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Section A :
Physique théorique.

A trace formula for semi-simple Lie algebras

by

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ABSTRACT. — A full set of invariants for a general semi-simple Lie group is constructed. The eigenvalues of these invariants (i. e. their images under the Harish-Chandra homomorphism) are determined by an explicit formula which is simple and easy to apply. The invariants constructed are natural generalizations of the universal Casimir element.

1. INTRODUCTION

The dermination of a full set of algebraically independent Casimir Invariants and their eigenvalues is a problem which has a long and distinguished history. Quadratic operators commuting with the elements of the universal enveloping algebra were first introduced by Casimir [*1*] in his purely algebraic treatment of the complete reducibility of finite dimensional representations. Later these operators, now no longer necessarily quadratic, came to be known as « Casimir operators ». By the early 1960's it was known, from previous work of Racah [*2*], Gel'fand [*3*] and Chevalley [*4*], that any semi-simple Lie algebra of rank n algebraically independent Casimir invariants. Such invariants, for the various classical groups, have since been explicitly constructed [*5*] and their eigenvalues on irreducible representations of the group determined [*6*].

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It is the aim of this paper to present a unified method for the construction of a full set of algebraically independent invariants for an arbitrary semi-simple Lie algebra. The eigenvalues of these invariants on representations admitting an infinitesimal character are determined by an explicit formula which is simple and easy to apply.

Let L be a semi-simple Lie algebra of rank l and let $V(\lambda)$ be a finite dimensional irreducible representation of L with highest weight λ. Our approach is intimately related to the nature of the tensor product space $V(\lambda) \otimes V$ where V is a representation (finite or infinite dimensional) admitting an infinitesimal character. The case where V is a Verma module has been treated in detail by Bernstein, Gel'fand and Gel'fand [*7*] while the more general case where V is a Harish-Chandra module has been treated by Kostant [*8*]. We shall be concerned with a certain matrix A, with entries from L, which may be viewed as an operator on the space $V(\lambda) \otimes V$. Traces of powers of A are shown to be Casimir invariants which (at least for the simple Lie algebras) generate the centre of the universal enveloping algebra. In determining the eigenvalues of these invariants we make use of Weyl's dimension formula along the lines suggested in recent work of Okubo [*9*] and Edwards [*10*] (and also compare with Louck and Biedenharn [*11*]). We remark however that our extension is nontrivial and exploits the Clebsch-Gordan formula originally obtained by Klimyk [*12*]. By this means we obtain a character formula which is a polynomial in the highest weight of the representation and hence extends (via Zariski continuity) to infinite dimensional representations.

2. PRELIMINARIES

Let L be a complex semi-simple Lie algebra, H a Cartan subalgebra, H^* the dual space to H and $\Phi \subset H^*$ the set of roots with respect to the pair (L, H). Let $\Phi^+ \subset \Phi$ be a system of positive roots, $\Delta \subset \Phi^+$ a base for Φ, δ the half sum of the positive roots and W the Weyl group for the pair (L, H). Let Λ denote the set of integral linear functions on H and $\Lambda^+ \subset \Lambda$ the set of dominant integral linear functions on H. Finally let (,) denote the inner product induced on H^* by the Killing form and for $\lambda \in H^*$, $\alpha \in \Phi$, set $\langle \lambda, \alpha \rangle = \dfrac{2(\lambda, \alpha)}{(\alpha, \alpha)}$.

For any $v \in H^*$ let τ_v denote the translation map defined by $\tau_v(\lambda) = \lambda + v$ for any λ in H^*. The translated Weyl group $\tilde{W}$ is defined as the conjugate $\tau_{-\delta} W \tau_\delta$ of W in the group of invertible affine transformations of H^*. Thus every element of $\tilde{W}$ is of the form $\tilde{\sigma} = \tau_{-\delta}\sigma\tau_\delta$ where $\sigma \in W$. $\tilde{W}$ therefore acts on H^* according to $\tilde{\sigma}(\lambda) = \sigma(\lambda + \delta) - \delta$ for any λ in H^*. Now let U be the universal enveloping algebra of L and let $U(H) \subset U$ denote the enveloping algebra of H. It is often convenient to identify U(H) with the ring of polynomial functions on H*. The Weyl group W acts on U(H) where if $\sigma \in \mathrm{W}$ and $h \in \mathrm{U(H)}$, $\lambda \in \mathrm{H}^*$, then

$$(\sigma h)(\lambda) = h(\sigma^{-1}\lambda).$$

Similarly the translated Weyl group acts on U(H) according to

$$(\tilde{\sigma} h)(\lambda) = h(\tilde{\sigma}^{-1}\lambda) = h(\sigma^{-1}(\lambda + \delta) - \delta).$$

Now let Z be the centre of U and let B be the nilpotent subalgebra of L spanned by root vectors $x_\alpha \in \mathrm{L}$ corresponding to roots $\alpha \in \Phi^+$. For any $z \in \mathrm{Z}$ it is known [*13*] that there is a unique element $f_z \in \mathrm{U(H)}$ such that

$$z - f_z \in \mathrm{UB} \tag{1}$$

Accordingly one obtains an algebra homomorphism $\mathrm{Z} \to \mathrm{U(H)}$ defined by $z \to f_z$. Following Harish-Chandra and Dynkin one in fact has the following result.

2.1. THEOREM. — For any $z \in \mathrm{Z}$ one has $f_z \in \mathrm{U(H)}^{\tilde{\mathrm{W}}}$ and the map $\mathrm{Z} \to \mathrm{U(H)}^{\tilde{\mathrm{W}}}$, $z \to f_z$, is an algebra isomorphism.

By virtue of this result one may identify the centre Z of U with the ring of $\tilde{\mathrm{W}}$-invariant polynomial functions on H*.

If $l = \operatorname{rank} \mathrm{L} = \dim \mathrm{H}$ then one knows that there exist $\tilde{f}_1, \ldots, \tilde{f}_l \in \mathrm{U(H)}^{\tilde{\mathrm{W}}}$ which are algebraically independent and such that $\mathrm{U(H)}^{\tilde{\mathrm{W}}}$ is generated as an algebra by the $\tilde{f}_i$;

$$\mathrm{U(H)}^{\tilde{\mathrm{W}}} = \mathbb{C}[\tilde{f}_1, \ldots, \tilde{f}_l].$$

In view of Theorem (2.1) one knows that there exist elements $z_1, \ldots, z_l \in \mathrm{Z}$ which may be identified with the polynomial functions $\tilde{f}_1, \ldots, \tilde{f}_l$ under the Harish-Chandra homomorphism. It is clear, from equation (1), that the z_i are uniquely determined by $f_{z_i} = \tilde{f}_i$. Hence, as a corollary to Theorem (2.1), we obtain

COROLLARY. — $\mathrm{Z} = \mathbb{C}[z_1, \ldots, z_l]$ where the z_i are algebraically independent.

By a character χ we shall mean an algebra homorphism of Z into the scalars $\mathbb{C}$. If $z_1, \ldots, z_l \in \mathrm{Z}$ are algebraically independent then a character χ is uniquely determined by the scalars $\chi(z_i)$ which may be arbitrary complex numbers. Thus if $c = (c_1, \ldots, c_l) \in \mathbb{C}^l$ there exists a unique character χ_c such that

$$c_i = \chi_c(z_i).$$

By this means one may set up a bijection between the characters over Z and elements of $\mathbb{C}^l$.

We say that a module M over U admits an infinitesimal character if the elements of the centre Z take constant values on M. Such a module determines an algebra homomorphism $\chi_M : Z \to \mathbb{C}$, $z \to \chi_M(z)$, where $\chi_M(z)$ is the eigenvalue of the central element z on M. In such a case we say that M admits the infinitesimal character χ_M. If v_0 is a maximal weight vector, of weight λ say, then v_0 determines an algebra homomorphism $\chi_\lambda : Z \to \mathbb{C}$, where $\chi_\lambda(z)$ is the eigenvalue of $z \in Z$ on v_0. In view of equation (1) we see that χ_λ is uniquely determined by

$$\chi_\lambda(z) = f_z(\lambda), \ \lambda \in H^* . \tag{2}$$

The characters χ_λ play a fundamental role in character analysis since it is a theorem of Harish-Chandra [*13*, *14*] that every character χ over Z is of the form $\chi = \chi_\lambda$ for some $\lambda \in H^*$. The character χ_λ does not characterize the weight λ uniquely since it may happen that $\chi_\lambda = \chi_\mu$, $\mu \in H^*$, but $\lambda \neq \mu$. One in fact has the following result due to Harish-Chandra.

2.2. THEOREM. — $\chi_\lambda = \chi_\mu$ if and only if λ and μ are $\tilde{W}$-conjugate. Two such weights are called linked and we write $\lambda \sim \mu$.

Remark. — When $z = c_L$ is the universal Casimir element one obtains from equations (1) and (2) the well known formula

$$\chi_\lambda(c_L) = (\lambda, \lambda + 2\delta) \tag{3}$$

3. REDUCTION OF THE PRODUCT SPACE $V(\lambda) \otimes V(\mu)$

3.1. Regular Weights.

The reduction of a finite dimensional representation V into irreducible representations frequently reduces to a determination of the infinitesimal characters occurring in V. If the character χ_μ, $\mu \in H^*$, occurs in V there must exist a finite dimensional irreducible representation, $V(\nu)$ say, which admits χ_μ as an infinitesimal character. By virtue of Theorem (2) the weights μ and ν must, under such circumstances, be conjugate under $\tilde{W}$. Thus a character χ_μ can only occur if the weight μ is $\tilde{W}$-conjugate to a dominant integral weight. This leads one to investigate conditions under which a given integral weight is linked to a dominant integral weight.

For $\alpha \in \Phi^+$ we define the translated hyperplanes

$$\tilde{P}_\alpha = \{ \lambda \in H^* \mid (\lambda + \delta, \alpha) = 0 \} .$$

The translated hyperplanes intersect in the translated origin $-\delta$ and partition H* into finitely many regions (the translated Weyl chambers). The correspondence $\lambda \to (\lambda + \delta, \alpha)$ for $\lambda \in H^*$ determines a polynomial function on H*. Consequently the translated hyperplanes are closed in the Zariski topology on H* (see [13], p. 133). Since Φ is finite it follows that the set $\bigcup_{\alpha \in \Phi^+} \tilde{P}_\alpha$ is also Zariski closed. Its complement in H*, denoted $\tilde{R}$, is therefore open in the Zariski topology. We call elements of $\tilde{R}$ regular. Since all open sets (non empty) are dense in the Zariski topology it follows that the set of regular elements is Zariski dense in H*.

Suppose now that $\mu \in \Lambda$ and let δ be the half sum of the positive roots. One knows (see for example Humphreys [13]), since $\mu + \delta \in \Lambda$, that $\mu + \delta$ is W-conjugate to a unique dominant integral weight, ν say. The following conditions are equivalent.

$$
\begin{array}{lrll}
i) & \langle \mu + \delta, \alpha \rangle \neq 0 & \text{for all} & \alpha \in \Phi^+ \\
ii) & \langle \mu + \delta, \alpha \rangle \neq 0 & \text{for all} & \alpha \in \Phi \\
iii) & \langle \nu, \alpha \rangle \neq 0 & \text{for all} & \alpha \in \Phi \\
iv) & \langle \nu, \alpha \rangle > 0 & \text{for all} & \alpha \in \Phi^+ \\
v) & \langle \nu, \alpha \rangle > 0 & \text{for all} & \alpha \in \Delta
\end{array}
\tag{4}
$$

The equivalence of *i*) and *ii*) is trivial while the equivalence of *ii*) and *iii*) follows from the fact that the Weyl group acts on the root system Φ by permuting the roots among themselves. The remaining eigenvalences follow from the definition of dominant integral weight and the fact that every positive root can be written as a linear combination of elements from Δ with coefficients being non-negative integers.

Since $\mu + \delta$ is W-conjugate to $\nu \in \Lambda^+$ it is clear that μ is conjugate under $\tilde{W}$ to $\nu - \delta$. Suppose now that μ satisfies part *i*) of equation (4). One knows that $\langle \delta, \alpha \rangle = 1$ for $\alpha \in \Delta$ and, in view of *v*) of equation (4), $\nu - \delta$ must satisfy $\langle \nu - \delta, \alpha \rangle \geqslant 0$ for $\alpha \in \Delta$. By definition the weight $\nu - \delta$ is dominant integral. By virtue of the equivalences (4) we therefore have the following easy result.

LEMMA. — Let $\mu \in \Lambda$. Then μ is conjugate under $\tilde{W}$ to a dominant integral weight if and only if $\mu \in \tilde{R}$.

Clearly then the set of integral weights conjugate under $\tilde{W}$ to a dominant integral weight is just the set $\tilde{\Lambda}$ of regular integral weights. One has $\Lambda^+ \subset \tilde{\Lambda} = \Lambda \cap \tilde{R}$. It is a well known property of the Zariski topology (see [13], p. 134) that the weight lattice Λ is Zariski dense in H*. Hence $\tilde{\Lambda}$, being a non-empty intersection of a Zariski dense subset Λ and a Zariski open set $\tilde{R}$, is Zariski dense in H*.

3.2. Klimyk's Formula.

In this section we consider the reduction of the product space $V(\lambda)\otimes V(\mu)$ where $V(\lambda)$ and $V(\mu)$ denote irreducible modules over U with highest weights λ and μ respectively. Let $\{\lambda_1, \ldots, \lambda_k\}$ be the set of *distinct* weights in $V(\lambda)$ and suppose the weight λ_i occurs with multiplicity $n(i)$. It is known [*8*, *13*, *14*] that the infinitesimal characters admitted are of the form $\chi_{\mu+\lambda_i}$. Now let v_i be the unique dominant integral weight W-conjugate to $\mu + \lambda_i + \delta$ via $\sigma_i \in W$; $v_i = \sigma_i(\mu + \lambda_i + \sigma)$. If $(\mu + \lambda_i + \delta, \alpha) = 0$ for some $\alpha \in \Phi$ (i. e. if $\mu + \lambda_i + \delta$ is fixed by some element in W) then section (3.1) implies that the infinitesimal character $\chi_{\mu+\lambda_i}$ cannot occur. On the other hand if $\mu + \lambda_i \in \tilde{\Lambda}$ then $v_i - \delta \in \Lambda^+$ and $\mu + \lambda_i$ is $\tilde{W}$-conjugate to $v_i - \delta$. In this case the representation $V(v_i - \delta)$ occurs in the Clebsch-Gordan decomposition of $V(\lambda) \otimes V(\mu)$ with multiplicity which may be obtained by comparison with Steinberg's formula [*15*]. By this means we obtain the ansatz originally due to Klimyk [*12*]. The reduction of the space $V(\lambda) \otimes V(\mu)$ is probably best represented in terms of formal characters (see [*13*], p. 124). Let ch_μ, $\mu \in \Lambda^+$, denote the formal character of $V(\mu)$ and let $\mathrm{ch}_{\lambda\otimes\mu}$ denote the formal character of $V(\lambda) \otimes V(\mu)$. Application of the above considerations yields the formula

$$\mathrm{ch}_{\lambda\otimes\mu} = \sum_{i=1}^{k} m(i)\,\mathrm{ch}_{v_i-\delta} \tag{5}$$

where the multiplicites are given by

$$\begin{aligned} m(i) &= \mathrm{sn}\,(\sigma_i)n(i), &&\text{when} \quad \mu + \lambda_i \in \tilde{\Lambda} \\ &= 0, &&\text{otherwise} \end{aligned}$$

where $\mathrm{sn}\,(\sigma_i) = \pm 1$ is the sign of the Weyl group element σ_i.

Remark. — Formula (5) may be written in the more familiar form as

$$\mathrm{ch}_{\lambda\otimes\mu} = \sum_{i=1}^{k} n(i)t(\mu + \lambda_i + \delta)\,\mathrm{ch}_{\mu+\lambda_i}$$

where $t(\mu) = 0$ if $\mu \notin \tilde{\Lambda}$ and if $\mu \in \tilde{\Lambda}$, $t(\mu) = \mathrm{sn}\,(\sigma)$ where σ is the unique element of W such that $\sigma(\mu) \in \Lambda^+$. We note further that the highest weights occurring in $V(\lambda) \otimes V(\mu)$ are necessarily of the form $\mu + \lambda_i$ (see [*13*, *14*]). Hence the representations occurring in $V(\lambda) \otimes V(\mu)$ are of the form $V(\mu + \lambda_i)$, for $\mu + \lambda_i \in \Lambda^+$, with multiplicity

$$m(i) = \sum_{\substack{j \\ \mu+\lambda_j \sim \mu+\lambda_i}} \mathrm{sn}\,(\sigma_j)n(j)$$

where σ_j is the unique element of W such that $\sigma_j(\mu + \lambda_j + \delta) = \mu + \lambda_i + \delta$.

4. CASIMIR INVARIANTS

Following the notation of section (3.2) we shall henceforth let $V(\lambda)$ be a finite dimensional irreducible module over U with highest weight $\lambda \in \Lambda^+$ and let $\{ \lambda_1, \ldots, \lambda_k \}$ denote the set of distinct weights in $V(\lambda)$ occurring with multiplicities $n(1), \ldots, n(k)$ respectively.

Now U may be embedded in the algebra $U \otimes U$ by the diagonal homomorphism

$$d : U \to U \otimes U$$

defined for $x \in L$ by $dx = 1 \otimes x + x \otimes 1$. In general du for arbitrary $u \in U$ is a more complicated expression which may be written

$$du = \sum_i u_i \otimes v_i \tag{6}$$

where $u_i, v_i \in U$. Now let Y be the algebra $Y = [\mathrm{End}\, V(\lambda)] \otimes U$ and let π_λ be the representation afforded by $V(\lambda)$. Following Kostant [8] we consider the map

$$\partial : U \to Y$$

defined for $x \in L$ by

$$\partial(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes x$$

which we extend to an algebra homomorphism to all of U. More generally if $u \in U$ with du as in equation (6) then

$$\partial(u) = \sum_i \pi_\lambda(u_i) \otimes v_i \tag{7}$$

When z is an element of the centre Z Kostant shows that the operator $\partial(z)$ satisfies a certain polynomial identity over the centre of $\partial(U)$. We consider here the operator $A = -\frac{1}{2} [\partial(z) - \pi_\lambda(z) \otimes 1 - 1 \otimes z]$. For our purposes no loss of generality is incurred by choosing z to be the universal Casimir element c_L. We henceforth let A denote the operator

$$A = -\frac{1}{2} [\partial(c_L) - \pi_\lambda(c_L) \otimes 1 - 1 \otimes c_L] \tag{8}$$

Now let

$$\tau : Y \to U \tag{9}$$

be the map defined by

$$\tau : \sum_i \rho_i \otimes u_i \to \sum_i t_r[\rho_i] u_i$$

where $\rho_i \in \mathrm{End}\, V(\lambda)$, $u_i \in U$ and where $t_r[\rho_i]$ denotes the trace of the endomorphism $\rho_i \in \operatorname{End} V(\lambda)$. In particular for $u \in U$, $\partial(u)$ as in equation (7), we have

$$\tau\partial(u) = \sum_i t_r[\pi_\lambda(u_i)]v_i .$$

In view of the properties of trace it is immediate that τ is well defined and linear. In fact τ is a U-module homomorphism as one can see from

$$\tau[(1 \otimes u)y] = u\tau(y)$$
$$\tau[y(1 \otimes u)] = \tau(y)u$$

for all $y \in Y$, $u \in U$. Also for $u \in U$ we have $\tau(1 \otimes u) = d(\lambda)u$, where $d(\lambda) = \dim V(\lambda)$, which shows that τ is surjective. Moreover, from the properties of trace, one obtains

$$\tau[(\rho \otimes 1)y] = \tau[y(\rho \otimes 1)] \qquad (10)$$

for $\rho \in \operatorname{End} V(\lambda)$, $y \in Y$.

Now let C denote the centralizer of $\partial(U)$ in Y. If $w \in C$ we have, for arbitrary $x \in L$,

$$\partial(x)w = w\partial(x)$$

which may be rearranged to give

$$(1 \otimes x)w - w(1 \otimes x) = w(\pi_\lambda(x) \otimes 1) - (\pi_\lambda(x) \otimes 1)w .$$

Applying τ to both sides of this equation, using the fact that τ is a U-homomorphism, we obtain, in view of equation (10),

$$x\tau(w) - \tau(w)x = 0 .$$

It follows then that $\tau(C) \subseteq Z$. In view of the surjectivity and linearity of τ we in fact obtain $\tau(C) = Z$.

For $y \in Y$ let us call $\tau(y)$ the trace of y. Now the operator A defined by equation (8) and all of its powers belong to C. Hence traces of arbitrary polynomials in A belong to the centre Z. Of particular importance to us are the fundamental invariants

$$I_m(\lambda) = \tau(A^m) . \qquad (11)$$

Remark. — Let $\{x_1, \ldots, x_n\}$ ($n = \dim L$) be a basis for L and let $\{x^1, \ldots, x^n\}$ be the corresponding dual basis with respect to the Killing form of L. Now fix a basis for the reference representation $V(\lambda)$ and let $\pi_\lambda(x) \in \operatorname{End} V(\lambda)$, $x \in L$, denote the matrix representing x in this basis. With respect to this basis the entries of the matrix A are given by

$$A_{ij} = -\sum_{r=1}^{n} \pi_\lambda(x^r)_{ij}x_r , \qquad i,j = 1, \ldots, d(\lambda) = \dim V(\lambda) .$$

In particular A is a matrix with entries from L. Polynomials in A may be defined recursively according to

$$(\mathrm{A}^{m+1})_{ij} = \sum_{k=1}^{d(\lambda)} \mathrm{A}_{ik}(\mathrm{A}^m)_{kj} = \sum_{k=1}^{d(\lambda)} (\mathrm{A}^m)_{ik}\mathrm{A}_{kj}$$

for positive integral m. The invariants $\mathrm{I}_m(\lambda)$ defined by equation (11) may then be written

$$\mathrm{I}_m(\lambda) = \sum_{i=1}^{d(\lambda)} (\mathrm{A}^m)_{ii}$$

From this expression it can be seen that these invariants are generalizations of the well known Gel'fand invariants of $\mathrm{O}(n)$ and $\mathrm{U}(n)$. Note that the actual basis chosen for $\mathrm{V}(\lambda)$ is immaterial as far as the invariants $\mathrm{I}_m(\lambda)$ are concerned.

One may define an adjoint $\bar{\mathrm{A}}$ of A by

$$\bar{\mathrm{A}} = \sum_{r=1}^{n} \pi_\lambda^{\mathrm{T}}(x^r) \otimes x_r$$

where $(\ )^{\mathrm{T}}$ denotes transposition. In a given basis for $\mathrm{V}(\lambda)$ the matrix elements of A and $\bar{\mathrm{A}}$ are related by $\bar{\mathrm{A}}_{ij} = -\mathrm{A}_{ji}$. As for the matrix A one may define polynomials in the matrix $\bar{\mathrm{A}}$. One also obtains a set of invariants for the Lie algebra by taking traces of polynomials in the matrix $\bar{\mathrm{A}}$. In particular one may consider the adjoint invariants

$$\bar{\mathrm{I}}_m(\lambda) = \sum_{i=1}^{d(\lambda)} (\bar{\mathrm{A}}^m)_{ii}$$

It is clear that the adjoint invariant $\bar{\mathrm{I}}_m(\lambda)$ is the image of the invariant $\mathrm{I}_m(\lambda)$ under the principal antiautomorphism of U (see [*14*], p. *73*).

5. EIGENVALUES OF THE $\mathrm{I}_m(\lambda)$

Throughout this section we let $d(\mu)$ denote Weyl's dimension function which is the polynomial function on H^* defined by

$$d(\mu) = \prod_{\alpha\in\Phi^+} \frac{(\mu + \delta, \alpha)}{(\delta, \alpha)}, \qquad \mu \in \mathrm{H}^* .$$

The translated Weyl group acts on d according to

$$\tilde{\sigma}d(\mu) = d(\sigma^{-1}(\mu + \delta) - \delta) = \prod_{\alpha\in\Phi^+} \frac{(\mu + \delta, \sigma(\alpha))}{(\delta, \alpha)}$$

where we have used the fact that (,) is invariant under the Weyl group. One knows, moreover, that σ permutes the roots in Φ and if $l(\sigma)$ denotes the number of factors in the reduced expression for σ ($l(\sigma)$ is called the length of σ) then the number of positive roots sent to negative roots is $l(\sigma)$. It follows then $\tilde{\sigma}d = \mathrm{sn}\,(\sigma)d$, where $\mathrm{sn}\,(\sigma) = (-1)^{l(\sigma)}$ is the sign of the Weyl group element σ. We say that d is skew under $\tilde{W}$.

It shall be convenient to consider the polynomial functions d_i, $i=1, \ldots, k$, obtained from d by translation with respect to the distinct weights $\lambda_1, \ldots, \lambda_k$ of $V(\lambda)$;

$$d_i(\mu) = d(\mu + \lambda_i) \qquad i = 1, \ldots, k\,.$$

The translated Weyl group acts on the functions d_i according to $\tilde{\sigma}d_i(\mu) = \mathrm{sn}\,(\sigma)d(\mu + \sigma(\lambda_i))$. Recall however that σ just permutes the distinct weights λ_i; $\sigma(\lambda_i) = \lambda_{\pi(i)}$ where π is a permutation of the numbers $1, \ldots, k$. Thus $\tilde{\sigma}d_i = \mathrm{sn}\,(\sigma)d_{\pi(i)}$. Accordingly it follows that any symmetric combination of the polynomial functions d_i will be skew under the translated Weyl group.

Suppose now that $V(\mu)$ is a finite dimensional irreducible module over U with highest weight $\mu \in \Lambda^+$ and let π_μ be the representation afforded by $V(\mu)$. One may extend π_μ to an algebra homomorphism

$$\tilde{\pi}_\mu : [\mathrm{End}\ V(\lambda)] \otimes U \rightarrow \mathrm{End}\ V(\lambda) \otimes \mathrm{End}\ V(\mu)$$

defined by

$$\tilde{\pi}_\mu : \sum_i \rho_i \otimes u_i \rightarrow \sum_i \rho_i \otimes \pi_\mu(u_i)$$

where $\rho_i \in \mathrm{End}\ V(\lambda)$ and $u_i \in U$. In particular if A is the operator defined by equation (8) then

$$\tilde{\pi}_\mu(A) = -\frac{1}{2}\,[\pi_\lambda \otimes \pi_\mu(c_L) - \pi_\lambda(c_L) \otimes 1 - 1 \otimes \pi_\mu(c_L)]\,.$$

$\tilde{\pi}_\mu(A)$ is clearly an operator on the product space $V(\lambda) \otimes V(\mu)$ whose entries are endomorphisms of $V(\mu)$. In the following we shall identify A and $\tilde{\pi}_\mu(A)$. A basis may be chosen for $V(\lambda) \otimes V(\mu)$ for which A is represented by a diagonal matrix. The eigenvalues of A are uniquely determined by the characters occurring in the reduction of $V(\lambda) \otimes V(\mu)$. If $V(\nu)$, $\nu \in \Lambda^+$, is an irreducible module occurring in the decomposition of $V(\lambda) \otimes V(\mu)$ one knows, in view of section (3.2), that $V(\nu)$ admits an infinitesimal character $\chi_{\mu+\lambda_i}$ for some $i = 1, \ldots, k$. On $V(\nu)$ the operator A therefore takes the constant value $-\frac{1}{2}[\chi_{\mu+\lambda_i}(c_L) - \chi_\lambda(c_L) - \chi_\mu(c_L)]$ which, by virtue of equation (3), equals

$$\frac{1}{2}(\lambda, \lambda + 2\delta) - \frac{1}{2}(\lambda_i, 2(\mu + \delta) + \lambda_i).$$

The correspondence $\mu \to \frac{1}{2}(\lambda, \lambda + 2\delta) - \frac{1}{2}(\lambda_i, 2(\mu + \delta) + \lambda_i)$ determines a linear polynomial function on H*. We henceforth denote this polynomial function by a_i;

$$a_i(\mu) = \frac{1}{2}(\lambda, \lambda + 2\delta) - \frac{1}{2}(\lambda_i, 2(\mu + \delta) + \lambda_i) \tag{12}$$

Remark. — From the above considerations it is an easy matter to deduce that in the representation afforded by $V(\mu)$ the operator A satisfies the polynomial identity

$$\prod_{i=1}^{k}(A - a_i(\mu)) = 0$$

This result in fact follows from the easily established fact that a diagonal matrix D with distinct eigenvalues $d_1, \ldots, d_k$ satisfies the polynomial identity

$$\prod_{i=1}^{k}(D - d_i) = 0$$

Applying the same considerations to the adjoint matrix $\bar{A}$ ones arrives at the adjoint identity

$$\prod_{i=1}^{k}(\bar{A} - \bar{a}_i(\mu)) = 0$$

where

$$\bar{a}_i(\mu) = \frac{1}{2}(\lambda, \lambda + 2\delta) + \frac{1}{2}(\lambda_i, 2(\mu + \delta) - \lambda_i) \tag{13}$$

The translated Weyl group acts on the polynomial functions a_i according to

$$\begin{aligned}\tilde{\sigma} a_i(\mu) &= a_i(\sigma^{-1}(\mu + \delta) - \delta) \\ &= \frac{1}{2}(\lambda, \lambda + 2\delta) - \frac{1}{2}(\sigma(\lambda_i), 2(\mu + \delta) + \sigma(\lambda_i))\end{aligned}$$

where we have used the fact that (,) is invariant under the Weyl group. Moreover, since $\sigma \in W$ just permutes the distinct weights of $V(\lambda)$, we have $\sigma(\lambda_i) = \lambda_{\pi(i)}$ where π is a permutation of the numbers $1, \ldots, k$. Hence we obtain

$$\tilde{\sigma} a_i = a_{\pi(i)}$$

and it follows that any symmetric combination of the a_i determines a $\tilde{W}$-invariant polynomial function.

Following the notation of section (3.2) let ν_i be the unique dominant integral weight which is W conjugate to $\mu + \lambda_i + \delta$ and let σ_i be the corresponding Weyl group element; $\sigma_i(\nu_i) = \mu + \lambda_i + \delta$. We may then write the Clebsch-Gordan reduction of the product space $V(\lambda) \otimes V(\mu)$ formally as

$$V(\lambda) \otimes V(\mu) = \bigoplus_{i=1}^{k} m(i)V(\nu_i - \delta) \tag{14}$$

where

$$\begin{aligned} m(i) &= \operatorname{sn}(\sigma_i)n(i) \qquad &&\text{if} \quad \mu + \lambda_i \in \tilde{\Lambda} \\ &= 0 &&\text{otherwise}. \end{aligned}$$

Now consider the linear map

$$\operatorname{tr}_\mu : \operatorname{End} V(\lambda) \otimes U \to \mathbb{C}$$

defined by

$$\operatorname{tr}_\mu : \sum_i \rho_i \otimes u_i \to \sum_i (t_r\rho_i)(t_r[\pi_\mu(u_i)])$$

where $\rho_i \in \operatorname{End} V(\lambda)$, $u_i \in U$ and where $t_r\rho_i$, $t_r[\pi_\mu(u_i)]$ denotes the trace of the endomorphisms ρ_i and $\pi_\mu(u_i)$ respectively. It was shown earlier that on each of the spaces $V(\nu_i - \delta)$ occurring in (14) the operator A takes the constant value $a_i(\mu) = -\frac{1}{2}[\chi_{\mu+\lambda_i}(c_L) - \chi_\mu(c_L) - \chi_\lambda(c_L)]$ where we have used the fact that $V(\nu_i - \delta)$ admits the infinitesimal character $\chi_{\mu+\lambda_i}$. More generally the operator A^m takes the constant value $a_i(\mu)^m$ on the space $V(\nu_i - \delta)$. It follows immediately that

$$\operatorname{tr}_\mu(A^m) = \sum_{i=1}^{k} m(i)a_i(\mu)^m d(\nu_i - \delta) \tag{15}$$

On the other hand let t_μ denote the linear map

$$t_\mu : U \to \mathbb{C}$$

defined by

$$t_\mu(u) = t_r[\pi_\mu(u)], \qquad u \in U.$$

Also let

$$\tau : [\operatorname{End} V(\lambda)] \otimes U \to U$$

denote the map defined by equation (9). Then it is easily shown that tr_μ is the composite map $\operatorname{tr}_\mu = t_\mu\tau$. From this we obtain

$$\begin{aligned} \operatorname{tr}_\mu(A^m) = t_\mu[\tau(A^m)] &= t_\mu[I_m(\lambda)] \\ &= t_r(\pi_\mu[I_m(\lambda)]) \\ &= \chi_\mu[I_m(\lambda)]d(\mu) \end{aligned} \tag{16}$$

where $I_m(\lambda)$ are the Casimir invariants defined by equation (11) and χ_μ is the infinitesimal character of $V(\mu)$. Comparing equations (15) and (16) one obtains

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} m(i)a_i(\mu)^m \frac{d(v_i - \delta)}{d(\mu)}$$

Note that this expression is well defined since $d(\mu) \neq 0$ for $\mu \in \Lambda^+ \subset \tilde{\Lambda}$.

Now $v_i - \delta$ is $\tilde{W}$-conjugate, via $\tilde{\sigma}_i$, to $\mu + \lambda_i + \delta$. Hence one obtains, in view of the fact that d is skew under $\tilde{W}$,

$$\begin{aligned} d(v_i - \delta) = d(\tilde{\sigma}_i^{-1}(\mu + \lambda_i)) &= \tilde{\sigma}_i d(\mu + \lambda_i) \\ &= \operatorname{sn}(\sigma_i) d(\mu + \lambda_i)\,. \end{aligned}$$

Thus we obtain, after substituting for Weyl's dimension formula, the explicit result

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} m(i)a_i(\mu)^m \operatorname{sn}(\sigma_i) \prod_{\alpha\in\Phi^+} \frac{(\mu + \lambda_i + \delta, \alpha)}{(\mu + \delta, \alpha)}$$

However the multiplicities $m(i)$ are given by $m(i) = \operatorname{sn}(\sigma_i)n(i)$ or else $m(i) = 0$ which only occurs when $(\mu + \lambda_i + \delta, \alpha) = 0$ for some $\alpha \in \Phi^+$. We therefore finally obtain the character formula

$$\chi_\mu[I_m(\lambda)] = \sum_{i=1}^{k} n(i)a_i(\mu)^m \prod_{\alpha\in\Phi^+} \frac{(\mu + \lambda_i + \delta, \alpha)}{(\mu + \delta, \alpha)} \tag{17}$$

Using the fact that W acts on the λ_i by permuting them among themselves and the fact that W-conjugate weights occur with the same multiplicity one may prove directly that equation (17) in fact determines a $\tilde{W}$-invariant polynomial function as required by Harish-Chandra's Theorem. One may make formula (17) fully explicit using the Kostant multiplicity formula [*16*].

Note that we have strictly only proved this formula for weights $\mu \in \Lambda^+$. However this is sufficient to prove the formula for arbitrary $\mu \in H^*$ (see for example Kostant [*8*]). Alternatively one may extend formula (17) to $\tilde{\Lambda}$ using the fact that elements of $\tilde{\Lambda}$ are $\tilde{W}$-conjugate to dominant integral weights. Then, using the fact that $\tilde{\Lambda}$ is Zariski dense in H^*, extend via continuity to all of H^*. To be more explicit let g_m be the polynomial function on H^* defined by

$$g_m(\mu) = \sum_{i=1}^{k} n(i)a_i(\mu)^m d_i(\mu)\,.$$

One may prove directly that g_m is skew under $\tilde{W}$. In terms of the polynomial functions g_m and d we may rewrite equation (17) as

$$\chi_\mu[I_m(\lambda)] = g_m(\mu)/d(\mu), \qquad \mu \in \Lambda^+ .$$

Suppose now that $\mu \in \tilde{\Lambda}$. Then according to lemma (3.1) μ is $\tilde{W}$-conjugate to a unique dominant integral weight $\nu \in \Lambda^+$. Write $\mu = \tilde{\sigma}(\nu)$, $\tilde{\sigma} \in \tilde{W}$. Since μ and ν are $\tilde{W}$-conjugate we may write

$$\chi_\mu[I_m(\lambda)] = \chi_\nu[I_m(\lambda)] = g_m(\nu)/d(\nu).$$

But

$$\frac{g_m(\nu)}{d(\nu)} = \frac{g_m(\tilde{\sigma}^{-1}(\mu))}{d(\tilde{\sigma}^{-1}(\mu))} = \frac{\tilde{\sigma}g_m(\mu)}{\tilde{\sigma}d(\mu)} = \frac{g_m(\mu)}{d(\mu)}$$

where the last equality follows from the fact that both g_m and d are skew under $\tilde{W}$. This shows that formula (17) extends to arbitrary $\mu \in \tilde{\Lambda}$. Note that (17) is well defined for such μ since $d(\mu)$ cannot vanish for $\mu \in \tilde{\Lambda}$.

Now let f_m be the unique $\tilde{W}$-invariant polynomial function determined by the Casimir $I_m(\lambda)$ under the Harish-Chandra homomorphism;

$$\chi_\mu[I_m(\lambda)] = f_m(\mu), \qquad \mu \in H^* .$$

We have proved then that

$$f_m(\mu) = \frac{g_m(\mu)}{d(\mu)} \qquad \text{for} \quad \mu \in \tilde{\Lambda} .$$

To avoid singularities we express this by writing

$$d(\mu)f_m(\mu) - g_m(\mu) = 0 \qquad \text{for all} \quad \mu \in \tilde{\Lambda} .$$

Thus the polynomial function $df_m - g_m$ vanishes on a Zariski-dense subset of H^*. Since polynomial functions are continuous in the Zariski topology on H^* we may write $df_m = g_m$. In particular the polynomial function d necessarily divides g_m and it makes formal sense to write $f_m = g_m/d$. This proves the character formula (17) for arbitrary $\mu \in H^*$.

Note that formula (17) is strictly only correct for all regular elements of H^*. However for weights μ lying on a translated hyperplane care must be taken in order to avoid singularities. In such a case formula (17) must first be expanded into a polynomial.

6. In the notation of section (4) let $\bar{A}$ denote the adjoint matrix of A and let $\bar{I}_m(\lambda)$ denote the corresponding adjoint invariants. Proceeding as before one may determine the eigenvalues of the invariants $\bar{I}_m(\lambda)$. However in this case it suffices to note that there exists a unique element τ of W which sends Φ into $-\Phi$ (see e. g. [*13*] and [*14*]). If $z \in Z$ one may deduce (by [*14*], p. 246) for arbitrary $\mu \in H^*$,

$$\chi_{\mu+\delta}(z) = \chi_{-\tau(\mu)+\delta}(\bar{z})$$

where $\bar{z}$ is the image of z under the principal anti-automorphism of U. Hence

$$\chi_\mu(z) = \chi_{-\tau(\mu-\delta)+\delta}(\bar{z}).$$

In particular

$$\chi_\mu[\mathrm{I}_m(\lambda)] = \chi_{-\tau(\mu-\delta)+\delta}[\bar{\mathrm{I}}_m(\lambda)].$$

One may deduce from this the formula

$$\chi_\mu[\bar{\mathrm{I}}_m(\lambda)] = \sum_{i=1}^{k} n(i)\bar{a}_i(\mu)^m \prod_{\alpha\in\Phi^+} \frac{(\mu - \lambda_i + \delta, \alpha)}{(\mu + \delta, \alpha)}$$

where $\bar{a}_i(\mu)$ denotes the polynomial function defined by equation (13).

7. GENERALIZATIONS

In previous sections we worked, for simplicity, with the universal Casimir element c_L of U. One may equally well work with any (non-trivial) element of Z. If $z \in \mathrm{Z}$ is any central element one may show, by the same techniques, that the operator

$$\mathrm{A}(z) = -\frac{1}{2}[\partial(z) - \pi_\lambda(z) \otimes 1 - 1 \otimes z]$$

satisfies a polynomial identity

$$\prod_{i=1}^{k} (\mathrm{A}(z) - a_i(\mu)) = 0$$

where the a_i are the polynomial functions over H* given by

$$a_i(\mu) = -\frac{1}{2}[\chi_{\mu+\lambda_i}(z) - \chi_\mu(z) - \chi_\lambda(z)] \tag{18}$$

By taking traces of powers of the matrix A(z) we obtain a set of invariants $\mathrm{I}_m(\lambda) = \tau[\mathrm{A}(z)]^m$. The eigenvalues of these invariants are given still by formula (17) except now the polynomial functions a_i are given by equation (18) instead of (12).

Applications.

The character formula (17) in many ways reflects the nature of the tensor product space $\mathrm{V}(\lambda) \otimes \mathrm{V}(\mu)$. In fact one may obtain Klimyk's multiplicity formula by examining equation (17). It is hoped therefore that this formula may be of use in obtaining useful information concerning the more general tensor product space $\mathrm{V}(\lambda) \otimes \mathrm{V}$ where V is an infinite dimensional module admitting an infinitesimal character χ_ν, $\nu \in \mathrm{H}^*$. Kostant [*8*] has shown that when V is a Harish-Chandra module that $V(\lambda) \otimes V$ admits a composition series whose factors admit infinitesimal characters $\chi_{\nu+\lambda_i}$. However it is clear that not all infinitesimal characters $\chi_{\nu+\lambda_i}$ need be admitted. It would therefore be of interest to determine precisely which characters do occur (the infinite dimensional analogue of the Clebsch-Gordan series). It is suggestive that such considerations are closely related to the structure of maximal ideals in the universal enveloping algebra [*14*].

From the point of view of applications to physics one sees that the polynomial identities satisfied by the matrix A (see remarks following equation (12)) are useful for the construction of projection operators:

$$P[i] = \prod_{j \neq i} \left[\frac{A - a_j(\mu)}{a_i(\mu) - a_j(\mu)} \right]$$

The matrix elements of such projection operators, in unitary representations of the group, are bilinear combinations of Clebsch-Gordan coefficients. This opens up the possibility of a complete determination of the multiplicity free Wigner coefficients of a semi-simple Lie group by a close examination of the projection operators $P[i]$. For applications along these lines see [*17*] (and references quoted therein).

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