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An integral formula for cocycles of Lie groups

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ABSTRACT. — The relationship between cohomology of Lie groups and of Lie algebras is studied, locally as well globally, by means of a formalism associating some differential forms with C^∞ cochains. An integral formula is constructed which allows to define an isomorphism from the Lie algebra cohomology (relative or not) onto the group cohomology (global or local), and furnishes an explicit expression for cocycles.

Cohomology of Lie groups, in the sense of Eilenberg and MacLane [*1*], was first considered in a series of papers by W. T. Van Est [*2*], [*3*], [*4*], then by other authors [*5*]. Later on the theory was extended to group germs by S. Swierczkowski [*6*]. One of the main problems consists to relate the cohomology of a given group (defined by C^∞ cochains) to that of its Lie algebra. For group germs the two cohomologies are isomorphic [*6*], while globally the Lie algebra cohomology must be replaced by the relative cohomology with respect to a maximal compact subgroup [*3*]. The methods used in these papers are of abstract nature, mainly founded on the consideration of double complexes and of spectral sequences. We intend to give here a more elementary and perhaps more constructive analysis of the problem. The key is the setting up of an integral formula for cocycles, valid locally or globally.

The present paper is divided into two parts, respectively devoted to local and global theory, whose the developments are parallel. We begin in defining and studying some differential forms we associate with cochains. As it is known, certain of these forms allow to define a homomorphism between the group cohomology and the Lie algebra cohomology. The next step consists in constructing a set of chains (called *standard chains*) either in the neighbourhood of the neutral element or in the quotient space of the group with respect to a maximal compact subgroup. These chains serve to construct a homomorphism from the Lie algebra cohomology (relative or not) into the group cohomology (based on a set of cochains reduced or not with the help of a maximal compact subgroup) which is a right inverse of the preceding one. The isomorphism is finally proved with the help of the integral formula. The scheme of the construction of that formula is founded on a set of recursion relations satisfied by the differential forms we have introduced, and on the properties of standard chains. The theory is developed in the case where the vector space in which the cochains take their values is finite-dimensional. But an extension to infinite-dimensional spaces seems, in some measure, possible, and an example is given in an Appendix.

I. LOCAL COHOMOLOGY

I.1. Definitions and lemmas.

Let G be a Lie group, and $\omega \to T_\omega$ a linear representation of G in a finite-dimensional real vector space E. With any element X of the Lie algebra A of G is associated the generator

$$T_X = (XT_\omega)_{\omega=e}, \qquad \forall X \in A \tag{I.1}$$

The generators satisfy the relations

$$[T_X, T_Y] = T_{[X,Y]}, \qquad \forall X, Y \in A \tag{I.2}$$

$$XT_\omega = T_\omega T_X, \qquad \forall \omega \in G,\ \forall X \in A \tag{I.3}$$

The space of differential forms on G with values in E will be denoted by $\mathscr{A}(G; E)$. For any $\phi \in \mathscr{A}(G; E)$ we define the form $T\phi$ by the formula

$$(T\phi)_\omega = (T_\omega\phi)_\omega = T_\omega\phi_\omega, \qquad \forall \omega \in G \tag{I.4}$$

The operator D is then defined by

$$D = T^{-1} \circ d \circ T \tag{I.5}$$

where d is the usual operator of exterior differentiation. For a m-form ϕ and any elements $X^{(1)}, \ldots, X^{(m+1)}$ of A, we explicitly have

$$\text{(I.6)}\quad D\phi(X^{(1)}, \ldots, X^{(m+1)}) = \frac{1}{m+1}\left\{ \sum_k (-1)^{k-1} T_{X^{(k)}}\phi(\ldots \hat{X}^{(k)} \ldots) + \sum_k (-1)^{k-1} X^{(k)}(\phi(\ldots \hat{X}^{(k)} \ldots)) + \sum_{i<j} (-1)^{i+j}\phi([X^{(i)}, X^{(j)}], \ldots \hat{X}^{(i)} \ldots \hat{X}^{(j)} \ldots) \right\}$$

The space $\Lambda(A^*) \otimes E$ of multilinear alternating forms on A with values in E will be identified with $\mathscr{A}_g(G; E)$, the space of left invariant forms. From the relation

$$\text{(I.7)}\qquad \gamma_\omega^* T\phi = T_\omega T\gamma_\omega^*\phi, \qquad \forall\omega \in G,\ \forall\phi \in \mathscr{A}(G; E)$$

where γ_ω denotes the left translation by ω in G, it follows that D commutes with γ_ω^* and then that $\Lambda(A^*) \otimes E$ is stable by D. This is equally seen from Formula (I.6), in which the second term disappears when ϕ belongs to $\Lambda(A^*) \otimes E$. We evidently have $D^2 = 0$, and the corresponding cohomology space will be denoted by H(A, E).

Let us denote by $C(G, E) = \bigoplus_{m=0}^{\infty} C^m(G, E)$ the space of non-homogeneous C^∞ cochains defined on G with values in E. The corresponding cohomology space deduced from the usual operator δ associated with the representation T [*1*] will be denoted by H(G, E). For any vector field $X \in A$ we denote by X_i the corresponding differential operator acting on the variable $\omega_i \in G$; with any family $X^{(1)}, \ldots, X^{(m)}$, $m > 0$, of elements of A we associate the differential operator $D_{X^{(1)},\ldots,X^{(m)}}$ on $C^m(G, E)$ which is defined by

$$\text{(I.8)}\quad (D_{X^{(1)},\ldots,X^{(m)}}f)(\omega_1, \ldots, \omega_m) = \frac{1}{m!}\sum_{p\in S_m} \varepsilon_p X_1^{(p(1))} \cdots\cdots X_m^{(p(m))} f(\omega_1, \ldots, \omega_m)$$

in which S_m denotes the symmetric group of degree m and ε_p the signature of the permutation p. The notation $D^e_{X^{(1)},\ldots,X^{(m)}}(f)$ will then mean the value of the left hand side of (I.8) for $\omega_1 = \ldots = \omega_m = e$.
For $f \in C^m(G, E)$ and $0 \leqslant k \leqslant m$ we introduce again the cochain

$$f\,|_{\omega_1,\ldots,\omega_k} \in C^{m-k}(G, E)$$

which is given by

$$f\,|_{\omega_1,\ldots,\omega_k}(\omega_{k+1}, \ldots, \omega_m) = f(\omega_1, \ldots, \omega_m)$$

If $m > 0$ we then define the forms $[f] \in \Lambda^m(\mathrm{A}^*) \otimes \mathrm{E} \equiv \mathscr{A}_g^{(m)}(\mathrm{G}; \mathrm{E})$ and $\langle f \rangle \in \mathscr{A}^{(m-1)}(\mathrm{G}; \mathrm{E})$ by the following formulas

$$(\mathrm{I}.9) \qquad [f](\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)}) = \mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m)}}(f)$$

$$(\mathrm{I}.10) \qquad \langle f \rangle (\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m-1)})_\omega = \mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m-1)}}(f\,|_\omega)$$

If $m = 0$ we put $[f] = f$ and $\langle f \rangle = 0$. The following relation is easily verified, for $m > 0$,

$$(\mathrm{I}.11) \quad [f](\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)}) = \frac{1}{m}\sum_{k=1}^{m}(-1)^{k-1}\mathrm{X}^{(k)}(\langle f \rangle(\ldots \hat{\mathrm{X}}^{(k)} \ldots))_e$$

LEMMA 1. — For $f \in \mathrm{C}^m(\mathrm{G}, \mathrm{E})$ the two forms $[\delta f]$ and $\langle \delta f \rangle$ are given by

$$(\mathrm{I}.12) \qquad [\delta f] = \mathrm{D}[f]$$

$$(\mathrm{I}.13) \qquad \langle \delta f \rangle = \mathrm{T}[f] - d\langle f \rangle, \qquad d^\circ f = m > 0$$

Proof. — We first demonstrate (I.13). By applying the operator

$$\frac{1}{m!}\sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m)\mathrm{X}_2^{(k_1)}, \ldots, \mathrm{X}_{m+1}^{(k_m)}\,|_{\omega_2 = \ldots = \omega_{m+1} = e}$$

to the two members of the relation

$$\delta f(\omega_1, \ldots, \omega_{m+1}) = \mathrm{T}_{\omega_1} f(\omega_2, \ldots, \omega_{m+1}) - f(\omega_1\omega_2, \omega_3, \ldots, \omega_{m+1}) + \sum_{p=2}^{m}(-1)^p f(\omega_1, \ldots, \omega_p\omega_{p+1}, \ldots) + (-1)^{m+1} f(\omega_1, \ldots, \omega_m)$$

and by using the left invariance of the $\mathrm{X}^{(k)}$'s, we find

$$\begin{aligned}(\mathrm{I}.14) \quad \langle \delta f \rangle (\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)})_{\omega_1} &= (\mathrm{T}[f])(\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)})_{\omega_1} \\ &- \frac{1}{m}\sum_k (-1)^{k-1}\mathrm{X}^{(k)}(\langle f \rangle(\ldots \hat{\mathrm{X}}^{(k)} \ldots))_{\omega_1} \\ &+ \frac{1}{m!}\sum_{p=2}^{m}(-1)^p \sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m)\mathrm{X}_2^{(k_1)} \ldots (\mathrm{X}^{(k_{p-1})}\mathrm{X}^{(k_p)})_p \\ &\qquad \ldots \mathrm{X}_m^{(k_m)} f(\omega_1, \ldots, \omega_m)\,|_{\omega_2 = \ldots = \omega_m = e}\end{aligned}$$

The last term is transformed as follows: let us decompose the permutation represented by the sequence $(k_1, \ldots, k_m)$ into the product

$$(1, 2, \ldots, m) \to (k_{p-1}, k_p, 1, \ldots, m) \to (k_1, \ldots, k_m)$$

and let $\eta(k_{p-1}, k_p)$ and $\varepsilon_p(k_1, \ldots, k_m)$ be the signatures of these two successive permutations. We have

$$\eta(k_{p-1}, k_p) = \begin{cases} (-1)^{k_{p-1}+k_p-1} & \text{if } k_{p-1} < k_p \\ (-1)^{k_{p-1}+k_p} & \text{if } k_{p-1} > k_p \end{cases}$$

while $\varepsilon_p(k_1, \ldots, k_m)$ equals the signature of the permutation transforming the sequence $(1, 2, \ldots, m)$ not containing k_{p-1} and k_p into the sequence $(k_1, \ldots, \hat{k}_{p-1}, \hat{k}_p, \ldots, k_m)$. Let $\varepsilon(k_1, \ldots, k_{p-2}, 0, k_{p+1}, \ldots, k_m)$ be the signature of the permutation which transforms the sequence $(0, 1, 2, \ldots, m)$ not containing k_{p-1} and k_p into $(k_1, \ldots, k_{p-2}, 0, k_{p+1}, \ldots, k_m)$; we have

$$\varepsilon(k_1, \ldots, k_{p-2}, 0, k_{p+1}, \ldots, k_m) = (-1)^{p-2}\varepsilon_p(k_1, \ldots, k_m)$$

That gives for the last term in the right hand side of (I.14)

$$\begin{aligned} &\frac{1}{m!}\sum_{p=2}^{m}(-1)^p \sum_{k_{p-1},k_p} \eta(k_{p-1}, k_p) \sum_{(k_1,\ldots,\hat{k}_{p-1},\hat{k}_p,\ldots,k_m)} (-1)^{p-2}\varepsilon(k_1, \ldots, k_{p-2}, 0, k_{p+1}, \ldots, k_m) \\ &\qquad \times X_2^{(k_1)} \ldots (X^{(k_{p-1})}X^{(k_p)})_p \ldots X_m^{(k_m)} f(\omega_1, \ldots, \omega_m)\big|_{\omega_2 = \ldots = \omega_m = e} \\ &= \frac{1}{m!}\sum_{i,j}\eta(i, j)\sum_{p=2}^{m}\sum_{\substack{(k_1,\ldots,k_{p-2},0,k_{p+1},\ldots,k_m)\\ k_l \neq i,j}} \varepsilon(k_1, \ldots, k_{p-2}, 0, k_{p+1}, \ldots, k_m) \\ &\qquad \times X_2^{(k_1)} \ldots (X^{(i)}X^{(j)})_p \ldots X_m^{(k_m)} f(\omega_1, \ldots, \omega_m)\big|_{\omega_2 = \ldots = \omega_m = e} \\ &= -\frac{1}{m}\sum_{i<j}(-1)^{i+j}\langle f \rangle([X^{(i)}, X^{(j)}], \ldots, \hat{X}^{(i)}, \ldots, \hat{X}^{(j)}, \ldots)_{\omega_1} \end{aligned}$$

Taking that result in (I.14) we obtain (I.13).

Let us now write (I.13) with f replaced by δf, $d^0 f \geqslant 0$; we obtain

$$0 = \langle \delta^2 f \rangle = T[\delta f] - d\langle \delta f \rangle$$

If $d^0 f > 0$ the last term may be calculated from (I.13) that gives (I.12); if $d^0 f = 0$ this last formula remains valid since we have then $[f] = f$ and $\langle \delta f \rangle_\omega = \delta f(\omega) = T_\omega f - f$ that is $\langle \delta f \rangle = T[f] - f$. q. e. d.

Formula (I.12) extends by linearity to C(G, E). It expresses that the mapping $[\ \ ] : C(G, E) \to \Lambda(A^*) \otimes E$ is a homomorphism of differential spaces, and thus induces a homomorphism $[\ \ ]_\# : H(G, E) \to H(A, E)$. Formula (I.13) may be extended for all degrees in the form

$$\langle \delta f \rangle + d\langle f \rangle = T[f] - \mathscr{I}_0 f$$

in which $\mathscr{I}_0 : C(G, E) \to \mathscr{A}(G ; E)$ is the mapping which cancels the components of non-vanishing degree in C(G, E) and leaves invariant $C^0(G, E) = E = \mathscr{A}_g^{(0)}(G; E)$. It expresses that $\langle \quad \rangle$ is a homotopy operator for the homomorphisms $T[\quad]$ and $\mathscr{I}_0$ of the differential spaces $(C(G, E), \delta)$ and $(\mathscr{A}(G; E), d)$.

LEMMA 2. — For $f \in C^m(G, E)$ and $3 \leqslant k \leqslant m$ the following identity is valid

$$(I.15)\quad \langle \delta f |_{\omega_1,\dots,\omega_{k-1}} \rangle = T_{\omega_1} \langle f |_{\omega_2,\dots,\omega_{k-1}} \rangle + \sum_{l=1}^{k-2} (-1)^l \langle f |_{\omega_1,\dots,\omega_l\omega_{l+1},\dots,\omega_{k-1}} \rangle + (-1)^{k-1} \gamma^*_{\omega_{k-1}} \langle f |_{\omega_1,\dots,\omega_{k-2}} \rangle + (-1)^k d \langle f |_{\omega_1,\dots,\omega_{k-1}} \rangle$$

For $2 = k \leqslant m$ that identity is reduced to

$$(I.16)\qquad \langle \delta f |_{\omega_1} \rangle = T_{\omega_1} \langle f \rangle - \gamma^*_{\omega_1} \langle f \rangle + d \langle f |_{\omega_1} \rangle$$

Proof. — Let us first assume $3 \leqslant k \leqslant m$. According to the definition of δf, we may write

$$(I.17)\quad \delta f(\omega_1, \dots, \omega_{m+1}) = T_{\omega_1} f|_{\omega_2,\dots,\omega_{k-1}}(\omega_k, \dots, \omega_{m+1}) + \sum_{l=1}^{k-2} (-1)^l f |_{\omega_1,\dots,\omega_l\omega_{l+1},\dots,\omega_{k-1}}(\omega_k, \dots, \omega_{m+1}) + (-1)^{k-1} f |_{\omega_1,\dots,\omega_{k-2}}(\omega_{k-1}\omega_k, \dots, \omega_{m+1}) + \sum_{l=k}^{m} (-1)^l f |_{\omega_1,\dots,\omega_{k-1}}(\omega_k, \dots, \omega_l\omega_{l+1}, \dots, \omega_{m+1}) + (-1)^{m+1} f |_{\omega_1,\dots,\omega_{k-1}}(\omega_k, \dots, \omega_m)$$

The sum of the last two terms may be replaced by

$$(-1)^{k-1} \{ \delta(f |_{\omega_1,\dots,\omega_{k-1}})(\omega_k, \dots, \omega_{m+1}) - T_{\omega_k} f |_{\omega_1,\dots,\omega_{k-1}}(\omega_{k+1}, \dots, \omega_{m+1}) \}$$

Let us now apply to the two members of (I.17) the operator $D_{X^{(1)},\dots,X^{(m-k+1)}}$ acting on the variables $\omega_{k+1}, \dots, \omega_{m+1}$, and take the value for

$$\omega_{k+1} = \dots = \omega_{m+1} = e ;$$

we find

$$\langle \delta f |_{\omega_1,\dots,\omega_{k-1}} \rangle = T_{\omega_1} \langle f |_{\omega_2,\dots,\omega_{k-1}} \rangle + \sum_{l=1}^{k-2} (-1)^l \langle f |_{\omega_1,\dots,\omega_l\omega_{l+1},\dots,\omega_{k-1}} \rangle + (-1)^{k-1} \gamma^*_{\omega_{k-1}} \langle f |_{\omega_1,\dots,\omega_{k-2}} \rangle + (-1)^{k-1} \{ \langle \delta(f |_{\omega_1,\dots,\omega_{k-1}}) \rangle - T[f |_{\omega_1,\dots,\omega_{k-1}}] \}$$

The last term is transformed by (I.13) that gives (I.15).

For $2 = k \leqslant m$ we write

$$\delta f(\omega_1, \ldots, \omega_{m+1}) = \mathrm{T}_{\omega_1} f(\omega_2, \ldots, \omega_{m+1}) - f(\omega_1\omega_2, \ldots, \omega_{m+1})$$
$$+ \sum_{l=2}^{m} (-1)^l f\,|_{\omega_1}(\omega_2, \ldots, \omega_l\omega_{l+1}, \ldots, \omega_{m+1}) + (-1)^{m+1} f\,|_{\omega_1}(\omega_2, \ldots, \omega_m)$$
$$= \mathrm{T}_{\omega_1} f(\omega_2, \ldots, \omega_{m+1}) - f(\omega_1\omega_2, \ldots, \omega_{m+1})$$
$$- \{\, \delta(f\,|_{\omega_1})(\omega_2, \ldots, \omega_{m+1}) - \mathrm{T}_{\omega_2} f|_{\omega_1}(\omega_3, \ldots, \omega_{m+1}) \,\}$$

and, by applying the operator $\mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m-1)}}$ acting on the variables $\omega_3, \ldots, \omega_{m+1}$, we obtain

$$\langle\, \delta f\,|_{\omega_1} \rangle = \mathrm{T}_{\omega_1} \langle\, f \,\rangle - \gamma^*_{\omega_1} \langle\, f \,\rangle - \{\, \langle\, \delta(f\,|_{\omega_1}) \rangle - \mathrm{T}[f\,|_{\omega_1}] \,\}$$

By Formula (I.13) we obtain (I.16). q. e. d.

Remark. — The definitions (I.9) and (I.10), and the formulas (I.12), (I.13), (I.15) and (I.16) are similar to that introduced in ref. [*2*] in the definition of the bicochains and of the cochain transformation, except that we use here left invariant fields and differentiations on the *last variables* of the f's. This is the reason for the appearance of the usual operator d in these formulas instead of an operator like D.

I.2. **Homomorphism** $\Lambda(\mathrm{A}^*) \otimes \mathrm{E} \to \mathrm{C}_{\mathrm{loc}}(\mathrm{G}, \mathrm{E})$.

Let $\mathrm{C}_{\mathrm{loc}}(\mathrm{G}, \mathrm{E})$ be the space of the germs of cochains at the neutral element e, and $\mathrm{H}_{\mathrm{loc}}(\mathrm{G}, \mathrm{E})$ the corresponding cohomology space. For

$$f \in \mathrm{C}^m_{\mathrm{loc}}(\mathrm{G}, \mathrm{E})$$

the formula (I.9) still defines an element of $\Lambda(\mathrm{A}^*) \otimes \mathrm{E}$, while the formula (I.10) defines a germ of form. The relations (I.12) and (I.13) then remain valid and keep the same meaning as previously. In particular the first one allows to define a homomorphism $\mathrm{H}_{\mathrm{loc}}(\mathrm{G}, \mathrm{E}) \to \mathrm{H}(\mathrm{A}, \mathrm{E})$, which we will again denote by $[\ \ ]_{\#}$. On the contrary, the formulas (I.15) and (I.16) as they stand, the ω's being fixed, have no meaning for a germ of cochain. However they make sense for any representative of a given germ so that, for two such representatives, the corresponding relations become identical term by term when the ω's belong to a suitable neighbourhood of e. Any cochain defined on a given neighbourhood of e will be called a local cochain. The consideration of local cochains is sufficient to handle the germs of cochains since, if f is defined on the neighbourhood V, the differential δf is defined on any neighbourhood W taken such that $\mathrm{W}^2 \subset \mathrm{V}$. A local cocycle will then be a local cochain f whose the differential δf vanishes on a neighbourhood of e, while a local coboundary will be a local cochain g such that, for some local cochain h, we have the equality $g = \delta h$ on a neighbourhood of e.

We will construct a homomorphism from $\Lambda(A^*) \otimes E$ into $C_{loc}(G, E)$. Let $\omega \to (\omega^\alpha)$ be a coordinate system defined on a neighbourhood V_0 of e, for which the coordinates of e vanish. In addition, V_0 is assumed *star-shaped* with respect to that system, that is such that, for any $t \in [0, 1]$, the relation $\omega \in V_0$ implies the relation $t\omega \in V_0$, where $t\omega$ denotes the element whose coordinates are $t\omega^\alpha$. Let $H : [0, 1] \times V_0 \to V_0$ be the homotopy defined by $H(t, \omega) = t\omega$, $0 \leqslant t \leqslant 1$, $\omega \in V_0$.

We first define recursively a set of chains in the neighbourhood of e. The 0-chain $C(\omega)$ is, by definition,

$$C(\omega) = \{ \omega \} \tag{I.18}$$

If the $(m-1)$-chain $C(\omega_1, \ldots, \omega_m)$ is defined, we define the m-chain $C(e, \omega_1, \ldots, \omega_m)$ by

$$C(e, \omega_1, \ldots, \omega_m) = H([0, 1] \times C(\omega_1, \ldots, \omega_m)) \tag{I.19}$$

where the product of chains $[0, 1] \times C(\omega_1, \ldots, \omega_m)$ in $\mathbb{R} \times G$ is oriented by taking as first coordinate the coordinate t on $[0, 1]$. We then define the m-chain $C(\omega_1, \ldots, \omega_{m+1})$ by

$$C(\omega_1, \ldots, \omega_{m+1}) = \gamma_{\omega_1} C(e, \omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}) \tag{I.20}$$

More precisely we have the following lemma:

LEMMA 3. — Let W_k and V_k be two sequences of neighbourhoods of e, respectively symmetrical and star-shaped, satisfying the relations $W_0 \subset V_0$ and

$$W_k \subset V_k\,, \quad W_k^2 \subset W_{k-1}\,, \quad W_k V_k \subset V_{k-1}\,, \qquad \forall k \geqslant 1 \tag{I.21}$$

Through the relations (I.18), (I.19) and (I.20) the m-chains $C(\omega_1, \ldots, \omega_{m+1})$ are then defined when $\omega_1, \ldots, \omega_{m+1} \in W_{2m}$. If $\omega_1, \ldots, \omega_{m+1} \in W_{2m+k}$, $k \geqslant 0$, the set of points of any of these chains is contained in V_k.

If $\omega, \omega_1, \ldots, \omega_{m+1} \in W_{2m+k}$, $k \geqslant 1$, the following relation is satisfied

$$\gamma_\omega C(\omega_1, \ldots, \omega_{m+1}) = C(\omega\omega_1, \ldots, \omega\omega_{m+1}) \tag{I.22}$$

Proof. — It is easy to construct two sequences W_k and V_k satisfying (I.21) (the star-shaped neighbourhoods may be, for example, open balls in the given coordinate system). In particular, these sequences are decreasing. The lemma is evidently verified for the 0-chains. Let us assume it true for the $(m-1)$-chains, $m \geqslant 1$. If $\omega_1, \ldots, \omega_{m+1} \in W_{2m+k}$, $k \geqslant 0$, we have $\omega_1^{-1}\omega_p \in W_{2m+k-1}$, $2 \leqslant p \leqslant m+1$; then $C(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1})$ is defined and contained in V_{k+1}. By Formula (I.19) the same is true for

$$C(e, \omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1})$$

since V_{k+1} is star-shaped. Formula (I.20) then shows that $C(\omega_1, \ldots, \omega_{m+1})$ is defined and contained in $W_{2m+k}V_{k+1} \subset W_{k+1}V_{k+1} \subset V_k$.

Finally let us assume $\omega, \omega_1, \ldots, \omega_{m+1} \in W_{2m+k}$, $k \geqslant 1$. The definition gives

$$\gamma_\omega C(\omega_1, \ldots, \omega_{m+1}) = \gamma_\omega \gamma_{\omega_1} C(e, \omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}) = \gamma_{\omega\omega_1} C(e, (\omega\omega_1)^{-1}\omega\omega_2, \ldots, (\omega\omega_1)^{-1}\omega\omega_{m+1})$$

Since, for $1 \leqslant p \leqslant m+1$, we have $\omega\omega_p \in W_{2m+k-1}$ and $k-1 \geqslant 0$, that expression equals the right member of (I.22). q. e. d.

The chains $C(\omega_1, \ldots, \omega_m)$ will be called *standard chains* associated with the given coordinate system. Let us furthermore demonstrate the following formula for the boundary of a standard chain of dimension $\geqslant 1$

$$\text{(I.23)} \quad \partial C(\omega_1, \ldots, \omega_m) = \sum_{k=1}^{m} (-1)^{k-1} C(\omega_1, \ldots, \hat{\omega}_k, \ldots, \omega_m), \qquad \forall \omega_1, \ldots, \omega_m \in W_{2(m-1)}$$

If, for any chain C in G, we set $\hat{C} = [0,1] \times C$, a classical lemma reads

$$\text{(I.24)} \qquad \partial\hat{C} + \widehat{\partial C} = j_1 C - j_0 C$$

in which $j_t : G \to \mathbb{R} \times G$ is the mapping defined by $j_t(\omega) = (t, \omega)$. Assuming (I.23) at the order $m \geqslant 2$, taking

$$C = C(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}), \qquad \omega_1, \ldots, \omega_{m+1} \in W_{2m},$$

in (I.24) we find

$$\begin{aligned}\partial\hat{C}(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}) &= j_1 C(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}) - j_0 C(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}) \\ &+ \sum_{k=2}^{m+1} (-1)^{k-1} \hat{C}(\omega_1^{-1}\omega_2, \ldots, \widehat{\omega_1^{-1}\omega_k}, \ldots, \omega_1^{-1}\omega_{m+1})\end{aligned}$$

By applying H to the two members of that equation, and noting the relations $H \circ j_1 = id_{V_0}$ and $Im(H \circ j_0) = \{ e \}$, we obtain

$$\begin{aligned}\partial C(e, \omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}) &= C(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_{m+1}) \\ &+ \sum_{k=2}^{m+1} (-1)^{k-1} C(e, \omega_1^{-1}\omega_2, \ldots, \widehat{\omega_1^{-1}\omega_k}, \ldots, \omega_1^{-1}\omega_{m+1})\end{aligned}$$

With the help of (I.20) and (I.22), that relation implies (I.23) at the order $m+1$. Moreover, (I.23) is immediately verified for $m = 2$. For $m = 1$ we evidently have $\partial C(\omega) = 0$.

Let now ϕ be any element of $\mathscr{A}^{(m)}(G; E)$, and let s_ϕ be the local m-cochain which is defined by

$$\text{(I.25)} \qquad s_\phi(\omega_1, \ldots, \omega_m) = \int_{C(e, \omega_1, \omega_1\omega_2, \ldots, \omega_1\omega_2 \ldots \omega_m)} T\phi$$

The corresponding homogeneous cochain S_ϕ (see ref. [1]) is given by

$$\text{(I.26)} \qquad S_\phi(\omega_0, \omega_1, \ldots, \omega_m) = \int_{C(\omega_0,\omega_1,\ldots,\omega_m)} T_{\omega_0}\gamma^*_{\omega_0^{-1}}T\phi$$

With the notations of Lemma 3, the latter is defined for $\omega_0, \ldots, \omega_m \in W_{2m}$. Thus s_ϕ is defined on a neighbourhood of e.

LEMMA 4. — For ω in the neighbourhood of e, the local cochain s_ϕ satisfies the identity

$$\text{(I.27)} \qquad \delta s_\phi\,|_\omega = T_\omega s_{\phi - \gamma^*_\omega\phi} + s_{D\phi}\,|_\omega$$

Proof. — From (I.26) and (I.23) we deduce, if $\omega_0, \ldots, \omega_{m+1} \in W_{2(m+1)}$,

$$\delta S_\phi(\omega_0, \ldots, \omega_{m+1}) = \sum_{k=0}^{m+1} (-1)^k S_\phi(\omega_0, \ldots, \hat{\omega}_k, \ldots, \omega_{m+1})$$
$$= \int_{C(\omega_1,\ldots,\omega_{m+1})} T_{\omega_1}\gamma^*_{\omega_1^{-1}}T\phi + \sum_{k=1}^{m+1}(-1)^k \int_{C(\omega_0,\ldots,\hat{\omega}_k,\ldots,\omega_{m+1})} T_{\omega_0}\gamma^*_{\omega_0^{-1}}T\phi$$
$$= \int_{C(\omega_1,\ldots,\omega_{m+1})} T_{\omega_1}\gamma^*_{\omega_1^{-1}}T\phi + \int_{\partial C(\omega_0,\ldots,\omega_{m+1}) - C(\omega_1,\ldots,\omega_{m+1})} T_{\omega_0}\gamma^*_{\omega_0^{-1}}T\phi$$

and then, by Stokes' theorem,

$$\delta S_\phi(\omega_0, \ldots, \omega_{m+1}) = \int_{C(\omega_1,\ldots,\omega_{m+1})} T_{\omega_1}\gamma^*_{\omega_1^{-1}}[T\phi - T_{\omega_1^{-1}\omega_0}\gamma^*_{\omega_0^{-1}\omega_1}T\phi]$$
$$+ \int_{C(\omega_0,\ldots,\omega_{m+1})} T_{\omega_0}\gamma^*_{\omega_0^{-1}}dT\phi$$

With the help of (I.7), and by using the definition of D, that gives

$$\delta S_\phi(\omega_0, \ldots, \omega_{m+1}) = S_{\phi - \gamma^*_{\omega_0^{-1}\omega_1}\phi}(\omega_1, \ldots, \omega_{m+1}) + S_{D\phi}(\omega_0, \ldots, \omega_{m+1})$$

Written for the non-homogeneous cochain s_ϕ, that relation is identical with (I.27) on some neighbourhood of e. q. e. d.

The formula (I.25) defines an element of $C^m_{\text{loc}}(G, E)$ which we will again denote by s_ϕ. By linearity that defines a mapping

$$s : \mathscr{A}(G\,;E) \to C_{\text{loc}}(G, E)\,.$$

LEMMA 5. — For $\phi \in \mathscr{A}^{(m)}(G\,;E)$ the form $[s_\phi]$ and the germ of form $\langle s_\phi \rangle$ are given by

$$\text{(I.28)} \qquad [s_\phi](X^{(1)}, \ldots, X^{(m)}) = \phi(X^{(1)}, \ldots, X^{(m)})_e$$

$$\text{(I.29)} \qquad \langle s_\phi \rangle = hT\phi$$

where h is the homotopy operator associated with the homotopy H, and where $hT\phi$ stands for the corresponding germ.

Proof. — It suffices to demonstrate (I.28) and (I.29) for the local cochain (I.25) taken on a suitably restricted neighbourhood of e. We will simultaneously demonstrate (I.28) and (I.29) by induction. These formulas are evident for $m = 0$ since we then have

$$[s_\phi] = s_\phi = \phi(e)$$
$$\langle s_\phi \rangle = 0\,, \qquad h\mathrm{T}\phi = 0$$

To demonstrate them at any order we first establish the following auxiliary formula

$$\langle s_\phi \rangle_\omega = [s_{\mathrm{T}^{-1}\gamma_\omega^* h\mathrm{T}\phi}]_\omega \tag{I.30}$$

Let us recall the definition of h [7] :

$$h\psi = \int_0^1 dt j_t^* i_\mathrm{T} \mathrm{H}^*\psi\,, \qquad \forall\psi \in \mathscr{A}(\mathrm{V}_0\,;\, \mathrm{E})$$

where T is the vector field $\mathrm{T} = \frac{d}{dt}$ on $\mathbb{R}$. According to the definition of $\langle\ \ \rangle$, we have, for any $\phi \in \mathscr{A}^{(m)}(\mathrm{G}\,;\,\mathrm{E})$,

$$\langle s_\phi \rangle (\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m-1)})_\omega = \mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m-1)}} \int_{\mathrm{C}(e,\omega,\omega\omega_2,\ldots,\omega\omega_2\ldots\ldots\omega_m)} \mathrm{T}\phi$$

in which the operators $\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m-1)}$ act on the variables $\omega_2, \ldots, \omega_m$. Since the standard chain which occurs in that formula is contained in V_0, we may replace $\mathrm{T}\phi$ by its restriction on V_0, keeping for convenience the same notation. Thus we have, with the help of (I.19) and (I.22),

$$\begin{aligned}
\langle s_\phi \rangle (\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m-1)})_\omega &= \mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m-1)}} \int_{\mathrm{H}([0,1]\times \mathrm{C}(\omega,\omega\omega_2,\ldots,\omega\omega_2\ldots\ldots\omega_m))} \mathrm{T}\phi \\
&= \mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m-1)}} \int_{[0,1]\times\gamma_\omega \mathrm{C}(e,\omega_2,\ldots,\omega_2\ldots\ldots\omega_m)} \mathrm{H}^*\mathrm{T}\phi \\
&= \mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m-1)}} \int_{\gamma_\omega \mathrm{C}(e,\omega_2,\ldots,\omega_2\ldots\ldots\omega_m)} \int_0^1 dt j_t^* i_\mathrm{T} \mathrm{H}^*\mathrm{T}\phi \\
&= \mathrm{D}^e_{\mathrm{X}^{(1)},\ldots,\mathrm{X}^{(m-1)}} s_{\mathrm{T}^{-1}\gamma_\omega^* h\mathrm{T}\phi}(\omega_2, \ldots, \omega_m) \\
&= [s_{\mathrm{T}^{-1}\gamma_\omega^* h\mathrm{T}\phi}](\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m-1)})
\end{aligned}$$

what demonstrates (I.30). Let us now assume that (I.28) and (I.29) are satisfied when $d^0\phi \leqslant m$, and let $\phi \in \mathscr{A}^{(m+1)}(\mathrm{G}\,;\,\mathrm{E})$. Formula (I.30) gives at first, with the help of (I.28) at the order m,

$$\begin{aligned}
\langle s_\phi \rangle (\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)})_\omega &= [s_{\mathrm{T}^{-1}\gamma_\omega^* h\mathrm{T}\phi}](\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)}) \\
&= (\mathrm{T}^{-1}\gamma_\omega^* h\mathrm{T}\phi)(\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)})_e = (h\mathrm{T}\phi)(\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m)})_\omega
\end{aligned}$$

that demonstrates (I.29) at the order $m + 1$. We after get, due to (I.11),

$$[s_\phi](X^{(1)}, \ldots, X^{(m+1)}) = \frac{1}{m+1} \sum_{k=1}^{m+1} (-1)^{k-1} X^{(k)}((hT\phi)(\ldots, \hat{X}^{(k)}, \ldots))_e$$
$$= d(hT\phi)(X^{(1)}, \ldots, X^{(m+1)})_e$$
$$- \frac{1}{m+1} \sum_{i<j} (-1)^{i+j} (hT\phi)([X^{(i)}, X^{(j)}], \ldots, \hat{X}^{(i)}, \ldots, \hat{X}^{(j)}, \ldots)_e$$

By introducing the homotopy formula

$$dh\psi + hd\psi = \psi, \qquad d^0\psi \geqslant 1, \ \psi \in \mathscr{A}(V_0 ; E)$$

and the relation (*) $(h\psi)_e = 0$, we find

$$[s_\phi](X^{(1)}, \ldots, X^{(m+1)}) = (T\phi)(X^{(1)}, \ldots, X^{(m+1)})_e$$

that is (I.28) at the order $m + 1$. q. e. d.

PROPOSITION 1. — The map s restricted on $\Lambda(A^*) \otimes E$ induces a homomorphism $s_\#$ from H(A, E) into $H_{loc}(G, E)$ such that

$$\text{(I.31)} \qquad [\ ]_\# \circ s_\# = id_{H(A,E)}$$

Proof. — Let ξ be a left invariant form on G. From (I.27) we get

$$\delta s_\xi = s_{D\xi}$$

that means that $s : \Lambda(A^*) \otimes E \to C_{loc}(G, E)$ is a homomorphism. Moreover, Formula (I.28) becomes

$$[s_\xi] = \xi$$

that implies (I.31). q. e. d.

To demonstrate the isomorphism of H(A, E) with $H_{loc}(G, E)$, it suffices now to show that $s_\#$ is surjective or, what comes to the same thing, that any local cocycle f is equivalent to $s_{[f]}$. This will be achieved in the next section by means of an explicit formula for local cocycles.

I.3. **Integral expression for local cocycles.**

Let f be a local cocycle of degree $m \geqslant 3$ and let $3 \leqslant k \leqslant m$. Formula (I.15) gives, on some neighbourhood of e,

$$\text{(I.32)} \quad d\langle f|_{\omega_1,\ldots,\omega_{k-1}}\rangle = (-1)^{k-1} \Big\{ T_{\omega_1} \langle f|_{\omega_2,\ldots,\omega_{k-1}}\rangle + \sum_{l=1}^{k-2} (-1)^l \langle f|_{\omega_1,\ldots,\omega_l\omega_{l+1},\ldots,\omega_{k-1}}\rangle + (-1)^{k-1} \gamma^*_{\omega_{k-1}} \langle f|_{\omega_1,\ldots,\omega_{k-2}}\rangle \Big\}$$

(*) That relation comes from the formula $dH_{(t,e)}(T) = 0$.

Let us integrate the two members of that relation on the chain

$$C(e, \omega_k, \omega_k\omega_{k+1}, \dots, \omega_k\omega_{k+1} \cdot \dots \cdot \omega_m).$$

By putting

$$(\mathrm{I}.33) \quad f^{(m-k+1)}(\omega_1, \dots, \omega_{m-1}) = \int_{C(e,\omega_{k-1},\omega_{k-1}\omega_k,\dots,\omega_{k-1}\omega_k\cdot\dots\cdot\omega_{m-1})} \langle f \mid_{\omega_1,\dots,\omega_{k-2}} \rangle$$

the sum of the first two terms in the right member of (I.32) gives

$$T_{\omega_1} f^{(m-k+1)}(\omega_2, \dots, \omega_m) + \sum_{l=1}^{k-2} (-1)^l f^{(m-k+1)}(\omega_1, \dots, \omega_l\omega_{l+1}, \dots, \omega_m)$$

The integral of the last term in (I.32) is transformed as follows:

$$\begin{aligned}
&\int_{\omega_{k-1}C(e,\omega_k,\dots,\omega_k\cdot\dots\cdot\omega_m)} \langle f \mid_{\omega_1,\dots,\omega_{k-2}} \rangle \\
&\quad = \int_{C(\omega_{k-1},\omega_{k-1}\omega_k,\dots,\omega_{k-1}\cdot\dots\cdot\omega_m)} \langle f \mid_{\omega_1,\dots,\omega_{k-2}} \rangle \\
&\quad = \int_{\substack{\partial C(e,\omega_{k-1},\dots,\omega_{k-1}\cdot\dots\cdot\omega_m) \\ +(-1)^{k-1} \sum_{l=k-1}^{m} (-1)^l C(e,\omega_{k-1},\dots,\widehat{\omega_{k-1}\cdot\dots\cdot\omega_l},\dots,\omega_{k-1}\cdot\dots\cdot\omega_m)}} \langle f \mid_{\omega_1,\dots,\omega_{k-2}} \rangle \\
&\quad = \int_{C(e,\omega_{k-1},\dots,\omega_{k-1}\cdot\dots\cdot\omega_m)} d\langle f \mid_{\omega_1,\dots,\omega_{k-2}} \rangle \\
&\qquad + (-1)^{k-1} \sum_{l=k-1}^{m-1} (-1)^l f^{(m-k+1)}(\omega_1, \dots, \omega_l\omega_{l+1}, \dots, \omega_m) \\
&\qquad + (-1)^{k-1}(-1)^m f^{(m-k+1)}(\omega_1, \dots, \omega_{m-1})
\end{aligned}$$

Equation (I.32) then gives the following recursion formula

$$(\mathrm{I}.34) \quad \int_{C(e,\omega_k,\dots,\omega_k\cdot\dots\cdot\omega_m)} d\langle f \mid_{\omega_1,\dots,\omega_{k-1}} \rangle = \int_{C(e,\omega_{k-1},\dots,\omega_{k-1}\cdot\dots\cdot\omega_m)} d\langle f \mid_{\omega_1,\dots,\omega_{k-2}} \rangle + (-1)^{k-1}\delta f^{(m-k+1)}(\omega_1, \dots, \omega_m)$$

For $k = 2 \leqslant m$, Formula (I.15) is replaced by (I.16), and a similar calculation gives

$$(\mathrm{I}.35) \quad \int_{C(e,\omega_2,\dots,\omega_2\cdot\dots\cdot\omega_m)} d\langle f \mid_{\omega_1} \rangle = \int_{C(e,\omega_1,\dots,\omega_1\cdot\dots\cdot\omega_m)} d\langle f \rangle - \delta f^{(m-1)}(\omega_1, \dots, \omega_m)$$

with

$$f^{(m-1)}(\omega_1, \ldots, \omega_{m-1}) = \int_{C(e,\omega_1,\ldots,\omega_1\ldots\omega_{m-1})} \langle f \rangle \tag{I.36}$$

Let now f be a local cocycle of degree $m \geqslant 2$; the form $\langle f |_{\omega_1,\ldots,\omega_{m-1}} \rangle$ has degree zero and is then reduced to the function $f |_{\omega_1,\ldots,\omega_{m-1}}$. We then have

$$f(\omega_1, \ldots, \omega_m) - f(\omega_1, \ldots, \omega_{m-1}, e) = \int_{C(e,\omega_m)} d(f |_{\omega_1,\ldots,\omega_{m-1}}) \quad .$$

$$= \int_{C(e,\omega_m)} d \langle f |_{\omega_1,\ldots,\omega_{m-1}} \rangle$$

Putting again

$$f^{(0)}(\omega_1, \ldots, \omega_{m-1}) = f(\omega_1, \ldots, \omega_{m-1}, e) = (-1)^m \delta f^{(0)}(\omega_1, \ldots, \omega_m) \tag{I.37}$$

and successively using (I.34) and (I.35), we get

$$f(\omega_1, \ldots, \omega_m) = \int_{C(e,\omega_1,\ldots,\omega_1\ldots\omega_m)} d \langle f \rangle + \sum_{k=0}^{m-1} (-1)^{m-k} \delta f^{(k)}(\omega_1, \ldots, \omega_m) \tag{I.38}$$

Since f is a cocycle, the relation (I.13) gives $d \langle f \rangle = \mathrm{T}[f]$ so that (I.38) becomes

$$f = s_{[f]} + \sum_{k=0}^{m-1} (-1)^{m-k} \delta f^{(k)} \tag{I.39}$$

That relation remains valid for $m = 1$ since we have then $\langle f \rangle = f$ and $f(e) = 0$, that gives

$$f(\omega_1) = \int_{C(e,\omega_1)} d \langle f \rangle = \int_{C(e,\omega_1)} \mathrm{T}[f] = s_{[f]}(\omega_1)$$

We then have $f^{(0)} = 0$. For $m = 0$, Equation (I.39) is replaced by

$$f = [f] = s_{[f]}$$

It is easy to see that all the preceding calculations are valid on a sufficiently restricted neighbourhood of e. We have then

PROPOSITION 2. — Any local cocycle f is represented, in the neighbourhood of e, by Formula (I.39) in which the functions $f^{(k)}$ are given by (I.33), (I.36) and (I.37).

Formula (I.39) shows that any local cocycle f is equivalent to $s_{[f]}$.

According to Proposition 1 and the remark at the end of the preceding section, that gives

PROPOSITION 3. — The cohomology spaces H(A, E) and $H_{loc}(G, E)$ are isomorphic.

II. GLOBAL COHOMOLOGY

II.1. Homogeneous formalism.

Let $\mathscr{C}(G, E) = \bigoplus_{m=0}^{\infty} \mathscr{C}^m(G, E)$ be the space of homogeneous C^∞ cochains on G. With any element F of $\mathscr{C}^m(G, E)$ we associate the function $\hat{F}$ by

(II.1) $$\hat{F}(\omega_1, \ldots, \omega_m) = F(e, \omega_1, \ldots, \omega_m)$$

and, for $\omega \in G$, the $(m-1)$-cochain $\tau_\omega F$ by

(II.2) $$\tau_\omega F(\omega_0, \ldots, \omega_{m-1}) = F(\omega_0, \omega_0\omega, \omega_0\omega\omega_0^{-1}\omega_1, \ldots, \omega_0\omega\omega_0^{-1}\omega_{m-1})$$

We have the relation

(II.3) $$\widehat{\tau_\omega F}(\omega_1, \ldots, \omega_{m-1}) = \hat{F}(\omega, \omega\omega_1, \ldots, \omega\omega_{m-1})$$

For $m > 0$, let $[F] \in \Lambda^m(A^*) \otimes E$ and $\langle F \rangle \in \mathscr{A}^{(m-1)}(G; E)$ be the differential forms which are defined by

(II.4) $$[F] = [\hat{F}]$$

(II.5) $$\langle F \rangle (X^{(1)}, \ldots, X^{(m-1)})_\omega = [\widehat{\tau_\omega F}](X^{(1)}, \ldots, X^{(m-1)})$$

For $m = 0$ we put $[F] = F(e)$ and $\langle F \rangle = 0$.

LEMMA 1'. — For $F \in \mathscr{C}^m(G, E)$ we have

(II.6) $$[\delta F] = D[F]$$

(II.7) $$\langle \delta F \rangle = T[F] - d\langle F \rangle, \qquad d^0 F > 0$$

Proof. — From the relation

$$\delta F(\omega_0, \ldots, \omega_{m+1}) = F(\omega_1, \ldots, \omega_{m+1}) - F(\omega_0, \omega_2, \ldots, \omega_{m+1}) + \sum_{k=2}^{m+1} (-1)^k F(\omega_0, \omega_1, \ldots, \hat{\omega}_k, \ldots)$$

the definition gives

$$\begin{aligned}\langle \delta F \rangle (X^{(1)}, \ldots, X^{(m)})_\omega &= D_{X^{(1)},\ldots,X^{(m)}} \cdot \widehat{\delta F}(\omega, \omega\omega_1, \ldots, \omega\omega_m)\,|_{\omega_1 = \ldots = \omega_m = e} \\ &= D_{X^{(1)},\ldots,X^{(m)}} [F(\omega, \omega\omega_1, \ldots, \omega\omega_m) - F(e, \omega\omega_1, \ldots, \omega\omega_m)]_{\omega_1 = \ldots = \omega_m = e}\end{aligned}$$

The first term is equal to

$$D^e_{X^{(1)},\ldots,X^{(m)}} T_\omega \hat{F}(\omega_1, \ldots, \omega_m) = T_\omega [F](X^{(1)}, \ldots, X^{(m)})$$

For the second term, we have

$$\frac{1}{m!} \sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) X_1^{(k_1)} \ldots . X_m^{(k_m)} \hat{F}(\omega\omega_1, \ldots, \omega\omega_m) |_{\omega_1 = \ldots = \omega_m = e}$$

$$= \frac{1}{m!} \sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) X_1^{(k_1)} \cdot X_2^{(k_2)} \ldots . X_m^{(k_m)} \hat{F}(\omega\omega_1, \omega, \ldots, \omega) |_{\omega_1 = e}$$

$$= \frac{1}{m!} \sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) \left[X_\omega^{(k_1)} - \sum_{p=2}^{m} X_p^{(k_1)} \right] X_2^{(k_2)} \ldots . X_m^{(k_m)} \hat{F}(\omega, \ldots, \omega)$$

$$= \frac{1}{m} \sum_{k_1} (-1)^{k_1 - 1} X_\omega^{(k_1)} \langle F \rangle (\ldots, \hat{X}^{(k_1)}, \ldots)_\omega$$

$$- \frac{1}{m!} \sum_{p=2}^{m} (-1)^p \sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) X_2^{(k_1)} \ldots (X^{(k_{p-1})} X^{(k_p)})_p \ldots . X_m^{(k_m)} \hat{F}(\omega, \omega\omega_2, \ldots, \omega\omega_m) |_{\omega_2 = \ldots = \omega_m = e}$$

By using the result of the calculation made for Lemma 1, the last term becomes

$$\frac{1}{m} \sum_{i<j} (-1)^{i+j} \langle F \rangle ([X^{(i)}, X^{(j)}], \ldots, \hat{X}^{(i)}, \ldots, \hat{X}^{(j)}, \ldots)_\omega$$

Formula (II.7) follows. Formula (II.6) is then demonstrated as in Lemma 1. q. e. d.

From (II.6) and (II.7) we evidently draw the same conclusion as from (I.12) and (I.13).

Remark. — It is to be noted that, in spite of the analogy between the definitions and properties of $[f]$ and $[F]$ or $\langle f \rangle$ and $\langle F \rangle$, these forms are not identical. For example, taking into account the relation between f and F, we obtain

$$[F](X^{(1)}, \ldots, X^{(m)}) = \frac{1}{m!} \sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) X_1^{(k_1)} \ldots . X_m^{(k_m)} . f(\omega_1, \omega_1^{-1}\omega_2, \ldots, \omega_{m-1}^{-1}\omega_m) |_{\omega_1 = \ldots = \omega_m = e}$$

$$= \frac{1}{m!} \sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m)(X_1^{(k_1)} - X_2^{(k_1)}) \ldots (X_{m-1}^{(k_{m-1})} - X_m^{(k_{m-1})}) X_m^{(k_m)} f(\omega_1, \ldots, \omega_m) |_{\omega_1 = \ldots = \omega_m = e}$$

If f is normalized that relation actually gives $[\mathrm{F}] = [f]$. Otherwise, *for a cocycle f* there exists a normalized cocycle g such that $f \sim g$. From the relations $[f] \sim [g]$, $[\mathrm{F}] \sim [\mathrm{G}]$ and $[\mathrm{G}] = [g]$ we deduce $[\mathrm{F}] \sim [f]$. Thus the homomorphisms $[\ \ ]_\#$ and $[\ \ ]_\#$ are identical.

The correspondence with Section I.1 is achieved by the following lemma:

LEMMA 2'. — For $\mathrm{F} \in \mathscr{C}^m(\mathrm{G}, \mathrm{E})$ and $3 \leqslant k \leqslant m$ the following relation is satisfied

$$(\mathrm{II}.8)\quad \langle \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_1} \delta \mathrm{F} \rangle = \mathrm{T}_{\omega_1} \langle \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_2} \mathrm{F} \rangle + \sum_{l=1}^{k-2} (-1)^l \langle \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_l \omega_{l+1}} \cdot \ldots \cdot \tau_{\omega_1} \mathrm{F} \rangle + (-1)^{k-1} \gamma^*_{\omega_{k-1}} \langle \tau_{\omega_{k-2}} \cdot \ldots \cdot \tau_{\omega_1} \mathrm{F} \rangle + (-1)^k d \langle \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_1} \mathrm{F} \rangle$$

For $2 = k \leqslant m$ that identity is reduced to

$$(\mathrm{II}.9)\qquad \langle \tau_{\omega_1} \delta \mathrm{F} \rangle = \mathrm{T}_{\omega_1} \langle \mathrm{F} \rangle - \gamma^*_{\omega_1} \langle \mathrm{F} \rangle + d \langle \tau_{\omega_1} \mathrm{F} \rangle$$

Proof. — From the definition of τ_ω we obtain

$$\tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_1} \delta \mathrm{F}(\omega_0, \omega_k, \ldots, \omega_{m+1}) = \delta \mathrm{F}(\omega_0, \omega_0\omega_1, \ldots, \omega_0\omega_1 \cdot \ldots \cdot \omega_{k-1}, \omega_0\omega_1 \cdot \ldots \cdot \omega_{k-1}\omega_0^{-1}\omega_k, \ldots, \omega_0\omega_1 \cdot \ldots \cdot \omega_{k-1}\omega_0^{-1}\omega_{m+1})$$

For $3 \leqslant k \leqslant m$ we write

$$\delta \mathrm{F}(\omega_0, \ldots, \omega_{m+1}) = \mathrm{F}(\omega_1, \ldots, \omega_{m+1}) + \sum_{l=1}^{k-2} (-1)^l \mathrm{F}(\ldots, \hat{\omega}_l, \ldots) + (-1)^{k-1} \mathrm{F}(\ldots, \hat{\omega}_{k-1}, \ldots) + \sum_{l=k}^{m+1} (-1)^l \mathrm{F}(\ldots, \hat{\omega}_l, \ldots)$$

That gives, after some rearrangements,

$$\begin{aligned} \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_1} \delta \mathrm{F}(\omega_0, \omega_k, \ldots, \omega_{m+1}) &= \mathrm{T}_{\omega_0\omega_1\omega_0^{-1}} \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_2} \mathrm{F}(\omega_0, \omega_k, \ldots, \omega_{m+1}) \\ &+ \sum_{l=1}^{k-2} (-1)^l \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_l\omega_{l+1}} \cdot \ldots \cdot \tau_{\omega_1} \mathrm{F}(\omega_0, \omega_k, \ldots, \omega_{m+1}) \\ &+ (-1)^{k-1} \tau_{\omega_{k-2}} \cdot \ldots \cdot \tau_{\omega_1} \mathrm{F}(\omega_0, \omega_0\omega_{k-1}\omega_0^{-1}\omega_k, \ldots, \omega_0\omega_{k-1}\omega_0^{-1}\omega_{m+1}) \\ &+ (-1)^{k-1} \{ \delta \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_1} \mathrm{F}(\omega_0, \omega_k, \ldots, \omega_{m+1}) \\ &\qquad - \mathrm{T}_{\omega_k} \tau_{\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_1} \mathrm{F}(e, \omega_k^{-1}\omega_{k+1}, \ldots, \omega_k^{-1}\omega_{m+1}) \} \end{aligned}$$

By making the replacement $(\omega_0, \omega_k, \ldots, \omega_{m+1}) \to (e, \omega, \omega\omega_k, \ldots, \omega\omega_m)$ and applying the operator $\mathrm{D}^e_{\mathrm{X}^{(1)}, \ldots, \mathrm{X}^{(m-k+1)}}$ acting on the variables $\omega_k, \ldots, \omega_m$ we obtain (II.8).

For $k = 2 \leqslant m$ we write

$$\delta F(\omega_0, \ldots, \omega_{m+1}) = F(\omega_1, \ldots, \omega_{m+1}) - F(\omega_0, \omega_2, \ldots, \omega_{m+1}) + \sum_{l=2}^{m+1} (-1)^l F(\omega_0, \omega_1, \ldots, \hat{\omega}_l, \ldots)$$

to obtain

$$\begin{aligned}\tau_{\omega_1} \delta F(\omega_0, \omega_2, \ldots, \omega_{m+1}) &= T_{\omega_0 \omega_1 \omega_0^{-1}} F(\omega_0, \omega_2, \ldots, \omega_{m+1}) \\ &- F(\omega_0, \omega_0 \omega_1 \omega_0^{-1} \omega_2, \ldots, \omega_0 \omega_1 \omega_0^{-1} \omega_{m+1}) \\ &- \{ \delta \tau_{\omega_1} F(\omega_0, \omega_2, \ldots, \omega_{m+1}) - T_{\omega_2} \tau_{\omega_1} F(e, \omega_2^{-1} \omega_3, \ldots, \omega_2^{-1} \omega_{m+1}) \}\end{aligned}$$

By making the replacement $(\omega_0, \omega_2, \ldots, \omega_{m+1}) \to (e, \omega, \omega\omega_2, \ldots, \omega\omega_m)$ and applying the operator $D^e_{X^{(1)}, \ldots, X^{(m-1)}}$ acting on the variables $\omega_2, \ldots, \omega_m$ we find (II.9). q. e. d.

II.2. **Introduction of a maximal compact subgroup.**

From now the group G is assumed to be connected, and we denote by K a maximal compact subgroup of G. That subgroup is connected and the space G/K of left cosets of G with respect to K is diffeomorphic to an Euclidean space [*8*].

Let $[d\Omega]$ be the normalized invariant measure on K. With any cochain $F \in \mathscr{C}^m(G, E)$ we associate the cochain $\bar{F} \in \mathscr{C}^m(G, E)$ by the formula

$$\text{(II.10)} \quad \bar{F}(\omega_0, \ldots, \omega_m) = \int_{K^{m+1}} F(\omega_0 \Omega_0, \ldots, \omega_m \Omega_m) [d\Omega_0] \ldots [d\Omega_m]$$

The mapping $F \to \bar{F}$ extends by linearity into an idempotent mapping from $\mathscr{C}(G, E)$ onto $\bar{\mathscr{C}}(G, E)$, the subspace of homogeneous C^∞ cochains which satisfy the condition

$$\text{(II.11)} \quad F(\omega_0, \ldots, \omega_m) = F(\omega_0 \Omega_0, \ldots, \omega_m \Omega_m), \qquad \forall \Omega_0, \ldots, \Omega_m \in K$$

Furthermore, this mapping commutes with δ, and it was proved in [*3*] that the induced homomorphism between the cohomology spaces of $\mathscr{C}(G, E)$ and of $\bar{\mathscr{C}}(G, E)$ is an isomorphism. That means that for any cocycle F of $\mathscr{C}(G, E)$ we have $F \sim \bar{F}$, and that any coboundary in $\mathscr{C}(G, E)$ which belongs to $\bar{\mathscr{C}}(G, E)$ is also a coboundary in $\bar{\mathscr{C}}(G, E)$. Thus, in what follows, we shall be exclusively dealing with $\bar{\mathscr{C}}(G, E)$.

We also need to introduce the subspace of $\mathscr{A}(G; E)$ constituted by the forms which satisfy the two conditions

$$i_Y \psi = L_Y \psi = 0, \qquad \forall Y \in A_K$$

in which i_Y and L_Y respectively denote the contraction and the Lie derivative, and A_K the Lie algebra of K. This subspace will be denoted by

$\mathscr{A}(G;E)|_K$. One can show (see ref. [3]) that $\mathscr{A}(G;E)|_K$ is isomorphic to $\mathscr{A}(G/K;E)$ so that, if π denotes the projection $G \to G/K$, for any $\psi \in \mathscr{A}(G;E)|_K$ there exists an unique element $\tilde{\psi}$ of $\mathscr{A}(G/K;E)$ such that $\psi = \pi^*\tilde{\psi}$. This correspondence clearly commutes with the exterior differentiation and the left translations.

Let now F be an element of $\bar{\mathscr{C}}^m(G, E)$, $m > 0$. Due to (II.11) we have

$$F(\omega_0, \ldots, \Omega, \ldots, \omega_m) = F(\omega_0, \ldots, e, \ldots, \omega_m), \qquad \forall \Omega \in K$$

what entails the relation $Y_k F|_{\omega_k = e} = 0$, $\forall Y \in A_K$. From the definition of $[F]$ this implies at first

$$i_Y[F] = 0, \qquad \forall Y \in A_K$$

That relation remains evidently true for $m = 0$. By contracting the two members of (II.6) with Y we then find

$$i_Y D[F] = 0, \qquad \forall Y \in A_K$$

By using the Cartan relation $L_Y = i_Y d + d i_Y$, and noting that $i_Y T = T i_Y$, it is seen that these two conditions are equivalent to

$$\text{(II.12)} \qquad T[F] \in \mathscr{A}(G;E)|_K$$

The set of forms ξ such that

$$\text{(II.13)} \qquad \xi \in \Lambda(A^*) \otimes E, \qquad T\xi \in \mathscr{A}(G;E)|_K$$

to which $[F]$ belongs, will be denoted by $\Lambda(A^*) \otimes E|_K^T$. It is stable under D and, as in [3], the corresponding cohomology space will be denoted by $H(A \text{ mod. } A_K, E)$.

In the same way we prove the relation

$$\text{(II.14)} \qquad \langle F \rangle \in \mathscr{A}(G;E)|_K, \qquad \forall F \in \bar{\mathscr{C}}(G, E)$$

Finally let us note that (II.14) remains valid for $F' = \tau_{\omega_k} \cdot \ldots \cdot \tau_{\omega_1} F$, $F \in \bar{\mathscr{C}}(G, E)$. The cochain F′ satisfies in fact the condition (II.11) with the restriction $\Omega_0 = e$, while $\delta F'$ satisfies the same condition with $\Omega_0 = \Omega_1 = e$. This is sufficient to insure the relations

$$i_Y[F'] = i_Y \langle F' \rangle = i_Y \langle \delta F' \rangle = 0$$

With the help of (II.7) written for F′ (for $d^0 F' > 0$) we obtain $i_Y d \langle F' \rangle = 0$ and then

$$\text{(II.15)} \qquad \langle \tau_{\omega_k} \cdot \ldots \cdot \tau_{\omega_1} F \rangle \in \mathscr{A}(G;E)|_K, \qquad \forall F \in \bar{\mathscr{C}}(G, E)$$

For $d^0 F' = 0$ we have $\langle F' \rangle = 0$, and the same relation is true.

II.3. Homomorphism $\Lambda(A^*) \otimes E|_K^T \to \bar{\mathscr{C}}(G, E)$.

Once more we begin by constructing a set of standard chains on G/K. Let $\theta \to (\theta^\alpha) \in \mathbb{R}^p$, $p = \dim G - \dim K$, be a coordinate system defined on the whole of G/K, for which the coordinates of $\dot{e} = K$ vanish. We again denote by $t\theta$ the element of G/K whose coordinates are $t\theta^\alpha$, and we introduce the homotopy $H' : [0,1] \times G/K \to G/K$ by $H'(t, \theta) = t\theta$. The 0-chains are defined by

$$C'(\theta) = \{\theta\} \tag{II.16}$$

If $C'(\theta_1, \ldots, \theta_m)$ is defined, we define $C'(\dot{e}, \theta_1, \ldots, \theta_m)$ by the integral (*)

$$C'(\dot{e}, \theta_1, \ldots, \theta_m) = \int_K [d\Omega]\gamma_\Omega H'([0,1] \times C'(\Omega^{-1}\theta_1, \ldots, \Omega^{-1}\theta_m) \tag{II.17}$$

where γ_ω denotes the left action of $\omega \in G$ on G/K. The following relation is easily verified

$$\gamma_\Omega C'(\dot{e}, \theta_1, \ldots, \theta_m) = C'(\dot{e}, \Omega\theta_1, \ldots, \Omega\theta_m), \qquad \forall \Omega \in K \tag{II.18}$$

Finally, if $\dot{\omega}_1 = \omega_1 K$, we put

$$C'(\dot{\omega}_1, \theta_2, \ldots, \theta_{m+1}) = \gamma_{\omega_1} C'(\dot{e}, \omega_1^{-1}\theta_2, \ldots, \omega_1^{-1}\theta_{m+1}) \tag{II.19}$$

One verifies that, due to (II.18), that expression only depends on the coset $\dot{\omega}_1$. From these definitions we deduce the relation

$$\gamma_\omega C'(\theta_1, \ldots, \theta_m) = C'(\omega\theta_1, \ldots, \omega\theta_m) \tag{II.20}$$

The boundary of the standard chains is determined as in Section I.2. and we find

$$\partial C'(\theta_1, \ldots, \theta_m) = \sum_{k=1}^{m} (-1)^{k-1} C'(\ldots, \hat{\theta}_k, \ldots), \qquad m \geqslant 2 \tag{II.21}$$

Let ϕ be a differential form such that $T\phi \in \mathscr{A}^m(G; E)|_K$, and let

$$S'_\phi(\omega_0, \ldots, \omega_m) = \int_K [d\Omega] \int_{C'(\dot{\omega}_0, \ldots, \dot{\omega}_m)} T_{\omega_0\Omega}\gamma^*_{(\omega_0\Omega)^{-1}}\widetilde{T\phi} \tag{II.22}$$

From the properties of standard chains it follows that S'_ϕ belongs to $\mathscr{C}(G, E)$.

LEMMA 4'. — For $\xi \in \Lambda(A^*) \otimes E|_K^1$ the cochain S'_ξ satisfies the relation

$$\delta S'_\xi = S'_{D\xi} \tag{II.23}$$

(*) Subsequent formulas actually define currents in the de Rham's sense [9]. We will continue however to use the denomination of chains together with the corresponding integral notation for the application of a current to a differential form.

Proof. — From the definition (II.22) we obtain

$$\delta S'_\phi(\omega_0, \ldots, \omega_{m+1}) = \int_K [d\Omega] \int_{C'(\dot\omega_1,\ldots,\dot\omega_{m+1})} T_{\omega_1\Omega}\gamma^*_{(\omega_1\Omega)^{-1}}\widetilde{T\phi} + \int_K [d\Omega] \int_{\sum_{k=1}^{m+1}(-1)^k C'(\dot\omega_0,\ldots,\hat{\dot\omega}_k,\ldots)} T_{\omega_0\Omega}\gamma^*_{(\omega_0\Omega)^{-1}}\widetilde{T\phi}$$

By using (II.21) to transform the chain occuring in the second integral, and by the Stokes' theorem, that expression becomes

$$\text{(II.24)} \quad \delta S'_\phi(\omega_0, \ldots, \omega_{m+1}) = \int_K [d\Omega] \int_{C'(\dot\omega_1,\ldots,\dot\omega_{m+1})} (T_{\omega_1\Omega}\gamma^*_{(\omega_1\Omega)^{-1}}\widetilde{T\phi} - T_{\omega_0\Omega}\gamma^*_{(\omega_0\Omega)^{-1}}\widetilde{T\phi}) + \int_K [d\Omega] \int_{C'(\dot\omega_0,\ldots,\dot\omega_{m+1})} T_{\omega_0\Omega}\gamma^*_{(\omega_0\Omega)^{-1}}d\widetilde{T\phi}$$

Owing to the definition of D, the last term is equal to $S'_{D\phi}(\omega_0, \ldots, \omega_{m+1})$. Furthermore, if $\phi = \xi \in \Lambda(A^*) \otimes E|^T_K$, the first two terms in (II.24) cancel since the relation (I.7) implies

$$T_\omega\gamma^*_{\omega^{-1}}\widetilde{T\xi} = \widetilde{T_\omega\gamma^*_{\omega^{-1}}T\xi} = \widetilde{T\gamma^*_{\omega^{-1}}\xi} = \widetilde{T\xi}. \qquad \text{q. e. d.}$$

For any $\psi \in \mathscr{A}^{(m)}(G; E)|_K$ we define the function σ_ψ by

$$\text{(II.25)} \qquad \sigma_\psi(\omega_1, \ldots, \omega_m) = \int_{C'(\dot e,\dot\omega_1,\ldots,\dot\omega_m)} \tilde\psi$$

LEMMA 5′. — The form $[\sigma_\psi]$ is given by

$$\text{(II.26)} \quad [\sigma_\psi](X^{(1)}, \ldots, X^{(m)}) = \psi(X^{(1)}, \ldots, X^{(m)})_e, \qquad \forall\psi \in \mathscr{A}^{(m)}(G; E)|_K$$

Proof. — Formula (II.26) is true for $m = 0$ since we then have

$$\sigma_\psi = \int_{C'(\dot e)} \tilde\psi = \tilde\psi(\dot e) = \tilde\psi \circ \pi(e) = \pi^*\tilde\psi(e) = \psi(e)$$

To demonstrate that formula at any order we first establish a recursion formula for the functions σ_ψ. From the definition of standard chains, (II.25) may be written

$$\sigma_\psi(\omega_1, \ldots, \omega_m) = \int_{\int_K [d\Omega]\gamma_\Omega H'([0,1]\times C'(\Omega^{-1}\dot\omega_1,\ldots,\Omega^{-1}\dot\omega_m))} \tilde\psi$$
$$= \int_K [d\Omega] \int_{[0,1]\times\gamma_\Omega^{-1}C'(\dot\omega_1,\ldots,\dot\omega_m)} H'^*\gamma^*_\Omega\tilde\psi$$
$$= \int_K [d\Omega] \int_{\gamma_\Omega^{-1}C'(\dot\omega_1,\ldots,\dot\omega_m)} h'\gamma^*_\Omega\tilde\psi$$

where h' denotes the homotopy operator in $\mathscr{A}(G/K\,;E)$ associated with H'. By putting

$$(II.27)\qquad \bar{h}' = \int_{K} [d\Omega]\gamma^{*}_{\Omega^{-1}} h' \gamma^{*}_{\Omega}$$

the preceding formula becomes

$$\sigma_{\psi}(\omega_1, \ldots, \omega_m) = \int_{C'(\dot{\omega}_1,\ldots,\dot{\omega}_m)} \bar{h}'\tilde{\psi}$$

that is, with the help of (II.20),

$$(II.28)\qquad \sigma_{\psi}(\omega_1, \ldots, \omega_m) = \sigma_{\gamma^{*}_{\omega_1}\bar{h}'\psi}(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_m)$$

In this last formula we have retained the notation $\bar{h}'$ for the transported of the operator (II.27) in $\mathscr{A}(G\,;E)|_{K}$. We now obtain

$$[\sigma_{\psi}](X^{(1)}, \ldots, X^{(m)}) = \frac{1}{m!}\sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) X_1^{(k_1)} \ldots X_m^{(k_m)} \cdot \sigma_{\gamma^{*}_{\omega_1}\bar{h}'\psi}(\omega_1^{-1}\omega_2, \ldots, \omega_1^{-1}\omega_m)\,|_{\omega_1=\ldots=\omega_m=e}$$

$$= \frac{1}{m!}\sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) X_{\omega_1}^{(k_1)} \cdot X_1^{(k_2)} \ldots X_{m-1}^{(k_m)} \sigma_{\gamma^{*}_{\omega_1}\bar{h}'\psi}(\omega_1^{-1}, \ldots, \omega_1^{-1})\,|_{\omega_1=e}$$

$$= \frac{1}{m!}\sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m) X_{\omega_1}^{(k_1)} \cdot X_1^{(k_2)} \ldots X_{m-1}^{(k_m)} \sigma_{\gamma^{*}_{\omega_1}\bar{h}'\psi}(e, \ldots, e)\,|_{\omega_1=e}$$

$$+ \frac{1}{m!}\sum_{(k_1,\ldots,k_m)} \varepsilon(k_1, \ldots, k_m)\left(-\sum_{p=1}^{m-1} X_p^{(k_1)}\right) X_1^{(k_2)} \ldots X_{m-1}^{(k_m)} \sigma_{\bar{h}'\psi}(e, \ldots, e)$$

According to the calculation made for Lemma 1', the second term is equal to

$$\frac{1}{m}\sum_{i<j} (-1)^{i+j} [\sigma_{\bar{h}'\psi}]([X^{(i)}, X^{(j)}], \ldots, \hat{X}^{(i)}, \ldots, \hat{X}^{(j)}, \ldots)$$

so that we get

$$[\sigma_{\psi}](X^{(1)}, \ldots, X^{(m)}) = \frac{1}{m}\left\{ \sum_{k} (-1)^{k-1} X_{\omega}^{(k)} [\sigma_{\gamma^{*}_{\omega}\bar{h}'\psi}](\ldots, \hat{X}^{(k)}, \ldots) + \sum_{i<j} (-1)^{i+j} [\sigma_{\gamma^{*}_{\omega}\bar{h}'\psi}]([X^{(i)}, X^{(j)}], \ldots, \hat{X}^{(i)}, \ldots, \hat{X}^{(j)}, \ldots) \right\}_{\omega=e}$$

By introducing the recursion hypothesis

$$[\sigma_{\psi}](X^{(1)}, \ldots, X^{(p)}) = \psi(X^{(1)}, \ldots, X^{(p)})_e\,, \qquad 0 \leqslant p \leqslant m-1$$

that gives

$$[\sigma_\psi](X^{(1)}, \ldots, X^{(m)}) = \frac{1}{m}\left\{ \sum_k (-1)^{k-1} X^{(k)}_\omega (\bar{h}'\psi(\ldots, \hat{X}^{(k)}, \ldots)_\omega) + \sum_{i<j} (-1)^{i+j} \bar{h}'\psi([X^{(i)}, X^{(j)}], \ldots, \hat{X}^{(i)}, \ldots, \hat{X}^{(j)}, \ldots)_\omega \right\}_{\omega=e}$$

$$= d\bar{h}'\psi(X^{(1)}, \ldots, X^{(m)})_e = (\psi - \bar{h}'d\psi)(X^{(1)}, \ldots, X^{(m)})_e$$

As in Section I.2 the operator $\bar{h}'$ is such that $(\bar{h}'\psi)_e = 0$, that gives (II.26). Since (II.26) is verified for $m = 0$ it is verified for all m. q. e. d.

The definition of S'_ϕ now implies the relation

$$\hat{S}'_\phi = \sigma_{T\int_K [d\Omega]\gamma^*_\Omega \phi}$$

and then

$$[\hat{S}'_\phi](X^{(1)}, \ldots, X^{(m)}) = \int_K [d\Omega]\phi(X^{(1)}, \ldots, X^{(m)})_\Omega$$

For $\phi = \xi \in \Lambda(A^*) \otimes E\,|^T_K$ that formula is reduced to

(II.29) $$[S'_\xi] = \xi$$

The relations (II.23) and (II.29) lead to a conclusion analogous to that of Section I.2, namely that the mapping

$$S' : \Lambda(A^*) \otimes E\,|^T_K \to \bar{\mathscr{C}}(G, E)$$

induces a homomorphism

$$S'_\# : H(A \bmod . A_K, E) \to H(G, E)$$

such that

$$[\ \ ]_\# \circ S'_\# = id_{H(A \bmod A_K, E)}$$

The isomorphism will be proved as in I by setting up an integral formula for cocycles of $\bar{\mathscr{C}}(G, E)$.

Remark. — Lemma 5′ may be used as Lemma 5 to give an expression for $\langle S'_\phi \rangle$. We find here

$$\langle S'_\phi \rangle = \bar{h}'T \int_K [d\Omega]\gamma^*_\Omega \phi$$

II.4. Integral formula.

As in I the basic relations are those demonstrated in Lemma 2′. According to the remark at the end of Section II.2, for any $F \in \bar{\mathscr{C}}(G, E)$, all the forms occuring in Formulas (II.8) and (II.9) belong to $\mathscr{A}(G ; E)\,|_K$. These relations may then be transported in $\mathscr{A}(G/K ; E)$ and integrated on standard chains. The formal analogy with Lemma 2, and the analogous properties of standard chains, allow the same calculations as in I. We only need to change the arguments ω which occur in these formulas so as to directly introduce homogeneous cochains.

For $3 \leqslant k \leqslant m = d^0F$, by substituting the values $\omega_0^{-1}\omega_1, \omega_1^{-1}\omega_2, \ldots$ to $\omega_1, \omega_2, \ldots$ in (II.8), then applying the operator $T_{\omega_0}\gamma^*_{\omega_{k-1}^{-1}}$ and integrating on the chain $C'(\dot\omega_{k-1}, \dot\omega_k, \ldots, \dot\omega_m)$ we find, for a cocycle $F \in \mathscr{C}(G, E)$, instead of (I.34),

$$(II.30)\quad \int_{C'(\dot\omega_{k-1},\ldots,\dot\omega_m)} T_{\omega_0}\gamma^*_{\omega_{k-1}^{-1}} d\overline{\langle \tau_{\omega_{k-2}^{-1}\omega_{k-1}} \cdot \ldots \cdot \tau_{\omega_0^{-1}\omega_1} F \rangle}$$
$$= \int_{C'(\dot\omega_{k-2},\ldots,\dot\omega_m)} T_{\omega_0}\gamma^*_{\omega_{k-2}^{-1}} d\overline{\langle \tau_{\omega_{k-3}^{-1}\omega_{k-2}} \cdot \ldots \cdot \tau_{\omega_0^{-1}\omega_1} F \rangle} + (-1)^{k-1}\delta F^{(m-k+1)}(\omega_0, \ldots, \omega_m)$$

with

$$(II.31)\quad F^{(m-k+1)}(\omega_0, \ldots, \omega_{m-1}) = \int_{C'(\dot\omega_{k-2},\ldots,\dot\omega_{m-1})} T_{\omega_0}\gamma^*_{\omega_{k-2}^{-1}} \overline{\langle \tau_{\omega_{k-3}^{-1}\omega_{k-2}} \cdot \ldots \cdot \tau_{\omega_0^{-1}\omega_1} F \rangle}$$

Analogous calculations for $k = 2 \leqslant m$ give

$$(II.32)\quad \int_{C'(\dot\omega_1,\ldots,\dot\omega_m)} T_{\omega_0}\gamma^*_{\omega_1^{-1}} d\overline{\langle \tau_{\omega_0^{-1}\omega_1} F \rangle} = \int_{C'(\dot\omega_0,\ldots,\dot\omega_m)} T_{\omega_0}\gamma^*_{\omega_0^{-1}} d\overline{\langle F \rangle} - \delta F^{(m-1)}(\omega_0, \ldots, \omega_m)$$

with

$$(II.33)\quad F^{(m-1)}(\omega_0, \ldots, \omega_{m-1}) = \int_{C'(\dot\omega_0,\ldots,\dot\omega_{m-1})} T_{\omega_0}\gamma^*_{\omega_0^{-1}} \overline{\langle F \rangle}$$

To end it suffices to note the equality

$$(II.34)\quad F(\omega_0, \ldots, \omega_{m-1}, \omega) = T_{\omega_0}\gamma^*_{\omega_{m-1}^{-1}} \langle \tau_{\omega_{m-2}^{-1}\omega_{m-1}} \cdot \ldots \cdot \tau_{\omega_0^{-1}\omega_1} F \rangle_\omega$$

and, proceeding as in I, we finally get

$$(II.35)\quad F(\omega_0, \ldots, \omega_m) = \int_{C'(\dot\omega_0,\ldots,\dot\omega_m)} T_{\omega_0}\gamma^*_{\omega_0^{-1}} \overline{T[F]} + \sum_{k=0}^{m-1} (-1)^{m-k}\delta F^{(k)}(\omega_0, \ldots, \omega_m)$$

with

$$(II.36)\quad F^{(0)}(\omega_0, \ldots, \omega_{m-1}) = F(\omega_0, \ldots, \omega_{m-1}, \omega_{m-1})$$

Each of the terms appearing in the right member of (II.35) does not necessarily belong to $\mathscr{C}(G, E)$. By applying to that equation the averaging operation (II.10), what does not change the left member, we obtain the wanted relation

$$\text{(II.37)} \qquad F = S'_{[F]} + \sum_{k=0}^{m-1} (-1)^{m-k} \delta \overline{F^{(k)}}$$

For $m = 1$ and $m = 0$ that relation is reduced to $F = S'_{[F]}$.

To sum up we have obtained

PROPOSITION 4. — Any cocycle F of $\overline{\mathscr{C}}(G, E)$ is represented by (II.37), where the functions $F^{(k)}$ are defined by (II.31), (II.33) and (II.36).

PROPOSITION 5. — The cohomology spaces H(A mod. A_K, E) and H(G, E) are isomorphic.

APPENDIX

AN INFINITE-DIMENSIONAL CASE

Let M be a differentiable manifold on which the group G operates as a Lie transformation group. The transformed of $x \in M$ by $\omega \in G$ will be denoted by either of the three expressions

$$(A.1) \qquad T_\omega(x) = U_x(\omega) = x^\omega$$

The induced action on $\mathcal{F}(M)$, the space of C^∞ functions on M, will also be denoted by T_ω, so that we have

$$(A.2) \qquad (T_\omega f)(x) = f(x^{\omega^{-1}}), \qquad \forall x \in M, \ \forall \omega \in G$$

We will examine the application of the preceding theory to the case corresponding to the space $E = \mathcal{F}(M)$ endowed with the representation $\omega \to T_\omega$. To the generators T_X correspond the following vector fields on M (velocity fields)

$$(A.3) \qquad \mathcal{D}_X(x) = - dU_x(X_e)$$

the relation (I.2) being replaced by

$$(A.4) \qquad [\mathcal{D}_X, \mathcal{D}_Y] = \mathcal{D}_{[X,Y]}$$

and (I.3) by the Lie equations

$$(A.5) \qquad X(f \circ U_x \circ S) = \mathcal{D}_X f \circ U_x \circ S, \qquad \forall f \in \mathcal{F}(M)$$

where S denotes the symmetry $\omega \to \omega^{-1}$ on G.

The space $\mathcal{A}(G; E) = \mathcal{A}(G) \otimes E$ of differential forms with values in E will be interpreted here as the space $\mathcal{F}(M, \mathcal{A}(G))$ constituted by the families $\phi = (\phi_x)_{x \in M}$, $\phi_x \in \mathcal{A}(G)$, which correspond to C^∞ mappings $(x, \omega) \to \phi_{x,\omega}$ from $M \times G$ in $\Lambda(T^*G)$, the exterior algebra of the cotangent fiber space T^*G. For a form of degree m, it is equivalent to state that, for any $X^{(1)}, \ldots, X^{(m)} \in A$, the mapping $(x, \omega) \to \phi_x(X^{(1)}, \ldots, X^{(m)})_\omega$ is C^∞. The action (A.2) of the group is easily extended to $\mathcal{F}(M, \mathcal{A}(G))$ by the formula

$$(T_\omega \phi)_x = \phi_{x^{\omega^{-1}}}$$

while the operation T defined by (I.4) is generalized by

$$(T\phi)_{x,\omega} = \phi_{x^{\omega^{-1}},\omega}$$

By defining the operator d by $(d\phi)_x = d\phi_x$, and D by (I.5), Formula (I.6) remains valid after the formal replacement $T_X \to \mathcal{D}_X$. The left translations on $\mathcal{F}(M, \mathcal{A}(G))$ are evidently defined by $(\gamma_\omega^* \phi)_x = \gamma_\omega^* \phi_x$ so that the space $\Lambda(A^*) \otimes E$ previously considered is here replaced by $\mathcal{F}(M, \Lambda(A^*))$. Since $\Lambda(A^*)$ is finite-dimensional, that space is in fact isomorphic to $\Lambda(A^*) \otimes \mathcal{F}(M)$.

The space $C^m(G, \mathcal{F}(M))$ of non-homogeneous m-cochains we are then considering is the space of C^∞ functions : $M \times G^m \to \mathbb{R}$. Homogeneous cochains are analogously defined. It is convenient to consider $f \in C^m(G, \mathcal{F}(M))$ as a family $(f_x)_{x \in M}$, $f_x \in C^m(G, \mathbb{R})$. The definitions of Sections I and II are then easily translated in the present context, and we check that all the calculations and results there obtained remain valid.

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NOTE ADDED IN PROOF

A notion similar to the notion of standard chains presented in the text was previously introduced in case of semi-simple groups in: J.-L. DUPONT, *Topology*, t. **15**, 1976, p. 233; H. SHULMAN and D. TISCHLER, *J. Diff. Geom.*, t. **11**, 1976, p. 535. For the 2-chains in the local case see also: J.-C. HOUARD, *J. Math. Phys.*, t. **18**, 1977, p. 502. As a consequence of the integral formula, let us also mention the equivalence of C^∞ cocycles with analytic cocycles: J.-C. HOUARD, *C. R. Acad. Sci. Paris*, t. **290**, 1980, p. 61.

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