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R. F. ALVAREZ-ESTRADA

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The polaron model revisited: rigorous construction of the dressed electron

by

R. F. ALVAREZ-ESTRADA
Departamento de Física Teórica, Facultad de Ciencias Físicas,
Universidad Complutense, Madrid-3.

RÉSUMÉ. — Le modèle du polaron (c'est-à-dire, un électron non relativiste en interaction avec un champ quantifié de phonons optiques) est étudié du point de vue mathématique. Deux méthodes sont utilisées pour construire rigoureusement l'état d'un électron habillé (c'est-à-dire, le polaron). La première est basée sur une combinaison de transformations d'habillement, la théorie de Brillouin-Wigner et la méthode des approximations successives: elle est valable seulement pour des valeurs assez petites de la constante de couplage. La deuxième méthode est basée sur : *i*) une version modifiée et rigoureuse de la théorie de Tamm-Dancoff, spécialement modifiée pour résoudre les problèmes posés par le modèle du polaron pour des grandes énergies des phonons, *ii*) des méthodes itératives, de Fredholm et de la théorie de la dispersion pour plusieurs particules, combinées avec des théorèmes de point fixe. Il est démontré que cette deuxième méthode est valable pour des valeurs croissantes de la constante de couplage, pour lesquelles la première méthode n'est plus valable, et que, d'autre part, elle fournit des fondaments pour faire des analyses non perturbatives.

ABSTRACT. — The polaron model, which describes a non-relativistic quantum electron, interacting with a quantized optical phonon field, is studied from a mathematical standpoint. Two methods are used in order to construct rigorously the dressed one-electron state (the polaron). The first combines a dressing transformation, the Brillouin-Wigner approach and the contraction-mapping principle, and works only for rather small values of the coupling constant. The second method is based upon: *i*) a modified and rigorous version of the Tamm-Dancoff approach, specially adapted to cope with the peculiarities of the polaron model at large phonon

energies, *ii*) iterative, Fredholm and multiparticle scattering techniques, together withfixed-point theorems. The second approach can be shown to work for increasing values of the coupling constant, when the first method breaks down, and to provide a mathematical basis for non-perturbative studies.

1. INTRODUCTION

The large polaron model describes a slow conduction electron interacting with a quantized optical phonon field in an ionic crystal. References [*1-9*] provide comprehensive accounts of the different approaches, results and open problems in the subject. The polaron model is interesting for, at least, the following reasons: *i*) it describes physical phenomena in threedimensional space, *ii*) it is free of divergences, at the level of perturbation theory, in the one electron subspace, *iii*) it constitutes an excellent testing ground for estimating the convergence of perturbative methods and developing non-perturbative techniques. There is a vast literature about field-theoretic models related more or less to the polaron one, from the standpoint of Mathematical Physics (see [*10-27*]).

However, to the author's knowledge and with the exception of a rather short discussion by Ginibre in [*28*], no rigorous study of the polaron model seems have been reported so far. It turns out that the polaron model is characterized by a phonon energy $\omega(k)$ and a cut-off or vertex function $v(k)$ having a peculiar dependence on the threemomentum k (see section 2), which requires a special treatment.

The purposes of this paper are: 1) to treat rigorously a class of field-theoretic models which includes the standard large-polaron one, and, very specially, 2) to construct mathematically the dressed or renormalized one-electron state. The methods to be used and the corresponding main results are the following.

a) The dressing transformation used by Gross [*24*] and Nelson [*25*] will be combined with the Brillouin-Wigner method (see, for instance, [*29*]) and the contraction-mapping principle (section 3 and Appendices A, B). The main result will be the rigorous determination of the dressed electron state via convergent iteration-perturbation methods, for rather restricted values of the coupling constant.

b) A new method will be presented in sections 4, 5 and Appendix C, which avoids the use of dressing transformations. Its starting point is a variant of the Tamm-Dancoff approach [*30-31*], specially adapted to cope with the energy $\omega(k)$ and vertex function $v(k)$ characterizing the models under consideration (section 4). One of the main results is the rigorous construction of the dressed electron state for a limited range of values for the coupling constant (subsection 5.A), through a new convergent perturbation-iteration method which is different from the one presented before in section 3. In the case of the standard large polaron model, a numerical analysis indicates that convergence now holds for values of the coupling constant which may be one to two orders of magnitude larger than the upper limit obtained in section 3 for convergence using dressing transformations. Another main result (subsections 5.B and 5.C) is the rigorous determination of the dressed electron state for increasing values of the coupling constant (actually larger than the ones previously considered in subsection 5.A), by combining perturbation-iteration, Fredholm and multi-particle scattering methods. This allows for the possibility of exploring, in a systematic and mathematically controllable way, the non-perturbative regime of values for the coupling constant. A rather qualitative numerical estimate, which illustrates the last statement, is also presented.

2. CHARACTERIZATION OF MODELS

We consider a non-relativistic spinless quantum electron with bare mass m_0 and position and threemomentum operators $\bar{x} = (x_l)$, $\bar{p} = (p_j)$, $l, j = 1, 2, 3$ ($[x_l, p_j] = i\delta_{lj}$), and an indefinite number of longitudinal optical phonons, regarded as scalar bosons. Let $|0\rangle$ be the phonon vacuum and $a(\bar{k})$, $a^+(\bar{k})$ be the standard destruction and creation operators for a phonon with (continuously-varying) threemomentum $\bar{k}$ and energy $\omega(k)$ ($[a(\bar{k}), a^+(\bar{k}')] = \delta^{(3)}(\bar{k} - \bar{k}')$, $k = |\bar{k}|$). The bare one-electron state with threemomentum $\bar{q}$ is denoted by $\psi(\bar{q})$, so that the set of all bare electron-phonon states

$$\psi(\bar{q}; \bar{k}_1 \dots \bar{k}_n) = \psi(\bar{q}) \otimes \left[\frac{1}{(n!)^{1/2}} a^+(\bar{k}_1) \dots a^+(\bar{k}_n) |0\rangle \right] \tag{2.1}$$

constitutes a basis for the actual Hilbert space.

By assumption, the total hamiltonian is:

$$H = H_0 + H_1, \qquad H_1 = \frac{\bar{p}^2}{2m_0} + \int d^3\bar{k}\omega(k) a^+(\bar{k}) a(\bar{k}) \tag{2.2}$$

$$H_1 = f \int d^3\bar{k} \, [v(k) a(\bar{k}) \exp(i\bar{k}.\bar{x}) + v(k)^* a^+(\bar{k}) \exp(-i\bar{k}.\bar{x})] \tag{2.3}$$

The total threemomentum operator is

$$\bar{P}_{\text{tot}} = \bar{p} + \int d^3\bar{k} . \bar{k} a^+(\bar{k}) a(\bar{k}), \qquad ([H, \bar{P}_{\text{tot}}] = 0)$$

f and $v(k)$ are, respectively, a real coupling constant and a complex cut-off function. They satisfy the following assumptions:

1) $\omega(k) \geqslant \omega_0 > 0$,

2) $\lim_{k \to +\infty} \frac{\omega(k)}{k^{1/2}}$ either vanishes or is a finite constant,

3) $\int_{k \leqslant k_0} d^3\bar{k} \mid v(k) \mid^n < +\infty$, for any $k_0 < +\infty$ and $n = 1, 2$,

4) $v(k) \underset{k \to +\infty}{\to} \frac{v_0}{k}$, v_0 being a non-vanishing complex constant.

The above assumptions 1)-4) define a slight generalization of the physically interesting large polaron model, characterized by $\omega(k) = \omega_0 > 0$ and $v(k) = \frac{i}{k}$ for any phonon threemomentum $\bar{k}$, which clearly satisfies them. We shall apply all our latter developments to the large polaron model, with $f = \frac{[2^{1/2}\alpha]^{1/2}}{2\pi} \cdot \left[\frac{\omega_0^3}{m_0}\right]^{1/4}$, α being the usual dimensionless coupling constant [32].

Let $\mathcal{H}_{\bar{\pi}}$ be the subspace of all kets ψ such that $(\bar{\mathrm{P}}_{\mathrm{tot}} - \bar{\pi})\psi = 0$ with $\frac{\bar{\pi}^2}{2m_0} < \omega_0$, in order to avoid unwanted « Cerenkov effects » [8].

The set of all bare states $\psi(\bar{q}_{\bar{\pi}}; \bar{k}_1, \ldots \bar{k}_n)$, $\bar{q}_{\bar{\pi}} = \bar{\pi} - \sum_{j=1}^{n} \bar{k}_j$, constitutes a complete orthonormal set in $\mathcal{H}_{\bar{\pi}}$ with respect to the restricted scalar product:

$$(\psi(\bar{q}'_{\bar{\pi}}; \bar{k}'_1 \ldots \bar{k}'_{n'}), \psi(\bar{q}_{\bar{\pi}}; \bar{k}_1 \ldots \bar{k}_n))_{\bar{\pi}} = \\ = \frac{\delta_{nn'}}{n!} \sum_{\nu(1),\ldots,\nu(n)} \delta^{(3)}(\bar{k}'_{\nu(1)} - \bar{k}_1) \ldots \delta^{(3)}(\bar{k}'_{\nu(n)} - \bar{k}_n) \quad (2.4)$$

In Eq. (2.4), $\sum_{\nu(1),\ldots,\nu(n)}$ denotes the usual sum over the $n!$ permutations $(\nu(1), \ldots, \nu(n))$ of $(1, \ldots, n)$.

Notice that the ordinary scalar product of the same states in the full Hilbert space equals the restricted one, given in Eq. (2.4), times a volume divergent factor, namely, $\delta^{(3)}(\bar{0})$.

In $\mathcal{H}_{\bar{\pi}}$, the restricted norms of a state ψ belonging to it and an operator A such that $[\mathrm{A}, \bar{\mathrm{P}}_{\mathrm{tot}}] = 0$ are defined respectively as $\| \psi \|_{\bar{\pi}} = [(\psi, \psi)_{\bar{\pi}}]^{1/2}$ and $\| \mathrm{A} \|_{\bar{\pi}} =$ least upper bound of $(\| \mathrm{A}\Phi \|_{\bar{\pi}} / \| \Phi \|_{\bar{\pi}})$, as Φ varies throughout $\mathcal{H}_{\bar{\pi}}$.

Let $\psi_+ = \psi_+(\bar{\pi})$ the ground state of H (the dressed electron or polaron) with physical energy $\mathrm{E} = \mathrm{E}(\bar{\pi})$ and small total threemomentum $\bar{\pi}$:

$$(\mathrm{H} - \mathrm{E})\psi_+ = 0, (\bar{\mathrm{P}}_{\mathrm{tot}} - \bar{\pi})\psi_+ = 0, \frac{\bar{\pi}^2}{2m_0} < \omega_0 .$$

Both ψ_+ and E are finite in each order of perturbation theory, as explicit calculations, patterned after those in [4] [8] [33], show. Explicitly, for the large-polaron model it has been conjectured that perturbation theory converges for $\alpha \lesssim 0.5$ (see D. Pines in [3]). It is not straightforward to prove the convergence of the whole perturbation series for ψ_+, E, even for very small α or f, and, to author's knowledge, no such proof has been published so far. Actually, the standard arguments (see, for instance, [23]) implying that H_I is a small perturbation of H_0 and the usual convergence proof for the Neumann expansion of $(z - H)^{-1}$ or ψ_+, for very small f, all run into difficulties since $\int d^3\bar{k} \mid v(k) \mid^2 = \infty$. Thus, a rigorous determination of ψ_+, E and estimates of the values of f for which convergence holds seem desirable.

3. CONSTRUCTION OF DRESSED ELECTRON USING DRESSING TRANSFORMATION AND BRILLOUIN-WIGNER APPROACH

We perform the dressing transformation implemented by the unitary operator exp T, where

$$T = \int d^3\bar{k}\,[\beta(k)a(\bar{k})\exp(i\bar{k}\bar{x}) - \beta(k)^* a^+(\bar{k})\exp(-i\bar{k}\bar{x})] \tag{3.1}$$

$$\beta(k) = -\frac{f[1 - \theta(\Lambda - k)]v(k)}{\omega(k) + (k^2/2m_0)} \tag{3.2}$$

Λ being a non-negative fixed constant and $\theta(x) = 1$ if $x > 0$, $\theta(x) = 0$ for $x < 0$.

A lengthy calculation, similar to those in [24-25], yields:

$$[\exp T].H.[\exp(-T)] = H' + E', \quad E' = -f^2.\int d^3\bar{k}\frac{\mid v(k)\mid^2[1 - \theta(\Lambda - k)]}{\omega(k) + (k^2/2m_0)} \tag{3.3}$$

$$H' = H_0 + H'_{I,1} + H'_{I,2}, \quad [\exp T].\bar{P}_{tot}.[\exp(-T)] = \bar{P}_{tot} \tag{3.4}$$

$$H'_{I,1} = f\int d^3\bar{k}\theta(\Lambda - k)[v(k)a(\bar{k})\exp(i\bar{k}\bar{x}) + v(k)^* a^+(\bar{k})\exp(-i\bar{k}\bar{x})] \tag{3.5}$$

$$H'_{I,2} = \frac{1}{2m_0}[\bar{A}^2 + (\bar{A}^+)^2 + 2\bar{A}^+\bar{A} + 2(\bar{p}\bar{A} + \bar{A}^+\bar{p})] \tag{3.6}$$

$$\bar{A} = -\int d^3\bar{k}\beta(k).\bar{k}a(\bar{k})\exp(i\bar{k}\bar{x}) \tag{3.7}$$

Notice that $-\infty < E' < 0$. The usefulness of this type of transformation, in order to renormalize a field-theoretic model more singular than the one studied here, was established in [*24-25*]. Its potential usefulness for the polaron model was pointed out by Ginibre in [*28*].

The new hamiltonian H′ can be given a mathematical sense, by extending to it the rigorous analysis of Nelson [*25*] directly. Appendix A contains some rigorous results which will be useful later in this work.

After the dressing transformation, the dressed electron state is

$$\psi'_+ = \psi'_+(\bar{\pi}) = [\exp \mathrm{T}] . \psi_+(\bar{\pi}) .$$

It belongs to $\mathscr{H}_{\bar{\pi}}$ and fulfills $[\mathrm{H}' - (\mathrm{E} - \mathrm{E}')]\psi'_+ = 0$. We shall impose the normalization $(\psi(\bar{\pi}), \psi'_+)_{\bar{\pi}} = 1$ and try to construct ψ'_+ from $\psi(\bar{\pi})$, by regarding the latter and all $\psi(\bar{q}_{\bar{\pi}}; \bar{k}_1 \dots \bar{k}_n)$ as unperturbed kets and $\mathrm{H}'_{\mathrm{I},1} + \mathrm{H}'_{\mathrm{I},2}$ as perturbation.

One could construct $(z - \mathrm{H}')^{-1}$ and ψ'_+ by using the perturbation theory for quadratic forms and the projection techniques presented in [*34*] (see also [*26*]). However, it is easier to use the Brillouin-Wigner approach [*29*], which leads directly to:

$$\psi'_+ = \psi(\bar{\pi}) + (\mathbb{1} - \mathrm{Q}_{\bar{\pi}})\mathrm{G}_0(\mathrm{E} - \mathrm{E}')(\mathrm{H}'_{\mathrm{I},1} + \mathrm{H}'_{\mathrm{I},2})\psi'_+ \tag{3.8}$$

$$\mathrm{E} = \mathrm{E}' + \frac{\bar{\pi}^2}{2m_0} + (\psi(\bar{\pi}), (\mathrm{H}'_{\mathrm{I},1} + \mathrm{H}'_{\mathrm{I},2})\psi'_+)_{\bar{\pi}} \equiv \mathrm{M}(\mathrm{E}) \tag{3.9}$$

Here, $\mathbb{1}$ is the unit operator, $\mathrm{Q}_{\bar{\pi}}$ is the projector upon $\psi(\bar{\pi})$ inside $\mathscr{H}_{\bar{\pi}}$ and $\mathrm{G}_0(z) = (z - \mathrm{H}_0)^{-1}$.

The basic polycy will consist in: *i*) solving the linear Eq. (3.8) for ψ'_+, regarding E as a parameter, *ii*) plugging the resulting solution for ψ'_+ into the right-hand-side of Eq. (3.9) and solving for E.

We introduce

$$\psi_0(\mathrm{E}) = (\mathbb{1} - \mathrm{Q}_{\bar{\pi}})[-\mathrm{G}_0(\mathrm{E} - \mathrm{E}')]^{1/2}.(\mathrm{H}'_{\mathrm{I},1} + \mathrm{H}'_{\mathrm{I},2}).\psi(\bar{\pi}) \tag{3.10}$$

$$g_1(\mathrm{E}_1, \mathrm{E}_2) = (\mathbb{1} - \mathrm{Q}_{\bar{\pi}})[-\mathrm{G}_0(\mathrm{E}_1 - \mathrm{E}')]^{1/2} .(\mathrm{H}'_{\mathrm{I},1} + \mathrm{H}'_{\mathrm{I},2})[-\mathrm{G}_0(\mathrm{E}_2 - \mathrm{E}')]^{1/2}(\mathbb{1} - \mathrm{Q}_{\bar{\pi}}) \tag{3.11}$$

$$g_2(\mathrm{E}_1, \mathrm{E}_2) = (\mathbb{1} - \mathrm{Q}_{\bar{\pi}})[-\mathrm{G}_0(\mathrm{E}_1 - \mathrm{E}')]^{1/2} .[-\mathrm{G}_0(\mathrm{E}_2 - \mathrm{E}')]^{1/2}.(\mathbb{1} - \mathrm{Q}_{\bar{\pi}}) \tag{3.12}$$

The series formed by all successive iterations of Eq. (3.8) can be cast into:

$$\psi'_+ = \psi(\bar{\pi}) - (\mathbb{1} - \mathrm{Q}_{\bar{\pi}})[-\mathrm{G}_0(\mathrm{E} - \mathrm{E}')]^{1/2}.\sum_{l=0}^{\infty}[-g_1(\mathrm{E}, \mathrm{E})]^l\psi_0(\mathrm{E}) \tag{3.13}$$

Majorizing Eqs. (3.13) and (3.10), one finds

$$\|\psi'_+\| \leqslant 1 + \|(\mathbb{1} - \mathrm{Q}_{\bar{\pi}})[-\mathrm{G}_0(\mathrm{E} - \mathrm{E}')]^{1/2}\|_{\bar{\pi}} . \frac{\|\psi_0(\mathrm{E})\|_{\bar{\pi}}}{1 - \|g_1(\mathrm{E}, \mathrm{E})\|_{\bar{\pi}}} \tag{3.14}$$

$$\|\psi_0(\mathrm{E})\|_{\bar{\pi}} \leqslant \frac{1}{|\omega_0 + \mathrm{E}' - \mathrm{E}|^{1/2}} . [\lambda_1 + |\bar{\pi}| . x_1^{1/2}] + \frac{m_0 . x_1}{[2|2\omega_0 + \mathrm{E}' - \mathrm{E}|]^{1/2}} \tag{3.15}$$

with $x_1 = \int d^3\bar{k} \frac{k^2 |\beta(k)|^2}{m_0^2}$, and λ_1 being given in Eq. (A.3).

Next, we consider the mapping $\mathrm{M}: \mathrm{E} \to \mathrm{M(E)}$, where M(E) is given by the right-hand-side of Eq. (3.9), as well as two values E_i, $i = 1, 2$, and their associates $\mathrm{M(E}_i)$ under the mapping. Some majorations yield:

$$\left| \mathrm{M(E)} - \mathrm{E}' - \frac{\bar{\pi}^2}{2m_0} \right| \leqslant \frac{[\, \| \psi_0(\mathrm{E}) \|_{\bar{\pi}}]^2}{1 - \| g_1(\mathrm{E}, \mathrm{E}) \|_{\bar{\pi}}} \tag{3.16}$$

$$| \mathrm{M(E_1)} - \mathrm{M(E_2)} | \leqslant \eta_1(\mathrm{E_1}, \mathrm{E_2}) . | \mathrm{E_1} - \mathrm{E_2} | \tag{3.17}$$

$$\begin{aligned} \eta_1(\mathrm{E_1}, \mathrm{E_2}) &= \| g_2(\mathrm{E_1}, \mathrm{E_2}) \|_{\bar{\pi}} \\ &\cdot \left\{ \| \psi_0(\mathrm{E_1}) \|_{\bar{\pi}} . \left[\| \psi_0(\mathrm{E_2}) \|_{\bar{\pi}} + \frac{\| g_1(\mathrm{E_2}, \mathrm{E_1}) \|_{\bar{\pi}} . \| \psi_0(\mathrm{E_1}) \|_{\bar{\pi}}}{1 - \| g_1(\mathrm{E_1}, \mathrm{E_1}) \|_{\bar{\pi}}} \right] + \| \psi_0(\mathrm{E_2}) \|_{\bar{\pi}} \right. \\ &\left. \cdot \left[\frac{\| g_1(\mathrm{E_2}, \mathrm{E_1}) \|_{\bar{\pi}} . \| \psi_0(\mathrm{E_2}) \|_{\bar{\pi}}}{1 - \| g_1(\mathrm{E_2}, \mathrm{E_2}) \|_{\bar{\pi}}} + \frac{(\| g_1(\mathrm{E_2}, \mathrm{E_1}) \|_{\bar{\pi}})^2 \| \psi_0(\mathrm{E_1}) \|_{\bar{\pi}}}{[1 - \| g_1(\mathrm{E_1}, \mathrm{E_1}) \|_{\bar{\pi}}][1 - \| g_1(\mathrm{E_2}, \mathrm{E_2}) \|_{\bar{\pi}}]} \right] \right\} \end{aligned} \tag{3.18}$$

One sees easily that for any complex E, $\mathrm{E_1}$, $\mathrm{E_2}$ and any real E, $\mathrm{E_1}$, $\mathrm{E_2}$ smaller than ω_0:

i) $\| \psi_0(x) \|_{\bar{\pi}} < + \infty$, $x = \mathrm{E}, \mathrm{E_1}, \mathrm{E_2}$

ii) $\| (\mathbb{1} - \mathrm{Q}_{\bar{\pi}})[- \mathrm{G_0}(\mathrm{E} - \mathrm{E}')]^{1/2} \|_{\bar{\pi}} < + \infty$, $\| g_2(\mathrm{E_1}, \mathrm{E_2}) \|_{\bar{\pi}} < + \infty$

due to the projector $\mathbb{1} - \mathrm{Q}_{\bar{\pi}}$. Moreover, using results from Appendix A, we give a bound for $\| g_1(\mathrm{E_1}, \mathrm{E_2}) \|_{\bar{\pi}}$ in Appendix B (Eq. (B.2)), which shows that $\| g_1(\mathrm{E_1}, \mathrm{E_2}) \|_{\bar{\pi}} < + \infty$ for fixed Λ, under the same conditions for $\mathrm{E_1}$, $\mathrm{E_2}$.

Then, for sufficiently small f, $| \bar{\pi} |$: *a)* there is a domain D in the complex E-plane, containing $\frac{\bar{\pi}^2}{2m_0}$ and $\frac{\bar{\pi}^2}{2m_0} + \mathrm{E}'$, which maps into itself under M, *b)* $\| g_1(\mathrm{E}, \mathrm{E}) \|_{\bar{\pi}} < 1$ for E inside D, *c)* $\eta_1(\mathrm{E_1}, \mathrm{E_2}) < 1$ for $\mathrm{E_1}$, $\mathrm{E_2}$ belonging to D. Then, the contraction mapping principle (see, for instance, [*35*]) ensures the existence of a unique fixed point E of M, E = M(E), which belongs to D and can be found by successive iterations of Eq. (3.9). Moreover, for the fixed point, ψ'_+ is given by the convergent series (3.13). The analysis of Nelson [*25*] shows that exp ($\pm$ T) are well defined unitary operators. Then, the true polaron state is unambiguously given by

$$\psi_+ = [\exp(- \mathrm{T})] . \psi'_+ .$$

Let us summarize some qualitative estimates of the convergence conditions for the contraction mapping principle to apply, in the case of the standard large polaron model (see section 2) for $\bar{\pi} = 0$. One has:

$$\lambda_1 = \left(\frac{2^{1/2} \alpha \Lambda}{\pi} \right)^{1/2} . \left(\frac{\omega_0^3}{m_0} \right)^{1/4} \tag{3.19}$$

$$\begin{aligned} x_1 &= \frac{2 . 2^{1/2} \alpha}{\pi} . \left(\frac{\omega_0^3}{m_0} \right)^{1/2} \\ &\cdot \left[\frac{1}{(2m_0\omega_0)^{1/2}} \left(\frac{\pi}{2} - \mathrm{arctn} \frac{\Lambda}{(2m_0\omega_0)^{1/2}} \right) + \frac{\Lambda}{2m_0\omega_0 + \Lambda^2} \right] \end{aligned} \tag{3.20}$$

We have found that the condition $\| g_1(E, E) \|_{\underline{\pi}} < 1$ is roughly fulfilled when $\Lambda = 10(m_0\omega_0)^{1/2}$ for values of α up to about 0.009, which is almost the order of magnitude which characterizes semiconductors of type II-VI [6]. The condition $\eta_1 < 1$ is also essentially satisfied under the same conditions. This upper limit for convergence, $\alpha \lesssim 0.009$, is about one to two orders of magnitude smaller than the one previously conjectured (see D. Pines in [3]): its smallness is due to the Λ-dependence of λ_1, x_1 which, in turn, comes from the dressing transformation. The new method to be presented in the following sections will not rely on dressing transformations and will allow one to improve the convergence conditions, at least in principle.

The actual treatment can be generalized when an external homogeneous magnetic field $\bar{h} = (0, 0, h)$, $h > 0$, along the x_3-axis is present. Let $\bar{A}_h = (-hx_2, 0, 0)$, e, c be the standard vector potential (see, for instance, [36]), the electron electric charge and the velocity of light in vacuum. Then, if the sustitution $\bar{p} \to \bar{p} - \frac{e}{c}\bar{A}_h$ is done, Eqs. (2.2-3) remain valid. Notice that the two operators $P'_{tot,j} = p_j + \int d^3\bar{k}\,.\,k_j a^+(\bar{k})a(\bar{k})$, $j = 1, 3$ conmute with the actual H, but $p_2 + \int d^3\bar{k}\,.\,k_2 a^+(\bar{k})a(\bar{k})$ does not. Now, the basic bare states are $(r = 0, 1, 2, \ldots)$

$$\psi(\underline{q}; r; \bar{k}_1 \ldots \bar{k}_n) = \exp\left[i(q_1x_1 + q_3x_3)\right] . \frac{H_r\left[\left(\frac{|e|h}{c}\right)^{1/2}(x_2 - x_{2,0})\right] . \left(\frac{|e|h}{c}\right)^{1/4}}{(\pi^{1/2}2^r . r!)^{1/2}}$$
$$. \exp\left[-\frac{|e|h}{2c}(x_2 - x_{2,0})^2\right] \otimes \left[\frac{1}{(n!)^{1/2}} a^+(\bar{k}_1) \ldots a^+(\bar{k}_n) | 0 \rangle\right] \quad (3.21)$$

$x_{2,0} = -\frac{c . q_1}{|e| . h}$, $\underline{q} = (q_1, q_3)$ and the H_r's being the standard Hermite polynomials. Let $\mathscr{H}(\underline{\pi})$ be the subspace of all states ψ such that

$$(P'_{tot,j} - \pi_j)\psi = 0\,, \qquad j = 1, 3\,.$$

The set of all bare states

$$\psi(\underline{q}_{\underline{\pi}}; r; \bar{k}_1 \ldots \bar{k}_n)\,, \qquad \underline{q}_{\underline{\pi}} = \underline{\pi} - \sum_{i=1}^{n} \underline{k}_i\,, \qquad \underline{k} = (k_1, k_3)$$

is a complete set for $\mathscr{H}(\underline{\pi})$ with respect to the new restricted scalar product:

$$(\psi(\underline{q}'_{\underline{\pi}}; r'; \bar{k}'_1 \ldots \bar{k}'_{n'}), \psi(\underline{q}_{\underline{\pi}}; r; \bar{k}_1 \ldots \bar{k}_n))_{\underline{\pi}}$$
$$= \frac{\delta_{r'r}\delta_{n'n}}{n!} \sum_{\nu(1),\ldots,\nu(n)} \delta^{(3)}(\bar{k}'_{\nu(1)} - \bar{k}_1) \ldots \delta^{(3)}(\bar{k}'_{\nu(n)} - \bar{k}_n) \quad (3.22)$$

Eqs. (3.1-3), (3.5-7) and the first Eq. (3.4) remain valid provided that $\bar{p} \to \bar{p} - \frac{e}{c}\bar{A}_h$, while the second equation (3.4) should be replaced by

$$[\exp T].P'_{tot,j}.\ [\exp(-T)] = P'_{tot,j}, \qquad j = 1, 3.$$

By using $\bar{p} \to \bar{p} - \frac{e}{c}\bar{A}_h$, as well as the restricted norms for vectors and operators and the quadratic form (analogous to F of Appendix A) induced by (3.22) and the new H′, one can show that the analogue of (A.1) holds, with the same e_1, e_2. Let $\psi(\bar{\pi})$, $\frac{\bar{\pi}^2}{2m_0}$ in Eqs. (3.8-9) be replaced respectively by the ground state and the energy of the electron in the external magnetic field, namely, $\psi(\underline{\pi}\,;0)$, $\frac{1}{2}\cdot\frac{|e|h}{c} + \frac{(\pi_3)^2}{2m_0}$. After these substitutions, the Brillouin-Wigner equations remain valid and determine the ground state ψ_+ and the energy E of the dressed electron in the external magnetic field. The applicability of the contraction mapping principle and the convergence proofs can be established as we did before in this section, when $h = 0$. The polaron model in presence of an external magnetic field has been studied previously by several authors [*37-38*]: our brief discussion above tried to provide a rigorous justification of those works (at least, for the ground state).

4. MODIFIED TAMM-DANCOFF APPROACH AND A USEFUL BOUND

The polaron state ψ_+ can be expanded into bare states as:

$$\psi_+ = \sum_{n=0}^{+\infty}\int d^3\bar{k}_1 \ldots d^3\bar{k}_n \frac{b_n(\bar{k}_1 \ldots \bar{k}_n)}{|e_n(\bar{k}_1 \ldots \bar{k}_n)|^{1/2}}\psi(\bar{q}_{\bar{\pi}}\,;\bar{k}_1 \ldots \bar{k}_n) \tag{4.1}$$

$$e_n(\bar{k}_1 \ldots \bar{k}_n) = E - \sum_{i=1}^{n}\omega(k_i) - \frac{\left(\bar{\pi} - \sum_{i=1}^{n}\bar{k}_i\right)^2}{2m_0},$$

$$\|\psi_+\|_{\bar{\pi}}^2 = \sum_{n=0}^{+\infty}\int d^3\bar{k}_1 \ldots d^3\bar{k}_n \frac{|b_n(\bar{k}_1 \ldots \bar{k}_n)|^2}{|e_n(\bar{k}_1 \ldots \bar{k}_n)|} \tag{4.2}$$

Thus, $b_n/|e_n|^{1/2}$ is the unnormalized probability amplitude for finding n

phonons in the polaron, and is symmetric under interchanges of $\bar{k}$'s. The interest of having factored out $|e_n|^{-1/2}$ will be appreciated shortly. Upon replacing the expansion (4.1) into $(H - E)\psi_+ = 0$ and using Eqs. (2.2-4), one finds the basic recurrence relations ($e_n^{1/2} = e_n/|e_n|^{1/2}$) :

$$b_n(\bar{k}_1 \dots \bar{k}_n) = \frac{1}{e_n(\bar{k}_1 \dots \bar{k}_n)^{1/2}} \left\{ \frac{f}{n^{1/2}} \cdot \sum_{i=1}^{n} v(k_i)^* \cdot \frac{b_{n-1}(\bar{k}_1 \dots \bar{k}_{i-1}\bar{k}_{i+1} \dots \bar{k}_n)}{|e_{n-1}(\bar{k}_1 \dots \bar{k}_{i-1}\bar{k}_{i+1} \dots \bar{k}_n)|^{1/2}} + f(n+1)^{1/2} \int d^3\bar{k}\, \frac{v(k) b_{n+1}(\bar{k}\bar{k}_1 \dots \bar{k}_n)}{|e_{n+1}(\bar{k}\bar{k}_1 \dots \bar{k}_n)|^{1/2}} \right\} \tag{4.3}$$

for $n = 0, 1, 2, 3, \dots$, with $b_{-1} = 0$. We shall choose the normalization $b_0/|e_0|^{1/2} = 1$ and introduce, for later convenience:

$$B_1 = \begin{pmatrix} b_1(\bar{k}_1) \\ b_2(\bar{k}_1, \bar{k}_2) \\ \vdots \\ b_n(\bar{k}_1 \dots \bar{k}_n) \\ \vdots \end{pmatrix}, \qquad B_1^{(0)} = \begin{pmatrix} \frac{f\, v(k_1)^*}{e_1(\bar{k}_1)^{1/2}} \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \tag{4.4}$$

Then, Eq. (4.3) for $n = 0$ and the set of all Eqs. (4.3) for $n \geqslant 1$ become respectively

$$E = \frac{\bar{\pi}^2}{2m_0} + f \int d^3\bar{k}_1 \frac{v(k_1) b_1(\bar{k}_1)}{|e_1(\bar{k}_1)|^{1/2}} = M_1(E) \tag{4.5}$$

$$B_1 = B_1^{(0)} + W_1 . B_1 \tag{4.6}$$

Here, W_1 is the linear operator defined by the right-hand-side of all Eqs. (4.3) for $n = 1, 2, 3, \dots$

We shall introduce

$$\tau_1 = \left[f^2 \int \frac{d^3\bar{k}\, |v(k)|^2}{|e_1(\bar{k})|} \right]^{1/2},$$

$$\tau_n = \left[\operatorname*{Max}_{\bar{k}_1,\dots,\bar{k}_{n-1}} \frac{f^2 n}{|e_{n-1}(\bar{k}_1 \dots \bar{k}_{n-1})|} \int \frac{d^3\bar{k}\, |v(k)|^2}{|e_n(\bar{k}\bar{k}_1 \dots \bar{k}_n)|} \right]^{1/2} \quad n \geqslant 2 \tag{4.7}$$

as well as the L^2-norm of b_n:

$$\| b_n \|_2 = \left[\int d^3\bar{k}_1 \dots d^3\bar{k}_n \, | b_n(\bar{k}_1 \dots \bar{k}_n) |^2 \right]^{1/2}$$

(no confusion should arise between the notations $\| b_n \|_2$ and, say, $\| \psi \|_{\bar{\pi}}$) and the following continued fractions :

$$Z_r = \cfrac{1}{1 - \cfrac{|\tau_{r+1}|^2}{1 - \cfrac{|\tau_{r+2}|^2}{1 - \cfrac{|\tau_{r+3}|^2}{1 - \cdots}}}} \tag{4.8}$$

By recalling Eq. (4.7), the first Eq. (4.2) and assumptions 1) - 4) in section 2, one sees that, at least for small $|\bar{\pi}|$ and $|E|$: *i)* $\tau_n < +\infty$ for $n \geqslant 1$, *ii)* $\tau_n \to 0$ as $n \to \infty$, *iii)* there exists some $R > 0$ such that $Z_r > 0$ for any $r \geqslant R$.

For $n \geqslant R$, one proves the following bound :

$$\| b_n \|_2 \leqslant \tau_n . Z_n . \| b_{n-1} \|_2 \tag{4.9}$$

The inequality (4.9) can be proved in two alternative ways :

1) By integrating both sides of Eq. (4.3) over $\bar{k}_1 \ldots \bar{k}_n$ and using

$$\left[\int d^3\bar{k}_1 \ldots d^3\bar{k}_n \left| \frac{f}{e_n(\bar{k}_1 \ldots \bar{k}_n)^{1/2} . n^{1/2}} \sum_{i=1}^{n} v(k_i)^* \frac{b_{n-1}(\bar{k}_1 \ldots \bar{k}_{i-1}\bar{k}_{i+1} \ldots \bar{k}_n)}{|e_{n-1}(\bar{k}_1 \ldots \bar{k}_{i-1}\bar{k}_{i+1} \ldots \bar{k}_n)|^{1/2}} \right|^2 \right]^{1/2} \leqslant \tau_n . \| b_{n-1} \|_2 \tag{4.10}$$

$$\left[\int d^3\bar{k}_1 \ldots d^3\bar{k}_n \left| \frac{f(n+1)^{1/2}}{e_n(\bar{k}_1 \ldots \bar{k}_n)^{1/2}} \int d^3\bar{k}\, v(k) \frac{b_{n+1}(\bar{k}\bar{k}_1 \ldots \bar{k}_n)}{|e_{n+1}(\bar{k}\bar{k}_1 \ldots \bar{k}_n)|^{1/2}} \right|^2 \right]^{1/2} \leqslant \tau_{n+1} . \| b_{n+1} \|_2 \tag{4.11}$$

one derives the three-term recurrence of inequalities:

$$\| b_n \|_2 \leqslant \tau_n . \| b_{n-1} \|_2 + \tau_{n+1} . \| b_{n+1} \|_2 \tag{4.12}$$

The solution of the recurrence of inequalities (4.12) is outlined in Appendix C. By setting $\tau'_{0,n} = 0$, $\tau'_{1,n} \to \tau_n$, $Z'_n \to Z_n$, $X_n \to \| b_n \|_2$ in (C.3), one obtains (4.9).

2) Consider the set of all Eqs. (4.3) for $n \geqslant R > 0$, cast it into a matrix form similar to Eq. (4.6), containing b_{R-1} in the inhomogeneous term instead of $f v(k_1)^* e_1(\bar{k}_1)^{1/2}$, and iterate the corresponding matrix system, thereby generating an infinite series for b_n, $n \geqslant R$. By taking the L^2-norm

of the series for b_n and majorizing as in alternative 1), one finds a majorizing series for $\| b_n \|_2$, which can be summed into the continued fraction Z_n in (4.9), that is, it coincides with the series obtained when Z_n is expanded into power series in all $| \tau_{r+1} |^2$, $r \geqslant n$. For brevity, we omit details.

Notice that to have factored out $| e_n |^{-1/2}$ in the probability amplitudes has turned out to be crucial for the validity of (4.9). In fact, it leads to the inequalities (4.10-11) with τ_n, τ_{n+1} which, in turn, contain $| e_1 |^{-1}$, $| e_{n+1} |^{-1}$ inside the integrals and, hence, are finite in spite of assumption 4) in section 2 (and so on if the derivation of (4.9) proceeds through alternative 2)). We have been unable to derive a recurrence or a bound similar to (4.12) or (4.9) respectively for the full probability amplitudes $b_n / | e_n |^{1/2}$ due precisely to assumption 4) and the absence of factors $| e_n |^{-1}$ inside certain integrals analogous to τ_n, τ_{n+1}, which now diverge.

Remarks. — *i*) A Tamm-Dancoff approach to the dressed electron state was also proposed by Larsen in an interesting paper [*39*]. However, he did not factor out $| e_n |^{-1/2}$ and he did not present any bound or rigorous study.

ii) The Tamm-Dancoff approach is related to the so-called N-quantum approximation: accounts of the latter appear in [*40-41*].

5. CONSTRUCTION OF DRESSED ELECTRON STATE IN MODIFIED TAMM-DANCOFF APPROACH

We shall construct rigorously the amplitudes b_n, $n \geq 1$, and the polaron energy E in several cases, for successively increasing values of f.

5.A. R = 1.

We assume that $Z_r > 0$ for any $r \geqslant 1$, for given small values of f, in a certain domain D_1 for E. The bound (4.9) for $r \geqslant 2$ together with $\| b_1 \|_2 \leqslant \tau_1 . Z_1$ (which can be proved similarly, as in alternatives 1) or 2) in section 4) ensure that the series for all b_n's, $n \geqslant 1$, obtained by successive iterations of the system (4.6) converges in L^2-norm: $\| b_n \|_2 < \infty$, $n \geqslant 1$. Here, we shall make the E-dependence of b_n's and τ's explicit, frequently. The direct majoration of $M_1(E)$ (Eq. (4.5)) gives:

$$\left| M_1(E) - \frac{\bar{\pi}^2}{2m_0} \right| \leqslant \tau_1^2 . Z_1 \qquad (5.A.1)$$

Next, we shall study the mapping $M_1 : E \rightarrow M_1(E)$. We consider the set of all Eqs. (4.3) for $n \geqslant 1$ (or the system (4.6)) and Eq. (4.5) for two values E_1, E_2 such that $Z_r(E_i) > 0$, $r \geqslant 1$, $i = 1, 2$, and subtract them. By majorizing as in alternative 1) of section 4, one finds:

$$|M_1(E_1) - M_1(E_2)| \leqslant \tau_1(E_2) \,.\, \| b_1(E_1) - b_1(E_2) \|_2 + |E_1 - E_2| \,.\, \tau_0 \,.\, \| b_1(E_1) \|_2 \quad (5.A.2)$$

$$\| b_n(E_1) - b_n(E_2) \|_2 \leqslant \tau'_n(E_2) \| b_{n-1}(E_1) - b_{n-1}(E_2) \|_2 + \tau'_{n+1}(E_2) \| b_{n+1}(E_1) - b_{n+1}(E_2) \|_2 + \tau'_{0,n}, \qquad n \geqslant 1 \quad (5.A.3)$$

$$\tau_0 = \left[f^2 \int \frac{d^3\bar{k} \, | v(k) |^2}{| e_1(\bar{k}, E_1) | \,.\, | e_1(\bar{k}, E_2) | \,.\, [| e_1(\bar{k}, E_1) |^{1/2} + | e_1(\bar{k}, E_2) |^{1/2}]} \right]^{1/2} \quad (5.A.4)$$

$$\tau'_1(E) = 0, \qquad \tau'_n(E) = \tau_n(E) \qquad n \geqslant 2 \quad (5.A.5)$$

$$\tau'_{0,1} = | E_1 - E_2 | \,.\, [\tau_0 + \varepsilon_{2,1} \,.\, \| b_2(E_1) \|_2] \quad (5.A.6)$$

$$\tau'_{0,n} = | E_1 - E_2 | \,.\, [\varepsilon_{1,n} \| b_{n-1}(E_1) \|_2 + \varepsilon_{2,n} \| b_{n+1}(E_1) \|_2] \qquad n \geqslant 2 \quad (5.A.7)$$

$$\underset{n \geqslant 2}{\varepsilon_{1,n}} = \Bigg\{ f^2 . n . \underset{\bar{k}_1 \ldots \bar{k}_{n-1}}{\mathrm{Max}} \int \frac{d^3\bar{k} \, | v(k) |^2}{| e_n(\bar{k}\bar{k}_1 \ldots \bar{k}_{n-1}, E_2) | \,.\, | e_{n-1}(\bar{k}_1 \ldots \bar{k}_{n-1}, E_1) |} \cdot \Bigg[\frac{1}{[| e_n(\bar{k}\bar{k}_1 \ldots \bar{k}_{n-1}, E_1) |^{1/2} + | e_n(\bar{k}\bar{k}_1 \ldots \bar{k}_{n-1}, E_2) |^{1/2}]} \cdot \frac{1}{| e_n(\bar{k}\bar{k}_1 \ldots \bar{k}_{n-1}, E_1) |^{1/2}} + \frac{1}{[| e_{n-1}(\bar{k}_1 \ldots \bar{k}_{n-1}, E_1) |^{1/2} + | e_{n-1}(\bar{k}_1 \ldots \bar{k}_{n-1}, E_2) |^{1/2}]} \cdot \frac{1}{| e_{n-1}(\bar{k}_1 \ldots \bar{k}_{n-1}, E_2) |^{1/2}} \Bigg]^2 \Bigg\}^{1/2} \quad (5.A.8)$$

$$\underset{n \geqslant 1}{\varepsilon_{2,n}} = \Bigg\{ f^2 . (n+1) . \underset{\bar{k}_1 \ldots \bar{k}_n}{\mathrm{Max}} \int \frac{d^3\bar{k} \, | v(k) |^2}{| e_n(\bar{k}_1 \ldots \bar{k}_n, E_1) | \,.\, | e_{n+1}(\bar{k}\bar{k}_1 \ldots \bar{k}_n, E_2) |} \Bigg[\frac{1}{[| e_{n+1}(\bar{k}\bar{k}_1 \ldots \bar{k}_n, E_1) |^{1/2} + | e_{n+1}(\bar{k}\bar{k}_1 \ldots \bar{k}_n, E_2) |^{1/2}]} \cdot \frac{1}{| e_{n+1}(\bar{k}\bar{k}_1 \ldots \bar{k}_n, E_1) |^{1/2}} + \frac{1}{[| e_n(\bar{k}_1 \ldots \bar{k}_n, E_1) |^{1/2} + | e_n(\bar{k}_1 \ldots \bar{k}_n, E_2) |^{1/2}]} \cdot \frac{1}{| e_n(\bar{k}_1 \ldots \bar{k}_n, E_2) |^{1/2}} \Bigg]^2 \Bigg\}^{1/2} \quad (5.A.9)$$

The solution of the recurrence of inequalities (5.A.3) for $\| b_n(E_1) - b_n(E_2) \|_2$

is also given in Appendix C. By combining (5.A.2-3), (5.A.6-7) and (C.3) one gets:

$$| M_1(E_1) - M_1(E_2) | \leqslant \eta_2 . | E_1 - E_2 | \qquad (5.A.10)$$

$$\eta_2 = \tau_1(E_2).Z'_1(E_2). \left\{ \tau_0 + \varepsilon_{2,1} . \prod_{i=1}^{2} \tau_i(E_1).Z_i(E_1) \right.$$
$$+ \tau_1(E_1).Z_1(E_1). \sum_{l=1}^{+\infty} \left(\prod_{h=2}^{l} \tau_h(E_1)Z_h(E_1) \right) . \left[\varepsilon_{2,1+l} \left(\prod_{r=1}^{2} \tau_{l+r}(E_1)Z_{l+r}(E_1) \right) \right.$$
$$\left. \left. + \varepsilon_{1,1+l} \right] . \left[\prod_{s=1}^{l} \tau'_{1+s}(E_2).Z'_{1+s}(E_2) \right] \right\} + \tau_0.\tau_1(E_1).Z_1(E_1) \qquad (5.A.11)$$

with the convention $\prod_{h=2}^{l} \tau_h.Z_h = 1,\ \tau_2.Z_2$ if $l = 1, 2$ respectively. Z'_n is given by Eq. (C.2), with $\tau'_{1,n} \to .\tau'_n(E_2)$, $\tau'_{2,n} \to \tau'_{n+1}(E_2)$. By looking at the ratio of the $(l + 1)$-th term over the l-th one in the series on the right-hand-side of Eq. (5.A.11) and noticing that $\tau_n \to 0$ as $n \to \infty$, one shows that such a series converges.

From (5.A.1), (4.7-8), (5.A.8-11), (5.A.4) and (C.2) we see that for given f and small fixed $| \bar{\pi} |$, there is a domain D_2 of values for E such that: *i*) it contains $\bar{\pi}^2/2m_0$, *ii*) all continued fractions Z_n, Z'_n are strictly positive, *iii*) it maps into itself under M_1, *iv*) $\eta_2 < 1$ for any E_1, E_2 belonging to it.

Again, the contraction mapping principle guarantees that there exists a unique fixed point $E = M_1(E)$ lying inside D_2, which can be found by iterating Eq. (4.5). For this fixed point, all amplitudes b_n, $n \geqslant 1$, are given by the convergent series generated through the iterations of (4.6).

We shall consider the standard large polaron model (recall section 2) for $\bar{\pi} = 0$ and study numerically the range of validity of the above rigorous construction of the dressed electron state, for increasing values of the dimensionless coupling constant α.

Now, τ_n $(n \geqslant 1)$ can be evaluated explicitly. Then, in what follows, it will be understood that τ_1 and τ_n, $n \geqslant 2$, are replaced respectively by $\alpha^{1/2}.(1 - E)^{-1/4}$ and $(n - E)^{-1/4}.\ [\alpha n/(n - 1 - E)]^{1/2}$ (with $\omega_0 = m_0 = 1$). We have carried out several types of numerical estimates:

a) We have studied the validity of $Z_r > 0$ for $r \geqslant 1$ and real values of E. Let $\tilde{E}_1 \leqslant 0$ be some energy (not necessarily the highest one) such that $Z_r(E) > 0$ for $r \geqslant 1$ and any $E \leqslant \tilde{E}_1$.

b) We have studied the inequality (5.A.1) with $M_1(E) = E$ and $\bar{\pi} = 0$. For this purpose, we have obtained the solutions $\tilde{E}_2$ of

$$- \tilde{E}_2 = [\tau_1(\tilde{E}_2)]^2.Z_1(\tilde{E}_2),$$

which fulfill $\tilde{E}_2 \leqslant \tilde{E}_1$.

For $\alpha < 0.1$, the rigorous construction of the dressed electron state outlined in this subsection can be proved to converge. We shall not present detailed numerical results for such a range. Rather, we shall discuss, in some detail, the more interesting case $\alpha \geqslant 0.1$.

Table I summarizes our numerical results for $\tilde{E}_1$ and $\tilde{E}_2$ if $\alpha \geqslant 0.1$.

TABLE I

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$-\tilde{E}_1$	0.08	0.1	0.27	0.3	0.475	0.831	1.227
$-\tilde{E}_2$	0.10984	0.24236	0.40240	0.59496	0.82495	1.09500	1.40883

We have shown numerically that $Z_r > 0$ for any $r \geqslant 1$ ceases to be true for $E > \tilde{E}_1$ if $\alpha = 0.5$, 0.6 and 0.7.

Moreover, we have seen that the inequality (5.A.1), with $M_1(E) = E$ and $\bar{\pi} = 0$ is satisfied for $\tilde{E}_2 \leqslant E \leqslant \tilde{E}_1$. This means that $\tilde{E}_2$ can be regarded as a lower bound for the exact polaron energy. Other polaron lower bounds have been obtained, using different techniques, in [*42*].

We have studied the validity of the contraction-mapping condition $\eta_2 < 1$ (recall (5.A.11)) for given α and values of E_1, E_2 in the range $\tilde{E}_2 \leqslant E_1$, $E_2 \leqslant \tilde{E}_1$. It turns out that $\eta_2 < 1$ is fulfilled for $\alpha \leqslant 0.3$. For $\alpha = 0.4$, our estimates indicate that η_2 is close to 1. For $\alpha \geqslant 0.5$ we find that η_2 is appreciably larger than 1 and, moreover, that it increases as α does.

Notice that the polaron self-energy obtained from standard perturbation theory up to second order [*8*], [*33*], E_{PT}, and Feynman's upper bound for it [*8-9*], E_F, satisfy: *i)* $\tilde{E}_1 > E_F \geqslant E_{PT} > \tilde{E}_2$ for $0.1 \leqslant \alpha \leqslant 0.5$, *ii)* $E_F > E_{PT} > \tilde{E}_1 > \tilde{E}_2$ if $\alpha \geqslant 0.6$.

The main conclusion from the above analysis is the following. The rigorous construction of the dressed electron presented in this subsection does certainly converge for $\alpha \leqslant 0.3$. Quite probably, it also converges for $\alpha = 0.4$. The last statement is motivated by our previous numerical findings and by the fact that our majorations and the contraction mapping principle only give sufficient conditions for convergence. The above conclusions provide a partial answer to the conjecture by Pines [*3*] commented in section 2. Thus the present method, which avoids the use of dressing transformations, allows one to establish the mathematical existence of the polaron for values of α which characterize semiconductors of type III-V [*6*] and which are one and half orders of magnitude larger than the upper limit for convergence ($\alpha \leqslant 0.009$) obtained in section 3. It is uncertain (and hard to establish numerically) whether the mathematical construction of this subsection actually converges for $\alpha > 0.4$. Thus, for $\alpha = 0.5$, the corresponding value for $\tilde{E}_1$ in Table I could allow for convergence to occur: however, we remark that $\tilde{E}_2$ is rather large and that η_2 is appreciably larger than 1 in this case. Unless important cancellations occur in the formal solutions obtained by successive iterations, such convergence seems more and more doubtful as α increases above 0.4, since the corresponding values for $\tilde{E}_1$ and $\tilde{E}_2$ and η_2 become large. All this is in agreement, essentially, with the conjectures formulated by Pines [3].

5.B. Another viewpoint and study of the case R = 2.

Throughout this subsection, we shall assume $Z_r > 0$ for any $r \geqslant 2$, for given $|\bar{\pi}|$, f and a certain domain of values for E.

We cast the set of all equations (4.3) for $n \geqslant 2$ into a matrix form similar to (4.4), (4.6), namely:

$$B_2 = B_2^{(0)} + W_2 . B_2 \tag{5.B.1}$$

Here, B_2 is the column vector formed by all $b_2(\bar{k}_1, \bar{k}_2), \ldots, b_n(\bar{k}_1 \ldots \bar{k}_n) \ldots$ (that is, it is obtained by dropping $b_1(\bar{k}_1)$ in B_1), $B_2^{(0)}$ is a column vector whose elements vanish identically except the first one, which equals

$$\frac{f}{e_2(\bar{k}_1\bar{k}_2)^{1/2} . 2^{1/2}} \cdot \sum_{i=1}^{2} v(k_i)^* \cdot \frac{b_1(\bar{k}_j)}{|e_1(\bar{k}_j)|^{1/2}}$$

$j = 1, 2, j \neq i$ and W_2 is the corresponding kernel. Throughout this subsection, the bound (4.9) remains valid for $n \geqslant 2$, so that the series for each b_n, $n \geqslant 2$ obtained by iterating Eq. (5.B.1) converges in L^2-norm. Upon considering Eq. (4.3) for $n = 1$ and replacing b_2 in it by the right-hand-side of Eq. (4.3) for $n = 2$, one finds the following linear integral equation for b_1:

$$b_1(\bar{k}_1) = b_{1,in}(\bar{k}_1) + \int d^3\bar{k}_1' l(\bar{k}_1, \bar{k}_1') b_1(\bar{k}_1') \tag{5.B.2}$$

$$b_{1,in}(\bar{k}_1) = \sigma_1(\bar{k}_1) . \left\{ \frac{f v(k_1)^*}{e_1(\bar{k}_1)^{1/2}} + \frac{f^2 . 6^{1/2}}{e_1(\bar{k}_1)^{1/2}} \int \frac{d^3\bar{k}_1' d^3\bar{k}_1'' v(k_1') v(k_1'') b_3(\bar{k}_1'\bar{k}_1''\bar{k}_1)}{e_2(\bar{k}_1''\bar{k}_1) | e_3(\bar{k}_1'\bar{k}_1''\bar{k}_1) |^{1/2}} \right\} \tag{5.B.3}$$

$$l(\bar{k}_1, \bar{k}_1') = \sigma_1(\bar{k}_1) . \frac{f^2 v(k_1)^* v(k_1')}{e_1(\bar{k}_1)^{1/2} e_2(\bar{k}_1\bar{k}_1') | e_1(\bar{k}_1') |^{1/2}} \tag{5.B.4}$$

$$\sigma_n(\bar{k}_1 \ldots \bar{k}_n) = \left[1 - \frac{f^2}{e_n(\bar{k}_1 \ldots \bar{k}_n)} \int \frac{d^3\bar{k}' | v(k') |^2}{e_{n+1}(\bar{k}'\bar{k}_1 \ldots \bar{k}_n)} \right]^{-1}, \quad n \geqslant 1 \tag{5.B.5}$$

First, we consider the case of small f (so that $Z_1 > 0$ as well) and present a viewpoint alternative and complementary to that adopted in subsection 5.A. Now, we have a mapping $M_1(E \to M_1(E))$ essentially equivalent to the one in subsection 5.A, that is, M_1 is defined by the right-hand-side of Eq. (4.5), and the series formed by all iterations of Eqs. (5.B.1-2). One has:

$$\left| M_1(E) - \frac{\bar{\pi}^2}{2m_0} \right| \leqslant \tau_1 . \| b_1 \|_2 ,$$

but a slightly different for $\| b_1 \|_2$ will be obtained. One finds:

$$| b(\bar{k}_1) | \leqslant | b_{1,in}(\bar{k}_1) | + \frac{1}{\Delta_m} \int d^3\bar{k}_1' N_m(\bar{k}_1, \bar{k}_1') \, | b_{1,in}(\bar{k}_1') | \quad (5.B.6)$$

$$N_m(\bar{k}_1, \bar{k}_1') = f^2 \left[\operatorname*{Max}_{\bar{k}_2, \bar{k}_3} \frac{1}{| e_2(\bar{k}_2, \bar{k}_3) |} \right] \cdot \frac{| v(k_1) | . | v(k_1') | . \sigma_1(\bar{k}_1)}{| e_1(\bar{k}_1) |^{1/2} . | e_1(\bar{k}_1') |^{1/2}} ,$$

$$\Delta_m = 1 - \int d^3\bar{k}_1 N_m(\bar{k}_1, \bar{k}_1) \quad (5.B.7)$$

The inequality (5.B.6) can be proved by iterating Eq. (5.B.2), using

$$| l(\bar{k}_1, \bar{k}_1') | \leqslant N_m(\bar{k}_1, \bar{k}_1')$$

and summing the resulting geometric series. Taking L^2-norms in (5.B.3), (5.B.6), using (4.9) for $n = 2, 3$ and solving for $\| b_1 \|_2$, one arrives at the new bound:

$$\| b_1 \|_2 \leqslant \left[1 + \left(\left(\int d^3\bar{k}_1 d^3\bar{k}_1' \, | N_m(\bar{k}_1 \bar{k}_1') |^2 \right)^{1/2} / | \Delta_m | \right) \right] \cdot \gamma_1$$
$$\cdot \left\{ 1 - \left[1 + \left(\left(\int d^3\bar{k}_1 d^3\bar{k}_1' | N_m(\bar{k}_1 \bar{k}_1') |^2 \right)^{1/2} / | \Delta_m | \right) \right] \cdot \gamma_2 \tau_2 \cdot \tau_3 \cdot Z_2 \cdot Z_3 \right\}^{-1} \quad (5.B.8)$$

$$\gamma_1 = \left[f^2 \int d^3\bar{k} \, \frac{| \sigma_1(\bar{k}) v(k) |^2}{| e_1(\bar{k}_1) |} \right]^{1/2} \quad (5.B.9)$$

$$\gamma_2 = \left[\operatorname*{Max}_{\bar{k}_1} \frac{f^4 \cdot 6 \, | \sigma_1(\bar{k}_1) |^2}{| e_1(k_1) |} \cdot \int \frac{d^3\bar{k}_1' d^3\bar{k}_1'' \, | v(k_1') |^2 \, | v(k_1'') |^2}{| e_2(\bar{k}_1'' \bar{k}_1) |^2 \cdot | e_3(\bar{k}_1', \bar{k}_1'', \bar{k}_1) |} \right]^{1/2} \quad (5.B.10)$$

Assumptions 1)-4) in section 2 imply the finiteness of γ_1 and γ_2. One can obtain another inequality, similar to (5.A.10), with a new positive constant η_2', whose lengthy expression will be omitted. As in previous cases, the contraction mapping principle can be applied to the mapping M_1 when $\eta_2' < 1$, which leads to construct unique solutions for $E = M_1(E)$ and all b_n's, $n \geqslant 1$.

As f increases, the conditions $Z_1 > 0$ and $\eta_2 < 1, \eta_2' < 1$ can be expected to break down. Then, the inequality (4.9) would become meaningless for $n = 1$, and the convergence of the series for b_1 obtained by iterating (4.6) or (5.B.2) would not be warranted. In order to solve this problem, we start by noticing that $l(\bar{k}_1, \bar{k}_1')$ is a Hilbert-Schmidt kernel

$$\left(\| l \|_2 = \left[\int d^3\bar{k}_1 d^3\bar{k}_1' \mid l(\bar{k}_1, \bar{k}_1') \mid^2 \right]^{1/2} < +\infty \right)$$

by virtue of assumptions 1)-4) in section 2. Then, the modified Fredholm theory [*43*] leads to:

$$b_1(\bar{k}_1) = b_{1,in}(\bar{k}_1) + \frac{1}{\Delta} \int d^3\bar{k}_1' \mathrm{N}(\bar{k}_1, \bar{k}_1') b_{1,in}(\bar{k}_1') \tag{5.B.11}$$

$$\mathrm{N}(\bar{k}_1, \bar{k}_1') = l(\bar{k}_1, \bar{k}_1') + \sum_{n=1}^{+\infty} \mathrm{N}^{(n)}(\bar{k}_1, \bar{k}_1') \tag{5.B.12}$$

$$\mathrm{N}^{(n)}(\bar{k}_1, \bar{k}_1') = \frac{(-1)^n}{n!} \int \prod_{j=1}^{n} d^3\bar{k}_j'' \det \begin{pmatrix} l(\bar{k}_1\bar{k}_1') & l(\bar{k}_1\bar{k}_1'') & l(\bar{k}_1\bar{k}_2'') \cdots\cdots l(\bar{k}_1\bar{k}_n'') \\ l(\bar{k}_1''\bar{k}_1') & 0 & l(\bar{k}_1''\bar{k}_2'') \cdots\cdots l(\bar{k}_1''\bar{k}_n'') \\ \cdots & \cdots & \cdots \\ l(\bar{k}_n''\bar{k}_1') & l(\bar{k}_n''\bar{k}_1'') & l(\bar{k}_n''\bar{k}_2'') \cdots\cdots \quad 0 \end{pmatrix} \tag{5.B.13}$$

$$\| \mathrm{N}^{(n)} \|_2 = \left[\int d^3\bar{k}_1 d^3\bar{k}_1' \mid \mathrm{N}^{(n)}(\bar{k}_1\bar{k}_1') \mid^2 \right]^{1/2} \leqslant \frac{\exp\left((n+1)/2\right)}{n^{n/2}} \cdot (\| l \|_2)^{n+1}, \qquad n \geqslant 1 \tag{5.B.14}$$

$$\| \mathrm{N} \|_2 = \left[\int d^3\bar{k}_1 d^3\bar{k}_1' \mid \mathrm{N}(\bar{k}_1, \bar{k}_1') \mid^2 \right]^{1/2} \leqslant \| l \|_2 + \sum_{n=1}^{+\infty} \| \mathrm{N}^{(n)} \|_2 \tag{5.B.15}$$

$$\Delta = 1 + \sum_{n=1}^{+\infty} \Delta^{(n)} \tag{5.B.16}$$

$$\Delta^{(n)} = \frac{(-1)^n}{n!} \int \prod_{j=1}^{n} d^3\bar{k}_j'' \det \begin{pmatrix} 0 & l(\bar{k}_1''\bar{k}_2'') & l(\bar{k}_1''\bar{k}_3'') \cdots\cdots l(\bar{k}_1''\bar{k}_n'') \\ l(\bar{k}_2''\bar{k}_1'') & 0 & l(\bar{k}_2''\bar{k}_3'') \cdots\cdots l(\bar{k}_2''\bar{k}_n'') \\ \cdots & \cdots & \cdots \\ l(\bar{k}_n''\bar{k}_1'') & l(\bar{k}_n''\bar{k}_2'') & l(\bar{k}_n''\bar{k}_3'') \cdots\cdots \quad 0 \end{pmatrix} \tag{5.B.17}$$

$$\mid \Delta_n \mid \leqslant \frac{\exp(n/2)}{n^{n/2}} \cdot (\| l \|_2)^n, \qquad n \geqslant 1 \tag{5.B.18}$$

According to Smithies [43], the following properties are valid: *i*) the series (5.B.12), (5.B.16) for $N(\bar{k}_1, \bar{k}'_1)$ and Δ (the modified first Fredholm minor and Fredholm determinant, respectively) always converge, by virtue of the bounds (5.B.14-15) and (5.B.18), and $\| N \|_2 < +\infty$, *ii*) by assuming $\Delta \neq 0$, the right-hand-side of Eq. (5.B.11) gives the unique solution of Eq. (5.B.2), which is valid even when the series formed by all the iterations of Eq. (5.B.2) diverges. The bound (5.B.8) is also valid, provided that $N_m(\bar{k}_1, \bar{k}'_1)$ and Δ_m be replaced by $N(\bar{k}_1, \bar{k}'_1)$ and Δ respectively. Eqs. (5.B.1), (5.B.3), (5.B.11) and the right-hand-side of (4.5) define a new mapping, also denoted by $M_1 : E \to M_1(E)$. As in previous cases, the application of fixed-point theorems to the actual mapping allows to construct solutions for $E = M_1(E)$ and all b_n's, $n \geqslant 1$.

We shall apply the developments of this subsection to the large polaron model:

a) A numerical solution of $-\tilde{E}'_2 = \tau_1(\tilde{E}'_2)\cdot \| b_1(\tilde{E}'_2) \|_2$, $\| b_1(\tilde{E}'_2) \|_2$ being replaced by the right-hand-side of (5.B.8), yields values for $-\tilde{E}'_2$ which are slightly smaller than those for $-\tilde{E}_2$ (Table I): thus, for $\alpha = 0.3$, 0.4 and 0.5, we get respectively $-\tilde{E}'_2 = 0.362$, 0.551, 0.795. This leads to, essentially, the same conclusions as in subsection 5.A: the method based upon Eq. (4.5) and the series formed by the iterations of Eqs. (5.B.1-2) does converge if $\alpha \leqslant 0.3$, its convergence being quite plausible for $\alpha = 0.4$, and uncertain for $\alpha > 0.4$ (increasingly doubtful as α increases).

b) A numerical study for $\alpha > 0.4$, based upon Eqs. (4.5), (5.B.1) and Fredholm theory is more complicated. For this reason, we shall limit ourselves to some qualitative estimates. We use separable approximations for $l(\bar{k}_1, \bar{k}'_1)$ of the type

$$l(\bar{k}_1, \bar{k}_1) \simeq \frac{\sigma_1(\bar{k}_1)\cdot 2^{1/2}\cdot \alpha}{(2\pi)^2 e_1(\bar{k}_1)^{1/2} k_1} \cdot \left[\frac{2 - E}{2 - E + (k_1^2/2)}\right]^{n/2} \frac{1}{k'_1 \cdot | e_1(\bar{k}'_1) |^{1/2}\cdot (2 - E + (k_1'^2/2))} \equiv l_s(\bar{k}_1, \bar{k}'_1)$$

with $n = 1$ or 2 ($m_0 = \omega_0 = 1$). Even if it is hard to estimate the error of such an approximation, we expect the latter to be moderately adequate when $l_s(\bar{k}_1, \bar{k}'_1)$ acts upon functions whose $\bar{k}'_1$-behavior ressembles that of the exact $b_1(\bar{k}'_1)$, in a limited range above $\alpha = 0.4$. Here, one finds

$$N(\bar{k}_1, \bar{k}'_1) = l_s(\bar{k}_1, \bar{k}'_1), \qquad \Delta = 1 - \int d^3\bar{k}'_1 l_s(\bar{k}'_1 \bar{k}'_1)$$

With these, we have carried out a numerical analysis similar to the one outlined in *a*) above. Instead of $-\tilde{E}'_2$, we now find out numerical solutions $-\tilde{E}_3 = 0.515$, 0.755, for $\alpha = 0.4$, 0.5, respectively (essentially the same for both $n = 1$ and $n = 2$), which are systematically a bit smaller than $-\tilde{E}_2$ or $-\tilde{E}'_2$. This fact could be perhaps regarded as a numerical hint of the reliability of Fredholm theory. We stress that $Z_r > 0$ for $r \geqslant 2$ is always fulfilled and believe that $\tilde{E}_3$ is also a lower bound for the true polaron energy. Here, the main qualitative conclusion is: the method based upon Eq. (4.5), the series of iterations for Eq. (5.B.1) and Eqs. (5.B.11-18) can be expected to converge for $0.4 \leqslant \alpha \leqslant 0.5$ (while that of subsection 5.A and the one based upon the iterations of Eq. (5.B.2) are of uncertain validity in such a range), at least.

5.C. R = 3.

Here, we assume that $Z_r > 0$ if $r \geqslant 3$, in a certain domain of E and for given $\bar{\pi}$ and somewhat larger values of f. We allow for (4.9) to break down for $n = 1.2$ so that we do not expect that either b_1 or b_2 could be found by successive iterations. The set of all Eqs. (4.3) for $n \geqslant 3$ can be cast into the matrix system $B_3 = B_3^{(0)} + W_3 \cdot B_3$ (B_3 being a column vector obtained from B_1, Eq. (4.4), by dropping b_1 and b_2, etc.): by iterating the latter, one obtains a series for b_n $n \geqslant 3$, which converges in L^2-norm, as the bound (4.9) is meaningful for $n \geqslant 3$.

We shall concentrate in displaying and solving a new difficulty regarding b_2, which did not appear when R = 2, omitting unnecessary details. By considering Eq. (4.3) for $n = 2$ and replacing in it b_3 by the right-hand-side of Eq. (4.3) for $n = 3$, one finds the linear integral equation:

$$b_2(\bar{k}_1, \bar{k}_2) = b_{2,in}(\bar{k}_1, \bar{k}_2) + \int d^3\bar{k}'_1 l_1(\bar{k}_1, \bar{k}_2; \bar{k}'_1) b_2(\bar{k}'_1 \bar{k}_2) + \int d^3\bar{k}'_2 l_2(\bar{k}_1 \bar{k}_2; \bar{k}'_2) b_2(\bar{k}_1 \bar{k}'_2) \quad (5.C.1)$$

$$b_{2,in}(\bar{k}_1, \bar{k}_2) = \sigma_2(\bar{k}_1 \bar{k}_2) \left\{ \frac{f}{e_2(\bar{k}_1 \bar{k}_2)^{1/2} \cdot 2^{1/2}} \sum_{i=1}^{2} v(k_i)^* \frac{b_1(\bar{k}_j)}{|e_1(\bar{k}_j)|^{1/2}} + \frac{f^2 \cdot (12)^{1/2}}{e_2(\bar{k}_1 \bar{k}_2)^{1/2}} \int d^3\bar{k}' d^3\bar{k}'' \frac{v(k') v(k'') b_4(\bar{k}' \bar{k}'' \bar{k}_1 \bar{k}_2)}{e_3(\bar{k}'' \bar{k}_1 \bar{k}_2) \, | e_4(\bar{k}' \bar{k}'' \bar{k}_1 \bar{k}_2) |^{1/2}} \right\} \quad (5.C.2)$$

$$l_i(\bar{k}_1 \bar{k}_2; \bar{k}'_i) = \frac{\sigma_2(\bar{k}_1 \bar{k}_2) \cdot f^2 v^*(k_i) v(k'_i)}{e_2(\bar{k}_1 \bar{k}_2)^{1/2} \cdot e_3(\bar{k}'_i \bar{k}_1 \bar{k}_2) \cdot | e_2(\bar{k}'_i \bar{k}_j) |^{1/2}} \equiv l_i, \quad i = 1, 2 \quad i \neq j \quad (5.C.3)$$

where $j = 1, 2, j \neq i$ in Eqs. (5.C.2-3) and σ_2 is given in Eq. (5.B.5). Symbolically, we shall rewrite Eq. (5.C.1) as $b_2 = b_{2,in} + \left(\sum_{i=1}^{2} l_i \right) \cdot b_2$. The kernel $\sum_{i=1}^{2} l_i$ is not Hilbert-Schmidt in all threemomenta $\bar{k}_1$, $\bar{k}_2$, $\bar{k}'_1$, $\bar{k}'_2$, due to the structure of the right-hand-side of Eq. (5.C.1), so that the modified

Fredholm theory cannot be used to solve (5.C.1) as it stands. To solve this difficulty, we shall apply rearrangement techniques typical of multi-particle scattering theory [*44*]. Using symbolic notation partially, the basic new equations read:

$$b_2 = b_{2,in} + \sum_{i=1}^{2} d_i, \qquad d_i = (\xi_i + \xi_i\xi_j) b_{2,in} + \xi_i\xi_j d_i \qquad i,j = 1, 2,\ i \neq j \quad (5.C.4)$$

$$\begin{aligned} \xi_1 &= \xi_1(\bar{k}_1\bar{k}_2; \bar{k}_1') = l_1(\bar{k}_1, \bar{k}_2; \bar{k}_1') + \int d^3\bar{k}_1'' l_1(\bar{k}_1\bar{k}_2; \bar{k}_1'')\xi_1(\bar{k}_1''\bar{k}_2; \bar{k}_1') \\ \xi_2 &= \xi_2(\bar{k}_1\bar{k}_2; \bar{k}_2') = l_2(\bar{k}_1\bar{k}_2; \bar{k}_2') + \int d^3\bar{k}_2'' l_2(\bar{k}_1\bar{k}_2; \bar{k}_2'')\xi_2(\bar{k}_1\bar{k}_2''; \bar{k}_2') \end{aligned} \quad (5.C.5)$$

Notice that either by iterating eqs. (5.C.5) and the second Eq. (5.C.4) and inserting the resulting series into the first Eq. (5.C.4), or by iterating directly Eq. (5.C.1), one arrives at the same formal series. The main properties of Eqs. (5.C.4-5) are:

a) The kernel l_i is Hilbert-Schmidt in $\bar{k}_i$, $\bar{k}_i'$, for fixed $\bar{k}_j$, $j \neq i$. Then, the solution of Eq. (5.C.5) is given by the modified Fredholm theory, that is, by performing suitable replacements in Eqs. (5.B.11-17). Symbolically, we writte: $\xi_i = l_i + \frac{1}{\Delta_i} \mathrm{N}_i l_i$, $i = 1, 2$. Notice that Δ_i depends only on $\bar{k}_j$, $j \neq i$.

b) We notice that the kernel $\xi_i\xi_j$, $i \neq j$, is defined for any $\varphi = \varphi(\bar{k}_1', \bar{k}_2')$ as follows:

$$\xi_1\xi_2\varphi = \int d^3\bar{k}_1' d^3\bar{k}_2' \xi_1(\bar{k}_1\bar{k}_2; \bar{k}_1')\xi_2(\bar{k}_1'\bar{k}_2; \bar{k}_2')\varphi(\bar{k}_1'\bar{k}_2'),$$

and so on for $\xi_2 \cdot \xi_1$. In what follows, we shall assume that

$$\hat{\lambda}_i = \operatorname*{Max}_{\bar{k}_j} \frac{1}{|\Delta_i(\bar{k}_j)|} < +\infty, \qquad i \neq j.$$

Let

$$\|\varphi\|_2 = \left[\int d^3\bar{k}_1 d^3\bar{k}_2 |\varphi(\bar{k}_1\bar{k}_2)|^2\right]^{1/2},$$

$$\|l_1(\bar{k}_2)\|_2 = \left[\int d^3\bar{k}_1 d^3\bar{k}_1' |l_1(\bar{k}_1\bar{k}_2; \bar{k}_1')|^2\right]^{1/2}, \qquad l_{1,2} = \operatorname*{Max}_{\bar{k}_2} \|l_1(\bar{k}_2)\|_2,$$

and so on for $\|l_2(\bar{k}_1)\|_2$, $l_{2,1}$. Notice that $l_{1,2} < +\infty$, $l_{2,1} < +\infty$. We can prove that $\xi_i\xi_j = \left(l_i + \frac{1}{\Delta_i}\mathrm{N}_i l_i\right)\left(l_j + \frac{1}{\Delta_j}\mathrm{N}_j l_j\right)$ is a bounded operator.

In fact, by using the analogues of (5.B.12) and (5.B.14), some direct majorations give:

$$\left\| \frac{1}{\Delta_1} N_1 l_1 \frac{1}{\Delta_2} N_2 l_2 \varphi \right\|_2 \leqslant \tilde{\lambda}_1 \tilde{\lambda}_2 (l_{1,2} \cdot l_{2,1})^2$$

$$\cdot \left\{ 1 + \sum_{n=1}^{+\infty} \frac{\exp \frac{1}{2}(n+1)}{n^{n/2}} \cdot (l_{1,2})^n + \sum_{r=1}^{+\infty} \frac{\exp \frac{1}{2}(r+1)}{r^{r/2}} \cdot (l_{2,1})^r \right.$$

$$\left. + \sum_{n,r=1}^{+\infty} \frac{\exp \frac{1}{2}(n+1)}{n^{n/2}} \cdot \frac{\exp \frac{1}{2}(r+1)}{r^{r/2}} (l_{1,2})^n \cdot (l_{2,1})^r \right\} \| \varphi \|_2 \quad (5.C.6)$$

The series in (5.C.6) converge, and the proof for the remaining contributions to $\xi_i \xi_j$ is similar. Moreover, since

$$\operatorname{Max}_{\bar{k}_2} \int d^3\bar{k}_1 d^3\bar{k}'_1 \, | \xi_1(\bar{k}_1 \bar{k}_2 ; \bar{k}'_1) |^2 < + \infty$$

and so on for ξ_2 and

$$\| b_{2,in} \|_2 \leqslant \left[\operatorname{Max}_{\bar{k}} \cdot \frac{2f^2}{| e_1(\bar{k}) |} \int d^3\bar{k}' \frac{| \sigma_2(\bar{k}\bar{k}') |^2 \cdot | v(k') |^2}{| e_2(\bar{k}\bar{k}') |} \right]^{1/2} \cdot \| b_1 \|_2$$

$$+ \left[\operatorname{Max}_{\bar{k}_1 \bar{k}_2} \left(\frac{12 f^4}{| e_2(\bar{k}_1 \bar{k}_2) |} \int d^3\bar{k}' d^3\bar{k}'' \frac{| v(k') |^2 \cdot | v(k'') |^2}{| e_3(\bar{k}''\bar{k}_1 \bar{k}_2 |^2 \cdot | e_4(\bar{k}'\bar{k}''\bar{k}_1 \bar{k}_2) |} \cdot | \sigma_2(\bar{k}_1 \bar{k}_2) |^2 \right) \right]^{1/2} \cdot \| b_4 \|_2 \quad (5.C.7)$$

one finds easily a finite bound for $\| \xi_i + \xi_i \xi_j) b_{2,in} \|_2$ in terms of $\| b_1 \|_2$ and $\| b_4 \|_2$.

c) The kernel $\xi_i \xi_j$ is Hilbert-Schmidt:

$$\| \xi_1 \xi_2 \|_2 = \left[\int d^3\bar{k}_1 d^3\bar{k}_2 d^3\bar{k}'_1 d^3\bar{k}'_2 \, | \xi_1(\bar{k}_1 \bar{k}_2 ; \bar{k}'_1) |^2 \cdot | \xi_2(\bar{k}'_1 \bar{k}_2 ; \bar{k}'_2) |^2 \right]^{1/2} < + \infty$$

and similarly for $\xi_2 \xi_1$. Let us sketch a direct proof. By using the analogues of (5.B.12), namely, $N_i = l_i + \sum_{n=1}^{+\infty} N_{i,n}$, $i = 1, 2$, one finds:

$$\| \xi_1 \xi_2 \|_2 \leqslant \tilde{\lambda}_1 \tilde{\lambda}_2 \left\{ \| l_1 l_2 \|_2 + \| l_1 l_2^2 \|_2 + \| l_1^2 l_2 \|_2 + \| l_1^2 l_2^2 \|_2 \right.$$

$$+ \sum_{n=1}^{+\infty} \| l_1 N_{2,n} l_2 \|_2 + \sum_{n=1}^{+\infty} \| l_1^2 N_{2,n} l_2 \|_2$$

$$\left. + \sum_{r=1}^{+\infty} \| N_{1,r} l_1 l_2 \|_2 + \sum_{r=1}^{+\infty} \| N_{1,r} l_1 l_2^2 \|_2 + \sum_{n,r=1}^{+\infty} \| N_{1,r} l_1 N_{2,n} l_2 \|_2 \right\} \quad (5.C.8)$$

Let $\tilde{l}(\bar{k}_2;\bar{k}_1'')=\left[\int d^3\bar{k}_1'''|l_1(\bar{k}_1'''\bar{k}_2;\bar{k}_1')|^2\right]^{1/2}$. Then, by using appropriately the Schwartz inequality and the Mean-Value theorem, one gets:

$$
\begin{aligned}
(\| N_{1,r}l_1N_{2,n}l_2 \|_2)^2 \leqslant & \int d^3\bar{k}_1''(\| l_2(\bar{k}_1'') \|_2)^2 \\
& \cdot\left\{\int d^3\bar{k}_1 d^3\bar{k}_1' d^3\bar{k}_2 d^3\bar{k}_2' \,|\, N_{2,n}(\bar{k}_1''\bar{k}_2;\bar{k}_2')\,|^2\cdot|\, N_{1,r}(\bar{k}_1,\bar{k}_2;\bar{k}_1')\,|^2 \right. \\
& \left. \cdot\,|\,\tilde{l}_1(\bar{k}_2;\bar{k}_1'')\,|^2\right\} \\
\leqslant & \int d^3\bar{k}_1''(\| l_2(\bar{k}_1'') \|_2)^2\cdot|\,\tilde{l}_1(\bar{k}_{2,mv}(\bar{k}_1'');\bar{k}_1'')\,|^2 \\
& \left[\int d^3\bar{k}_2 d^3\bar{k}_2'\cdot|\, N_{2,n}(\bar{k}_1'',\bar{k}_2;\bar{k}_2')\,|^2\cdot\frac{\exp(r+1)}{r^r}(\| l_1(\bar{k}_2) \|_2)^{2(r+1)}\right] \\
\leqslant & \int d^3\bar{k}_1''\,|\,\tilde{l}_1(\bar{k}_{2,mv}(\bar{k}_1'');\bar{k}_1'')\,|^2 \\
& \cdot\frac{\exp(r+1)}{r^r}(\| l_1(\bar{k}_{2,mv}) \|_2)^{2(r+1)}\cdot\frac{\exp(n+1)}{n^n}\cdot(\| l_2(\bar{k}_1'') \|_2)^{2(n+2)} \qquad (5.C.9)
\end{aligned}
$$

where use is also made of (5.B.14), and the subscript « mv » denotes suitable intermediate threemomenta, which arise due to the application of the Mean-Value theorem and depend, in each case, on the unintegrated threemomentum. One has: $\| l_2(\bar{k}_1'') \|_2 \to 0$ as $|\bar{k}_1''| \to \infty$, which implies $\| N_{1,r}l_1N_{2,n}l_2 \|_2 < +\infty$, for any n larger than some n_0 and any $r \geqslant 1$. Similar arguments, with slight modifications, imply the finiteness of $\| l_1^h N_{2,n}l_2 \|_2$, $h = 1, 2$, for $n > n_0'$, of $\| N_{1,r}l_1l_2^h \|_2$, $h = 1, 2$, for $r > r_0$, and of $\| N_{1,r}l_1N_{2,n}l_2 \|_2$ for $n \leqslant n_0$ and $r > r_0'$. The finiteness of the first four terms in (5.C.8) and, eventually, of the remaining first few terms in the series of (5.C.8) (say, of $\| l_1^h N_{2,n}l_2 \|_2$, $h = 1, 2$, for $n \leqslant n_0'$, etc.) follows readily by power-counting arguments. All this leads easily to $\| \xi_1\xi_2 \|_2 < +\infty$ and so on for $\| \xi_2\xi_1 \|_2$.

d) Consequently, both $d_i(\bar{k}_1,\bar{k}_2)$, $i = 1, 2$, can be constructed by applying the modified Fredholm method [*43*] to the second Eq. (5.C.4).

Now, we have a new mapping, still denoted by M_1, $E \to M_1(E)$, which is defined through the series of all iterations of $B_3 = B_3^{(0)} + W_3\cdot B_3$, Eqs. (5.B.11-17), Eqs. (5.C.4-5) and the right-hand-side of Eq. (4.5). One can obtain bounds for $\| b_1 \|_2$, $\| b_2 \|_2$ which generalize (5.B.8). Again, the application of fixed-point theorems to the mapping M_1 would lead to construct solutions for E and all b_n's, $n \geqslant 1$. For brevity, we shall omit details, and the discussion of extensions to cases with $R > 3$.

Generalizations to scattering processes and to other models of Quantum Field Theory are under study.

APPENDIX A

RIGOROUS DEFINITION OF H'USING QUADRATIC FORMS

For any ψ_1, ψ_2 in $\mathcal{H}_{\bar{\pi}}$, let us define the quadratic form $F(\psi_1, \psi_2) = (\psi_1, (H'_{1,1} + H'_{1,2})\psi_2)_{\bar{\pi}}$. Then, for any ψ belonging both to $\mathcal{H}_{\bar{\pi}}$ and the domain of $H_0^{1/2}$ ($\| \psi \|_{\bar{\pi}} < +\infty$), one proves, by extending the rigorous analysis in [*25*], that:

$$| F(\psi, \psi) | \leqslant e'_1(\| H_0^{1/2}\psi \|_{\bar{\pi}})^2 + e'_2(\| \psi \|_{\bar{\pi}})^2 \tag{A.1}$$

$$e'_1 = \frac{\varepsilon\lambda_1}{\omega_0^{1/2}} + \frac{\lambda_2^2}{m_0} + 2\lambda_2 + \frac{\lambda_3}{m_0}\left(\frac{1}{\omega_0^{1/2}} + \frac{\varepsilon}{2}\right) \tag{A.2}$$

$$e'_2 = \lambda_1\left(2 + \frac{1}{\varepsilon\omega_0^{1/2}}\right) + \frac{\lambda_3}{2\varepsilon m_0}, \qquad \lambda_1 = f\cdot\left[\int d^3\bar{k}\,\theta(\Lambda - k)\,|\,v(k)\,|^2\right]^{1/2} \tag{A.3}$$

$$\lambda_2 = \left[\int d^3\bar{k}\, \frac{k^2\,|\,\beta(k)\,|^2}{\omega(k)}\right]^{1/2}, \qquad \lambda_3 = \int d^3\bar{k}\, \frac{k^2\,|\,\beta(k)\,|^2}{\omega(k)^{1/2}} \tag{A.4}$$

ε being any strictly positive quantity with dimension $(\text{energy})^{-1/2}$ (for instance, $\omega_0^{-1/2}$). For fixed Λ, one has $\lambda_i < +\infty$ and $\lambda_i \to 0$ if $f \to 0$, $i = 1, 2, 3$.

Let us assume $e'_1 < 1$, which holds for suitably small f. Then, one shows, as in [*25*], that H' is a self-adjoint operator, whose domain is contained inside that of $H_0^{1/2}$ and that it is bounded below by $-e'_2$. Moreover, for any ψ_1, ψ_2 in the domain of H' and $\mathcal{H}_{\bar{\pi}}$, one has $(\psi_1, H'\psi_2) = (H_0^{1/2}\psi_1, H_0^{1/2}\psi_2) + F(\psi_1, \psi_2)$.

APPENDIX B

UPPER BOUND FOR $\| g_1(E_1, E_2) \|_{\bar{\pi}}$.

The bound (A.1) and lemma 3.1, page 336 in [*34*] ensures the existence of a bounded operator g_3 that

$$F((\mathbb{1} - Q_{\bar{\pi}})\psi_1, (\mathbb{1} - Q_{\bar{k}})\psi_2) = \left(\left(H_0 + \frac{e'_2}{e'_1}\right)^{1/2} (\mathbb{1} - Q_{\bar{\pi}})\psi_1, g_3\left(H_0 + \frac{e'_2}{e'_1}\right)^{1/2} (\mathbb{1} - Q_{\bar{\pi}})\psi_2\right)_{\bar{\pi}} \quad (B.1)$$

for any ψ_1, ψ_2 in $\mathcal{H}_{\bar{\pi}}$, and $\| g_3 \|_{\bar{\pi}} < e'_1$. Since the left-hand-side of Eq. (B.1) also equals

$$((H_0 + E' - E_1)^{1/2}(\mathbb{1} - Q_{\bar{\pi}})\psi_1, g_1(E_1, E_2)(H_0 + E' - E_2)^{1/2}(\mathbb{1} - Q_{\bar{\pi}})\psi_2)_{\bar{\pi}}$$

we derive easily:

$$\| g_1(E_1, E_2) \|_{\bar{\pi}} \leqslant \left\| (H_0 + E' - E_1)^{-1/2}\left(H_0 + \frac{e'_2}{e'_1}\right)^{1/2} (\mathbb{1} - Q_{\bar{\pi}}) \right\|_{\bar{\pi}} \cdot \left\| (\mathbb{1} - Q_{\bar{\pi}})\left(H_0 + \frac{e'_2}{e'_1}\right)^{1/2} (H_0 + E' - E_2)^{-1/2} \right\|_{\bar{\pi}} \cdot e'_1 \quad (B.2)$$

APPENDIX C

SOLUTION OF A RECURRENCE OF INEQUALITIES

Let us consider the following recurrence of inequalities for the unknowns x_n:

$$x_n \leqslant \tau'_{1,n} \cdot x_{n-1} + \tau'_{2,n} \cdot x_{n+1} + \tau'_{0,n}, \qquad n \geqslant 1 \tag{C.1}$$

where $\tau'_{1,n}$, $\tau'_{2,n}$ and $\tau'_{0,n}$ are known positive quantities. Let

$$Z'_r = \cfrac{1}{1 - \cfrac{\tau'_{1,r+1} \cdot \tau'_{2,r}}{1 - \cfrac{\tau'_{1,r+2} \cdot \tau'_{2,r+1}}{1 - \cfrac{\tau'_{1,r+3} \cdot \tau'_{2,r+2}}{1 - \ddots}}}}$$

Some lengthy, but direct, calculations show that the recurrence (C.1) implies the bounds:

$$x_n \leqslant Z'_n \cdot \left\{ \tau'_{1,n} \cdot x_{n-1} + \tau'_{0,n} + \sum_{l=1}^{+\infty} \tau'_{0,n+l} [(\tau'_{2,n} \cdot Z'_{n+1}) \cdot (\tau'_{2,n+1} \cdot Z'_{n+2}) \dots (\tau'_{2,n+l-1} \cdot Z'_{n+l})] \right\} \tag{C.3}$$

It is easy to check that the bound (C.3) for x_{n+1} in terms of x_n and the inequality (C.1) imply the bound (C.3) for x_n in terms of x_{n-1}.

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