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Infinitesimal holonomy group structure and geometrization

by

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ABSTRACT. — The role of the infinitesimal holonomy group (IHG) and associated curvature structure equations are considered in connection with programs involving geometrization. We argue that the IHG formalism provides a natural setting for discussions of the geometrization program and gauge formulations of gravity. In particular, we show that by considering the IHG structure of Riemannian curvature the Bianchi identities can be cast into the form of basic gauge field equations for the Riemannian curvature gauge fields. We point out that the underlying fibre bundle structure associated with the IHG formalism is that of the holonomy bundles which are reduced subbundles of the bundle of orthonormal frames. The IHG formalism is illustrated for Einstein vacuum and Einstein-Maxwell space-times, and Einstein-Yang-Mills space-times are briefly discussed.

1. INTRODUCTION

In the last few years a considerable amount of renewed interest and effort has been directed toward the unification problem with various authors proceeding from fairly diverse points of view. The goal of these attempts is the ultimate unification of the gravitational interaction with the weak, electromagnetic and the strong interactions. The well-known success of the gauge theoretic approach in unifying the weak and electromagnetic interactions [1] has provided much of the encouragement for the more ambitious unification

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problem ([1]). Most of the papers relating to this problem could be characterized as attempts ranging from schemes of total or partial geometrization to some type of natural synthesis of coupled gauge theories including the important concept of spontaneous symmetry breaking. The most important and unifying element in all of these attempts relating to gauge formulations is the proper characterization of these theories in terms of connections in fibre bundles. In this regard it should be noted, in accord with Trautman's observation [4], that it appears that a generalized Higgs type mechanism may always be operative and fundamental in the reduction of the principal fibre bundles associated with the physical theories.

An important first step in the direction of this unification problem is a suitable gauge formulation and understanding of the gravitational field. Indeed, many papers have appeared in the last few years dealing with the gauge formulation of Einstein and Einstein-Cartan gravitational theories and their extensions. While the considerations of this paper are relevant to these papers, it is not our purpose here to comment in detail on the specific results and conclusions they provide.

The primary purpose of this paper is to stress the fundamental importance of the holonomy bundle-infinitesimal holonomy group (IHG) structure especially for any gauge theoretic program involving total or partial geometrization. In particular, we will stress the important role of the structure equations that relate to this IHG structure with emphasis on the case of Riemannian space-time. Many of the important detailed features of the IHG structure of C^∞ Riemannian space-times have been largely ignored in the literature ([2]). While we put major emphasis in this paper on the IHG of Riemannian space-times we wish to stress that our general considerations relating to the IHG structure of C^∞ connections are applicable to a very

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([1]) In connection with the general gauge unification program, see, for example, Salam [3] Also, see [2] for a survey introducing gauge theories.

([2]) The infinitesimal holnomy group has been used to study and classify Einstein spaces by Schell [5] and Goldberg and Kerr [6]. In addition Loos [7] has used properties of the holonomy group of curved space-times in connection with a geometrical particle theory and related notions of symmetry breaking. Also the internal holonomy groups for Yang-Mills fields in flat space-time have been studied by Loos [8]. While our primary concern is with Riemannian geometry in this paper the IHG is well-defined for all C^∞ linearly connected geometries [11, 12] and in particular for the Riemann-Cartan geometries that are sometimes taken as the geometrical arena for gauge theories of gravitation. Although the decomposition (1) will not hold for a Riemann-Cartan curvature tensor a more general decomposition obtains [10] in which an expression of the form (1) does hold for the V_4 part of the full U_4 curvature tensor. A study of the relationship of torsion in U_4 geometries to the associated holonomy group structures would be an important extension of the present discussion. In particular Schouten [10], p. 361-362, points out that the non-homogeneous holonomy group relates to torsion as well ascurvature.

general range of theories which includes most of the theories currently being discussed ([3]).

2. THE IHG STRUCTURE OF RIEMANNIAN CURVATURE

Before proceeding we present the following theorem, due to Nijenhuis [9] and Schouten [10], which gives the structure of Riemannian curvature fundamentally in terms of the IHG structure of space-time ([4,5]).

THEOREM I (Nijenhuis [9] and Schouten [10]). — *Let* (M, g) *be an analytic space-time and let* $R_{\nu\mu\lambda}{}^{\varkappa}$ *be the Riemannian curvature tensor expressed in local coordinates. If the set* $\{ C_{a\lambda}{}^{\varkappa} \}$, $a = 1, 2, \ldots, r = \dim$ (IHG), *of tensors is a basis of generators of the IHG then*

$$(1) \qquad\qquad R_{\nu\mu\lambda}{}^{\varkappa} = R^{a}{}_{\nu\mu} C_{a\lambda}{}^{\varkappa} = R^{ab} C_{a\nu\mu} C_{b\lambda}{}^{\varkappa}$$

where R^{ab} *is an* $r \times r$ *symmetric matrix.*

The set $\{ C_{a\lambda}{}^{\varkappa} \}$ of tensors is at each $x \varepsilon M$ a realization of the Lie algebra

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([3]) For a discussion of ECSK theory and its extensions (*i. e.*, theories thought to be in the framework of Poincaré gauge theories) see, for example, Hehl *et al.* [13, 14]. Atkins *et al.* [15] and Davis *et al.* [16] give a brief discussion of the variational principles underlying these theories and their relationships to each other including a covering theory. In the case of Weyl's theory and its extensions (*i. e.*, theories thought to be in the framework of conformal and superconformal gauge theories) see, for example, Kaku *et al.* [17] and Mansouri [18]. See Yasskin [19] for a treatment of coupled Einstein-Yang-Mills fields. Also, we note Borechsennius [20] extends Einstein's nonsymmetric unified field theory to include Yang-Mills fields.

([4]) Throughout this paper we mainly follow the notations and definitions used by Schouten [10]. In particular ∇_{μ} will denote the operation of covariant differentiation and square brackets around indices will denote the operation of antisymmetrization. The dual $^{*}A^{\mu\lambda}$ of a skew symmetric tensor $^{*}A^{\mu\lambda}$ is defined to be $^{*}A^{\mu\lambda} = 1/2\eta^{\mu\lambda\sigma\rho}A_{\sigma\rho}$ where $\eta^{\mu\lambda\sigma\rho}$ is the permutation tensor with $\eta^{0123} = - (- g)^{-1/2}$. The Ricci tensor is given by the contraction $R_{\mu\lambda} = R_{\nu\mu\lambda}{}^{\nu}$.

For a general discussion of the holonomy groups associated with connections see, for example, Kobayashi and Nomizu [11], Lichnerowicz [12]. In this connection we remark that the infinitesimal holonomy group can only be defined if both the manifold and connection are C^{∞}.

([5]) In this paper we mainly consider analytic space-times in order to simplify our discussion of the holonomy group structures. However, we remark that many of the features of our discussion of the properties of the IHG, in particular Theorem I, continue to hold in C^{∞} space-times or regions of C^{∞} space-times for which the dimension of the IHG is constant. Moreover, when dim (IHG) is not constant singularities of Riemannian curvature characterized as « points singular for infinitesimal holonomy » [9, 11], could be of importance in a study of singularities of space-time structure.

of the r-parameter ($r \leqq 6$) IHG as linear transformations of the tangent space T_xM and satisfy the commutation relations [9, 10]

$$(2) \qquad [C_a, C_b]_\lambda{}^\varkappa \overset{d}{=} C_{a\lambda}{}^\mu C_{b\mu}{}^\varkappa - C_{b\lambda}{}^\mu C_{a\mu}{}^\varkappa = \varepsilon_{ab}{}^c C_{c\lambda}{}^\varkappa.$$

The $\varepsilon_{ab}{}^c$ are at each $x \varepsilon M$ the structure constants of the IHG. For C^∞ Riemannian space-times the IHG is a subgroup of the homogeneous Lorentz group $SO(1, 3)$.

The above theorem shows that the details of the structure of Riemannian curvature of a given analytic space-time (M, g) is contained in the symmetric matrix $R^{ab} = R^{ab}(x)$ together with the set of generators $\{ C_{a\lambda}{}^\nu \}$. When the set of bivectors $C_{a\lambda\nu} = C_{a\lambda}{}^\mu g_{\mu\nu}$ is composed of simple bivectors [5] the factorization (1) above of the curvature tensor tells us which 2-planes at each point contain non-zero Riemannian curvature. The classical formula [10]

$$(3) \qquad \delta u^\lambda = - 1/2\ R_{\nu\mu\varkappa}{}^\lambda u^\varkappa df^{\nu\mu}$$

for the variation of a vector u^λ due to parallel transport around the simple infinitesimal contravariant bivector $df^{\nu\mu}$ is a measure of the curvature in the 2-plane of $df^{\nu\mu}$. We emphasize that the factorization (1) of the curvature tensor into a quadratic sum of generators of the IHG is a property peculiar to the metrical nature of Riemannian curvature and does not hold for a general linearly connected geometry [9, 10].

Our first major point of emphasis is this quadratic bivector structure of the Riemann curvature tensor. By allowing basis transformations of the generators of the IHG of the form

$$(4) \qquad C_{a\lambda}{}^\varkappa \to \overline{C}_{a\lambda}{}^\varkappa = S_a^b(x) C_{b\lambda}{}^\varkappa \quad , \quad \det S_a^b(x) \neq 0$$

we may elevate the labeling indices $a, b = 1, 2, \ldots, r$ to tensorial type indices [9, 10] and hence make the expression (1) basis independent. By using standard bivector identities one may show that for any analytic space-time the Ricci tensor can be put in the form

$$(5) \qquad R_{\mu\lambda} = 1/2 R^{ab}(C_{a\nu\mu}C_{b\lambda}{}^\nu + {}^*C_{a\nu\mu}{}^*C_{b\lambda}{}^\nu) + 1/4 g_{\mu\lambda}(R^{ab}C_{a\alpha\beta}C_b{}^{\beta\alpha}).$$

For analytic space-times with vanishing Ricci scalar ($R \equiv R_\nu^\nu = 0$) the last term in (5) vanishes and the Ricci tensor reduces to the form of a generalized Einstein-Maxwell type stress-energy-momentum tensor. This result, which we will return to below, clearly suggests that the IHG structure of analytic space-times may be useful in the geometrization program for theories involving stress-energy-momentum tensors that are quadratic in the (bivector) gauge fields. Note that because of the transformation freedom (4) the $C_{a\lambda}{}^\varkappa$ need not be simple bivectors.

There exists another unique property of the curvature structure of Riemannian space-times that relates fundamentally to notions of geometrization and gauge theoretical considerations. This property can perhaps best be exhibited in terms of the IHG structure of Riemannian curvature using the

Cartan structure equations. Using an orthonormal basis of 1-forms ω^h the Cartan structure equations may be written in the form [10] ([6])

(6) $$d\omega^h = \omega^i \wedge \Gamma_i^h$$

(7) $$1/2 R_i{}^h = d\Gamma_i^h + \Gamma_j^h \wedge \Gamma_i^j$$

where $\Gamma_i^h = \Gamma_{ji}^h \omega^j$ is the matrix of Riemannian connection 1-forms. By considering only special classes of frames and coframes ([6]) the Γ_i^h may be expressed as

(8) $$\Gamma_i^h = 1/2 C_{ai}{}^h \theta^a$$

where the $C_{ai}{}^h$ are constants satisfying the commutation relations (2) above. The structure equations (6) and (7) may then be rewritten as

(9) $$d\omega^h = 1/2 C_{ai}{}^h \omega^i \wedge \theta^\alpha$$

(10) $$R^a = d\theta^a - 1/4 \varepsilon_{bc}{}^a \theta^b \wedge \theta^c \; ; \; R_i^h = R^a C_{ai}{}^h.$$

The $\theta^a = \theta_\mu^a dx^\mu$ are the connection 1-forms and the $R^a = R_{\mu\lambda}^a dx^\mu \wedge dx^\lambda$ the curvature 2-forms, or curvature gauge fields.

Using the above IHG formalism it is not too difficult to show that the Bianchi identities may be written in the form ([4])

(11) $$\nabla_\mu {}^*R^{a\lambda\mu} = - 1/2 \varepsilon_{bc}{}^{a} {}^*R^{b\lambda\mu} \theta_\mu^c.$$

The unique property of Riemannian geometries that we have alluded to above is that the first contraction of the Bianchi identities may be written in the form (see the appendix for the derivation)

(12) $$\nabla_\mu R^{a\lambda\mu} = - 1/2 \varepsilon_{bc}{}^{a} R^{b\lambda\mu} \theta_\mu^c + K^{a\lambda}$$

where $K^{a\lambda} C_{a\nu\mu} = 2\nabla_{[\nu} R_{\mu]}^\lambda$.

We stress that the existence of this second identity (12) in terms of θ^a and R^a is unique to Riemannian geometries ([7]).

The existence of these two curvature identities shows that one may view the curvature structure of Riemannian geometry itself as providing what might be called the basic gauge field structure equations. In particular we remark that equations (11) and (12) are formally analogous to standard

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([6]) In equations (6) through (10) we mainly follow Schouten [10] although we use here the more standard symbol $\wedge$ for the exterior product of forms. For a discussion of the special frames used in writing equations (8) through (10) see Schouten [10], p. 375-376, and our discussion of the holonomy bundles that follows below.

([7]) Equation (12) is closely related to the Bianchi identities when they are written in terms of the Weyl conformal tensor $C^{\nu\mu\lambda\kappa}$ as $\nabla_\kappa C^{\kappa\mu\lambda\nu} = \nabla^{[\nu} R^{\mu]\lambda} + 1/6 g^{\lambda[\mu} \nabla^{\nu]} R$. Some authors, for example Hawking and Ellis [21], have interpreted this conformal tensor equation « in a sense... as field equations for the Weyl tensor giving that part of the curvature at a point that depends on the matter distribution at other points ».

gauge theory equations [2, 22]. In terms of gauge theory analogy, one would call

$$J_{(M)}^{a\lambda} \stackrel{d}{=} -1/2\varepsilon_{bc}{}^{a*}R^{b\lambda\mu}\theta_{\mu}^{c} \quad \text{and} \quad J_{(E)}^{a\lambda} \stackrel{d}{=} -1/2\varepsilon_{bc}{}^{a}R^{b\lambda\mu}\theta_{\mu}^{c}$$

the generalized « magnetic » and « electric » gauge currents, respectively, carried by the curvature gauge fields themselves. Moreover these equations suggest the geometrical identification of source « matter currents » as $J_{(S)}^{a\lambda} \stackrel{d}{=} K^{a\lambda}$. In this context a specific theory of Riemannian curvature (gravitation) (e. g., Einstein's standard theory) coupled to source matter would be determined by any particular specification of these « matter currents » $J_{(S)}^{a\lambda}$ in terms of matter source fields together with a set of coupled field equations for these source fields (^{8}).

Such a geometrical identification of source « matter currents » is certainly not at odds with the spirit of Einstein's program of the geometrization of physics. Initially, a central idea in geometrization is that matter should tell geometry how to curve. The above identification of the coupling of « matter currents » in Riemannian space-times is clearly in harmony with this idea in that the index « a » in $J_{(S)}^{a\lambda} = K^{a\lambda}$ refers to the basis of generators of the IHG from which the curvature tensor is built. Note in particular that if dim (IHG) $= r$, $0 < r < 6$ with $a = 1, \ldots, r$, then there exists at least one 2-plane at each point in which there is no curvature and thus the currents $J_{(S)}^{a\lambda} = K^{a\lambda}$ do not couple to these non-curved 2-planes. Further, we remark that any particular identification $J_{(S)}^{a\lambda} = K^{a\lambda}$ would be amenable to a variational formulation in which the source matter currents would arise due to functional variations of a matter source Lagrangian $L_{(S)}$ with respect to the gauge potentials θ_{λ}^{a}.

3. HOLONOMY BUNDLES

The IHG structure of Riemannian curvature we have described above can be given a formal fibre bundle interpretation in terms of a connection on a reduced subbundle of the orthonormal frame bundle 0(M). The construction of the holonomy bundles, which we shall presently sketch, shows clearly the central importance of the IHG to the structure of space-time curvature. In particular Theorem I above shows that the detailed structure of Riemannian curvature is contained in the IHG structure of space-time. If follows that the reduction of 0(M) to the holonomy bundles, in which for simply connected analytic space-times the IHG plays the role of structure

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(^{8}) Note that the choice $K^{a\lambda} = 0$ implies $\nabla^{[\mu}R^{\lambda]\nu} = 0$ which are the equations given by Yang [23] for « pure spaces » in his integral formalism for gauge fields.

group, is a fundamental step in studying the structure of space-time curvature.

Using the process of parallel transport of fibres of 0(M) based on the Riemannian connection one may define [11, 12] the holonomy group with reference point $u\varepsilon 0(M)$, denoted $\Phi(u) \subseteq O(1, 3)$, and the corresponding reduced principal subbundle P(u). Specifically, for an orthonormal frame $u\varepsilon 0(M)$ at $x = \pi(u)$ where $\pi: 0(M) \to M$ is the projection, $\Phi(u)$ is the subgroup of SO(1, 3) induced by the group of isomorphisms $\Phi(x)$ of $\pi^{-1}(x)$ onto itself induced by parallel tranport along all closed loops at x. If one denotes by P(u) the set of points of 0(M) that can be connected to u by horizontal curves then it can be shown [11] that P(u) is a reduced subbundle of 0(M) with structure group $\Phi(u)$, and the connection in 0(M) is reducible to a connection in P(u). If we further assume that M is simply connected and that (M, g) is analytic, then it is known [11] that $\Phi(x)$ and the IHG at x, denoted $\Phi'(x)$, coincide at all points of M and hence the IHG at x is isomorphic with the structure group of P(u) where $x = \pi(u)$.

Using these reduced principal holonomy bundles with their reduced connections we can pull back the curvature structure equations to the base space M using local gauges (section) of P(u) to obtain local expressions. The special classes of frames referred to prior to equation (8) above are local sections of these bundles ([6]). Hence the structure equations (9) and (10) above are local expressions for the curvature structure of the holonomy bundles.

4. SIMPLE EXAMPLES

It is instructive to consider the above IHG curvature structure in the context of its specialization in terms of Einstein's standard gravitational theory in which matter is coupled to geometry *via* the Einstein field equations $G_{\mu\lambda} = 8\pi T_{\mu\lambda}$. We shall consider two classes of space-times, namely the Einstein vacuum ($T_{\mu\lambda} = 0$) and Einstein-Maxwell space-times.

In a 1961 paper Schell [5] has presented a classification of rotation groups for space-times of signature $(-, +, +, +)$. Schell's classification involves fifteen possible, mutually exclusive classes of rotation groups (including the trivial group which is the IHG of flat space-time). Each rotation group class is given in terms of a canonical basis of generators. We refer the interested reader to Schell's paper for the details.

Schell has further classified solutions of the Einstein vacuum equations [5] $R_{\mu\lambda} = 0$ according to which rotation group classes the IHG of such space-times can belong. He found that the IHG of the Einstein vacuum can belong to only three of the fourteen possible nontrivial rotation group classes corresponding to two-parameter, four-parameter and six-parameter (SO(1, 3)) groups. It is clear that in the Einstein vacuum the equations (11) and (12)

above are specialized by setting $K^{a\lambda} = 0$. Hence, the curvature gauge field structure equations in the Einstein vacuum reduce to

$$(13) \qquad D_\mu{}^*R^{a\lambda\mu} \equiv \nabla_\mu{}^*R^{a\lambda\mu} + 1/2\varepsilon_{bc}{}^a{}^*R^{b\lambda\mu}\theta_\mu^c = 0$$

$$(14) \qquad D_\mu R^{a\lambda\mu} \equiv \nabla_\mu R^{a\lambda\mu} + 1/2\varepsilon_{bc}{}^a R^{b\lambda\mu}\theta_\mu^c = 0.$$

In this sense the Einstein vacuum solutions can be thought of as solutions of the source-free Yang-Mills equation corresponding to the IHG curvature gauge fields ([9]). As a specific example we mention that the two-parameter IHG vacuum solutions are the plane fronted wave solutions [6] and the groups are abelian. For these solutions equations (13) and (14) reduce to $\nabla_\mu{}^*R^{a\lambda\mu} = \nabla_\mu R^{a\lambda\mu} = 0$ with $({}^*R^1, {}^*R^2) = (R^2, -R^1)$ and $R^a = d\theta^a$.

The standard theory of gravitation coupled to source-free electromagnetism is the Einstein-Maxwell theory. Rainich [25] and Misner and Wheeler [26] (RMW) have given a geometrized version of this theory in their already unified field theory. We recall that in the already unified theory RMW give a set of necessary and sufficient conditions that (M, g) must satisfy in order that the space-time be a solution of the source-free Einstein-Maxwell equations. Moreover RMW give a prescription for the construction of the Maxwell fields once these conditions are satisfied. We have studied Einstein-Maxwell space-times using the IHG formalism discussed above together with Schell's classification scheme for rotation group classes. The main result of our analysis, which we shall now briefly sketch (see Norris and Davis [29] for the details), is that we are able to give a more complete geometrical picture of Einstein-Maxwell space-times by identifying the geometrical role of the physical Maxwell fields. For simplicity we discuss only the non-null case.

Using techniques similar to those used by Schell [5] we first classified non-null source-free Einstein-Maxwell space-times according to the possible rotation group classes the IHG of such space-times could belong. It is a tedious but straight forward exercise to show that the IHG of non-null source-free Einstein-Maxwell space-times can only belong to three of the fourteen possible nontrivial rotation group classes corresponding to rotation groups of two, four and six parameters. The two-parameter class differs from the two-parameter class of the vacuum, while the four and six-parameter classes agree with the four and six-parameter classes of the vacuum (there is only one four-parameter and one six-parameter class). Looking

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([9]) It has been shown that any solution of the Einstein vacuum equations is a double self dual solution of the vacuum Yang-Mills type equations [24]. In this connection it should be noted that the IHG results sketched above show the explicit curvature structure of these gauge fields without any need of considering the double self dual nature of the curvature tensor.

at the details of the IHG structure in each of the three possible cases we found that the Maxwell fields $F_{\mu\lambda}$ and $*F_{\mu\lambda}$ could in each case be identified with two of the generators of the corresponding IHG. More precisely the Maxwell field tensors $F_{\mu\lambda}$ and $*F_{\mu\lambda}$ define in each of the three possible cases a two-dimensional (abelian) subalgebra of the Lie algebra of the IHG of the space-time, a result which characterizes the geometrical role played by the physical Maxwell fields.

This result can be made plausible by recalling that, as mentioned above, for $R_\mu^\mu = 0$ space-times the Ricci tensor takes the form

$$(15) \qquad R_{\mu\lambda} = 1/2 R^{ab}(C_{a\nu\mu}C_{b\lambda}{}^\nu + *C_{a\nu\mu}*C_{b\lambda}{}^\nu).$$

When the space-time under consideration is a non-null source-free Einstein-Maxwell space-time the (necessary) RMW algebraic conditions imply the reduction of this quadratic form to the Maxwell stress-energy-momentum tensor and two of the generators of the IHG can be identified with the Maxwell fields $F_{\mu\lambda}$ and $*F_{\mu\lambda}$.

The results of our analysis of the structure of Einstein-Maxwell space-times together with the form (15) of the Ricci tensor for $R_\mu^\mu = 0$ space-times suggests that it might be possible to extend the RMW already unified theory to certain classes of Yang-Mills fields by making use of the IHG formalism [10]. For comparison we note that the Ricci tensor for (massless) Einstein-Yang-Mills space-times takes the general form [19]

$$(16) \qquad R_{\mu\lambda} = \gamma_{AB}\left(F_{\mu\nu}^A F_\lambda^{B\nu} + *F_{\mu\nu}^A *F_\lambda^{B\nu}\right) \qquad A, B = 1, 2, \ldots, N$$

where the $F_{\mu\nu}^A$ are the (in general non-abelian) gauge fields for some N-parameter gauge group and where γ_{AB} is an invariant metric for the group. In any case it appears that the IHG for the given type of geometry for space-time must be at least as large as the gauge group for the type of Yang-Mills field considered if there is to be any possibility of a formulation approaching the demands of complete geometrization.

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[10] In this connection we note that Eguchi [27] has studied the classification of unquantized Yang-Mills fields in terms of their IHG structures following the type of classification used for vacuum gravitational fields mentioned above. In connection with this work see also the more general classification scheme for Yang-Mills fields by Castillejo *et al.* [28]. Yasskin [19] has considered the extension of the already unified field theory of RMW [25, 26] to Yang-Mills fields and finds a problem which indicates that the RMW program cannot be extended. In particular, he notes that a theory of this type would have physically distinct solutions that would correspond to the same geometry. To within the size limitation of the IHG of space-time to accommodate the gauge group of the given Yang-Mills field (as indicated above), we believe that Yasskin's conclusion [19] is not justified in that these physically distinct Yang-Mills fields can be realized as distinct features of the IHG of space-time.

5. CONCLUSIONS

The results and discussion presented above relating to the IHG of Rie-mannian space-times indicate that the IHG structure may be essential to a gauge formulation of the gravitational field. Certainly, considerations of the IHG structure are essential to any type of geometrization program involving the curvature of space-time. We stress that the results and dis-cussion presented above are only a first step in the study of the role of the holonomy group structures in connection with the general unification pro-blem. Indeed, extensions of our discussion of the IHG to the local, homo-geneous and non-homogeneous holonomy groups associated with general connections [4] would be essential. At present, it appears that an investi-gation of these structures in connection with the unification problem should be sufficiently general to include the affine frame bundles [11, 12].

APPENDIX

Equation (12) can be derived from the first contraction of the Bianchi identities $\nabla_{[\sigma}R_{\nu\mu]\lambda}{}^{\kappa} = 0$ by first writing the contracted equation in the form

(A-1)
$$\nabla_{\kappa}R^{\lambda\kappa}{}_{\nu\mu} = 2\nabla_{[\nu}R^{\lambda}_{\mu]}$$

where we have used the symmetry $R_{\nu\mu\lambda\kappa} = R_{\lambda\kappa\nu\mu}$ of the Riemann curvature tensor. Next expand (A-1) using (1) into the form

(A-2)
$$(\nabla_{\kappa}R^{a\lambda\kappa})C_{a\nu\mu} + R^{a\lambda\kappa}(\nabla_{\kappa}C_{a\nu\mu}) = 2\nabla_{[\nu}R^{\lambda}_{\mu]}.$$

Using (8) in the transformation formula for connection coefficients ([11]) one can show that

(A-3)
$$\omega^{i}_{\lambda}\nabla_{\mu}\omega^{\kappa}_{i} = 1/2C_{a\lambda}{}^{\kappa}\theta^{a}_{\mu}$$

where

(A-4)
$$C_{a\lambda}{}^{\kappa} \equiv C_{ai}{}^{h}\omega^{i}_{\lambda}\omega^{\kappa}_{h}$$

and where $\omega_{i} = \omega^{\lambda}_{i}\partial_{\lambda}$ is an orthonormal frame field dual to ω^{i}.

A direct calculation shows, upon making use of (A-3), (A-4) and the commutation relations (2) for the constant matrices $C_{ai}{}^{h}$, that

(A-5)
$$\nabla_{\kappa}C_{a\nu\mu} = g_{\lambda\mu}C_{ai}{}^{h}\nabla_{\kappa}\left(\omega^{i}_{\nu}\omega^{\lambda}_{h}\right) = \left(1/2\varepsilon_{ab}{}^{d}\theta^{b}_{\kappa}\right)C_{d\nu\mu}.$$

Equation (12) will now follow from (A-2) upon substituting (A-5) for $\nabla_{\kappa}C_{a\nu\mu}$ on the left hand side of (A-2).

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