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by

Z. HABA (*) (**)

ABSTRACT. — We consider local perturbations $\exp\left(-\int_{\Lambda} U\right)$ of the generalized free Gaussian measure defined by the covariance

$$B = \int d\sigma(s)(-\Delta + m^2 + s^2)^{-1}$$

with

$$\int d\sigma(s)(m^2 + s^2)^{-1} < \infty$$

in $d < 3$ dimensions. Lattice approximation is defined. It is shown that in the lattice approximation the field theory is equivalent to a continuous spin non-nearest neighbour Ising ferromagnet. The infinite volume limit of the Schwinger functions is obtained via Griffiths correlation inequalities. We can get in this way theories with a non-canonical short distance behaviour.

I. INTRODUCTION

Euclidean Gaussian fields satisfying Osterwalder-Schrader positivity condition [1] should have the covariance [2]

$$B(x, y) = \int d\sigma(s)(-\Delta + m^2 + s^2)^{-1}(x, y) \qquad (\text{I.1})$$

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(with some substractions if $\int d\sigma(s)(m^2 + s^2)^{-1}$ is infinite). Then local perturbations $\exp\left(- \int_\Lambda U(\varphi(x))dx \right)$ of the Gaussian measure μ_B preserve the Osterwalder-Schrader positivity if only Λ is invariant under time reflection $x_0 \rightarrow - x_0$. For the perturbed measure

$$d\mu_\Lambda = \left(\int \exp\left(- \int_\Lambda U \right)d\mu_B \right)^{-1} \exp\left(- \int_\Lambda U \right)d\mu_B \qquad (I.2)$$

to exist and to have finite all moments it is sufficient that

$$\exp\left(- \int_\Lambda U \right) \in L_p(\mu_B).$$

If $\int d\sigma(s) < \infty$ then

$$B \le \int d\sigma(s)(- \Delta + m^2)^{-1}$$

and $\left\| \exp\left(- \int_\Lambda U \right) \right\|_p < \infty$ follows from the classical results [3] [4] on polynomial and exponential [5] interactions via the conditional comparison theorem of ref. [3] (quoted later on as GRS). In Appendix we show by detailed examination of Nelson estimates [4] that $\exp\left(- \int_\Lambda U \right) \in L_p$ even if $\int d\sigma(s)$ is divergent. In such a case the sort distance behaviour of $B(x - y)$ is more singular. We show that for polynomial interactions of order $2n$ in two dimensions $\exp\left(- \int_\Lambda P(\varphi) \right) \in L_p$ if $B(x - y) \sim |x - y|^{-\eta}$ for short distances, where $\eta < (n^2 + n)^{-1}$. These estimates are true also for $m = 0$ provided $\int d\sigma(s)s^{-2} < \infty$.

In order to get a Euclidean invariant theory we have to perform the infinite volume limit. This can be done in an easy way [3] if we have the Griffiths correlation inequalities [6] [7]. We define the lattice approximation of the measures (1) (2) and show that under the assumption

$$\int d\sigma(s)(m^2 + s^2)^{-1} < \infty$$

these measures in the lattice approximation describe a continuous spin Ising *ferromagnet*. Then correlation inequalities result. In order to get the infinite volume limit we define an analogue of the half-Dirichlet measure. Then we show that the half-Dirichlet Schwinger functions increase monotonically, as a result of the correlation inequalities, to a finite infinite volume

limit. The resulting Schwinger functions fulfil all the Osterwalder-Schrader axioms.

We find models constructed by perturbation of the generalized free measure interesting because of the possibility of obtaining a more singular short distance behaviour. The freedom in the choice of the singularity of the covariance (1) should give some insight into the problem of renormalizability. At the same time the short distance behaviour determines some continuity properties of the Euclidean fields. So, we get an example of more singular sample paths (see ref. [8]) than those resulting from $P(\varphi)_2$ models.

As we learned after completing this paper the perturbations of the generalized free measure are studied in ref. [9] (by means of different methods as far as we know), where the problem of continuous symmetry breaking in less than three dimensions is discussed. $\Big($There is a symmetry breaking if

$$m = 0 \text{ and } \int_0^1 dp\, \tilde{\mathrm{B}}(p) < \infty \Big).$$

II. LATTICE APPROXIMATION

We are going to define the lattice approximation to the finite volume Schwinger functions

$$S_k^\Lambda(x_1, \ldots, x_k) = \left[\int d\mu_\mathrm{B}(\varphi) \exp\left(- \int_\Lambda \mathrm{U}(\varphi) \right) \right]^{-1}$$

$$\int d\mu_\mathrm{B}(\varphi) \exp\left(- \int_\Lambda \mathrm{U}(\varphi) \right) \varphi(x_1) \ldots \varphi(x_k) \quad \text{(II.1)}$$

Following GRS [3] we introduce lattice fields φ_n (for simplicity we write all formulae for $d = 2$ then $n = (n_1, n_2) \in \mathbb{Z}^2$) as Gaussian random variables with the covariance

$$\mathrm{B}_{nn'} = \int d\mu_\mathrm{B}(\varphi)\varphi_n\varphi_{n'}$$

$$= \frac{\delta^2}{(2\pi)^2} \int_{-\frac{\pi}{\delta}}^{\frac{\pi}{\delta}} dp \int d\sigma(s) [\delta^{-2}(4 - 2\cos p_1\delta - 2\cos p_2\delta) + m^2 + s^2]^{-1} e^{ip(n-n')\delta}$$
$$\text{(II.2)}$$

$\Big($here $\int d\sigma(s)(m^2 + s^2)^{-1} < \infty \Big).$

Then φ_n can be expressed in terms of φ

$$\varphi_n = \frac{1}{(2\pi)^2} \int_{-\frac{\pi}{\delta}}^{\frac{\pi}{\delta}} dp\, e^{ipn\delta} \left(\frac{\tilde{\mathrm{B}}_\delta(p)}{\tilde{\mathrm{B}}(p)} \right)^{\frac{1}{2}} \tilde{\varphi}(p) \quad \text{(II.3)}$$

where $\tilde{\mathrm{B}}(p)$ is the Fourier transform of $\mathrm{B}(x)$ (eq. (I.1)) and $\tilde{\mathrm{B}}_\delta(p)$ is the

Fourier transform of $B(n - n')$ (eq. (II.2)). $(B_{nn'})$ is an infinite dimensional matrix, which determines a bounded operator B on the space l^2 of sequences. It is easy to check that the inverse operator B^{-1} has the matrix elements

$$(B^{-1})_{nn'} = \frac{\delta^2}{(2\pi)^2} \int_{-\frac{\pi}{\delta}}^{\frac{\pi}{\delta}} dp e^{ip(n-n')\delta}$$

$$\left\{ \int d\sigma(s) [\delta^{-2}(4 - 2\cos p_1\delta - 2\cos p_2\delta) + m^2 + s^2]^{-1} \right\}^{-1} \quad (II.4)$$

Denote $\lambda = 4\delta^{-2} + m^2 + s^2$, $a = 4\delta^{-2} + m^2$, $\tilde{\sigma}(\lambda) = \sigma(s)$ and

$$x = 2\delta^{-2}(\cos p_1\delta + \cos p_2\delta).$$

Then $(B^{-1})_{nn'}$ is the Fourier transform of the inverse of

$$f(x) = \int_a^\infty \frac{d\tilde{\sigma}(\lambda)}{\lambda - x} \quad (II.5)$$

$f(z)$ considered as a function of a complex variable z is an analytic function in the upper half-plane with a positive imaginary part there, i. e. it is a Pick function [10]. Moreover, $f(z)$ admits analytic continuation across the interval $(-\infty, a)$ by reflection into the lower half-plane, i. e. it is a Pick function of class $(-\infty, a)$. Clearly

$$| f(z) | \geq \int_a^\infty \frac{d\tilde{\sigma}(\lambda)}{| \lambda - | z | |} \neq 0.$$

Then, $-\dfrac{1}{f(z)}$ is also a Pick function of the same class. From the general representation theorem for Pick functions [10] we get

$$-\left(\int_a^\infty \frac{d\tilde{\sigma}(\lambda)}{\lambda - x} \right)^{-1} = \alpha x + \beta + \int_a^\infty \left(\frac{1}{\lambda - x} - \frac{1}{\lambda^2 + 1} \right) d\mu(\lambda) \quad (II.6)$$

where $\alpha \geq 0$ and μ is a measure on R with $\int d\mu(\lambda)(\lambda^2 + 1)^{-1} < \infty$. We can compute now the integral in eq. (II.4) directly using eq. (II.6) or expand first the r. h. s. of eq. (II.6) in powers of x. We get then

$$-\left(\int_a^\infty \frac{d\tilde{\sigma}(\lambda)}{\lambda - x} \right)^{-1} = \alpha_0 x + \beta_0 + \sum_{n > 1} a_n x^n \quad (II.7)$$

where $\alpha_0 \geq 0$ and

$$a_n = \int_a^\infty \lambda^{-n-1} d\mu(\lambda) \geq 0 \quad (II.8)$$

From eq. (II.7) and the formula $\int_0^\pi dx \cos mx \cos^n x \geq 0$ we can see

that $(B^{-1})_{nn'}$ is non-positive for $n \neq n'$. In fact, the expansion (II.7) is all what we needed in order to derive the result. The argument using the theory of Pick functions has been first applied by Glimm and Jaffe [11] to prove that the Fourier transform $- \Gamma(x)$ of the inverse of $\tilde{B}(p)$ is non-positive (for $x \neq 0$) in the continuum case.

We can define the Wick powers: φ_n^r : by means of the Wick powers of φ through eq. (II.3). Assume that g_k are functions with a compact support and define

$$\varphi_\delta(g) = \sum_n \delta^2 \varphi_n g(n\delta)$$

and

$$U_\delta^\Lambda = \sum_{n \in Z^2 \cap \Lambda} \delta^2 : P(\varphi_n) : \qquad (\text{II}.9)$$

We have then

THEOREM II.1. — For all spectral functions σ such that $\exp\left(- \int_\Lambda U \right) \in L_p$

i) $\varphi_\delta(g) \rightarrow \varphi(g)$

ii) $U_\delta^\Lambda \rightarrow \int_\Lambda U$

iii) $\exp(- U_\delta^\Lambda) \rightarrow \exp\left(- \int_\Lambda U \right)$ in L_p

iv) $S_k^{\delta,\Lambda}(g_1, \ldots, g_k) = \langle \exp(- U_\delta^\Lambda) \rangle^{-1} \langle \exp(- U_\delta^\Lambda)\varphi_\delta(g_1) \ldots \varphi_\delta(g_k) \rangle$

$$\rightarrow \left\langle \exp\left(- \int_\Lambda U \right) \right\rangle^{-1} \left\langle \exp\left(- \int_\Lambda U \right)\varphi(g_1) \ldots \varphi(g_k) \right\rangle$$

$$= S_k^\Lambda(g_1, \ldots, g_k)$$

Proof. — Follows from GRS [3]. In i) and ii) we perform the explicit computations, use pointwise convergence and dominated convergence theorem. In iii) we apply the inequality $|e^{-a} - e^{-b}| \leqslant |a - b| |e^{-a} + e^{-b}|$ the result ii) and the assumption $\exp\left(- \int_\Lambda U \right) \in L_p$. iv) follows from i)-iii).

Now the integral over the lattice fields φ_n in Theorem II.1 iv), is an integral over a cylinder function with a finite base (see Ref. [12]). Hence it is equal to the finite dimensional integral

$S_k^{\delta,\Lambda}(g_1, \ldots, g_k)$

$$= Z^{-1} \int \prod_i dx_i e^{-U(x_i)} e^{-\frac{1}{2}\sum_{i,j} x_i(B_\Lambda^{-1})_{ij}x_j} (\vec{x}, \vec{g}_1) \ldots (\vec{x}, \vec{g}_k) \quad (\text{II}.10)$$

where $(\vec{x}, \vec{g}) = \Sigma x_n \ g(n\delta)$ and B_Λ^{-1} is the inverse of the matrix B_Λ being the restriction of B (eq. (II.2)) to a finite dimensional submatrix correspon-

ding to $n, n' \in Z^2 \cap \Lambda$. It can be shown similarly as in Simon's book [3] that

$$B_\Lambda^{-1} = (B^{-1})_\Lambda + C_\Lambda \qquad (II.11)$$

where C_Λ is a matrix with non-positive matrix elements. So, B_Λ has also non-positive off-diagonal matrix elements, i. e. the Gibbs factor in eq. (II.10) is *ferromagnetic*. From eqs. (II.6)-(II.7) and the formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dx \cos^l x \cos kx = \begin{cases} 0 & l < k \quad \text{or} \quad l - k \text{ odd} \\ \dfrac{l!}{2^l \left(\dfrac{l+k}{2}\right)! \left(\dfrac{l-k}{2}\right)!} & l - k = 2r \geqslant 0 \quad (II.12) \end{cases}$$

it can be seen that either $(B^{-1})_{nn'}$ vanishes for $|n - n'| > 1$ (if $\mu = 0$), or all the spins x_n are coupled. From the point of view of the lattice systems it is interesting to know the behaviour of $(B^{-1})_{nn'}$ for large $|n - n'|$. We can prove the following

THEOREM II.2. — i) If $m > 0$ then for certain K

$$|(B^{-1})_{nn'}| \leq Ke^{-m|n-n'|\delta} \qquad (II.13)$$

ii) If $m = 0$ and

$$\left(\int_0^\infty \frac{d\sigma(s)}{1 + \alpha s^2 - x}\right)^{-1} = -\Sigma b_n x^n$$

with $b_n \geq Cn^{-\gamma}$ for large n then

$$|(B^{-1})_{nn'}| \geq A|n - n'|^{-\beta} \qquad (II.14)$$

where $\beta \geq 1 + 2\gamma$ for $d = 1$ and $\beta \geq 3 + 2\gamma$ if $d = 2$ for certain directions in the Z^2-plane.

Proof. — i) follows from the representation (II.6) and from the exponential decay of the similar integral (II.2). ii) can be shown using the formula (II.11). Consider first $d = 2$. Then clearly for $n \neq n'$ and any N

$$|(B^{-1})_{nn'}| \geq b_{2N} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dy_1 \int_{-\pi}^{\pi} dy_2 \cos(n_1 - n_1')y_1 \cos(n_2 - n_2')y_2$$
$$\cdot \left(\frac{1}{2}\cos y_1 + \frac{1}{2}\cos y_2\right)^{2N}$$
$$\geq \frac{b_{2N}(2N)!}{2^{2N}(N!)^2} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dy_1 \cos(n_1 - n_1')y_1 \cos^N y_1$$
$$\cdot \int_{-\pi}^{\pi} dy_2 \cos(n_2 - n_2')y_2 \cos^N y_2 \qquad (II.15)$$

where

$$\frac{1}{2^{2N}}\frac{(2N)!}{(N!)^2} = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(N+\frac{1}{2}\right)}{\Gamma(N+1)}.$$

We choose the direction $n_1 - n'_1 = n_2 - n'_2 = k$ and denote $2r = N - k$. Then according to eq. (II.12) we are to investigate the behaviour for large k of the expression

$$I = \frac{\Gamma(k + 2r + 1)}{2^{k+2r}\Gamma(k + r + 1)\Gamma(r + 1)} = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(r + \frac{k+1}{2}\right)\Gamma\left(r + \frac{k}{2} + 1\right)}{\Gamma(k + r + 1)\Gamma(r + 1)}. \quad \text{(II.16)}$$

Let us take $r = k^2$. Then using the asymptotic representation of the Γ function we get

$$I = \frac{1}{\sqrt{\pi e}}\frac{1}{k}\left(1 + 0\left(\frac{1}{k}\right)\right)$$

This together with

$$\frac{\Gamma\left(N+\frac{1}{2}\right)}{\Gamma(N + 1)} \sim \frac{1}{\sqrt{N}}$$

gives $|(B^{-1})_{nn'}| \geq Ak^{-3-2\gamma}$. The proof for $d = 1$ is similar.

Theorem (II.2) shows that if $m = 0$ we are dealing with forces decreasing rather slowly with the distance $|n - n'|$ between spins. This suggests that in this case the infinite volume limit may depend on the boundary conditions.

III. INFINITE VOLUME LIMIT

We will show in this section the existence of a unique infinite volume limit of the finite volume Schwinger functions with certain boundary conditions. From the uniqueness of the limit the Euclidean invariance follows and the Osterwalder-Schrader positivity present in a finite volume is preserved by the limit. So, the Osterwalder-Schrader axioms will be satisfied with the possible exception of the cluster property (we have the cluster property for exponential interactions). As is well-known [6] from the Griffiths inequalities for ferromagnetic Ising model it follows that the addition of further neighbour interactions increases the correlation functions and decreases the mass gap. So the non-nearest neighbour system is more susceptible to phase transitions.

First consider the case where we can show the cluster property.

THEOREM III.1. — Consider the following models (with $\int d\sigma(s)s^{-2} < \infty$ if $m = 0$).

i) $U(y) = g \int e^{-\alpha y} d\rho(\alpha)$

with $\rho(\alpha) = \rho(-\alpha)$ and $|\alpha| \int d\sigma(s) \leqslant 2\sqrt{\pi}$ in two-dimensions.

ii) $U(y) = \Sigma b_{2k} y^{2k}$,

$b_{2k} \geq 0$ and B(0) finite in one-dimension.

Then the Schwinger functions (II.1) are decreasing as $\Lambda \to R^d$ to a non-trivial infinite volume limit. They are decreasing functions of the coupling constants and therefore bounded by the correlation functions of the generalized free measure μ_B. The mass gap is not less than m and all the Osterwalder-Schrader axioms are fulfilled (possibly without the cluster property if $m = 0$).

Proof. — The proof of this theorem is now standard thanks to refs. [3], [4], [5]. The restriction $|\alpha| \int d\sigma(s) \leqslant 2\sqrt{\pi}$ comes from the requirement

$$\int_\Lambda : e^{\alpha\varphi(x)} : d\rho(\alpha) dx \in L_2(\mu_B).$$

That S_k^Λ decrease as Λ or the coupling constants increase can be seen from Griffiths inequalities by variation of Λ and coupling constants. The mass gap is not less than m because $S_2(x) \leq B(x)$. It remains to be shown that the infinite volume Schwinger functions are not zero. It will be shown below that the infinite volume Schwinger functions of models i)-ii) are not less than finite volume Schwinger functions with an analog of half-Dirichlet boundary conditions.

We are going now to define certain quadratic form $B_\Lambda \leq B$ confined to the region Λ. For this purpose let us note the inequalities

$$\int_0^r d\sigma(s)(p^2 + r^2 + m^2)^{-1} \leqslant \int d\sigma(s)(p^2 + s^2 + m^2)^{-1}$$

$$\leqslant \int d\sigma(s)(m^2 + s^2)^{-1} = M^{-1} \quad \text{(III.1)}$$

Hence the quadratic form $B(x - y)$ (I.1) corresponds to a bounded operator B bounded from below by

$$\int_0^r d\sigma(s)(-\Delta + r^2 + m^2)^{-1}$$

with certain $r > 0$. Let χ_{Λ^c} be the multiplication operator by the characteristic function of the region $\Lambda^c = R^d - \Lambda$. Then by the operator inequalities

of ref. [13] (p. 330) we get from eq. (III.1) for any $\omega \geq 0$ (with $a = \int_0^r d\sigma(s)$)

$$a(-\Delta + r^2 + \omega a \chi_{\Lambda^c})^{-1} \leq (B^{-1} + \omega \chi_{\Lambda^c})^{-1} \leq (M + \omega \chi_{\Lambda^c})^{-1} \quad (III.2)$$

But $(B^{-1} + \omega \chi_{\Lambda^c})^{-1}$ is monotonically decreasing in ω. So the limit

$$B_\Lambda = \lim_{\omega \to \infty} (B^{-1} + \omega \chi_{\Lambda^c})^{-1} \quad (III.3)$$

exists. This limit is not zero because from the left side of eq. (III.2) we get

$$B_\Lambda \geq a \lim_{\omega \to \infty} (-\Delta + r^2 + a \omega \chi_{\Lambda^c})^{-1} = a(-\Delta_D + r^2 + m^2)^{-1} \quad (III.4)$$

where Δ_D is the Laplacian with Dirichlet boundary conditions on $\partial\Lambda$. The equality in eq. (III.4) can be seen from the Feynman-Kac formula

$$(-\Delta + r^2 + m^2 + a\omega \chi_{\Lambda^c})^{-1}(x, y)$$
$$= \int_0^\infty dt\, e^{-t(m^2 + r^2)} \int dW_{(x,y)}^t(x(.)) \exp\left(-a\omega \int_0^t \chi_{\Lambda^c}(x(\tau))d\tau\right) \quad (III.5)$$

(here $dW_{(x,y)}^t(x(.))$ means integration over Brownian paths with $x(0) = x$ and $x(t) = y$) if we notice that $\omega \to \infty$ forces all paths to be inside Λ. From the right side of eq. (III.2) we get

$$(f, B_\Lambda f) = 0 \quad \text{if supp } f \subset \Lambda^c \quad (III.6)$$

Owing to the existence of the limit (III.3) we get

$$\lim_{\omega \to \infty} \lim_{\Omega \to \mathbb{R}^d} \left(\int d\mu_B(\varphi) \exp\left[-\frac{1}{2}\omega \int_{\Omega - \Lambda} :\varphi^2:\right]\right)^{-1}$$
$$\int d\mu_B(\varphi) \exp\left[-\frac{1}{2}\omega \int_{\Omega - \Lambda} :\varphi^2:\right] \exp i(\varphi, f)$$
$$= \lim_{\omega \to \infty} \exp\left[-\frac{1}{2}(f, (B^{-1} + \omega \chi_{\Lambda^c})^{-1}f)\right] = \exp\left[-\frac{1}{2}(f, B_\Lambda f)\right]$$
$$\equiv \int d\mu_{B_\Lambda}(\varphi) \exp i(\varphi, f) \quad (III.7)$$

Define now (with $\Lambda \subset \Lambda' \subset \Omega$, $f_i \geq 0$, supp $f_i \subset \Lambda$)

$$S_k^{\Lambda, HD}(f_1, \ldots, f_k)$$
$$= \lim_{\omega \to \infty} \lim_{\Omega \to \mathbb{R}^d} \left(\int d\mu_B(\varphi) \exp\left[-\int_{\Lambda'} U(\varphi) - \frac{\omega}{2} \int_{\Omega - \Lambda} :\varphi^2:\right]\right)^{-1}$$
$$\int d\mu_B(\varphi) \exp\left[-\int_{\Lambda'} U(\varphi) - \frac{\omega}{2} \int_{\Omega - \Lambda} :\varphi^2:\right] \varphi(f_1) \ldots \varphi(f_k) \quad (III.8)$$

The sequence in the limit (III.8) is monotonically decreasing in Ω and ω as a result of Griffiths inequalities. The existence of a non-trivial limit

follows from eq. (III.7). It is easy to see that this limit does not depend on $\Lambda' \supset \Lambda$. To see it let us note that

$$\exp\left(- \int_{\Lambda'} U \right) = \exp\left(- \int_{\Lambda} U \right) \exp\left(- \int_{\Lambda'-\Lambda} U \right)$$

and

$$\left| \exp\left(- \int_{\Lambda'-\Lambda} U \right) - 1 \right| \le \left| \int_{\Lambda'-\Lambda} U \right| \left(1 + \exp\left(- \int_{\Lambda'-\Lambda} U \right) \right)$$

In the limit $\omega \to \infty$ the integral over $\int_{\Lambda'-\Lambda} U$ vanishes due to eq. (III.6) and only the variables with support in Λ give contribution to $S_k^{\Lambda,HD}$. Applying Griffiths inequalities to the correlation functions (III.8) we get

$$S_k^{\Lambda_1,HD}(x_1, \ldots, x_k) \le S_k^{\Lambda_2,HD}(x_1, \ldots, x_k) \le S_k^{\Lambda'}(x_1, \ldots, x_k) \quad \text{(III.9)}$$

if $\Lambda_1 \subset \Lambda_2 \subset \Lambda'$.

Letting $\Lambda' \to R^d$ with Λ_1 fixed we get the *non-triviality* part of Theorem III.1.

Consider now the infinite volume limit of the half-Dirichlet Schwinger functions (III.8). According to eq. (III.9) these Schwinger functions are monotonically increasing as $\Lambda \to R^d$. Therefore to show that they converge it is sufficient to get a bound from above. We have from the Griffiths first inequality and inequalities (III.9).

$$S_k^{\Lambda,HD}(f_1, \ldots, f_k) \le S_k^{\Lambda,HD}(|f_1|, \ldots, |f_k|)$$
$$\le S_k^{\Lambda}(|f_1|, \ldots, |f_k|) \le S_k^{\Lambda}(h, \ldots, h)$$
$$\le k! \left\langle \exp\left(- \int_{\Lambda} U \right) \right\rangle^{-1} \left\langle \exp\left(- \int_{\Lambda} U \right) \exp \varphi(h) \right\rangle \quad \text{(III.10)}$$

where $h = \sum_{i=1}^{k} |f_i|$.

In order to bound the right hand side of eq. (III.10) we will use the Feynman-Kac formula for the generalized free field. Let us assume first that $\sigma(s)$ is differentiable i. e. $d\sigma(s) = \sigma'(s)ds$ and denote by $a(x_{d+1})$ the Fourier transform of $(\sigma'(s))^{\frac{1}{2}}$. Then the generalized free field (I.1) can be expressed as an integral over the free field φ_m with mass m in $d + 1$ dimensions

$$\varphi(x) = \int a(x_{d+1}) \varphi_m(x, x_{d+1}) dx_{d+1} \quad \text{(III.11)}$$

Because both sides are Gaussian eq. (III.11) follows from the equality

$$B(x - x') = \int dx_{d+1} dx'_{d+1} a(x_{d+1}) a(x'_{d+1}) \int d\mu_m(\varphi_m)$$
$$\cdot \varphi_m(x, x_{d+1}) \varphi_m(x', x'_{d+1}) \quad \text{(III.12)}$$

where μ_m is the Gaussian measure with covariance $(-\Delta_{d+1} + m^2)^{-1}$. Using the representation (III.11) and the Feynman-Kac formula for the free field in $d + 1$-dimensions we get (we write only the formula for $d = 2$ (*))

$$\int d\mu_B(\varphi)\overline{F(\varphi\delta_0)} \exp\left(-\int_0^t dx_0 \int_{-l}^l dx_1 U(\varphi)\right) G(\varphi\delta_t)$$

$$= \int d\mu_m(\varphi_m)\overline{F\left(\int \varphi_m\delta_0 a\right)} \exp\left(-\int_0^t dx_0 \int_{-l}^l dx_1 U\left(\int \varphi_m a\right)\right) G\left(\int \varphi_m\delta_t a\right)$$

$$= (F, e^{-tH_l}G) \qquad\qquad\qquad\qquad (III.13)$$

where

$$H_l = H_0 + \int_{-l}^l U\left(\int \varphi_m(x_1, x_3)a(x_3)dx_3\right)dx_1 \qquad (III.14)$$

$\varphi_m(x_1, x_3)$ is the three-dimensional time-zero field and the scalar product on the right hand side of eq. (III.13) is with respect to the (time-zero) Gaussian measure with covariance $(-\Delta_2 + m^2)^{-\frac{1}{2}}$.

The formula (III.13) holds for all $\sigma(s)$ such that

$$\exp\left[-\int_0^t dx_0 \int_{-l}^l dx_1 U(\varphi(x))\right] \in L_p(\mu_B)$$

(see GRS [3] for explanation of the connection between Euclidean and Glimm and Jaffe Hamiltonian formalism ([14])). From the Feynman-Kac formula (III.13) the Nelson-symmetry follows. Next, it is a direct consequence of the Nelson symmetry that the energy per unit volume (1 is the time-zero Fock space vacuum)

$$\alpha_l = -\frac{1}{l}(1, H_l 1) \qquad\qquad\qquad (III.15)$$

is a non-decreasing function of l (see [15] for a simple proof). Using this fact and the Feynman-Kac formula one can show (chessboard estimates [16] [17]) that

$$\left\langle \exp\left(-\int_\Lambda U\right)\right\rangle^{-1} \left\langle \exp\left(-\int_\Lambda U\right)\exp\left(-\varphi(h)\right)\right\rangle$$

$$\leq \exp\left[\int dx_1(\alpha_\infty(h(x_1)) - \alpha_\infty(0))\right] \quad (III.16)$$

where $\alpha_\infty(\rho)$ is the limit when $l \to \infty$ of the energy per unit volume for the interaction $U(\varphi) + \rho\varphi$. If $\sigma(s)$ is not differentiable we derive the formula (III.16) first for a sequence σ_n of differentiable functions and then we get in the limit $\sigma_n \to \sigma$ the estimate (III.16) for general σ. So to bound

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(*) For $d = 1$ there is no integration over x_1 in eq. (III.13) and therefore no dependence of H_l on l.

the Schwinger functions $S_k^{\Lambda,\text{HD}}$ it is sufficient to show that α_∞ is finite. But using the spectral decomposition of H_l and Jensen inequality we get

$$\alpha_l = -\frac{1}{l}(1, H_l 1) = \frac{1}{lt}\ln e^{-t(1, H_l 1)} \leq \frac{1}{lt}\ln(1, e^{-tH_l}1) \qquad \text{(III.17)}$$

Applying the Feynman-Kac formula (III.13) we get (*)

$$\alpha_\infty \leq \lim_{\Lambda \to \mathbf{R}^d} |\Lambda|^{-1}\ln\int d\mu_B \exp\left(-\int_\Lambda U\right) = p_\infty \qquad \text{(III.18)}$$

It can be easily seen from the derivation of the estimates (III.16) and (III.18) that these estimates are also true for $d = 1$.

If $\int d\sigma(s) < \infty$ then owing to eq. (I.3), the conditional comparison theorem [3] and the finiteness of the pressure for $P(\varphi)_2$ and exponential interactions of refs. [3]-[5] we get $\alpha_\infty < \infty$. We formulate this result as

THEOREM III.2. — Assume $\int d\sigma(s) < \infty$ and $\int d\sigma(s)s^{-2} < \infty$ if $m = 0$. The Schwinger functions $S_k^{\Lambda,\text{HD}}$, for U being a polynomial bounded from below or the exponential interaction $\int d\rho(\alpha) : e^{-\alpha\varphi} : \left(\text{with } \rho(\alpha) = \rho(-\alpha)\right.$ and $|\alpha|\int d\sigma < 2\sqrt{\pi}\right)$, are monotonically increasing to a finite infinite volume limit. The resulting theory fulfills Osterwalder-Schrader axioms with the possible exception of the cluster property.

In Appendix it is shown that $\exp\left(-\int_\Lambda U\right) \in L_p(\mu_B)$ even if $\int d\sigma(s)$ is infinite. In order to bound α_∞ in this case, we will define a quadratic form B_N^Λ and the corresponding Gaussian measure $\mu_{B_N^\Lambda}$ such that

$$p_\infty \leq p_\infty^N \equiv \lim_{\Lambda \to \mathbf{R}^d} p_\Lambda^N = \lim_{\Lambda \to \mathbf{R}^d} |\Lambda|^{-1}\ln\int d\mu_{B_N^\Lambda}\exp\left(-\int_\Lambda U\right) < \infty \qquad \text{(III.19)}$$

Namely, define the quadratic form

$$B_N^\Lambda(x, y) = \int d\sigma(s)(-\Delta_N + m^2 + s^2)^{-1}(x, y)$$

$$\equiv \int d\sigma(s)G_N^\Lambda(x, y; m^2 + s^2) \qquad \text{(III.20)}$$

where Δ_N is the Laplacian with Neumann boundary conditions on $\partial\Lambda$. Consider two regions Λ_1 and Λ_2 and denote $\Lambda = \text{int}(\bar{\Lambda}_1 \cup \bar{\Lambda}_2)$. Let $L_2(\Lambda)$

(*) The bound of (III.16) by $p\infty$ could be derived directly using a general form of chessboard estimates obtained in Ref. [18]. We thank Dr. J. Bellissard for this remark.

be the Hilbert space of square integrable functions with the support in Λ. Then $L_2(\Lambda) = L_2(\Lambda_1) \oplus L_2(\Lambda_2)$ and as is shown in GRS ([3]) we have in the sense of quadratic forms

$$G \le G_N^\Lambda \le G_N^{\Lambda_1} \oplus G_N^{\Lambda_2} \tag{III.21}$$

where $G(x, y; m^2 + s^2) = (-\Delta + m^2 + s^2)^{-1}(x, y)$. Hence also

$$B_N^\Lambda \le B_N^{\Lambda_1} \oplus B_N^{\Lambda_2} \tag{III.22}$$

Then using the conditional comparison theorem we get

$$\int d\mu_{B_N^\Lambda} \exp\left(-\int_\Lambda U\right) \le \int d\mu_{B_N^{\Lambda_1}} \exp\left(-\int_{\Lambda_1} U\right)$$
$$\int d\mu_{B_N^{\Lambda_2}} \exp\left(-\int_{\Lambda_2} U\right) \tag{III.23}$$

From this submultiplicativity it follows (see GRS) that p_Λ^N is a decreasing function of Λ. Hence p_∞^N is bounded by p_Λ^N. Now, p_Λ^N is finite if

$$\exp\left(-\int_{\Lambda_2} U\right) \in L_1(\mu_{B_N^\Lambda}).$$

In one dimension one can compute G_N^Λ explicitly and show that

$$\exp\left(-\int_\Lambda U\right) \in L_1(\mu_{B_N^\Lambda}) \quad \text{if} \quad \exp\left(-\int_\Lambda U\right) \in L_1(\mu_B).$$

In two dimensions we can get a similar conclusion from the estimate $G_N^\Lambda(m^2 + s^2) \le CG(m^2 + s^2)$ in the sense of quadratic forms on $L_2(\Lambda)$ where C does not depend on Λ and (for dimensional reasons) is also independent of s. So, we conclude that on $L_2(\Lambda)$

$$B_N^\Lambda \le CB \tag{III.24}$$

Then $\exp\left(-\int_\Lambda U\right)$ is integrable with respect to $\mu_{B_N^\Lambda}$ if it is integrable with respect to μ_B again by the conditional comparison theorem. We show in Appendix that $\exp\left(-\int_\Lambda P(\varphi)\right) \in L_p(\mu_B)$ if $B(x - y) \sim |x - y|^{-\eta}$ for short distances with $\eta < (n^2 + n)^{-1}$ for $d = 2$ and $\eta < (2n^2 + 2n)^{-1}$ for $d = 1$ where $2n$ is the order of the polynomial P. Let us summarize the considerations above in

THEOREM III.3. — Assume $U(\varphi)$ is a polynomial of order $2n$, $\tilde{B}(p)$ is not singular at $p = 0$ and $B(x) \sim |x|^{-\eta}$ for small $|x|$ with $\eta < (n^2 + n)^{-1}$ for $d = 2$ and $\eta < (2n^2 + 2n)^{-1}$ if $d = 1$. Then finite volume Schwinger functions $S_k^{\Lambda,HD}$ (III.8) exist and are monotonically increasing as $\Lambda \to \mathbf{R}^d$

to a finite infinite volume limit satisfying Osterwalder-Schrader axioms with the possible exception of the cluster property.

The more singular short distance behaviour leads to more singular sample paths of the random field. It follows from classical results [19] on sample paths of Gaussian stochastic processes that the Gaussian stochastic process φ_t has discontinuous sample paths if

$$B(0) - B(t) \geq \frac{C}{|\ln t|}$$

for small t. In two-dimensional case one can show that if $B(x-y) \sim |x-y|^{-\eta}$ for small distances then the generalized free field $\varphi(x)$, $x \in \mathbb{R}^2$ smeared out with f in one coordinate $\varphi(\delta_t f)$ is Hölder continuous in t with the continuity index α arbitrarily close to $\frac{1}{2} - \frac{\eta}{2}$. We have proved [8] such a behaviour of sample functions also for interacting Euclidean fields under some assumptions on correlation functions. In this case continuity index α is determined by the short distance behaviour of the covariance of the interacting field. Our assumptions in ref. [8] are satisfied immediately if we have the GHS inequality [20]. The GHS inequality holds true in our ferromagnetic system with φ^4 or exponential interaction. It is to be expected that the short distance behaviour of the interacting field is the same as the short distance behaviour of the corresponding generalized free field. So, we get Hölder continuity of $\varphi(\delta_t f)$ with the continuity index $\frac{1}{2} - \frac{\eta}{2}$ (less than for the canonical Euclidean field).

ACKNOWLEDGMENTS

APPENDIX

In this Appendix we will repeat the classical estimates of Nelson [4] (see also GRS [3]) in order to extend their validity to more singular short distance behaviour than the canonical case.

LEMMA A.1. — Assume that $\tilde{B}(q)$ is not singular at $q = 0$ and $|\tilde{B}(q)| \leqslant A|q|^{-2+\eta}$ for large $|q|$ then

i) $\| : \varphi^r : (g)\|_2 \leq R \|g\|_2$ if $\eta < \dfrac{2}{r}$ and $d = 2\left(\eta - 1 < \dfrac{1}{r} \text{ for } d = 1\right)$.

ii) Define $\varphi_x(x) = \int h_x(x - y)\varphi(y)dy$ where $\tilde{h}_x(q) = \theta(\varkappa - |q|)$. We have for arbitrary $p > 0$

$$\| : \varphi^r : (g) - : \varphi_x^r : (g)\|_p \leq K(p - 1)^{\frac{r}{2}} \|g\|_2 \varkappa^{-\delta}$$

where $\delta = 1 - \dfrac{r}{2}\eta - \varepsilon$ for $d = 2\left(\delta = \dfrac{1}{2} - \dfrac{r}{2}(\eta - 1) - \varepsilon \text{ if } d = 1\right)$ and ε can be arbitrarily small.

Remarks. — 1) We assume above that $\eta \geqslant 1$ in one-dimension, because if $\eta < 1$ then Wick ordering and regularization are unnecessary.

2) We could replace the condition $|\tilde{B}(q)| \leqslant A|q|^{-2+\eta}$ by $\tilde{B} \in L_\beta$ with $\beta > \left(1 - \dfrac{\eta}{2}\right)^{-1}$.

Proof. — Let us compute

$\| : \varphi^r : (g) - : \varphi_x^r : (g)\|_2^2$

$= \dfrac{r!}{(2\pi)^{dr}} \int dq_1 \ldots dq_r |\tilde{g}(q_1 + \ldots + q_r)|^2 \, \tilde{B}(q_1) \ldots \tilde{B}(q_r)(1 - \tilde{h}_x(q_1) \ldots \tilde{h}_x(q_r))^2$

$\leq r \dfrac{r!}{(2\pi)^{dr}} \int_{|q_r|>\varkappa} |\tilde{g}|^2 * \tilde{B} * \ldots * \tilde{B}(q_r)\tilde{B}(q_r)dq_r$

$\leq C_r\left(\int_{|q|>\varkappa} |\tilde{B}(q)|^\alpha \, dq\right)^{\frac{1}{\alpha}} \| |\tilde{g}|^2 * \tilde{B} \ldots * \tilde{B}\|_{\frac{\alpha}{\alpha-1}}$

By Young's inequality we have

$$\| |\tilde{g}|^2 * B * \ldots * \tilde{B}\|_{s_{r-1}} \leq \| |\tilde{g}|^2 * \tilde{B} \ldots * \tilde{B}\|_{s_{r-2}} \|B\|_\beta$$
$$\leq \| |\tilde{g}|^2 * \tilde{B} \ldots * \tilde{B}\|_{s_{r-k}} \|B\|_\beta^{k-1}$$

where β, $s_i \geq 1$ and

$$\frac{1}{s_{r-k}} = 1 - \frac{1}{\beta} + \frac{1}{s_{r-k+1}} \qquad k = 2, \ldots, r$$

Assuming $s_0 = 1$ we get

$$\frac{1}{s_{r-1}} = 1 - (r - 1)\left(1 - \frac{1}{\beta}\right).$$

We must have $\beta(2 - \eta) > d$ if $\|\tilde{B}\|_\beta$ is to be finite. Take

$$\frac{1}{\beta} = \frac{2 - \eta}{d} - \varepsilon$$

then

$$\frac{1}{s_{r-1}} = 1 - \frac{1}{\alpha} = 1 - (r-1)\left(\frac{d-2+\eta}{d} + \varepsilon\right)$$

Now compute

$$\left(\int_{|q|>\varkappa} |\tilde{B}(q)|^{\alpha}dq\right)^{\frac{1}{\alpha}} \leqslant A\left(\int_{|q|>\varkappa} |q|^{(-2+\eta)\alpha+d-1}dq\right)^{\frac{1}{\alpha}} = A'\varkappa^{-2+\eta+\frac{d}{\alpha}}$$

This proves ii) for $p = 2$. The proof for arbitrary p follows from hypercontractive bounds [3] [4]. The part i) can be proved in a similar way and in fact follows from the proof above if we put $\tilde{h}_{\varkappa} = 0$, $\varkappa = \infty$ and notice that from ii) $\eta = \frac{2}{r}(1 - \varepsilon - \delta) < \frac{2}{r}$.

LEMMA A.2. — Let $U(\varphi)$ be a (Wick ordered) polynomial of order $2n$. Then for $d = 2$ and arbitrary $p > 0$ we have the estimate

$$\mu_{B}\left\{\varphi : U(g) \leq -C\|g\|_{1}\varkappa^{\eta n}\right\} \geq K^{p}(p-1)^{np}\|g\|_{2}^{p}\varkappa^{-\delta p}$$

(for $d = 1$ if $\eta \geq 1$ we replace η by $\eta - 1$ in the formula above).

Proof. — Follows from Lemma A.1, via Nelson argument [4] (but now $\langle \varphi_{\varkappa}^{2} \rangle \sim \varkappa^{\eta}$ instead of $\ln \varkappa$).

We choose now p depending on $\varkappa$ in such a way that the right hand side of the estimate above is minimal

$$p = \varkappa^{\frac{\delta}{an}} \qquad a > 1$$

Then

$$K^{p}(p-1)^{np}\|g\|_{2}^{p} T^{-\delta p} \leq e^{-b\varkappa^{\frac{\delta}{an}}}$$

for large $\varkappa$.

LEMMA A.3. — Let $U(\varphi)$ be a polynomial of order $2n$ then in two dimensions

$$\exp(-U(g)) \in L_{p}(\mu_{B}) \qquad \text{if} \qquad \eta < (n^{2}+n)^{-1}(\eta - 1 < (2n^{2}+2n)^{-1}$$

in one dimension).

Proof. — Denote $\lambda(x) = \mu_{B}\left\{\varphi : U(g) \leq x\right\}$ then

$$\int \exp(-U(g))d\mu_{B}(\varphi) = \int e^{-x}d\lambda(x)$$

and the estimate of Lemma A.2 shows that this integral is convergent if

$$\int \exp(|x| - |x|^{\frac{\delta}{a\eta n^{2}}})dx$$

is convergent, i. e. $\frac{\delta}{n^{2}a.\eta} > 1$. Together with Lemma A.1 this gives the restriction on η.

REFERENCES

[1] K. OSTERWALDER and R. SCHRADER, *Comm. Math. Phys.*, t. **31**, 1973, p. 83.

[2] G. C. HEGERFELDT, *Comm. Math. Phys.*, t. **35**, 1974, p. 155.

[3] F. GUERRA, L. ROSEN and B. SIMON, *Ann. Math.*, t. **101**, 1975, p. 111; *Ann. Inst. Henri Poincaré*, t. **25**, 1976, p. 231; B. SIMON, $P(\varphi)_{2}$ *Euclidean (Quantum) Field Theory*, Princeton, 1974.

[4] E. NELSON, *in* Constructive Quantum Field Theory, G. Velo and A. Wightman, Eds., Springer, *Lecture Notes in Physics*, t. **25**, 1973.
[5] S. ALBEVERIO, R. HOEGH-KROHN, *J. Funct. Analysis*, t. **16**, 1974, p. 39.
[6] R. GRIFFITHS, in *Statistical Mechanics and Quantum Field Theory*, C. de Witt and R. Stora, Eds., Les Houches, 1970, Gordon and Breach, 1971.
[7] J. GINIBRE, *Comm. math. Phys.*, t. **16**, 1970, p. 310.
[8] Z. HABA, *Comm. math. Phys.*, t. **69**, 1979, p. 247.
[9] J. FRÖHLICH, R. ISRAEL, E. H. LIEB and B. SIMON, *in preparation*.
[10] W. F. DONOGHUE, J. R., *Monotone Matrix Functions and Analytic Continuation*, Springer, Berlin, 1974.
[11] J. GLIMM and A. JAFFE, *Phys. Rev.*, *D. 11*, 1975, p. 2816.
[12] I. M. GELFAND and N. YA. VILENKIN, *Generalized Functions*, Vol. **IV**, Academic Press, New York, 1964.
[13] T. KATO, *Perturbation Theory for Linear Operators*, Springer, New York, 1966.
[14] J. GLIMM and A. JAFFE, *in* [6].
[15] F. GUERRA, L. ROSEN, and B. SIMON, *Comm. math. Phys.*, t. **27**, 1972, p. 10.
[16] J. FRÖHLICH, *Helv. Phys. Acta*, t. **47**, 1974, p. 265.
[17] E. SEILER and B. SIMON, *Ann. Phys.*, t. **97**, 1976, p. 470.
[18] J. FRÖHLICH and B. SIMON, *Ann. Math.*, t. **105**, 1977, p. 493.
[19] H. CRAMER and M. R. LEADBETTER, *Stationary and Related Stochastic Processes*, Wiley, New York, 1967.
[20] Ch. NEWMAN, *Comm. math. Phys.*, t. **41**, 1976, p. 1.