \Rightarrow \mathrm{P}_b \mathrm{P}_a = \mathrm{P}_a$
(3) $a \perp b \Rightarrow \mathrm{P}_a \mathrm{P}_b = 0$
(4) P_a is positively homogeneous, that is

$$\mathrm{P}_a(tg) = t\mathrm{P}_a g$$

for all $t \geqslant 0$ and $g \in \bar{\mathrm{P}}$.

Moreover, it can easily be seen that the implications in (2) and (3) can be reversed, that is

(2′) $\mathrm{P}_b \mathrm{P}_a = \mathrm{P}_a \Rightarrow a \leqslant b$
(3′) $\mathrm{P}_a \mathrm{P}_b = 0 \Rightarrow a \perp b$.

Indeed, let $p(a) = 1$, where $p \in \mathrm{P}$. (Note that it can be assumed without loss of generality that $a \neq 0$, so a pure state p satisfying $p(a) = 1$ exists by Axiom 2.2 *i*)). Then, clearly, $p = p_a \neq 0$, so that $\mathrm{P}_a p = p$, and therefore $p(b)p_b = \mathrm{P}_b p = \mathrm{P}_b \mathrm{P}_a p = \mathrm{P}_a p = p$, which leads to $p(b) = 1$. So, we have shown that $p(a) = 1$ implies always $p(b) = 1$ (where $p \in \mathrm{P}$); hence $a \leqslant b$ by Axiom 2.2 *ii*).

Similarly we prove that $\mathrm{P}_a \mathrm{P}_b = 0$ implies $a \perp b$. In fact, suppose that $p(b) = 1, p \in \mathrm{P}$. Then, as before, $\mathrm{P}_b p = p$, so we have $0 = \mathrm{P}_a \mathrm{P}_b p = \mathrm{P}_a p = p(a)p_a$, hence either $p(a) = 0$ or $p_a = 0$, the latter implying also $p(a) = (p : p_a) = 0$. Thus we have shown that $p(b) = 1$ leads always to $p(a') = 1$, which means, as before, that $b \leqslant a'$ or $a \perp b$.

Remark. — $\| \cdot \|$ in (1) stands for an arbitrary norm of V among those which were defined in Section 1, V being the real vector space spanned by states of a physical system, since all these norms coincide on the positive cone V_+ of V, hence also on $\bar{\mathrm{P}}$.

2.3. **Dual filters on the vector space spanned by atomic propositions.**

Let us now consider, after Gunson [*21*], the vector space (L) defined as the linear span of the image of the propositional logic L under the canonical embedding $q : \mathrm{L} \to \mathrm{V}'$ defined by $q : |(\mathrm{A}, \mathrm{E})| \to q_{(\mathrm{A},\mathrm{E})}$ (see Section 1), where V denotes the complete base-norm space spanned by states of a physical system. In the sequel we will omit q, so that any proposition $|(\mathrm{A}, \mathrm{E})| \in \mathrm{L}$ will be identified with the corresponding functional $q_{(\mathrm{A},\mathrm{E})}$.

Similarly, we denote by (L_f) the linear span of the set $L_f \subseteq L$ of all finite elements (4) of L. Since every finite proposition $a \in L_f$ can be written as a (finite) join of pairwise orthogonal atoms, we see that

$$(L_f) = (A(L)) = \text{the linear span of the set of all atoms in } L.$$

Finally, we shall denote by U the metric completion of $(A(L)) + R1$, i. e., the norm closure of $(A(L)) + R1$ in V'. (Note that 1, the greatest element of L, is here identified with the order-unit functional $d \in V'$).

Define now, after Gunson [*21*], the following pseudoproduct $\circ$ for atomic propositions:

$$e \circ f \doteq \frac{1}{2}(Q_e - Q_{e'} + I)f \in (L_f),$$

where Q_a $(a \in L)$ is defined by

$$Q_a e = p^e(a)s_a(e), \qquad e \in A(L),$$

with p^e defined as $p^e = s^{-1}(e)$, and I stands for the identity map.

Therefore,

$$e \circ f = \frac{1}{2}(p^f(e)e - p^f(e')(e \vee f - e) + f). \tag{2.4}$$

Notice the following properties of the pseudoproduct $\circ$:

PROPOSITION 2.1. — Let e, f be two atomic propositions from L. Then we have:

i) $e \circ e = e$

ii) $e \perp f$ iff $e \circ f = 0$

iii) $e \circ f = f \circ e$ iff $(p^e : p^f) = (p^f : p^e)$.

Proof. — The statement *i)* follows as an immediate consequence of (2.4).

In order to prove *ii)*, let us note that $e \perp f$ implies $p^f(e) = 0$ and $e \vee f - e = f$, so that $e \circ f = 0$. Conversely, let us assume that $e \circ f = 0$. Then we get

$$0 = p^e(e \circ f) = \frac{1}{2}(p^f(e)p^e(e) - p^f(e')p^e(e \vee f - e) + p^e(f)) = \frac{1}{2}(p^f(e) + p^e(f));$$

hence $p^e(f) = 0$, which leads to $e \perp f$, as claimed.

We shall finally prove *iii)*. Let us observe that

$$e \circ f = \frac{1}{2}(p^f(e)(e \vee f) - e \vee f + e + f),$$

so we have

$$e \circ f = f \circ e \quad \text{iff} \quad p^f(e) = p^e(f) \quad \text{iff} \quad (p^f : p^e) = (p^e : p^f),$$

as required. Our statement is therefore proved.

---

(4) A proposition $a \in L$ is said to be *finite* if it is a join of a finite number of atoms.

Every atomic proposition $e \in A(L)$ represents, as we know, some *pure test* which answers the experimental question: « Is the physical system in the pure state $p = s^{-1}(e)$? ». More precisely, the interpretation of an atomic proposition as a pure test requires in fact the operational treatment of this proposition in terms of the corresponding filtering procedure, and this consists of the following (see Subsection 2.2):

If, after a measurement performed on a physical system being initially in the pure state p the atomic proposition $e \in L$ (such that $p(e) > 0$) has been verified to be true, then the system is necessarily found to be in the pure state $p_e = s^{-1}(e)$. More generally, if the initial state of a physical system is given by the mixture $m = \sum_i t_i p_i$ (t_i are here positive real numbers such that $\left.\sum_i t_i = 1\right)$ and if the atomic proposition $e \in L$ (satisfying $m(e) > 0$) is verified to be true, then the subsequent (unnormalized) state of the system is clearly (see (2.5))

$$\sum_i t_i P_e p_i = \sum_i t_i p_i(e) s^{-1}(e) = t s^{-1}(e),$$

where $t = \sum_i t_i p_i(e)$, so that the normalized final state is $s^{-1}(e)$.

We thus see that here the final state of a physical system does not depend on its initial state m (we have in fact two possibilities for the final state: $s^{-1}(e)$ when $m(e) > 0$, and 0 when $m(e) = 0$), and this is the reason for which we often say that « the pure test $e \in L$ (or, to be more precise, the pure filter E_e corresponding to e) *prepares* the physical system in the pure state $p = s^{-1}(e)$ ».

Every linear combination $u = \sum_i t_i e_i$ of pure tests e_i with positive real coefficients $t_i > 0$ satisfying $\sum_i t_i = 1$ will be thought as a *mixed test* answering the question: « Is the physical system in the mixed state $m = \sum_i t_i p_i$ with $p_i = s^{-1}(e_i)$? ». More generally, each linear combination $u = \sum_i t_i e_i$ with positive coefficients which are not subjected to any additional condition, will be meant as an *unnormalized mixed test* preparing the physical system in the unnormalized mixed state $x = \sum_i t_i p_i$

with the intensity $\| x \| = \sum_i t_i$, where as before $p_i = s^{-1}(e_i)$. We then alternatively say that the physical system is in the pure state p_i with the probability $t_i \big/ \sum_k t_k$.

Now suppose that the physical system is initially in the unnormalized mixed state $m = \sum_i t_i p_i$, $t_i > 0$, prepared by the test $u = \sum_i t_i e_i$, i. e., that the system is initially found to be in the pure state $p_i = s^{-1}(e_i)$ with the probability $s_i = t_i \big/ \sum_k t_k$. Suppose next that after a measurement performed on the system the proposition $a \in \mathrm{L}$ has been verified to be true, so that we find as the possible final states of the system the pure states $\mathrm{E}_a p_i$, occuring clearly with the probabilities

$$s_i' = s_i(p_i : \mathrm{E}_a p_i) \Big/ \sum_k s_k(p_k : \mathrm{E}_a p_k) = t_i p_i(a) \Big/ \sum_k t_k p_k(a) .$$

In other words, the final (unnormalized) mixed state of the system is

$$m' = \sum_i t_i p_i(a) \mathrm{E}_a p_i , \tag{2.5}$$

and therefore the (unnormalized) mixed test preparing the system in the state (2.5) can be written as

$$u' = \sum_i t_i p_i(a) s(\mathrm{E}_a p_i) = \sum_i t_i \mathrm{Q}_a e_i$$

We shall denote this mixed test by $\hat{\mathrm{Q}}_a u$. Thus we have

$$\hat{\mathrm{Q}}_a u = \sum_i t_i \mathrm{Q}_a e_i$$

and since for every atomic proposition $e \in \mathrm{L}$ we have $\hat{\mathrm{Q}}_a e = \mathrm{Q}_a e$, the equality above can be rewritten as

$$\hat{\mathrm{Q}}_a \Big(\sum_i t_i e_i \Big) = \sum_i t_i \hat{\mathrm{Q}}_a e_i ,$$

so we have obtained in such a way an affine (i. e., additive and positively homogeneous) extension of Q_a.

Summarizing the heuristic considerations presented above we arrive at the following postulate:

AXIOM 2.4. — Each Q_a ($a \in L$) can be extended to an affine mapping $\hat{Q}_a : (L_f)_+ \to (L_f)_+$, where $(L_f)_+$ is the generating cone in (L_f) defined by $(L_f)_+ = \left\{ \sum_{i=1}^{n} t_i e_i : t_i \geqslant 0,\ e_i \in A(L),\ n = 1, 2, \ldots \right\}$.

Notice that $\hat{Q}_a$ can easily be extended to a linear mapping $T_a : (L_f) \to (L_f)$ by setting:

$$T_a u = \hat{Q}_a u_1 - \hat{Q}_a u_2$$

whenever $u = u_1 - u_2$, where $u_1, u_2 \in (L_f)_+$.

It is an easy matter to check that the definition above does not depend on any particular choice of $u_1, u_2 \in (L_f)_+$ in the decomposition of $u \in (L_f)$.

We shall call the mappings T_a the *dual filters* associated with propositions from L.

2.4. **Symmetry of the transition probability and the Segal algebra structure in U.**

It is a common belief of physicists that in the world of microphenomena there is no reason for the asymmetry of the transition probability between pure states. We thus accept the following postulate:

AXIOM 2.5. — For any pair p, q of pure states we have $(p : q) = (q : p)$.

Note that as a consequence of the axiom above one obtains the commutativity of the pseudoproduct $\circ$ (see Proposition 2.1).

Axiom 2.5 has several interesting consequences, and perhaps the most important is that the space U, defined as the norm closure of $(L_f) + R1$ in the order-unit space $(V', 1)$, becomes then a distributive Segal algebra.

More precisely, we shall prove the following statement:

THEOREM 2.2. — The pseudoproduct $\circ$ can be extended to a commutative product on U such that $(U, \circ, 1)$ becomes a distributive Segal algebra with 1 acting on it as the unit element.

Before proving the theorem we need some lemmas.

LEMMA 2.3. — Each $u \in (L_f)$ can be written in the form

$$u = \sum_{i=1}^{n} t_i e_i\,, \qquad t_i \in R\,,$$

with pairwise orthogonal atoms e_i, $i = 1, 2, \ldots$.

Remark. — The lemma above is due to Gunson [*21*], and in our axiomatic framework it can be proved by repeating the arguments of Gunson, so its proof will be omitted.

Note that as a consequence of Lemma 2.3 it follows that the positive cone $(L_f) \cap V'_+$ of (L_f) coincides with the cone $(L_f)_+$ defined on page 86.

LEMMA 2.4. — If $e \in L$ is an atomic proposition, then the linear mapping $W_e : (L_f) \to (L_f)$ defined by

$$W_e = \frac{1}{2}(T_e - T_{e'} + I)$$

has the following property:

$$W_e 1 = e, \qquad \text{whenever } 1 \in (L_f).$$

Proof. — Let us suppose that $1 \in (L_f)$. Then, by applying Lemma 2.3 one can write $1 = \sum_{i=1}^{n} t_i e_i$, where e_i are pairwise orthogonal atomic propositions, so that for $p_i = s^{-1}(e_i)$ we get

$$1 = p_i(1) = t_i, \qquad i = 1, 2, \ldots .$$

So, we have $1 = \sum_{i=1}^{n} e_i \in L_f$, and therefore by applying the orthomodularity of L we find that every $a \in L$ must be finite.

Now, let us note that $T_e 1 = e$. Indeed,

$$T_e 1 = \sum_{i=1}^{n} T_e e_i = \sum_{i=1}^{n} p_i(e)(e' \vee e_i - e') = \sum_{i=1}^{n} p_i(e) e,$$

where the last equality follows from the fact that $e_i \perp e$ implies $p_i(e) = 0$, but, owing to the symmetry of the transition probability, we get

$$p_i(e) = (p_i : p_e) = (p_e : p_i) = p_e(e_i),$$

so that

$$T_e 1 = \sum_{i=1}^{n} p_e(e_i) e = p_e(1) e = e.$$

We therefore obtain

$$W_e(1 - T_e 1) = \frac{1}{2}(T_e - T_{e'} + I) e' = 0$$

with the last equality derived as a consequence of the equalities $T_e e' = 0$ and $T_{e'} e' = e'$, which in turn depend essentially on the fact that e' is finite (see above), the latter implying that e' is a join of a finite number of pairwise orthogonal atoms.

But, at the same time,

$$W_e(1 - T_e 1) = W_e 1 - \frac{1}{2}(T_e - T_{e'} + I)T_e 1 = W_e 1 - \frac{1}{2}(T_e 1 + T_e 1) = W_e 1 - e,$$

so we obtain

$$W_e 1 = e,$$

as claimed. Our lemma is therefore proved.

We shall now pass on to the proof of Theorem 2.2, and we will follow the arguments used by Alfsen *et al.* [5].

Proof of Theorem 2.2. — First, we shall extend each W_e, e being an atom, onto the space $(L_f) + R1$ (where the part $R1$ is necessary when $1 \notin (L_f)$) by setting

$$W_e(u + t1) = W_e u + te, \quad u \in (L_f).$$

Note that W_e is well-defined, since the equality $u + t1 = v + s1$ (where $u, v \in (L_f)$) leads immediately to $t = s$, whenever $1 \notin (L_f)$, hence also $u = v$, and therefore $W_e(u + t1) = W_e(v + s1)$. Obviously, W_e is linear again.

Let now $u \in (L_f) + R1$, say, $u = \sum_i t_i e_i + t1$. (If $1 \in (L_f)$, we put $t = 0$).

Define the linear operator W_u acting from $(L_f) + R1$ into itself by setting

$$W_u = \sum_i t_i W_{e_i} + tI,$$

where I stands, as usually, for the identity operator.

Note that the correspondence $u \to W_u$ is well-defined. Indeed, for an arbitrary atom $e \in A(L)$ we have

$$\left(\sum_i t_i W_{e_i} + tI\right)e = \sum_i t_i(e_i \circ e) + te = W_e\left(\sum_i t_i e_i\right) + tW_e 1 = W_e u,$$

so that the result does not depend on any particular representation of u, and by linearity the same will hold for an arbitrary $v \in (L_f) + R1$ in place of e, since one easily gets for $v = \sum_j s_j f_j + s1$ that

$$\left(\sum_i t_i W_{e_i} + tI\right)v = \left(\sum_j s_j W_{f_j} + sI\right)u.$$

As a corollary one obtains

$$W_u v = W_v u$$

for all $u, v \in (L_f) + R1$.

Let us now define (for $u, v \in (L_f) + R1$):

$$u \circ v = W_u v .$$

The definition above is clearly an extension of the definition of the pseudo-product $\circ$ given previously for atoms only. Note also that for an arbitrary u from $(L_f) + R1$ we have $u \circ 1 = u$.

We shall now verify the norm requirements for the Segal algebra. Let $u = \sum_{i=1}^{n} t_i e_i + t1$, where $e_i (i = 1, 2, \ldots, n)$ are pairwise orthogonal atoms (see Lemma 2.3). Let us rewrite u as

$$u = \sum_{i=1}^{n} s_i e_i + s_{n+1} a ,$$

where $a = 1 - \sum_{i=1}^{n} e_i = \left(\bigvee_{i=1}^{n} e_i \right)'$, $s_i = t_i + t$, $s_{n+1} = t$, so that $e_1, \ldots, e_n, a$ are now all orthogonal.

We shall first prove that

$$\| u \| = \max \{ | s_i | : 1 \leqslant i \leqslant n + 1 \} ,$$

where $\| \cdot \|$ denotes the order-unit norm of the order-unit space $(V', 1)$. (Note that the subspace $((L_f) + R1, 1, \| \cdot \|)$ and its norm-closure U are then also order-unit spaces).

We have by definition (see Appendix A):

$$\| u \| = \inf \{ s > 0 : - s1 \leqslant u \leqslant s1 \} ,$$

but if $u = \sum_i t_i e_i + t1 \in s[-1, 1]$, then for every state $m \in S$ we get

$$-(s + t) \leqslant \left(\sum_i t_i e_i \right)(m) \leqslant s - t , \tag{2.6}$$

so that by setting $m = s^{-1}(e_i)(i = 1, 2, \ldots, n)$ one obtains for all $i = 1, 2, \ldots, n$

$$| s_i | = | t_i + t | \leqslant s ,$$

which leads to $\max \{ | s_i | : 1 \leqslant i \leqslant n \} \leqslant \inf \{ s \} = \| u \|$.

If we assume that $1 \notin (L_f)$, we then have $a \neq 0$, so that there exists $m \in S$

with $a(m) = 1$; hence $e_i(m) = 0$ for all $i = 1, 2, \ldots, n$, and one then finds by applying (2.6) that $|t| = |s_{n+1}| \leqslant s$. Therefore, we have

$$\max \{ |s_i| : 1 \leqslant i \leqslant n + 1 \} \leqslant \| u \| .$$

Let us denote the number $\max \{ |s_i| : 1 \leqslant i \leqslant n + 1 \}$ by M; we shall show that $\| u \| = \mathrm{M}$.

We have

$$- \mathrm{M}e_i \leqslant s_i e_i \leqslant \mathrm{M}e_i \quad \text{for all } i = 1, 2, \ldots, n,$$

and

$$- \mathrm{M}a \leqslant s_{n+1}a \leqslant \mathrm{M}a ,$$

hence

$$- \mathrm{M}1 = - \mathrm{M}\left(\sum_{i=1}^{n} e_i + a\right) \leqslant u = \sum_{i=1}^{n} s_i e_i + s_{n+1}a \leqslant \mathrm{M}\left(\sum_{i=1}^{n} e_i + a\right) = \mathrm{M}1,$$

which shows that

$$\| u \| \leqslant \mathrm{M} .$$

We thus have proved that $\| u \| = \mathrm{M}$, as claimed.

We shall now show that for every $u \in (\mathrm{L}_f) + \mathrm{R}1$

$$\| u^2 \| = \| u \|^2 . \tag{2.7}$$

Let u be represented as before, that is

$$u = \sum_{i=1}^{n} t_i e_i + t1 = \sum_{i=1}^{n} s_i e_i + s_{n+1}a$$

with pairwise orthogonal atoms $e_1, e_2, \ldots, e_n$.

Then

$$u^2 = \sum_{i=1}^{n} s_i^2 e_i + s_{n+1}^2 a , \tag{2.8}$$

so that by the result above

$$\| u^2 \| = \max \{ |s_i^2| : 1 \leqslant i \leqslant n+1 \} = (\max \{ |s_i| : 1 \leqslant i \leqslant n+1 \})^2 = \| u \|^2 .$$

Let us next observe that since all squares u^2, where $u \in (\mathrm{L}_f) + \mathrm{R}1$, are positive (see (2.8)), the inequality

$$\| u^2 - v^2 \| \leqslant \max \{ \| u^2 \|, \| v^2 \| \} , \qquad u, v \in (\mathrm{L}_f) + \mathrm{R}1 , \tag{2.9}$$

follows as a consequence of the following general inequality

$$\| x - y \| \leqslant \max \{ \| x \|, \| y \| \}$$

valid for all positive elements x, y of any order-unit space (Alfsen [*1*]).

The proof of the inequality

$$\| u \circ v \| \leqslant \| u \| . \| v \|$$

is now straightforward and follows the well-known path.

Indeed, since

$$u \circ v = \frac{1}{4}((u + v)^2 - (u - v)^2),$$

we get by using (2.9) and (2.7)

$$\| u \circ v \| \leqslant \frac{1}{4} \max \{ \| u + v \|^2, \| u - v \|^2 \} \leqslant \frac{1}{4}(\| u \| + \| v \|)^2 . \quad (2.10)$$

Suppose first that $\| u \|$, $\| v \| \leqslant 1$; then the inequality (2.10) implies

$$\| u \circ v \| \leqslant 1 .$$

Thus, for arbitrary (non-zero) u, $v \in (\mathrm{L}_f) + \mathrm{R}1$ one has

$$1 \geqslant \| (\| u \|^{-1} u) \circ (\| v \|^{-1} v) \| = \| u \|^{-1} \| v \|^{-1} \| u \circ v \|,$$

so that

$$\| u \circ v \| \leqslant \| u \| . \| v \|, \quad (2.11)$$

as required.

Now, as a direct consequence of the inequality

$$\| u^2 - v^2 \| = \| (u + v) \circ (u - v) \| \leqslant \| u + v \| . \| u - v \|$$

we obtain the norm continuity of the function $u \to u^2$.

It, finally, remains to be shown that the usual rules for operating with polynomials in a single variable are here valid, i. e., that if f, g, h are polynomials with real coefficients such that $f(g(t)) = h(t)$ for all real t, then $f(g(u)) = h(u)$ for all $u \in (\mathrm{L}_f) + \mathrm{R}1$, where

$$f(u) = 1 + \sum_{k \geqslant 1} s_k u^k \text{ if } f(t) = \sum_{k \geqslant 0} s_k t^k ,$$

and u^k is defined inductively as $u^k = u \circ u^{k-1}$. This is, however, almost obvious, for if we have $u = \sum_{i=1}^{n} s_i e_i + s_{n+1} a$ with pairwise orthogonal atoms $e_1, \ldots, e_n$ and with $a \in \mathrm{L}$ orthogonal to all e_i, then

$$f(g(u)) = \sum_{i=1}^{n} f(g(s_i))e_i + f(g(s_i))a = h(u) .$$

Note finally that by the inequality (2.11) the product $u \circ v$ becomes norm-continuous on $(\mathrm{L}_f) + \mathrm{R}1$, so it can be extended by continuity to $\mathrm{U} = ((\mathrm{L}_f) + \mathrm{R}1)^-$, and U is easily shown to be a distributive Segal algebra for this product.

The proof of the theorem is therefore complete.

2.5. Pool's axiom on conditional probability and the Jordan identity.

We shall assume in this subsection the Pool's axiom on conditional probability (see Subsection 2.2) for pure states only, that is, we postulate the following:

AXIOM 2.6. — If $a \leqslant b$ and $p(b) > 0$ (where $a, b \in L$, $p \in P$), then

$$p_b(a) = p(a)/p(b).$$

Note that the axiom above can equivalently be rewritten as follows:

$$a \leqslant b \Rightarrow p(a) = (P_b p)(a), \tag{2.12}$$

where the equality above is clearly valid for all $p \in P$.

Remark. — As a consequence of (2.12) one obtains for all $p \in P$ (see Subsection 2.2)

$$p(a) = (P_a p)(a) = \| P_a p \|. \tag{2.13}$$

Notice also the following equivalent form of the condition (2.12):

$$\| P_a p \| = \| P_a P_b p \| \quad \text{for all } p \in P. \tag{2.14}$$

Here $\| \cdot \|$ stands for the base-norm (« natural norm ») of the vector space V spanned by states of a physical system defined in Subsection 1.2.

Having assumed the validity of the Pool's axiom one can deduce several interesting consequences. For instance, we can prove the following statement.

THEOREM 2.3. — If $a, b \in L$ are compatible, $a \leftrightarrow b$, then for each pure state $p \in P$ one has

$$p(a \wedge b) = (P_a p)(b) = (P_b p)(a).$$

Proof. — Let $a, b \in L$ be compatible, that is

$$a = a_1 \vee c, \quad b = b_1 \vee c,$$

where a_1, b_1, c are pairwise orthogonal propositions from L.

Then

$$(P_a p)(b) = (P_a p)(b_1) + (P_a p)(c).$$

Since $b_1 \perp a$, we obtain

$$(P_a p)(b_1) = p(a) p_a(b_1) \leqslant p(a) p_a(a') = 0,$$

so that

$$(P_a p)(b_1) = 0,$$

and since $c \leqslant a$, we obtain by using (2.12)

$$(P_a p)(c) = p(c).$$

We thus have

$$(P_a p)(b) = p(c),$$

which proves the theorem, since $c = a \wedge b$ (see, e. g., Varadarajan [*61*]).

COROLLARY 2.4. — Axiom 2.6 is equivalent to the following assumption:

$$a \leftrightarrow b \Leftrightarrow \forall_{p \in P} \| P_a P_b p \| = \| P_b P_a p \|. \tag{2.15}$$

Proof. — The implication from Axiom 2.6 to (2.15) follows as a consequence of Theorem 2.3 and (2.13). The converse implication is almost trivial, since $a \leqslant b$ leads always to $a \leftrightarrow b$, and then also $P_b P_a = P_a$ (see Subsection 2.2), so we see that (2.15) implies (2.14), the latter being equivalent to Axiom 2.6.

Now we shall pass on to another important consequence of the Axiom 2.6. We shall show that this axiom implies the validity of the Jordan identity for U, the Segal algebra spanned by atomic propositions and 1.

Before proving this one needs some lemmas.

LEMMA 2.5. — If e is an atom, and $e \leqslant a$ ($e, a \in L$), then $P_e P_a = P_e$.

Proof. — One needs to show that

$$P_e P_a p = P_e p \tag{2.16}$$

for all pure states $p \in P$.

We shall consider two cases. If $P_e P_a p = 0$, then we obtain by applying (2.13) and (2.12)

$$0 = \| P_e P_a p \| = (P_a p)(e) = p(e);$$

hence

$$P_e p = p(e) p_e = 0,$$

so (2.16) holds.

If $P_e P_a p \neq 0$, we have $(p_a)_e \neq 0$ and $p_e \neq 0$ (the latter follows by applying (2.14)), so that $(p_a)_e(e) = 1$, which leads to $(p_a)_e = s^{-1}(e) = p_e$. Therefore

$$P_e P_a p = P_e(p(a)p_a) = p(a)p_a(e)(p_a)_e = p(a)p_a(e)p_e = p(e)p_e = P_e p,$$

since by (2.12)

$$p(a)p_a(e) = p(e),$$

and we see that (2.16) holds again.

Thus we have shown that for all $p \in P$

$$P_e P_a p = P_e p,$$

as required.

LEMMA 2.6. — If e, f are two orthogonal atoms, then $P_{e'} P_{f'} = P_{f'} P_{e'}$.

Proof. — We must show, as before, that for all $p \in P$

$$P_{e'} P_{f'} p = P_{f'} P_{e'} p,$$

i. e., that

$$p(f')p_{f'}(e')(p_{f'})_{e'} = p(e')p_{e'}(f')(p_{e'})_{f'} .$$

Note that $e \perp f$ implies, by Axiom 2.6, that $p(e) = p(f')p_{f'}(e)$, so we get

$$p(f')p_{f'}(e') = p(f')(1 - p_{f'}(e)) = p(f') - p(e) = p(f' \wedge e') .$$

By symmetry reason we obviously have

$$p(e')p_{e'}(f') = p(e' \wedge f') = p(f')p_{f'}(e') . \tag{2.17}$$

Let us now observe that if $P_{e'}P_{f'}p = 0$, then also $P_{f'}P_{e'}p = 0$, since $P_{e'}P_{f'}p = 0$ implies (see (2.17))

$$|| P_{e'}P_{f'}p || = p(e')p_{e'}(f) = p(f')p_{f'}(e') = || P_{f'}P_{e'}p || = 0 .$$

One can therefore assume that $P_{e'}P_{f'}p \neq 0$ (hence also $P_{f'}P_{e'}p \neq 0$), and in view of (2.17) it remains to be shown that $(p_{f'})_{e'} = (p_{e'})_{f'}$ for all p satisfying the requirement above.

We have

$$p_{f'} = s^{-1}((f \vee g) \wedge f') ,$$

where $g = s(p)$, so that

$$(p_{f'})_{e'} = s^{-1}((e \vee s^{-1}(p_{f'})) \wedge e') = s^{-1}((e \vee [(f \vee g) \wedge f']) \wedge e') ,$$

and hence

$$s((p_{f'})_{e'}) = (e \vee [(f \vee g) \wedge f']) \wedge e' \leqslant f' \wedge e' = (f \vee e)' ,$$

since $e \leqslant f'$.

But, on the other hand,

$$s((p_{f'})_{e'}) \leqslant e \vee f \vee g .$$

Therefore

$$s((p_{f'})_{e'}) \leqslant (e \vee f \vee g) \wedge (e \vee f)' = s(p_{(e \vee f)'}) ;$$

hence

$$s((p_{f'})_{e'}) = s(p_{(e \vee f)'}) ,$$

since $s((p_{f'})_{e'})$ and $s(p_{(e \vee f)'})$ are both atoms (remind that $(p_{f'})_{e'} \neq 0$), so that $(p_{f'})_{e'} = p_{(e \vee f)'}$.

By symmetry reason we also have $(p_{e'})_{f'} = p_{(f \vee e)'} = (p_{f'})_{e'}$, which completes the proof of the lemma.

As a direct consequence of lemmas 2.5 and 2.6 we obtain:

COROLLARY 2.7. — If $a \in L$ and if e is an atom such that $e \leqslant a$, then $T_eT_a = T_e$. If e, f are orthogonal atoms, then $T_{e'}T_{f'} = T_{f'}T_{e'}$.

Proof. — It is not difficult to check that for all $e \in A(L)$ and all $a, b \in L$

$$T_bT_ae = p^e(a)(E_ap^e)(b)s(E_bE_ap^e) ,$$

and the formula above can readily be rewritten as

$$T_bT_ae = \bar{s}(P_bP_ap^e) , \tag{2.18}$$

if we extend the support mapping s to the positively-homogeneous map $\bar{s} : \bar{\mathrm{P}} \to \mathrm{R}_+\mathrm{A(L)}$ by setting

$$\bar{s}(g) = \begin{cases} g(1)s(g/g(1)), & \text{if } g \neq 0 \\ 0 & , \text{if } g = 0, \end{cases}$$

where $g \in \bar{\mathrm{P}} = \mathrm{R}_+\mathrm{P}$.

By using (2.18) we can now deduce the statements of our corollary as immediate consequences of lemmas 2.5 and 2.6.

LEMMA 2.8. — If e, f are orthogonal atoms, then $\mathrm{W}_e\mathrm{W}_f = \mathrm{W}_f\mathrm{W}_e$.

Proof. — By using Corollary 2.7 and the fact that $e \perp f$ implies $\mathrm{T}_e\mathrm{T}_f = \mathrm{T}_f\mathrm{T}_e = 0$, we obtain

$$\mathrm{W}_e\mathrm{W}_f = \frac{1}{4}(\mathrm{T}_{e'}\mathrm{T}_{f'} - \mathrm{T}_{e'} - \mathrm{T}_{f'} + \mathrm{I}) = \mathrm{W}_f\mathrm{W}_e,$$

as claimed.

We shall now prove the following theorem, whose proof is essentially that of Alfsen *et al.* ([5], Prop. 6.11).

THEOREM 2.9. — If we assume the validity of the Pool's axiom (Axiom 2.6), then $\mathrm{U} = ((\mathrm{L}_f) + \mathrm{R}1)^-$ endowed with the product $\circ$ becomes a Jordan algebra.

Proof. — First we shall prove that (L_f) is a Jordan algebra with respect to the product $\circ$. We need to verify that the Jordan identity

$$(u^2 \circ v) \circ u = u^2 \circ (v \circ u)$$

holds for all u, $v \in (\mathrm{L}_f)$.

Let

$$u = \sum_i t_i e_i, \qquad v = \sum_j s_j f_j,$$

where e_i, $f_j \in \mathrm{A(L)}$, and assume that the atoms e_i ($i = 1, 2, \ldots, n$) are pairwise orthogonal (see Lemma 2.3). Then by using Proposition 2.1 we get

$$u^2 = \sum_i t_i^2 e_i,$$

so that

$$(u^2 \circ v) \circ u = \sum_{i,j,k} t_i^2 s_j t_k (e_i \circ f_j) \circ e_k$$

and

$$u^2 \circ (v \circ u) = \sum_{i,j,k} t_i^2 s_j t_k e_i \circ (f_j \circ e_k).$$

Hence

$$(u^2 \circ v) \circ u - u^2 \circ (v \circ u) = \sum_{i,j,k} t_i^2 s_j t_k (W_{e_k} W_{e_i} - W_{e_i} W_{e_k}) f_j ,$$

which is zero by Lemma 2.8.

Now it is an easy matter to verify that also $(L_f) + R1$ becomes a Jordan algebra (with identity 1) with respect to the product $\circ$, and finally, since the product $\circ$, being norm-continuous on $(L_f) + R1$, extends by continuity to $U = ((L_f) + R1)^-$, the Jordan identity is also easily verified for U. The proof of the theorem is thus complete.

Now, as a consequence of theorems 2.9 and 2.2 we obtain the main result of this subsection:

THEOREM 2.10. — $U = ((L_f) + R1)^-$ endowed with the product $\circ$ and with the order-unit norm inherited from V′, where V is the base-norm space spanned by states of a physical system, becomes a real Jordan-Banach algebra.

The result above is of great importance for quantum axiomatics, as there recently has been proved the GNS-type representation theorem for Jordan-Banach algebras (Alfsen *et al.* [*4*]). More precisely, it was shown in [*4*] that every real Jordan-Banach algebra A possesses a unique norm-closed Jordan ideal J such that A/J has a faithful representation as a Jordan algebra of self-adjoint operators on a complex Hilbert space, while every irreducible representation of A not anihilating J is onto M_3^8, the exceptional Jordan algebra consisting of 3×3 hermitean matrices over the octonions (Cayley numbers).

3. CONDITIONAL PROBABILITY IN THE ALGEBRAIC APPROACH

3.1. The linear structure in O_b.

We begin with introducing a linear structure in the set O_b of bounded observables by postulating some physically obvious axioms concerning the mean values of bounded observables (see Mackey [*40*]; also Emch [*18*]).

AXIOM 3.1. — If A, $B \in O_b$ and $\langle A, m \rangle = \langle B, m \rangle$ for all states $m \in S$, then A = B.

AXIOM 3.2. — *i*) For each pair A, $B \in O_b$ there exists an observable $A + B \in O_b$ such that

$$\langle A + B, m \rangle = \langle A, m \rangle + \langle B, m \rangle$$

for all states $m \in S$.

ii) For every bounded observable $A \in O_b$ and every real number $t \in R$ there is an observable $tA \in O_b$ satisfying

$$\langle tA, m \rangle = t \langle A, m \rangle$$

for all $m \in S$.

iii) There exist observables $O, I \in O_b$ such that

$$\langle O, m \rangle = 0, \qquad \langle I, m \rangle = 1$$

for all $m \in S$.

The first axiom above (Axiom 3.1) expresses the fact that the set S of all states is sufficiently large: there exist (by Axiom 3.1) sufficiently many states in S in order to distinguish the bounded observables by measurements of its mean values in all these states. Thus, owing to Axiom 3.1 every bounded observable $A \in O_b$ can be identified with the corresponding mean value functional L_A (see Section 1).

By Axiom 3.1 the observables $A + B$, tA, O, and I defined in Axiom 3.2 are determined uniquely, so that the operations of addition and multiplication by real scalars are well-defined. It is not difficult to verify that Axiom 3.2 introduces in the set O_b of bounded observables the structure of a real vector space. Moreover, after identifying each $A \in O_b$ with L_A we obtain in O_b the structure of a partially ordered vector space inherited from V′, the Banach dual of the space V spanned by states of a physical system, and O_b becomes in fact an order-unit space (with I acting as an order unit), since V was shown to be a base-norm space (see Section 1 for details).

Therefore, the problem of the Hilbert space representation for O_b can be formulated as follows (Guz [*25*]):

Under what assumptions about (X, C), an order-unit space (X, C) may be identified with a subspace of the real part of some C*-algebra (or, equivalently, some B*-algebra)?

Before we shall formulate one of the possible answers to the question above, we will need some definitions.

Let A be a *-algebra (i. e., a complex algebra with an involution $* : A \to A$) with the unit e, and let us denote

H(A) = the real part of A, i. e. the real vector space consisting of all self-adjoint elements of A,

$C_0(A)$ = the cone in H(A) consisting of all finite sums of the form $\sum_i x_i^* x_i$, where $x_i \in A$,

C(A) = $\{ x \in H(A) : f(x) \geqslant 0$ for all $f \in H(A)^*$ which are nonnegative on $C_0(A) \}$.

We shall say that A is a *D-algebra* (compare Miles [*46*]) if (H(A), C(A)) is an order-unit space with e acting as an order unit.

The following theorems hold (Miles [46], Naimark [47]):

(1) A is a D-algebra if and only if there exists a *-monomorphism j of A into some B*-algebra B such that

i) $j(e)$ is the unit of B,

ii) every linear functional defined on $j(A)$ and nonnegative on $j(C(A))$ extends to a positive functional on B.

(2) If A is a D-algebra, then A has the GNS representation as a dense subalgebra of some closed (in the operator norm) *-algebra $B \subseteq B(H)$ of bounded linear operators on a complex Hilbert space H, i. e., as a dense subalgebra of some C*-algebra.

Thus, in order to get a GNS representation for O_b it is sufficient to assume the following postulate:

The order-unit space (O_b, O_{b+}, I) is of the form $(H(A), C(A), e)$ for some *-algebra A with unit e.

Note, however, that although similar assumptions were accepted in quantum axiomatics (see, e. g., Emch [18]), the postulate above has in fact no physical justification (a little support for it can be obtained by relating the involution * to the particle - antiparticle conjugation), so it cannot be considered as necessary from the physical point of view.

Therefore, we should look for another axiom (or axioms), possessing more clear physical meaning, from which the desired Hilbert space (or C*-algebra) representation will follow as a consequence. To do this, we will follow closely the path described in Section 2, i. e. we will go to the desired result by introducing the concept of the conditional probability mapping, and then by establishing, step by step, the Jordan-Banach algebra structure in the space O_b of bounded observables, so that we will finally be in a position to appeal to GNS representation theorem proved for Jordan-Banach algebras by Alfsen *et al.* [4].

3.2. Conditional probability and filters.

After Pool [52] we assume the following (see also Section 2):

AXIOM 3.3. — With every non-zero proposition $a \in L$ there is associated a mapping E_a of the set S of all states into itself, whose domain is $D(E_a) = \{ m \in S : a(m) > 0 \}$, such that (5)

(1) $a(E_a m) = 1$ for all $m \in D(E_a)$;

(2) $E_a m = m$, whenever $a(m) = 1$.

The physical interpretation of the mapping E_a is straightforward (see Subsection 2.2) : If, after a measurement performed on a physical system

(5) Here we shall prefer the notation $a(m)$ in place of the more conventional $m(a)$. This is in accordance with the fact that L is here considered as a subset of the Banach dual V'.

being initially in the state m the proposition $a \in L$ is verified to be true, then the subsequent state of the system is $E_a m$, so, in other words, $E_a m$ describes the state of the system *conditioned* by the fact of the occurrence of an « event » $a \in L$.

According to the interpretation given above, we will call E_a the *conditional probability mapping* associated with the (non-zero) proposition $a \in L$.

It is not difficult to show the following properties of E_a:

(3) $0 < a \leqslant b \Rightarrow E_b E_a m = E_a m$ for all $m \in D(E_a)$.

(4) $a \perp b$ $(a, b \in L \setminus \{0\}) \Rightarrow b(E_a m) = 0$ for each $m \in D(E_a)$.

Now it will be convenient to pass on (similarly as it was done in Subsection 2.2) from E_a to the transformation $P_a : V_+ \to V_+$ defined by

$$P_a x = \begin{cases} a(x)E_a(x/\| x \|), & \text{when } a(x) > 0 \\ 0 & \text{, when } a(x) = 0 , \end{cases}$$

where $x \in V_+$.

Note that P_a is also defined when $a = 0$, and clearly $P_0 = 0$.

The physical significance of the map P_a is straightforward again (see Subsection 2.2): P_a transforms any beam (i. e., an unnormalized state) $x \in V_+$ into the new beam $P_a x$ with the intensity $\| P_a x \| = a(x)$, so that the ratio of intensities $\| P_a x \| / \| x \| = a(x)/1(x)$ gives us the probability of finding the « property » $a \in L$ for a particle from the beam x. Therefore, the physical interpretation of P_a is that P_a represents the filtering procedure corresponding to the proposition (event) $a \in L$, so that $P_a x$ describes the part of the beam x, which passed through the filter P_a (*Remark:* Identifying the set S of all states with its canonical image $\hat{S}$, we shall write in the sequel $P_a m$ instead of $P_a p_m$, where $m \in S$).

In accordance with its operational meaning emphasized above, P_a is called the *filter* associated with the proposition $a \in L$.

The number $b(E_a m)$ gives us the conditional probability that the « event » $b \in L$ will occur, provided an « event » $a \in L$ was found to occur for the system being initially in the state m, and therefore the number $b(P_a m) = a(m)b(E_a m)$ gives us the probability that the event a and next b will occur, provided the system was initially in the state m. It should be emphasized at this moment that the order in which a and b are expected to occur is very essential, because we have in general $b(P_a m) \neq a(P_b m)$.

PROPOSITION 3.1. — For all $a \in L$ and $x \in V_+$ we have:

i) $\| P_a x \| = a(x) = a(P_a x)$.

ii) P_a is positively-homogeneous, that is for every $x \in V_+$ and $t \geqslant 0$

$$P_a(tx) = tP_a x .$$

iii) $\| P_a x \| \leqslant \| x \|$, and $\| P_a x \| = \| x \|$ if and only if $P_a x = x$.

iv) $a \leqslant b$ $(a, b \in L)$ implies $P_b P_a = P_a$. In particular, all P_a are idempotents.

v) $a \perp b$ $(a, b \in L)$ implies $P_a P_b = 0$.

Proof. — *i*) The first half of *i*) follows immediately from the definition of $P_a x$. To prove the second half of *i*) let us observe that if $a(x) > 0$, then by applying Axiom 3.3 (1) we obtain $a(E_a(x/\|x\|)) = 1$, so we have $a(P_a x) = a(x)$, as required, and if $a(x) = 0$, then the second half of *i*) is satisfied trivially.

ii) As for $t = 0$ the statement *ii*) becomes trivial, one can assume that $t > 0$. Then, if $a(x) > 0$, we have

$$P_a(tx) = ta(x)E_a(tx/\|tx\|) = tP_a x,$$

and when $a(x) = 0$, we obtain

$$P_a(tx) = 0 = tP_a x.$$

iii) Applying *i*) we obtain

$$\|P_a x\| = a(x) \leqslant 1(x) = \|x\|.$$

Now let us suppose that $\|P_a x\| = \|x\|$ for some $x \in V_+$. If $a(x) > 0$, then by using *i*) one obtains $a(x) = \|x\|$, which leads, by Axiom 3.3 (2), to $E_a(x/\|x\|) = x/\|x\|$, so that $P_a x = a(x)x/\|x\| = x$. If $a(x) = 0$, then by using *i*) again one finds $\|P_a x\| = \|x\| = 0$, so we have $P_a x = x = 0$, and the statement *iii*) is proved.

iv) Assume that $a \leqslant b$, where $a, b \in L$. By *i*) we get for an arbitrary $x \in V_+$

$$\|P_b P_a x\| = b(P_a x) \geqslant a(P_a x) = \|P_a x\|.$$

Hence, by using the first half of *iii*) we obtain

$$\|P_b P_a x\| = \|P_a x\|,$$

which, by the second half of *iii*), implies

$$P_b P_a x = P_a x.$$

v) Let us assume that $a \perp b$, where $a, b \in L$, and let $x \in V_+$. By *i*) we have

$$\|P_a P_b x\| = a(P_b x) \leqslant b'(P_b x) = \|P_b x\| - b(P_b x) = 0,$$

so that

$$P_a P_b x = 0.$$

Statement *v*) is therefore proved, and the proof of the proposition is complete.

Our next postulate is the following:

AXIOM 3.4. — For each $a \in L$ and each bounded observable $A \in O_b$ there is a bounded observable $B \in O_b$ such that

$$\langle B, m \rangle = \langle A, P_a m \rangle$$

for all $m \in S$.

Obviously, B is necessarily unique by Axiom 3.1, and we denote it by $Q_a A$. The existence of the (bounded) observable $Q_a A$ postulated by

Axiom 3.4 expresses the physically obvious fact that the action of the mapping P_a can alternatively be described by the corresponding « dual » action in the space of bounded observables.

The latter view could be named the « Heisenberg picture », while the former description, involving the state change $m \to P_a m$, could be called the « Schrödinger picture ».

As a consequence of Axiom 3.4 it can readily be shown that the mapping $A \to Q_a A$ is linear and positive. The mapping $E \to Q_{|(A,E)|}I$, where A is an arbitrary observable and E runs over the Borel subsets of the real line R, will be called the *canonical spectral measure of A*.

PROPOSITION 3.2. — The bilinear form $\langle \cdot, \cdot \rangle : O_b \times V \to R$ establishes a separating order duality (6) between O_b and V.

Proof. — One needs to show that if the inequality $\langle A, x \rangle \leqslant \langle A, y \rangle$, where $x, y \in V$, is valid for every bounded observable A, then $x \leqslant y$.

Substituting $Q_{|(A,E)|}I$ in place of A in the inequality above, we obtain $\| P_{|(A,E)|}x \| \leqslant \| P_{|(A,E)|}y \|$, or, by Proposition 3.1 *i)*, $q_{(A,E)}(x) \leqslant q_{(A,E)}(y)$, but the latter inequality, valid for all $A \in O$ and $E \in B(R)$, means that $x \leqslant y$.

Our proposition is therefore proved.

By using Proposition 3.2 one can readily prove that the map $P_a : m \to P_a m$ is affine, i. e. that for all $m_1, m_2 \in S$ and all $t \in (0, 1)$

$$P_a(tm_1 + (1 - t)m_2) = tP_a m_1 + (1 - t)P_a m_2 .$$

Obviously, P_a may uniquely be extended to a linear mapping acting on the whole space V. It will be denoted by the same letter P_a, as this does not lead to a misunderstanding. Moreover, it can easily be shown by using Axiom 3.4 that all the mappings $P_a : V \to V$, where $a \in L$, are continuous with respect to the weak topology $\sigma(V, O_b)$ in V given by the duality $\langle \cdot, \cdot \rangle$, so that $Q_a = P_a^*$, where P_a^* denotes the linear operator in O_b, weakly dual to P_a. It is not difficult to see that both P_a and P_a^* are positive projections (see Proposition 3.1).

Furthermore, since $L_I = d$, where d stands as before for the functional $q_{(A,R)}$ (or, equivalently, $d(x) = \| x_1 \| - \| x_2 \|$, if $x = x_1 - x_2$, where $x_1, x_2 \in V_+$), and since the functional d was shown to be an order unit for the space V′, I becomes an order unit for $O_b \subseteq V'$ (the inclusion is here meant up to isomorphism, of course). Moreover, since $\langle \cdot, \cdot \rangle$ is defined by $\langle A, x \rangle = L_A(x)$, where $A \in O_b$, $x \in V$, and since $\| \cdot \|_d = \| \cdot \|_1$ (see Section 1; $\| \cdot \|_d$ stands for the order-unit norm in O_b, and $\| \cdot \|_1$ denotes here the standard norm in the Banach dual of $(V, \| \cdot \|_1)$), $\langle \cdot, \cdot \rangle$ establishes also a norm duality (6) between $(O_b, \| \cdot \|_d)$ and $(V, \| \cdot \|_1)$. Therefore, we have proved the following statement:

(6) See Appendix B for a definition.

THEOREM 3.3. — (O_b, I) and $(V, \hat{S})$ are, respectively, the order-unit and the base-norm spaces in separating order and norm duality.

Note that passing on from P_a to the dual filters $Q_a = P_a^*$ we obtain, as a direct consequence of Proposition 3.1, the following:

PROPOSITION 3.4. — For every $a \in L$ we have

i) $a = L_{Q_a I}$.

ii) $Q_a I \leqslant I$.

iii) $a \leqslant b$ $(a, b \in L)$ implies $Q_a Q_b = Q_a$; in particular, $Q_a^2 = Q_a$.

iv) $a \perp b$ $(a, b \in L)$ implies $Q_a Q_b = 0$.

3.3. **Compatibility.**

We shall say that two filters P_a, P_b are *compatible*, and write $P_a \leftrightarrow P_b$, if for each state $m \in S$

$$\| P_b(P_a + P_{a'})m \| = \| P_b m \| . \tag{3.1}$$

The definition above is formal rather than physically motivated, however, later on will be found several equivalent forms of this definition and its physical meaning will be clarified.

From the definition of compatibility it follows readily that

i) $P_a \leftrightarrow P_a$ for all $a \in L$;

ii) $P_a \leftrightarrow P_b \Rightarrow P_{a'} \leftrightarrow P_b$;

iii) $P_a \leftrightarrow P_b \Rightarrow P_a \leftrightarrow P_{b'}$.

Remark. — After we define the relation of compatibility by (3.1), then there is no *a priori* evidence for the symmetry of this relation. But this is indeed so, and the symmetry will be shown later on.

Now, passing on to dual filters $Q_a = P_a^*$ we define

$$Q_a \leftrightarrow Q_b \quad \text{iff} \quad P_a \leftrightarrow P_b .$$

Moreover, we shall define the compatibility of Q_a with a bounded observable $A \in O_b$, and this will be done in two steps.

If $A \geqslant 0$, then we put by definition

$$Q_a \leftrightarrow A \quad \text{iff} \quad Q_a A \leqslant A ,$$

and if A is arbitrary, we define

$Q_a \leftrightarrow A$ iff there is a decomposition $A = A_1 - A_2$, where $A_1, A_2 \in O_{b+}$, such that $Q_a \leftrightarrow A_i$, $i = 1, 2$.

Now, it is not difficult to see that the set of all bounded observables which are compatible with a given filter Q_a is a linear subspace of O_b containing the order unit I.

PROPOSITION 3.5. — Let $A \in O_b$. If Q_a is compatible with the canonical spectral measure of A in the sense that $Q_a \leftrightarrow Q_{|(A,E)|}$ for all Borel subsets $E \subseteq R$, then $Q_a \leftrightarrow A$.

Proof. — We shall show that

$$Q_aA + Q_{a'}A = A, \tag{3.2}$$

which implies immediately the desired result.

Indeed, if $A \geqslant O$, then $Q_{a'}A \geqslant O$, since $Q_{a'}$ is positive, so we have $A = Q_aA + Q_{a'}A \geqslant Q_aA$, that is, $Q_a \leftrightarrow A$. Now let A be an arbitrary bounded observable satisfying (3.2). Since I is an order unit in O_b, we get $A \in t[-I, I]$ for some $t > 0$, so that $A + tI \geqslant 0$. By using (3.2) and the fact that $Q_aI + Q_{a'}I = I$ we obtain

$$Q_a(A + tI) + Q_{a'}(A + tI) = A + tI.$$

Hence, by the result which we already proved,

$$Q_a \leftrightarrow A + tI,$$

so we have also

$$Q_a \leftrightarrow (A + tI) - tI = A.$$

Now we shall pass on to the proof of (3.2), and assume that $Q_a \leftrightarrow Q_{|(A,E)|}$ for all $E \in B(R)$. We then have for all $x \in V_+$ and $E \in B(R)$

$$\| P_{|(A,E)|}P_ax \| + \| P_{|(A,E)|}P_{a'}x \| = \| P_{|(A,E)|}(P_a + P_{a'})x \| = \| P_{|(A,E)|}x \|,$$

so that for all $x \in V_+$

$$\begin{aligned}\langle Q_aA + Q_{a'}A, x \rangle &= \int_{-\infty}^{\infty} td(P_ax)(A) + \int_{-\infty}^{\infty} td(P_{a'}x)(A) \\ &= \int_{-\infty}^{\infty} tdq_{(A,.)}(P_ax) + \int_{-\infty}^{\infty} tdq_{(A,.)}(P_{a'}x) \\ &= \int_{-\infty}^{\infty} td \| P_{|(A,.)|}P_ax \| + \int_{-\infty}^{\infty} td \| P_{|(A,.)|}P_{a'}x \| \\ &= \int_{-\infty}^{\infty} td \| P_{|(A,.)|}x \| = \int_{-\infty}^{\infty} tdx(A) = \langle A, x \rangle,\end{aligned}$$

which leads to

$$Q_aA + Q_{a'}A = A,$$

as claimed.

As a natural completion of Proposition 3.5 the following postulate will be assumed:

AXIOM 3.5. — If Q_a is compatible with $A \in O_b$, then Q_a is also compatible with the canonical spectral measure of A, i. e. $Q_a \leftrightarrow Q_{|(A,E)|}$ for all Borel subsets $E \subseteq R$.

PROPOSITION 3.6. — Let $A \in O_b$. The following three statements are equivalent:

i) $Q_a \leftrightarrow A$.

ii) $Q_a \leftrightarrow Q_{|(A,E)|}$ for all $E \in B(R)$.

iii) $Q_a A + Q_{a'} A = A$.

Proof. — The validity of the implication *i)* ⇒ *ii)* has been assumed as the content of Axiom 3.5, while the next implication, *ii)* ⇒ *iii)*, and the implication *iii)* ⇒ *i)* were shown in the proof of the Proposition 3.5. Thus, there is nothing to prove.

As a direct consequence of Proposition 3.6 one obtains:

COROLLARY 3.7. — $Q_a \leftrightarrow A$ (where $A \in O_b$) implies $Q_{a'} \leftrightarrow A$.

PROPOSITION 3.8. — Every P_a is a P-projection (7), and so is Q_a.

Proof. — We shall show that P_a and $P_{a'}$ are quasicomplementary (see Appendix B for a definition), and that so are the dual projections Q_a and $Q_{a'}$; then the desired result will follow from the result of Alfsen *et al.* ([*3*], Theorem 1.8).

We must therefore prove that $\text{im}^+ P_a = \ker^+ P_{a'}$ and $\text{im}^+ Q_a = \ker^+ Q_{a'}$, since by the symmetry reason we shall then have $\text{im}^+ P_{a'} = \ker^+ P_a$, and similarly for Q_a.

Let $x \in \text{im } P_a$; then $x = P_a x$, so that by using Proposition 3.1 *v)* we obtain $P_{a'}x = P_{a'}P_a x = 0$. We thus have shown that $\text{im } P_a \subseteq \ker P_{a'}$, hence also $\text{im}^+ P_a \subseteq \ker^+ P_{a'}$. To prove the converse inclusion, let us assume that $x \in \ker^+ P_{a'}$. By applying Proposition 3.1 *i)* we then get

$$\| x \| = 1(x) = a(x) + a'(x) = \| P_a x \| + \| P_{a'} x \| = \| P_a x \|,$$

so that we obtain by using Proposition 3.1 *iii)*

$$x = P_a x \in \text{im}^+ P_a,$$

and this completes the proof that $\text{im}^+ P_a = \ker^+ P_{a'}$.

Now we shall pass on to the proof that $\text{im}^+ Q_a = \ker^+ Q_{a'}$. Assume first that $A \in \text{im } Q_a$. Then $A = Q_a A$, so that for an arbitrary $x \in V_+$ we have $\langle A, x \rangle = \langle A, P_a x \rangle$, and by substituting $P_{a'}m$ (m being an arbitrary state) in place of x one obtains $\langle A, P_{a'}m \rangle = 0$, or, equivalently, $\langle Q_{a'}A, m \rangle = 0$.

The equality above, valid for all $m \in S$, implies $Q_{a'}A = 0$, so we have proved that $\text{im } Q_a \subseteq \ker Q_{a'}$, and hence also $\text{im}^+ Q_a \subseteq \ker^+ Q_{a'}$.

To prove the converse inclusion, assume that $A \in \ker^+ Q_{a'}$; then $Q_{a'}A = 0 \leqslant A$, so we have $Q_{a'} \leftrightarrow A$, and by using Proposition 3.6 *iii)* we get $A = Q_a A \in \text{im}^+ Q_a$, as desired.

The proof of our proposition is therefore complete.

(7) For the definition of the P-projection see Appendix B.

3.4. The spectral duality between O_b and V.

The family of all the P-projections on either V or O_b is of great importance for the axiomatic scheme developed here, and it will be shown later on that this family coincides (up to isomorphism, of course) with the propositional logic L.

Note also that the concept of the P-projection, introduced by Alfsen *et al.* [*3*], plays a crucial role in building up the non-commutative spectral theory.

We shall now introduce, after Alfsen *et al.* [*3*], the following notation (8):

$\mathscr{P}$ = the set of all P-projections on V,
$\mathscr{Q}$ = the set of all P-projections on O_b,
$\mathscr{F}$ = the set of all projective faces of $\hat{S} \subseteq V$,
$\mathscr{U}$ = the set of all projective units of O_b.

All of the sets introduced above are orthoposets (i. e. partially ordered and orthocomplemented), and they are all mutually orthoisomorphic. The partial orderings and the orthocomplementations of these sets are respectively given by (see Alfsen *et al.* [*3*]):

a) $P_1 \leqslant P_2$ iff im $P_1 \subseteq$ im P_2 iff $\text{im}^+ P_1 \subseteq \text{im}^+ P_2$,
P' = the unique quasicomplement of P defined by the requirement that $\text{im}^+ P' = \ker^+ P$, $\ker^+ P' = \text{im}^+ P$,

whenever P_1, P_2, P belong either to V or to O_b.

b) $F_1 \leqslant F_2$ iff $F_1 \subseteq F_2$,
$F' = (\ker P) \cap \hat{S}$, whenever $F = (\text{im } P) \cap \hat{S}$,

where $P \in \mathscr{P}$, and $F_1, F_2, F \in \mathscr{F}$.

c) For projective units we define

$$Q_1 I \leqslant Q_2 I \quad \text{iff} \quad Q_1 \leqslant Q_2 ,$$
$$(QI)' = Q'I = I - QI ,$$

where $Q_1, Q_2, Q \in \mathscr{Q}$.

Remark. — The orthoisomorphism between $\mathscr{P}$ and $\mathscr{Q}$ is clearly established by the correspondence $P \to P^*$ ($P \in \mathscr{P}$). The existence of the orthoisomorphism between $\mathscr{Q}$ and $\mathscr{U}$ is evident—this follows directly from the definition of the partial ordering and the orthocomplementation in $\mathscr{U}$. Finally, the orthoisomorphism between $\mathscr{Q}$ and $\mathscr{F}$ was shown by Alfsen *et al.* [*3*].

Moreover, it has been proved by Alfsen *et al.* [*3*] that for P-projections $Q, R \in \mathscr{Q}$ we have

$$Q \leqslant R \quad \text{iff} \quad RQ = Q \quad \text{iff} \quad QR = Q , \tag{3.3}$$

and

$$Q \perp R \quad \text{iff} \quad QR = 0 \quad \text{iff} \quad QI + RI \leqslant I . \tag{3.4}$$

(Recall that by definition $Q \perp R$ iff $Q \leqslant R'$).

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(8) For definitions of the new concepts involved here, see Appendix B.

The result above is clearly valid also for dual projections from $\mathscr{P}$.

We shall say that a P-projection $P \in \mathscr{P}$ is an *orthogonal sum* of pairwise orthogonal P-projections $P_1, P_2, \ldots, P_n \in \mathscr{P}$, and write

$$P = P_1 \dotplus P_2 \dotplus \ldots \dotplus P_n,$$

if for each $m \in S$

$$\| Pm \| = \sum_{i=1}^{n} \| P_i m \| . \tag{3.5}$$

Alternatively, for dual P-projections $Q, Q_1, \ldots, Q_n \in \mathscr{Q}$ we shall write

$$Q = Q_1 \dotplus Q_2 \dotplus \ldots \dotplus Q_n \quad \text{iff} \quad QI = \sum_{i=1}^{n} Q_i I . \tag{3.6}$$

The definitions above extend easily to countable orthogonal sequences $\{ P_i \} \subseteq \mathscr{P}$, $\{ Q_i \} \subseteq \mathscr{Q}$. In the first case (3.5) applies without any essential modification (the only change is that we set $n = \infty$); in the second case we must replace (3.6) by the requirement that $\langle QI, m \rangle = \sum_{i=1}^{\infty} \langle Q_i I, m \rangle$ for all $m \in S$.

Obviously, $P \geqslant P_i$ and $Q \geqslant Q_i$ for all $i = 1, 2, \ldots$, and it is not difficult to show that P (respectively, Q) is the least upper bound for $\{ P_i \}$ ($\{ Q_i \}$, respectively). Indeed, suppose that $P_0 \geqslant P_i$ for all i, where $P_0 \in \mathscr{P}$. Then, by applying (3.5) and (3.3) one obtains for an arbitrary $m \in S$

$$\| PP_0 m \| = \sum_i \| P_i P_0 m \| = \sum_i \| P_i m \| = \| Pm \| ,$$

and since P is norm contracting, we obtain for all $m \in S$

$$\| Pm \| \leqslant \| P_0 m \| ,$$

so that $P \leqslant P_0$.

The latter inequality shows that $P = \bigvee_i P_i$, as claimed, and by passing on to dual P-projections we obtain the « dual » statement: $Q = \bigvee_i Q_i$.

Now we shall pass on to the proof of the main result of this subsection, which states that the spaces O_b and V are in spectral duality. However, before proving this, we will need some lemmas.

LEMMA 3.9. — Let $A \in O_{b+}$, and let $\{ Q_i \}$ be a sequence of P-projections from $\mathscr{Q}$ defined by

$$Q_i = R_1 \dotplus R_2 \dotplus \ldots \dotplus R_i, \qquad i = 1, 2, 3, \ldots ,$$

where $\{ R_i \}$ is an orthogonal sequence of P-projections, for which we assume that $R_1 \dotplus R_2 \dotplus \dots \dotplus R_i \dotplus \dots$ exists in $\mathcal{Q}$. If for each $i = 1, 2, \dots$ we have $Q_i \leftrightarrow A$ and $A > 0$ on the projective face associated with Q_i, then also $A > 0$ on the projective face corresponding to $\bigvee_{i=1}^{\infty} Q_i = R_1 \dotplus R_2 \dotplus \dots$.

Proof. — Let $F_0, F_1, F_2, \dots$ denote the projective faces corresponding to $Q_0 = \bigvee_{i=1}^{\infty} Q_i, Q_1, Q_2, \dots$ respectively, i. e.,

$$F_i = \{ m \in S : \langle Q_i I, m \rangle = 1 \} = \{ m \in S : \| P_i m \| = 1 \} = \{ m \in S : m = P_i m \},$$

where $P_i = Q_i^*$, $i = 0, 1, 2, \dots$.

Note that $Q_i = Q_{i+1} Q_i$ and $Q_i A \leqslant A$, since $Q_i \leqslant Q_{i+1}$ and $Q_i \leftrightarrow A \geqslant 0$, so we have

$$Q_i A = Q_{i+1} Q_i A \leqslant Q_{i+1} A, \quad i = 1, 2, \dots,$$

and we similarly prove that for all i

$$Q_i A \leqslant Q_0 A.$$

Hence, for each $m \in S$ we get

$$\langle Q_0 A, m \rangle \geqslant \lim_i \langle Q_i A, m \rangle = \sup_i \langle Q_i A, m \rangle \geqslant 0.$$

Therefore, if $\langle Q_0 A, m \rangle = 0$, then $\langle Q_i A, m \rangle = 0$ for all $i = 1, 2, \dots$, or, equivalently, we have the implication

$$\langle A, P_0 m \rangle = 0 \Rightarrow \langle A, P_i m \rangle = 0 \quad \text{for all } i = 1, 2, \dots. \tag{3.7}$$

Now suppose, to the contrary, that $\langle A, m \rangle = 0$ for some $m \in F_0$. Since $m = P_0 m$, we find by using (3.7) that $\langle A, P_i m \rangle = 0$, $i = 1, 2, \dots$, which leads immediately to

$$P_i m = 0, \quad i = 1, 2, \dots. \tag{3.8}$$

Indeed, if $P_i m > 0$ for some i, then for $m_1 = P_i m / \| P_i m \|$ one gets $\langle A, m_1 \rangle = 0$ (see above), but the latter contradicts our assumption, since $m_1 \in F_i$.

But (3.8) implies $\| P_0 m \| = \sum_i \| P_i m \| = 0$, which contradicts the assumption that $m \in F_0$. Our lemma is therefore proved.

LEMMA 3.10. — For every nonzero $A \in O_{b+}$ there exists a projective face $F \in \mathcal{F}$ compatible with A such that $A \leqslant 0$ on F and $A > 0$ on F′. Moreover, F belongs to $\mathcal{L} = \{ F \in \mathcal{F} : F = (\text{im } P_a) \cap \hat{S} \text{ for some } a \in L \}$.

Proof. — Let us first note that $(\text{Sp } A) \cap R_+ \neq \{ 0 \}$, and define

$$s = \inf \{ t > 0 : t \in \text{Sp } A \} \geqslant 0.$$

We shall consider two cases.

CASE 1 : $s > 0$.

In this case we have $(0, s) \cap \text{Sp A} = \emptyset$, so that for $m \in F$, where F is the projective face corresponding to $P_{|(A,(-\infty,0])|} = P_{|(A,(-\infty,s))|}$, we get

$$\langle A, m \rangle = \int_{\text{sp A}} t dp(A, m, \cdot) \leqslant 0 ,$$

since $m \in F$ means that $p(A, m, (-\infty, 0]) = \| P_{|(A,(-\infty,0])|} m \| = 1$.

On the other hand, for $m \in F'$ (F' being the projective face associated with $P'_{|(A,(-\infty,s))|} = P_{|(A,[s,+\infty))|}$) we obtain

$$\langle A, m \rangle = \int_{\text{sp A}} t dp(A, m, \cdot) \geqslant s > 0 ,$$

so that our statement is proved.

CASE 2 :. $s = 0$.

The projective face F is chosen, as before, as the one corresponding to $P_{|(A,(-\infty,0])|}$. Let $\{ t_n \}$ be a decreasing sequence of positive real numbers from Sp A such that $t_n \to 0$, and let $R_1 = Q_{|(A,[t_1,+\infty))|}$, $R_n = Q_{|(A,[t_n,t_{n-1}))|}$, $n = 2, 3, \ldots$. Since for each i we have

$$R_1 \dotplus R_2 \dotplus \ldots \dotplus R_i = Q_{|(A,[t_i,+\infty))|} \leftrightarrow A$$

and $A > 0$ on F_i, the projective face associated with $R_1 \dotplus R_2 \dotplus \ldots \dotplus R_i$, we get by using Lemma 3.9 that $A > 0$ on F_0, the projective face corresponding to $R_1 \dotplus R_2 \dotplus \ldots = Q_{|(A,(0,+\infty))|}$, and the latter shows that $F_0 = F'$.

Obviously, on F we have $A \leqslant 0$, and F belongs clearly to $\mathscr{L}$.

Our lemma is therefore proved.

Having established Lemma 3.10 we are in a position to prove several important statements about the pair of spaces O_b, V and about the posets $\mathscr{P} \cong \mathscr{Q} \cong \mathscr{F}$. In particular, the following fact can then be established (Alfsen *et al.* [3]):

PROPOSITION 3.11. — Every exposed face (9) of S is projective; moreover, it belongs to $\mathscr{L}$.

As a consequence of Proposition 3.11 one finds that every projective face (being necessarily exposed) belongs to $\mathscr{L}$, so that $\mathscr{L} = \mathscr{F}$.

As a consequence of Proposition 3.11 it can also easily be deduced that $\mathscr{P}$ and $\mathscr{Q}$ (the latter being orthoisomorphic to $\mathscr{P}$) are orthocomplemented orthomodular lattices.

(9) By a *face* of S we mean a convex subset $F \subseteq S$ such that the following holds: if $m_1, m_2 \in S$, $0 < t < 1$, and $tm_1 + (1 - t)m_2 \in F$, then $m_1, m_2 \in F$.

A face F of S is said to be *exposed* (or, to be more precise, O_b*-exposed*—see Alfsen *et al.* [3], where an equivalent definition is given) if $F = \{ m \in S : \langle A, m \rangle = 0 \text{ for some } A \in O_{b_+} \}$.

PROPOSITION 3.12. — $\mathscr{P}$ is an orthocomplemented orthomodular lattice.

Proof. — We shall first prove the lattice property for $\mathscr{P}$.

Let $A = P_1^*I + P_2^*I$, where $P_1, P_2 \in \mathscr{P}$, and let F be an exposed face of S defined by $F = \{ m \in S : \langle A, m \rangle = 0 \}$. By Proposition 3.11 there exists $P \in \mathscr{P}$ such that $F = (\text{im } P) \cap \hat{S}$. We shall prove that $P' = P_1 \vee P_2$. First, it will be shown that $P' \geqslant P_1, P_2$ or, equivalently, that $P \leqslant P_1', P_2'$. Indeed, suppose that $\| Pm \| = 1$ for some $m \in S$; then $m = Pm \in (\text{im } P) \cap \hat{S} = F$, so that $\langle A, m \rangle = 0$, which implies $\| P_1 m \| = \| P_2 m \| = 0$ or, equivalently, $\| P_1' m \| = \| P_2' m \| = 1$. We thus have shown that $F \subseteq F_1 \cap F_2$, where F_i denotes the projective face associated with P_i', and this means that $P \leqslant P_1', P_2'$, as claimed.

Now suppose that $P_1, P_2 \leqslant P_0$ for some $P_0 \in \mathscr{P}$. To close the proof that P' is the l. u. b. of P_1 and P_2, we need to show that $P' \leqslant P_0$.

Let $m \in F_0$, where F_0 is the projective face corresponding to P_0'; then $\| P_0' m \| = 1$, which leads to $\| P_1 m \| = \| P_2 m \| = 0$, so we have $\langle A, m \rangle = 0$, and hence $m \in F$, so that $P_0' \leqslant P$, or $P_0 \geqslant P'$, as required.

Since $\mathscr{P}$ is orthocomplemented by the correspondence $P \to P'$, we see that $\mathscr{P}$ is a lattice. Finally, having established Proposition 3.11 we can prove the orthomodularity of $\mathscr{P}$ by repeating the arguments of Alfsen *et al.* ([*3*], Theorem 4.5).

The proof of the proposition is therefore complete.

Now we shall collect the statements proved by Alfsen *et al.* [*3*] under the assumption that every exposed face is projective and that $\mathscr{P}$ ($\cong \mathscr{Q} \cong \mathscr{F}$) is an orthocomplemented orthomodular lattice, which will be of special interest to our purposes.

PROPOSITION 3.13 ([10]). — For P-projections $Q, R \in \mathscr{Q}$ the following statements are equivalent:

i) $QR \in \mathscr{Q}$;

ii) $QR = Q \wedge R$, i. e. QR is the greatest lower bound of Q and R in $\mathscr{Q}$;

iii) $Q \leftrightarrow RI$;

iv) $R \leftrightarrow QI$;

v) $QR = RQ$.

PROPOSITION 3.14 ([11]). — Let $Q, R \in \mathscr{Q}$. Then the following three statements are equivalent:

i) Q and R are compatible in the sense of Mackey, that is

$$Q = Q_1 \vee S \quad \text{and} \quad R = R_1 \vee S,$$

where Q_1, R_1, S are pairwise orthogonal P-projections from $\mathscr{Q}$.

ii) $Q \leftrightarrow RI$.

iii) $Q \leftrightarrow R$.

([10]) See Alfsen *et al.* [*3*], Proposition 5.2.

([11]) See Alfsen *et al.* [*3*], Proposition 5.4.

Proof. — The equivalence *i*) $\Leftrightarrow$ *ii*) was shown by Alfsen *et al.* ([3], Proposition 5.4), so we need to prove that *ii*) $\Leftrightarrow$ *iii*).

Let us begin by proving the implication *ii*) $\Rightarrow$ *iii*). Since $Q \leftrightarrow RI$ implies $Q' \leftrightarrow RI$, we find by using Proposition 3.13 *v*) that $QR = RQ$ and $Q'R = RQ'$, so we have

$$(Q + Q')RI = R(Q + Q')I = RI, \tag{3.9}$$

where the last equality above is most easily derived by passing on to dual P-projections R^*, Q^*, and $Q'^* = Q^{*\prime}$. But (3.9) means that $Q \leftrightarrow R$, as required.

To prove the converse implication, *iii*) $\Rightarrow$ *ii*), let us observe that $Q \leftrightarrow R$, being equivalent to (3.9), implies obviously $QRI \leqslant RI$, which means that $Q \leftrightarrow RI$, as claimed.

As a direct consequence of Propositions 3.13 and 3.14 one obtains:

COROLLARY 3.15. — $Q \leftrightarrow R$ iff $QRI = RQI$ iff $\| Q^*R^*m \| = \| R^*Q^*m \|$ for all $m \in S$.

Also, the following statements about the compatibility relation can be proved (Alfsen *et al.* [3], Lemma 5.6, Proposition 5.7):

COROLLARY 3.16. — If Q_1, Q_2 are two compatible P-projections from $\mathcal{Q}$ such that $Q_i \leftrightarrow A$, $i = 1, 2$, where $A \in O_b$, then also $Q_1 \vee Q_2 \leftrightarrow A$ and $Q_1 \wedge Q_2 \leftrightarrow A$.

PROPOSITION 3.17. — If Q_i, $i = 1, 2, \ldots, n$, are pairwise orthogonal P-projections from $\mathcal{Q}$ compatible with $A \in O_b$, then $\left(\bigvee_{i=1}^{n} Q_i \right) A = \sum_{i=1}^{n} Q_i A$.

Now we are in a position to extend the result of Lemma 3.9 to an arbitrary bounded observable $A \in O_b$.

LEMMA 3.18. — Let $A \in O_b$, and let $\{ Q_i \}$ be a sequence of P-projections from $\mathcal{Q}$ defined by

$$Q_i = R_1 \dotplus R_2 \dotplus \ldots \dotplus R_n, \quad i = 1, 2, \ldots,$$

where $\{ R_j \}$ is a sequence of pairwise orthogonal P-projections such that $R_1 \dotplus R_2 \dotplus \ldots$ exists in $\mathcal{Q}$. If for each $i = 1, 2, \ldots$ we have $Q_i \leftrightarrow A$ and $A > t$ on the projective face corresponding to Q_i, then also $A > t$ on the projective face associated with $\bigvee_{i=1}^{n} Q_i = R_1 \dotplus R_2 \dotplus \ldots$.

Proof. — One can assume without loss of generality that $t = 0$, since one can always replace A by $A - tI$, which is still compatible with all Q_i.

Let $F_0, F_1, F_2, \ldots$ denote, as before, the projective faces corresponding to $Q_0 = \bigvee_{i=1}^{n} Q_i$, $Q_1, Q_2, \ldots$, respectively. Note that $Q_iA \geqslant 0$ for all i. Indeed, for an arbitrary $x \in V_+$ we have $\langle Q_iA, x \rangle = \langle A, y \rangle$, where $y = Q_i^* x$, so that for $y \neq 0$ we get $y/\| y \| \in (\text{im } Q_i^*) \cap \hat{S} = F_i$, which leads by the assumption to $\langle A, y/\| y \| \rangle > 0$, and we therefore obtain $\langle Q_iA, x \rangle > 0$. Thus, we have shown that $Q_iA \geqslant 0$, as claimed.

By orthomodularity and by Propositions 3.13 *ii*), 3.16 and 3.17 we obtain for $i \leqslant j$

$$Q_jA = (Q_i \vee (Q_j \wedge Q_i'))A = Q_iA + Q_i'Q_jA .$$

But by the result above $Q_jA \geqslant 0$, so that $Q_i'Q_jA \geqslant 0$, and we thus see that for $i \leqslant j$

$$Q_iA \leqslant Q_jA . \tag{3.10}$$

We shall now prove that for all $i = 1, 2, \ldots$

$$Q_iA \leqslant Q_0A . \tag{3.11}$$

First, let us choose a positive real number s such that $A \in s[-I, I]$ (remind that I acts as an order unit in O_b), so that $A + sI \in O_{b+}$. Then, since $\{ Q_i \}$ is an increasing sequence of P-projections compatible with $A + sI \in O_{b+}$, we obtain (see the proof of Lemma 3.9)

$$Q_i(A + sI) \leqslant Q_0(A + sI), \quad i = 1, 2, \ldots . \tag{3.12}$$

Now suppose, to the contrary, that for some i and $x \in V_+$

$$\langle Q_iA, x \rangle > \langle Q_0A, x \rangle ,$$

and denote

$$\varepsilon = \langle Q_iA, x \rangle - \langle Q_0A, x \rangle > 0 .$$

Since $Q_0I = w\text{-}\lim_n Q_nI$, we get

$$\exists_{n_0} \forall_{n \geqslant n_0} \langle Q_0I, x \rangle - \langle Q_nI, x \rangle < \varepsilon/s ,$$

so that

$$s \langle Q_0I, x \rangle - s \langle Q_nI, x \rangle < \langle Q_iA, x \rangle - \langle Q_0A, x \rangle .$$

Hence, for all $k \geqslant \max (i, n_0)$ we obtain (see also (3.10)):

$$s \langle Q_0I, x \rangle - s \langle Q_kI, x \rangle < \langle Q_kA, x \rangle - \langle Q_0A, x \rangle ,$$

so we have

$$\langle Q_0(A + sI), x \rangle < \langle Q_k(A + sI), x \rangle ,$$

which contradicts (3.12).

The inequality (3.11) is therefore proved.

Now, for each $x \in V_+$ we have

$$\langle Q_0 A, x \rangle \geqslant \lim_i \langle Q_i A, x \rangle = \sup_i \langle Q_i A, x \rangle \geqslant 0,$$

and the rest of the proof follows as that of Lemma 3.9.

Having established Lemma 3.18 we are now in a position to prove the main result of this subsection.

THEOREM 3.19. — The spaces O_b and V are in spectral duality, that is, for every nonzero $A \in O_b$ and every real number t there exists a unique projective face F compatible with A such that $A \leqslant t$ on F and $A > t$ on F'.

Proof. — a) *The existence part.*

We can assume without loss of generality that $t = 0$, and then the proof becomes in fact a repetition of that of Lemma 3.10, because the Lemma 3.9, which led previously to the desired conclusion, can now be replaced by Lemma 3.18.

b) *The uniqueness part.*

The uniqueness of F follows from the result of Alfsen *et al.* ([*3*], Lemma 7.1), since F is actually bicompatible with A. Indeed, F has the form (see Lemma 3.10)

$$F = \{ m \in S : \| P_{|(A,(-\infty,0])|} m \| = 1 \} = \{ m \in S : \langle Q_{|(A,(-\infty,0])|} I, m \rangle = 1 \},$$

so, by Axiom 3.5, F is compatible with all $G \in \mathscr{F}$ ($\cong \mathscr{L}$) compatible with A, that is, F is bicompatible with A.

The theorem is thus proved.

3.5. The Jordan structure of the space O_b of bounded observables.

In establishing the Jordan algebra structure in O_b we shall follow the path indicated by Alfsen *et al.* [*6*]. We shall begin with introducing the following postulate:

AXIOM 3.6. — The space O_b of bounded observables is pointwise monotone σ-complete, that is, for each increasing sequence $\{ A_i \} \subseteq O_b$ bounded above there exists an $A \in O_b$ such that $\langle A, m \rangle = \sup \langle A_i, m \rangle$ for all $m \in S$.

This is, of course, a technical assumption. If it does not hold, we can always extend O_b to a larger order-unit space, being pointwise monotone σ-complete, by passing on to the so-called monotone σ-complete envelope of O_b.

The Axiom 3.6 has several important consequences, the following three of which are of crucial significance for the axiomatic scheme developed here (for proofs and details, see Alfsen *et al.* [*3*]):

i) O_b is norm-complete.

ii) The lattice $\mathscr{P}$ (hence also $\mathscr{Q}$, $\mathscr{F}$ and $\mathscr{U}$) is σ-complete.

iii) O_b admits a functional calculus by bounded Borel functions, having the usual properties.

Clearly, in the derivation of the statements *i), ii), iii)* the spectral duality between O_b and V plays an essential role, so that these statements are in fact the consequences of all the axioms assumed.

Furthermore, making use of the theorems on the functional calculus in O_b (see Alfsen *et al.* [*3*]) we are able to show that almost all the Segal's axioms are satisfied in our scheme, except possibly the requirement of the norm continuity of the squaring operation $A \to A^2$, where $A \in O_b$. More precisely, we can verify the following:

(1) If for A, $B \in O_b$ we set by definition

$$A \circ B = \frac{1}{4}((A + B)^2 - (A - B)^2) \tag{3.13}$$

and define inductively $A^{k+1} = A^k \circ A$, and $p(A) = s_0 I + \sum_{k=1}^{n} s_k A^k$ for any real polynomial $p(t) = \sum_{k=0}^{n} s_k t^k$, then:

a) $A \circ I = A$ for all $A \in O_b$;

b) $p(q(A)) = (p \circ q)(A)$ for all real polynomials p, q, where $\circ$ denotes the ordinary composition of functions.

(2) O_b is norm-complete.

(3) $\| A^2 \| = \| A \|^2$ for all $A \in O_b$.

(4) $\| A^2 - B^2 \| \leqslant \max (\| A^2 \|, \| B^2 \|)$.

The statement (2) is, as we noticed, a consequence of Axiom 3.6 (see *i*) above). The property (1) follows as an immediate consequence of the rules of the functional calculus in O_b (see Alfsen *et al.* [*3*]). The property (3) is a direct consequence of the definition of the square A^2 and the equality sp $f(A) = f(\text{sp } A)$, which holds for all $A \in O_b$ and all continuous functions f: Sp A $\to$ R (Alfsen *et al.* [*3*], Proposition 8.5). Finally, the property (4) follows as a consequence of the following general inequality

$$\| x - y \| \leqslant \max (\| x \|, \| y \|)$$

valid for all positive elements x, y of an arbitrary order-unit space (see, e. g., Alfsen [*1*]).

The next axiom, introduced for the first time by Alfsen *et al.* [*6*], formulates the key physical property needed for obtaining the Jordan algebra structure in O_b.

AXIOM 3.7. — For each state $m \in S$ the probability of the exclusive disjunction of P_1 and P_2, where $P_1, P_2 \in \mathscr{P}$, defined by

$$\text{Prob } ((P_1 \ \& \ P'_2) \quad \text{or} \quad (P'_1 \ \& \ P_2))_m = \| P'_2 P_1 m \| + \| P_2 P'_1 m \|,$$

is independent of the order of P_1 and P_2, that is,

$$\text{Prob}\,((P_1 \,\&\, P_2') \quad \text{or} \quad (P_1' \,\&\, P_2))_m = \text{Prob}\,((P_2' \,\&\, P_1) \quad \text{or} \quad (P_2' \,\&\, P_1))_m .$$

The physical content of the axiom introduced above is very clear, and the details of the interpretation of Axiom 3.7 can be found in the paper by Alfsen *et al.* [*6*].

It has been shown by Alfsen *et al.* [*6*] that the property expressed by Axiom 3.7 is sufficient and necessary for the space O_b, being a pointwise monotone σ-complete order-unit space in spectral duality with the base-norm space V, in order to be a Jordan-Banach algebra with the Jordan product defined by the Segal's formula (3.13) or, equivalently, by

$$A \circ B = \frac{1}{2}((A + B)^2 - A^2 - B^2),$$

so that we can apply the GNS representation theorem proved for real Jordan-Banach algebras by Alfsen *et al.* [*4*] (see also Section 2) to obtain the Hilbert space representation for O_b.

APPENDIX A

Order-unit and base-norm spaces.

In this appendix we have collected the well known, but scattered in the literature, definitions and results which are basic for the theory developed in the text. For proofs we refer the reader to references *[17]*, *[60]*, *[33]*, *[1]*.

Let V be a vector space over the reals. A nonempty subset $C \subseteq V$ is called a *cone* if $C + C \subseteq C$ and $tC \subseteq C$ for $t \geqslant 0$. A cone C is said to be *proper* if $C \cap (-C) = \{0\}$. If $V = C - C$, we say that V is *positively generated* or equivalently, *generated by C*, and then we call C a *generating cone* in V.

Every proper cone $C \subseteq V$ defines a partial ordering in V if we put by definition $x \leqslant y$ if and only if $y - x \in C$, and conversely, so that the pair (V, C) consisting of a real vector space V together with a proper cone $C \subseteq V$ is called the *partially ordered vector space.* The *order dual* (V^p, C^*) of a partially ordered vector space (V, C) is now defined by $V^p = C^* - C^*$ with C^* denoting the cone of positive (i. e. taking nonnegative values on C) linear functionals on V. It is not difficult to show that the cone C^* is proper (so, it then induces the partial ordering in V^p) if and only if the cone C is generating.

The partial ordering defined by a proper cone $C \subseteq V$ is said to be *almost Archimedean* if $-ty \leqslant x \leqslant ty$ for some $y \in C$ and all $t > 0$ implies $x = 0$, and we call it *Archimedean* if $x \leqslant ty$ for some $y \in C$ and all $t > 0$ implies $x \leqslant 0$. An element $e \in C$ is called an *order unit* for (V, C) if for each $x \in V$ there exists $t > 0$ such that $x \in t[-e, e]$, where $[x_1, x_2]$ denotes, as usually, the order interval consisting of those elements of V which lie between x_1 and x_2. With every order unit $e \in C$ we can associate the seminorm $\|\cdot\|_e$ on V defined by

$$\|x\|_e = \inf\{t > 0 : x \in t[-e, e]\}.$$

It was shown that $\|\cdot\|_e$ is a norm if and only if (V, C) is almost Archimedean ordered, in which case (V, C, e) is said to be an *order-unit space.*

Note that if (V, C, e) admits an order unit $e \in C$, then C generates V. Note also that if d is another order unit for (V, C), then $\|\cdot\|_e$ and $\|\cdot\|_d$ are equivalent seminorms.

A nonempty subset $K \subseteq C$, where C is a proper cone in V, is said to be a *base* for C if for every nonzero $x \in C$ there is a unique positive real number $t > 0$ such that $x \in tK$. It is not difficult to show that K is a base for C if and only if there is a strictly positive linear functional e on V (i. e. satisfying $e(C \setminus \{0\}) > 0$) such that $K = \{x \in C : e(x) = 1\}$. Moreover, for each K there exists exactly one such a functional e. If for $x \in V$ we put by definition

$$\|x\|_K = \inf\{e(x_1 + x_2) : x_1, x_2 \in C,\ x_1 - x_2 = x\},$$

then, provided we assume that C generates V, $\|\cdot\|_K$ is a seminorm on V and

$$\|x\|_K = \inf\{t > 0 : x \in t \operatorname{conv}(K \cup (-K))\},$$

where conv $(K \cup (-K))$ stands for the convex hull of $K \cup (-K)$.

A triple (V, C, K), where C is a generating proper cone in V with the distinguished base K, is said to be a *base-norm space* provided $\|\cdot\|_K$ is a norm on V.

It was shown that the Banach dual of an order-unit space is a base-norm space and conversely, the Banach dual of a base-norm space is an order-unit space. More precisely, if (V, C, K) is a base-norm space, then the partially ordered vector space (V', C'), where V' is the Banach dual of V and C' stands for the cone of all positive $\|\cdot\|_K$-continuous linear functionals on V, has an order unit $e \in C'$ such that $K = \{ x \in C : e(x) = 1 \}$; namely, e is defined by $e(y) = \| y_1 \|_K - \| y_2 \|_K$ where y_1, y_2 are elements of C such that $y_1 - y_2 = y$, and it is easily seen that this definition does not depend on any particular choice of $y_1, y_2 \in C$ in the decomposition of $y \in V$. Moreover, the order-unit norm $\| \cdot \|_e$ of V′ coincides with the standard norm of V′ dual to $\| \cdot \|_K$. Conversely, if (V, C, e) is an order-unit space, then (V′, C′, K) with $K = \{ f \in C' : f(e) = 1 \}$ becomes a base-norm space, with K being a base for C′, such that the base-norm $\| \cdot \|_K$ coincides with the standard norm of V′ dual to $\| \cdot \|_e$.

APPENDIX B

P-projections and spectral duality.

For all the definitions and proofs of the results described in this appendix we refer the reader to references [*2*], [*3*].

Let (X, X_+) be a partially ordered real vector space with X_+ being its positive cone. We say that two positive projections P, Q on X are *quasicomplementary* (*q. c.*, in short) if $(\text{im } P) \cap X_+ = (\ker Q) \cap X_+$ and $(\ker P) \cap X_+ = (\text{im } Q) \cap X_+$.

Now let (X, X_+) and (Y, Y_+) be two partially ordered real vector spaces which are in *separating order duality* given by a nondegenerate bilinear form $(\cdot, \cdot) : X \times Y \to R$, i. e. we assume that for $x \in X$, $y \in Y$

$$\begin{cases} x \geqslant 0 \quad \text{iff} \quad (x, y) \geqslant 0 \text{ for all } y \geqslant 0, \\ y \geqslant 0 \quad \text{iff} \quad (x, y) \geqslant 0 \text{ for all } x \geqslant 0. \end{cases} \tag{B.1}$$

We say that two weakly continuous positive projections P, Q on X are *complementary* if P, Q are quasicomplementary and if so are the dual projections P^*, Q^*. (Other equivalent definitions of complementarity can be found in [*3*]).

It has been shown that in a pair P, Q consisting of two complementary projections the second member Q is uniquely determined by P and *vice versa*, so we write $Q = P'$ or $P = Q'$.

Now, let us consider an order-unit space (A, A_+, e) and a base-norm space (V, V_+, K) and assume that they are in separating order and *norm duality*, i. e. we shall assume (B.1) together with the following requirement (in which $a \in A$, $x \in V$):

$$\begin{cases} \| a \| \leqslant 1 \quad \text{iff} \quad |(a, x)| \leqslant 1 \text{ for all } x \text{ with } \| x \| \leqslant 1, \\ \| x \| \leqslant 1 \quad \text{iff} \quad |(a, x)| \leqslant 1 \text{ for all } a \text{ with } \| a \| \leqslant 1. \end{cases} \tag{B.2}$$

Let P be a weakly continuous positive projection on either A or V with norm at most 1. For such a projection its dual P^* will also be of norm at most 1 by virtue of (B.2).

We say that P is a *P-projection* if P admits a complement with norm at most 1. Clearly, P' is then a P-projection, since $P'' = P$.

It has been shown that a weakly continuous positive projection P on one of the spaces A or V is a P-projection if and only if the dual projection P^* is a P-projection on the other space. Then also $P^{*\prime} = P^{\prime *}$.

Moreover, if P is a P-projection on V, then

i) $\| Px \| + \| P'x \| = \| x \|$ for $x \in V_+$;

ii) P is neutral, that is if $\|Px\| = \| x \|$, where $x \in V_+$, then $Px = x$.

For each P-projection P on either A or V, either one of the two cones $\text{im}^+ P = (\text{im } P) \cap X_+$, $\ker^+ P = (\ker P) \cap X_+$, where X stands for either A or V, determines the other and the projection P. In particular, since for a P-projection P on V we have (see *ii*) above) $\text{im}^+ P = \{ x \in V_+ : \| Px \| = \| x \| \}$, we obtain as a corollary that $P = Q$ (here Q is another P-projection on V) if and only if $\{ x \in K : \| Px \| = 1 \} = \{ x \in K : \| Qx \| = 1 \}$.

For a given P-projection P on V the element P^*e lies clearly in the order interval $[0, e]$ and is called the *projective unit* of A associated with P. The set $F_P = (\text{im } P) \cap K$ is a face of K (i. e. it is a convex subset of K such that $tx + (1 - t)y \in F_p$, where $0 < t < 1$ and $x, y \in K$, implies $x, y \in F_p$), and the faces of this form are called the *projective faces* of K. Clearly, $F_P = \{ x \in K : (e, Px) = 1 \} = \{ x \in K : (P^*e, x) = 1 \}$.

A P-projection Q on A is said to be *compatible* with $a \in A$ if $Qa + Q'a = a$. We say that a projective face F_P is *compatible* with $a \in A$ if P^* is compatible with a.

The spaces A and V are said to be in *weak spectral duality* if for every $a \in A$ and every $t \in R$ there exists a projective face F compatible with a such that $(a, x) \leqslant t$ for $x \in F$ and $(a, x) > t$ for $x \in F'$, where $F' = F_{P'}$, provided $F = F_P$. If, in addition, F is unique, then we say that A and V are in *spectral duality.*

We say that A is *pointwise monotone σ-complete* if for each increasing sequence $\{ a_n \} \subseteq A$ which is bounded above there exists an $a \in A$ such that $(a, x) = \sup_n (a_n, x)$ for all $x \in K$.

It is shown in [*3*] that if (A, A_+, e) and (V, V_+, K) are in spectral duality, with A being pointwise monotone σ-complete, then

i) The family of all the projective faces of K is a σ-complete orthomodular lattice.

ii) Every $a \in A$ possesses a unique spectral resolution, so that A admits a functional calculus by bounded Borel functions from R to R, having the usual properties (see [*3*] for details).

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