ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 32, nº 2 (1980), p. 109-123
<http://www.numdam.org/item?id=AIHPA_1980__32_2_109_0>

Ann. Inst. Henri Poincaré
Vol. XXXII, n° 2, 1980, p. 109-123.

Section A :
Physique théorique.

On the harmonic analysis of the elastic scattering amplitude of two spinless particles at fixed momentum transfer

by

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ABSTRACT. — The harmonic analysis of the elastic scattering amplitude $F(s, t)$ of two spinless particles, at fixed $t < 0$, is here revisited using the non-euclidean Fourier analysis in the sense of Helgason, and the approach of Ehrenpresis to the special functions. With these techniques it is possible to derive the Fourier and Laplace transforms for the scattering amplitude. Indeed these transforms are obtained by projecting the amplitude on functions which play a role similar to that played by the exponentials on the real line; here we show how to construct these functions, using essentially geometrical tools. Since the harmonic analysis is a decomposition which separates the dynamics from the symmetry of the problem, we obtain an explicit geometrical characterization of those terms which reflect the symmetry.

1° INTRODUCTION

As is well known, the classical Fourier transform refers to the decomposition of a function, belonging to an appropriate space, into exponentials of the form e^{ikx} (k real), which can also be viewed as the irreducible unitary representations of the additive group of real numbers. However in the current interpretation, particularly in connection with noncommutative groups, the phrase harmonic analysis has lost its original function theoretic meaning, and this term refers not to functions but to representations. As it has been remarked by Furstenberg [*1*] « it became natural to regard irreducible representations as the basic building blocks of the theory in the place of the exponential functions ». However, there are examples, in the theory of semi-simple noncompact Lie groups, where the classical setup prevails [*1*], in the sense that one can find a class of functions which appear to play a role similar to that played by the exponentials on the real line. This fact is particularly significant in the Fourier analysis of the scattering amplitude on the Lorentz group. Indeed this theory can be viewed as a mathematical tool for separating the dynamics, which is described by the « partial-waves », from the symmetry, which is represented by these generalized « exponentials ».

Let us return to the real-line. If one considers functions which are not square-integrable on the line (for instance polynomials), then the classical Fourier transform does not work. One possibility is to extend the domain of the Fourier integral operator to include polynomially bounded functions, by recognizing that the range of the integral operator then contains generalized functions (distributions). The other possibility is a generalization of the Fourier transform which leads to the Laplace transform. This is achieved by continuing the exponentials e^{ikx} in the complex k-plane, away from the real line, to reach non unitary representations. Then, the image of a power bounded function is an analytic function, regular in an appropriate portion of a half plane. At this point the following question arises: is it possible, in the case of the harmonic analysis on the three-dimensional Lorentz group, to build functions which are in some sense « nearly » e^{ikx}, $k \in \mathbb{C}$? In this paper we shall give an answer to this question, at least in the case of spinless particles. The problem is quite relevant; indeed the requirement of square integrability over the group manifold implies a decrease of amplitudes faster than $s^{-1/2}$ as $s \to \infty$ (s is the energy squared in the center of mass system), which is well below the asymptotic bound indicated by experiments and by the Froissart bound [*2*]. In order to overcome this difficulty, Rühl [*3*], [*4*] developed distribution-valued transforms on Lorentz groups. On the other hand many authors [*2*], [*5*]-[*10*], proposed a Laplace-like transform; one of the intents of this note is precisely that of giving a more rigorous foundation to the theory of this transform. In this note we essentially link different ideas, which are dispersed in different mathematical frameworks. The hope is that if we tie things together, then we gain insight in various questions of the harmonic analysis of the scattering amplitude which appear, up to now, quite unclear.

The paper is organized as follows. In Section 2 we formulate the problem and characterize a representation space for the three-dimensional Lorentz group. In Section 3 we construct the Legendre functions following the method of Ehrenpreis. In Section 4 we derive the Harish-Chandra representation of the spherical Legendre functions and the non-euclidean Fourier analysis in the sense of Helgason; from the latter a Fourier transform for the scattering amplitude follows. In Section 5 we derive the Laplace transform. Finally we recall that some of the results of Section 4 have been previously given, in a preliminary form, in a note of the author [11].

2° POSITION OF THE PROBLEM AND PRELIMINARIES

Let us denote by $F(s, t)$ the elastic scattering amplitude of two spinless particles (s and t are the usual Mandelstam variables). We limit ourselves to consider the case of spinless particles, in order to make more evident the geometry involved in the problem. As is well known, if one takes $s > 0$ fixed and expands the scattering amplitude in functions of t, then one gets the usual phase-shift analysis. In this case the little group is SO(3), the basis of the expansion is given by the Legendre polynomials and the kinematics is described by the cosine of the scattering angle in the center of mass system; i. e. $\cos \vartheta = 1 + \dfrac{2t}{s - 4m^2}$ in the case of particles of equal mass m. Here we leave the usual partial-wave decomposition aside, since it has been analyzed in detail even in the relativistic case, and we turn our attention to the following problem: take $t < 0$ fixed, in the s-channel physical region, and decompose the scattering amplitude in functions of s. In this case the little group is the isotropy group of the space-like momentum transfer, i. e. the noncompact SO(1, 2) group. Furthermore the amplitude $F(s, t)$ can be written in terms of a hyperbolic function as follows: $F(s, t) \equiv f(\cosh \beta)$, where $\cosh \beta = \dfrac{2s}{4m^2 - t} - 1$ in the case of particles of equal mass. If the masses of the particles are arbitrary, then $\cosh \beta$ is given by [2]:

$$\cosh \beta = - \frac{t^2 + 2st - t(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{\lambda(t, m_1^2, m_3^2)\lambda(t, m_2^2, m_4^2)} \tag{2.1}$$

where λ is the square root of the following function:

$$\lambda^2(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx) \tag{2.2}$$

Henceforth we shall keep the scattering amplitude $f(\cosh \beta)$ as the typical function which we want to decompose in the sense of Fourier and Laplace. However we will attempt to refrain from physical terminology. Therefore we shall adopt hereafter the notations which are more proper from the mathematical point of view. Indeed the translations of the results from the mathematical to the physical language will be quite evident.

Let us consider the group of linear transformations of R^3 leaving the form $x_1^2 - x_2^2 - x_3^2$ invariant. We denote this group by G and by g its elements. Now we shall characterize a representation space of G. At this purpose let us recall that a representation of G on function f on R^3 can be written as follows:

$$T(g)f(x) = f(g^{-1}(x)), \qquad g \in G \tag{2.3}$$

where $T(g)$ is an operator function on G. Indeed, equality (2.3) determines a representation of G since, for any two elements g_1 and g_2 of G, we have:

$$T(g_1 g_2)f(x) = T(g_1)T(g_2)f(x) \tag{2.4}$$

Now the space of homogenous functions is invariant for shifts; indeed if a function $f(x)$ is homogeneous, then $f(g^{-1}(x))$ is also a homogeneous function of the same degree. Moreover, if we denote by $\square$ the wave operator $\left(\text{i. e. } \square \equiv \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right)$, then $\square$ commutes with any $g \in G$. Therefore G acts on the space formed by those solutions of the wave equation (i. e. $\square f = 0$) which satisfy a condition of the following form:

$$f(ax_1, ax_2, ax_3) = a^{-\nu} f(x_1, x_2, x_3) \tag{2.5}$$

for some complex ν (fixed), and for all real $a > 0$.

Now we replace R^3 by the two sheets hyperboloid: $x_1^2 - x_2^2 - x_3^2 = 1$; its points can be described by the following coordinates:

$$\begin{cases} x_1 = \cosh r \\ x_2 = \sinh r \sin \vartheta \\ x_3 = \sinh r \cos \vartheta \end{cases} \tag{2.6}$$

Then, following Ehrenpreis [*12*], we write formally the solutions of the wave equation (i. e. $\square h = 0$) as Fourier integrals of suitable measures on the light cone, as follows:

$$\begin{aligned} h(x_1, x_2, x_3) &= \int_{z_1^2 = z_2^2 + z_3^2} \exp(ix_1 z_1 + ix_2 z_2 + ix_3 z_3) d\mu(z_1, z_2, z_3) \\ &= \int_{z_1^2 = z_2^2 + z_3^2} \exp\left\{ i \left[(x_1, x_2, x_3) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right] \right\} d\mu(z_1, z_2, z_3). \end{aligned} \tag{2.7}$$

3° THE SPHERICAL LEGENDRE FUNCTIONS

The Fourier analysis on R^2 is a decomposition of functions into exponential plane waves. Moreover, if one makes use of polar coordinates and if the functions to be decomposed are radial, then in the Fourier integral the zero-order Bessel function appears. The latter is a continuous superposition of plane-waves, it has rotational symmetry and it is related to the matrix elements of the unitary representations of the group of motions of the plane. In order to obtain a Fourier transform on the space associated to the three-dimensional Lorentz group, we shall proceed in a direction analogous to that followed in the case of R^2. Moreover, since the scattering amplitudes are radial (as we shall see below), one must construct a function with properties resembling those of the zero-order Bessel function. It must be a continuous superposition of « plane-waves » (or more precisely of the non-euclidean analog of the plane-wave), and it must have rotational invariance. To this purpose, let us recall that SO(2) is the maximal compact subgroup of SO(1, 2). More specifically, the elements of the group of rotations about the x_1 axis, which leave invariant the form $x_2^2 + x_3^2$, can be represented as follows:

$$m_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \tag{3.1}$$

Then, following a procedure due to Ehrenpreis [*12*], we make the function h, given by formula (2.7), invariant under SO(2), by choosing for the measure μ of (2.7) the invariant measure of the orbit of z^0 under the rotation group SO(2), where z^0 is a point belonging to the light cone. Then starting from formula (2.7), and following closely Ehrenpreis [*12*], we write:

$$(m_\varphi h)(x_1, x_2, x_3) = \int_{z_1^2 = z_2^2 + z_3^2} \exp\left\{ i \left[(x_1, x_2, x_3) m'_\varphi \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right] \right\} d\mu(z_1, z_2, z_3) \tag{3.2}$$

where m'_φ is the adjoint of m_φ. Then the orbit of $z^0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (z^0 belongs to the light cone) is given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \sin\varphi \\ \cos\varphi \end{pmatrix} \tag{3.3}$$

Choosing for $d\mu$ a measure invariant on this orbit, we get:

$$h(x_1, x_2, x_3) = \frac{1}{2\pi} \int_0^{2\pi} \exp\{ i(x_1 + x_2 \sin\varphi + x_3 \cos\varphi) \} \, d\varphi \tag{3.4}$$

where $(2\pi)^{-1}$ is a normalization factor. Next we must impose the condition (2.5). This is achieved by taking the Mellin transform, i. e.

$$f(x_1, x_2, x_3) = (2\pi)^{-1}\int_0^{+\infty}\int_0^{2\pi} \exp(itx_1 + itx_2 \sin\varphi + itx_3 \cos\varphi)t^{(\nu-1)}d\varphi dt \quad (3.5)$$

Indeed one has:

$$f(ax_1, ax_2, ax_3) = (2\pi)^{-1}\int_0^{+\infty}\int_0^{2\pi} \exp(iatx_1 + iatx_2 \sin\varphi + iatx_3 \cos\varphi)t^{(\nu-1)}d\varphi dt \quad (3.6)$$

and putting: $at = r (a > 0)$, one obtains:

$$(2\pi)^{-1}\int_0^{+\infty}\int_0^{2\pi} \exp(irx_1 + irx_2 \sin\varphi + irx_3 \cos\varphi)\left(\frac{r}{a}\right)^{(\nu-1)} a^{-1}d\varphi dr = a^{-\nu}f(x_1, x_2, x_3) \quad (3.7)$$

which proves the statement. Now let us observe that it holds:

$$\int_0^{+\infty} e^{-a\lambda}a^{(\nu-1)}da = \lambda^{-\nu}\Gamma(\nu) \quad (3.8)$$

which simply derives from the integral representation of the gamma function, i. e. $\Gamma(\nu) = \int_0^{+\infty} e^{-a}a^{(\nu-1)}da$. Exchanging the order of integration in formula (3.5) and using (3.8) we obtain:

$$\begin{aligned} f(x_1, x_2, x_3) &= (2\pi)^{-1}\int_0^{2\pi}\left(\int_0^{+\infty} \exp(itx_1 + itx_2 \sin\varphi + itx_3 \cos\varphi)t^{(\nu-1)}dt\right)d\varphi \\ &= (2\pi)^{-1}\Gamma(\nu).\int_0^{2\pi} [-i(x_1 + x_2 \sin\varphi + x_3 \cos\varphi)]^{-\nu}d\varphi \quad (3.9) \end{aligned}$$

Finally substituting for the coordinates x_i ($i = 1, 2, 3$) the expressions (2.6), and omitting some inessential factors, we obtain the following integral representation of the spherical Legendre functions:

$$P_{-\nu}(\cosh r) = \frac{1}{2\pi}\int_0^{2\pi} [\cosh r + \sinh r \cos(\varphi - \vartheta)]^{-\nu}d\varphi \quad (3.10)$$

Moreover from the group multiplication law (2.4), the multiplication formula for the Legendre spherical functions follows (for a detailed proof see Vilenkin [*13*], p. 325), i. e.

$$\frac{1}{2\pi}\int_0^{2\pi} P_\nu(\cosh r_1 \cosh r_2 + \sinh r_1 \sinh r_2 \cos\varphi)d\varphi = P_\nu(\cosh r_1)P_\nu(\cosh r_2) \quad (3.11)$$

Finally we can say that the Legendre functions (3.10) are analogous to the exponentials on the real line, since the relationship (3.11) is the analogous of the following multiplication property: $e^{x+y} = e^x . e^y$.

4° THE FOURIER TRANSFORM

Now we want to show that the functions (3.10) are a continuous superposition of non-euclidean « plane-waves ». The geometrical nature of this part of the problem requires a more specific characterization of the space where we want to work. We can refer, for instance, to the SL(2, R) group, which is homomorphic with respect to SO(1, 2), and acts transitively on the upper half-plane Im $z > 0$, by means of the mappings:

$$z \to h.z = \frac{az + b}{cz + d}, \qquad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathrm{R}) \tag{4.1}$$

Then the symmetric space associated to SL(2, R) (i. e. SL(2, R)/(SO(2)) can be identified with the upper half-plane Im $z > 0$; its curvature is -1 and the Riemannian structure can be written, in terms of geodesic polar coordinates (ϑ, r), as follows [*14 a*]:

$$(dr)^2 + (\sinh r)^2 (d\vartheta)^2 \tag{4.2}$$

On the other hand, the upper half-plane is conformal to the interior of the unit disk by a linear fractional transformation. This mapping is linked with the existence of the isomorphism between SL(2, R) and SU(1, 1). Indeed to matrices, belonging to SU(1, 1), correspond linear fractional transformations which map the unit disk onto itself. Therefore we can also work in the symmetric space SU(1, 1)/SO(2) (i. e. in the non-euclidean disk). Hereafter we shall generally work in the non-euclidean disk.

We parametrize the points of the non-euclidean disk by the following coordinates:

$$\begin{cases} y_1 = \tanh\left(\dfrac{r}{2}\right) \sin \vartheta \\ y_2 = \tanh\left(\dfrac{r}{2}\right) \cos \vartheta \end{cases} \tag{4.3}$$

Then we write the following Riemannian structure:

$$ds^2 = \frac{4[(dy_1)^2 + (dy_2)^2]}{[1 - (y_1^2 + y_2^2)]^2} = (dr)^2 + (\sinh r)^2 (d\vartheta)^2 \tag{4.4}$$

which induces the usual non-euclidean distance on the open unit disk $D(y_1^2 + y_2^2 < 1)$; i. e.

$$d(0, z) = \log \frac{1 + |z|}{1 - |z|} \tag{4.5}$$

where: $z \equiv |z| e^{i\vartheta} = \tanh\left(\frac{r}{2}\right) e^{i\vartheta}$. Now we are prepared to construct the analog of the plane-wave using the Lobacevskij geometry of the non-euclidean disk.

Here we shall closely follow the geometric approach of Helgason [*14* (*a*)-(*e*)]. First of all one must look for the analogs to hyperplanes in $\mathbf{R}^n$. These shall be given by the orthogonal trajectories to a family of parallel geodesic. A pencil of « parallel straight-lines » is given by arcs of circles orthogonal to the unit circle, lying in its interior and intersecting the boundary B of the unit disk D (the horizon) at a common point $b = e^{i\varphi}$. The trajectories, orthogonal to this pencil of parallel geodesics, are the circles tangent from within to the horizon at the point b. These circles are the euclidean images of the horocycles and form the analogs to hyperplanes in $\mathbf{R}^n$, or more precisely to the oriented hyperplanes in $\mathbf{R}^n$, since the space of horocycles is antisymmetric. Then we refer to the point of contact b as the normal to the horocycle.

Now the level lines of the Poisson kernel $P(z, b)$ are circles tangent to the unit circle at the point b [*15*]. Furthermore $P(z, b)$ being the real part of $\frac{z+b}{z-b}$, is a harmonic function of z and $[P(z, b)]^\nu$, $\nu \in \mathbb{C}$ is an eigenfunction of the Laplace-Beltrami operator on D [*14 b*]. Finally $P(z, b)$ is invariant with respect to any transformation that preserves the unit disk [*15*]. All these considerations make possible to state that the analogs of the exponential plane-waves are given by:

$$e^{\nu\langle z,b\rangle} = [P(z, b)]^\nu = \left(\frac{1 - |z|^2}{1 + |z|^2 - 2|z|\cos(\vartheta - \varphi)}\right)^\nu, \qquad \nu \in \mathbb{C} \quad (4.6)$$

where $z = |z| e^{i\vartheta}$ and $b = e^{i\varphi}$. In fact $[P(z, b)]^\nu$ is constant on each horocycle of normal b, it is an eigenfunction of the Laplace-Beltrami operator on D, and finally $\langle z, b \rangle$ gives the non-euclidean distance from the center of the unit disk to the horocycle with normal b and passing through z ($\langle z, b \rangle$ is negative if the center of the unit disk falls inside the horocycle) [*14 b*]. Then substituting $z = \tanh\left(\frac{r}{2}\right) e^{i\vartheta}$ in the Poisson kernel we obtain:

$$e^{\nu\langle z,b\rangle} = \left(\frac{1}{\cosh r - \sinh r \cos(\vartheta - \varphi)}\right)^\nu, \qquad \nu \in \mathbb{C} \quad (4.7)$$

and therefore the Legendre functions (3.10) can be rewritten as follows:

$$P_{-\nu}(\cosh r) = \frac{1}{2\pi} \int_B e^{\nu\langle z,b\rangle} db \quad (4.8)$$

where db is the usual angular measure on the boundary B of D [*14 b*].

Formula (4.8) gives the Harish-Chandra representation of the Legendre spherical functions. Finally let us recall that $P_{-\frac{1}{2}+i\lambda}(\cosh r)$ $(\lambda \in \mathrm{R})$ correspond to the principal series of the irreducible unitary representation of the SU(1, 1) group.

REMARKS. — *i*) Let us write the Laplace-Beltrami operator, corresponding to the Riemannian structure (4.2):

$$\frac{\partial^2}{\partial r^2} + \frac{\cosh r}{\sinh r}\frac{\partial}{\partial r} + \frac{1}{(\sinh r)^2}\frac{\partial^2}{\partial \vartheta^2} \tag{4.9}$$

Next we consider those radial functions $\psi(r)$, which are eigenfunctions of the operator (4.9), i. e. the solutions of the following differential equation:

$$\frac{d^2\psi(r)}{dr^2} + \frac{\cosh r}{\sinh r}\frac{d\psi(r)}{dr} = \alpha\psi(r) \tag{4.10}$$

where α is a complex constant. This equation can be put in the usual Legendre form [*16*] by substituting: $\cosh r = z$, $\alpha = \nu(\nu + 1)$. Now $P_\nu(\cosh r)$ is the most general solution of eq. (4.10), which is 1 for $r = 0$. We can say that in this sense, also, the analogy between the Legendre spherical functions $P_\nu(\cosh r)$ and the exponentials on the real line is precise. Indeed the latter are the eigenfunctions $\psi(x)$ of all differential operators on R, with constant coefficients, normalized by $\psi(0) = 1$.

ii) There is another way of looking at the integrand of formula (3.10). If $a, d \in \mathbb{C}$, then the transformation

$$z \to \tau . z = \frac{az + d}{\bar{d}z + \bar{a}}, \qquad |a|^2 - |d|^2 = 1 \tag{4.11}$$

maps D onto itself. Let us specify τ by taking

$$a = \cosh\left(\frac{r}{2}\right), \qquad d = \sinh\left(\frac{r}{2}\right);$$

then evaluate the Jacobian of the mapping $b \to \tau . b$ $(b \in \mathrm{B})$ [*14 b*]

$$\left|\frac{d(\tau . b)}{db}\right| = (\cosh r + \sinh r \cos\varphi)^{-1} \tag{4.12}$$

It coincides with the integrand of formula (3.10), if in the latter we put $\vartheta = 0$.

iii) Let us recall that the Laplace-Beltrami operator in the non-euclidean disk is given by:

$$\Delta = \frac{1}{4}[1 - (y_1^2 + y_2^2)]^2\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right).$$

Therefore the euclidean harmonic functions coincide with the non-euclidean ones. Then the classical Poisson integral formula for a harmonic function u on D, with continuous boundary values $u(b)$ on B, can be rewritten as follows [*14 b*]:

$$u(z) = (2\pi)^{-1} \int_{\mathrm{B}} e^{\langle z,b \rangle} u(b) db \tag{4.13}$$

Now we come to the Fourier analysis on the non-euclidean disk. In analogy with the classical case, it is essentially a decomposition of a function into non-euclidean plane-waves. Moreover in order to give a natural domain to the mapping $f \to \hat{f}$ ($\hat{f}$ denotes the Fourier transform), one must specify the space for the functions f. It is convenient to start with functions f nice enough, say $f \in \mathrm{C}_c^\infty(\mathrm{D})$, then to see if the mapping can be extended to L^2 spaces. At this purpose a positive answer is given by the following fundamental theorem of Helgason.

THEOREM (Helgason [*14 c*]). — For $f \in \mathrm{C}_c^\infty(\mathrm{D})$ let $\hat{f}$ denote the « Fourier transform »

$$\hat{f}(\lambda, b) = \int_{\mathrm{D}} e^{(-i\lambda + \frac{1}{2})\langle z,b \rangle} f(z) dz\,, \qquad \lambda \in \mathrm{R}\,, \quad b \in \mathrm{B} \tag{4.14}$$

where $dz = \dfrac{4 dy_1 dy_2}{\{1 - (y_1^2 + y_2^2)\}^2}$ is the non-euclidean surface element on D.

Then

$$f(z) = \int_{\mathrm{R}} \int_{\mathrm{B}} e^{(i\lambda + \frac{1}{2})\langle z,b \rangle} \hat{f}(\lambda, b) d\mu(\lambda, b) \tag{4.15}$$

where

$$d\mu(\lambda, b) = (2\pi)^{-2} \lambda \tanh(\pi\lambda) d\lambda db \tag{4.16}$$

$d\lambda$ being the euclidean measure on R, db the angular measure on B. Moreover if R^+ denotes the set of positive reals, the mapping $f \to \hat{f}$ extends to an isometry of $\mathrm{L}^2(\mathrm{D}, dz)$ onto $\mathrm{L}^2(\mathrm{R}^+ \times \mathrm{B}, 2d\mu)$.

Formula (4.15) can be proved [*14 b*] by a simple reduction to the case of radial f, in which case it becomes the inversion formula for the Mehler transform with the conical Legendre functions (i. e. $\mathrm{P}_{-\frac{1}{2}+i\lambda}(\cosh r)$) as kernel [*16*]. As we have seen in Section 2, the scattering amplitude of spinless particles is simply $f(\cosh \beta)$, which can be rewritten as $f(\cosh r)$, identifying β (given by formula (2.1)) with the geodesic coordinate r. Then formula (4.14) becomes (omitting π factors):

$$\int_1^{+\infty} f(\cosh r) \mathrm{P}_{-\frac{1}{2}+i\lambda}(\cosh r) d(\cosh r) \tag{4.17}$$

and formula (4.15) reads as follows:

$$\int_0^{+\infty} \hat{f}(\lambda) \mathrm{P}_{-\frac{1}{2}+i\lambda}(\cosh r) \lambda \tanh(\pi\lambda) d\lambda \tag{4.18}$$

which are the classical Mehler transforms [*16*]. These formulae have been used in high energy scattering theory [*17*]-[*18*], but without a clear understanding of their geometrical meaning.

Now let us briefly mention the Radon transform and its relationship with the Fourier transform. In the euclidean case the Radon transform is simply the decomposition of a function $f \in C_c^\infty(\mathbf{R}^n)$ over the various hyperplanes in $\mathbf{R}^n$. More precisely if $\omega \in \mathbf{R}^n$ is a unit vector, $r \in \mathbf{R}$, and dm is the euclidean measure on the hyperplane $(x, \omega) = r$ then the function

$$\check{f}(\omega, r) = \int_{(\omega, x) = r} f(x) dm(x) \tag{4.19}$$

is called the Radon transform of f [*14 d*]. It is easy to show that the relationship between the Radon and the Fourier transform is given by [*19*]:

$$\hat{f}(\omega) = \int_{\mathbf{R}^n} f(x) e^{i(x,\omega)} dx = \int_{-\infty}^{+\infty} e^{ir} \check{f}(\omega, r) dr \tag{4.20}$$

Using once more the analogy between horocycles and hyperplanes, we write the Radon transform on the non-euclidean disk D, as follows [*14 c*]:

$$\check{f}(\xi) = \int_\xi f(z) d\sigma(z) \tag{4.21}$$

where ξ is any horocycle in D, $d\sigma$ the measure on ξ induced by the Riemannian structure of D and f is any function on D for which the integral (4.21) exists. Writing $\check{f}(\xi) = \check{f}(b, r)$ if ξ has normal b and distance r from the origin, then in analogy with the euclidean case we have [*14 c*]

$$\hat{f}(\lambda, b) = \int_{\mathrm{D}} e^{(-i\lambda + \frac{1}{2})\langle z, b\rangle} f(z) dz = \int_{-\infty}^{+\infty} e^{(-i\lambda + \frac{1}{2})r} \check{f}(b, r) dr \tag{4.22}$$

which gives the relationship between the non-euclidean Fourier and Radon transforms.

Finally, in order to characterize the space $\hat{C}_c^\infty(\mathrm{D})$, let us give a Paley-Wiener type theorem. We call a C^∞ function $\varphi(\lambda, b)$ on $\mathbb{C} \times \mathrm{B}$ a holomorphic function of uniform exponential type, if it is holomorphic in λ and if there exists a constant $\mathrm{A} \geqslant 0$ such that, for each integer $\mathrm{N} \geqslant 0$ [*14 d*]

$$\sup_{\lambda \in \mathbb{C},\, b \in \mathrm{B}} \exp\{-\mathrm{A} \,|\, \mathrm{Im}\, \lambda \,|\,\} (1 + |\,\lambda\,|^{\mathrm{N}}) \,|\, \varphi(\lambda, b) \,| < \infty \tag{4.23}$$

Then reducing a result of Helgason [*14 e*], written in a very general setting, to the specific case of the non-euclidean disk, we can state the following proposition.

PROPOSITION. — The Fourier transform $f(z) \to \hat{f}(\lambda, b)$ is a bijection of $C_c^\infty(D)$ onto the set of holomorphic functions of uniform exponential type satisfying the identity:

$$\int_B \hat{f}(\lambda, b) e^{(i\lambda+\frac{1}{2})\langle z,b\rangle} db = \int_B \hat{f}(-\lambda, b) e^{(-i\lambda+\frac{1}{2})\langle z,b\rangle} db \tag{4.24}$$

5° THE LAPLACE TRANSFORM

Let us recall once more that in the classical case the Laplace transform generalizes the Fourier transform making use of the exponentials of the form e^{ikx}, $k \in \mathbb{C}$, which are in general non unitary representations of the additive group of real numbers. Now we want to copy, as closely as possible, the classical procedure. Therefore we must find out a prescription in order to construct functions which are, in some sense, the analogue of e^{ikx}, $k \in \mathbb{C}$.

Then let us return to the representation (2.7). In Section 3 we have obtained the Legendre spherical functions by choosing for the measure μ of formula (2.7), the invariant measure of the orbit of a point z^0, belonging to the light cone, under the rotation group SO(2). At this purpose we used the unitary representations (3.1) of the group SO(2). Therefore a possibility which we have here is to choose a measure in formula (2.7), which is invariant under the orbit of z^0 generated by non-unitary representations in the place of the representations (3.1). Then we should choose μ as the invariant measure of an orbit of z^0 under SO(2, C) (that is, allow φ to be an arbitrary complex number). However such a μ is too large to yield a meaningful function [*12*]. A way for overcoming this difficulty is to consider another subgroup of SO(2, C), by taking φ pure imaginary: $\varphi = i\psi$. Therefore instead of matrix (3.1), we shall use the following one:

$$m_\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\psi & -i\sinh\psi \\ 0 & i\sinh\psi & \cosh\psi \end{pmatrix} \tag{5.1}$$

Then we choose for the measure μ, a measure which is invariant under the orbit

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\psi & -i\sinh\psi \\ 0 & i\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -i\sinh\psi \\ \cosh\psi \end{pmatrix} \tag{5.2}$$

instead of the orbit (3.3). Next, repeating the same steps made in Section 3, we obtain, instead of formula (3.10), the following one:

$$Q_{\nu-1}(\cosh r) = \int_0^{+\infty} (\cosh r + \sinh r \cosh\psi)^{-\nu} d\psi. \quad (\nu > 1) \tag{5.3}$$

which is an integral representation of the second-kind Legendre function.

Then from the group multiplication law (2.4), the following multiplication formula derives (see also ref. [6]):

$$\frac{1}{2}\int_{-\infty}^{+\infty} Q_\nu(\cosh r_1 \cosh r_2 + \sinh r_1 \sinh r_2 \cosh \psi)d\psi = Q_\nu(\cosh r_1)Q_\nu(\cosh r_2) \quad (5.4)$$

which is the analogue of formula (3.11).

REMARK. — Let us return to the eq. (4.10). We have seen that a solution is given by P_ν (cosh r), a second linearly independent solution, which presents a logarithmic singularity at $r = 0$, is given by Q_ν (cosh r). Furthermore the following asymptotic behaviour holds true [16]:

$$Q_\nu(\cosh r)\underset{|\nu|\to\infty}{\sim}\frac{\text{const.}}{\sqrt{\nu}}e^{-(\nu+\frac{1}{2})r}, \qquad r > 0. \quad (5.5)$$

Therefore if, in the place of the Mehler formula (4.17), we write:

$$\tilde{f}(\lambda) = \int_1^{+\infty} f(\cosh r)Q_\lambda(\cosh r)d(\cosh r) \quad (5.6)$$

we have a Laplace-like transform. Indeed, as it has been shown by Crönstrom and Klink [2], if $f(x)$ $(x \in (1, +\infty))$ is an arbitrarily locally integrable function such that the following integral exists

$$\int_1^{+\infty}(x^2-1)^{-1/4}x^{-p-\frac{1}{2}}\,|\,f(x)\,|\,dx < \infty \quad (5.7)$$

for some real values of $p > -\frac{1}{2}$, then $\tilde{f}(\lambda)$ is regular analytic in the half-plane Re $\lambda > p$. Let us recall that formula (5.6) has been originally introduced in high energy scattering theory by Gribov [20] long time ago: then it has been reproposed by many authors [5]-[10].

Finally let us rapidly sketch, for the sake of completeness, the inversion of formula (5.6), which has been derived by Crönstrom [9] for those amplitudes which belong to the class specified above by formula (5.7). One starts from the following integral representation of the second-kind Legendre functions:

$$Q_\lambda(\cosh r) = \int_r^{+\infty}\frac{e^{-(\lambda+\frac{1}{2})\varphi}}{(2\cosh\varphi - 2\cosh r)^{1/2}}d\varphi \quad (5.8)$$

Then the Laplace transform (5.6) can be rewritten as follows:

$$\tilde{f}(\lambda) = \int_0^{+\infty} e^{-(\lambda+\frac{1}{2})\varphi}g(\varphi)d\varphi \quad (5.9)$$

where

$$g(\varphi) = \int_1^{\cosh \varphi} \frac{f(x)}{(2 \cosh \varphi - 2x)^{1/2}} dx \qquad (5.10)$$

Eq. (5.9) is the usual Laplace transform of an auxiliary function $g(\varphi)$; eq. (5.10) is Abel's equation. Both the eqs. (5.9) and (5.10) can be inverted. Therefore inserting the inverse of eq. (5.9) into the inverse of eq. (5.10) one obtains the following formula:

$$f\,(\cosh r) = \frac{1}{2\pi i} \int_{\mathrm{Re}\,\lambda = p} (2\lambda + 1)\tilde{f}(\lambda)\mathrm{R}_\lambda\,(\cosh r) d\lambda \qquad (5.11)$$

where the line of integration runs to the right of the singularities of $\tilde{f}(\lambda)$, and

$$\mathrm{R}_\lambda\,(\cosh r) = \frac{1}{\pi} \int_0^r \frac{e^{(\lambda + \frac{1}{2})\varphi}}{(2 \cosh r - 2 \cosh \varphi)^{1/2}} d\varphi \qquad (5.12)$$

The contour in the representation (5.11) can be pushed into the left half of the λ-plane (i. e. the complex angular momentum plane), provided proper account is taken of the singularities of $\tilde{f}(\lambda)$ for Re $\lambda < p$.

ACKNOWLEDGMENTS

I am very grateful to Prof. K. Chadan for his kind hospitality at the Laboratoire de Physique Théorique et Hautes Énergies d'Orsay, where this paper has been completed. In particular at Orsay I had the opportunity of long informative and very helpful discussions with Prof. F. Lurçat, to whom I am deeply indepted. I want also to thank Prof. S. Helgason for his generous and constant help in clarifying to me many points of his theory of the non-euclidean Fourier analysis. Last but not least I am grateful to Prof. C. Cecchini, E. Eberle, T. Regge and M. Toller for useful and stimulating discussions.

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(*Manuscrit révisé*, reçu le 15 janvier 1980)