

ANNALES DE L'I. H. P., SECTION A

IZU VAISMAN

A coordinatewise formulation of geometric quantization

Annales de l'I. H. P., section A, tome 31, n° 1 (1979), p. 5-24

http://www.numdam.org/item?id=AIHPA_1979__31_1_5_0

© Gauthier-Villars, 1979, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

A coordinatewise formulation of geometric quantization

by

Izu VAISMAN

Department of Mathematics,
University of Haifa, Israel

SUMMARY. — In this paper, geometric quantization via a complex integrable polarization is rediscussed. The main idea is to use extensively the local real-complex coordinates defined by the « integration » of the polarization. The main new result is the construction of a distinguished trivialization of the Kostant-Souriau line bundle, which allows a simple characterization of the wave functions. These will be square integrable and analytic in a part of the variables. Another new result is a pairing formula for two complex polarizations which are really transverse but have a common complex part. Metilinear structures are introduced only in an elementary form. The paper ends by applying the considered formulation of the geometric quantization to the harmonic oscillator, which simplifies Simms' discussion in [5].

MOS Subject classification 1970: 81-00; 81 A 12, 81 A 81.

The aim of this paper is to give a formulation of the geometric quantization procedure which uses local coordinates with respect to a specially adapted atlas on the basic symplectic manifold.

Except for a number of aspects, this formulation is equivalent to the standard geometric quantization with respect to a complex integrable polarization but, in our opinion, it is simpler to handle with in concrete cases.

The main new result is the construction of a *distinguished local triviali-*

zation of the Kostant-Souriau line bundle, allowing a definition of the wave functions which makes no direct use of differential operators. We can therefore consider *square-integrable wave functions* which is important in physical applications. Another new result is a pairing formula for two complex polarizations which are really transverse but have a common complex part.

We are referring to [6] for a standard exposition of geometric quantization and for bibliographical references which we fail to give here.

Except for the general wave functions, everything will be in the C^∞ category and this is without any further notice.

1. COMPLEX INTEGRABLE DISTRIBUTIONS AND ASSOCIATED HILBERT SPACES

Let M be a d -dimensional differentiable manifold, $T(M)$ its tangent bundle and $T^c(M) = T(M) \otimes C$. An m -dimensional complex distribution S on M is a field of m -subspaces of the fibers of $T^c(M)$. Then the field $\tilde{S} = S + \bar{S}$ can be constructed pointwise and S is called (*complex*) *integrable* if: *i*) S is closed by brackets, *ii*) \tilde{S} is a regular distribution and, again, it is closed by brackets.

Our basic starting point is

THEOREM (Nirenberg [4]). — *The distribution S is integrable iff every point $x \in M$ has a coordinate neighbourhood with coordinates (y^1, \dots, y^d) such that S be spanned by the local vector fields*

$$(1.1) \quad \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^{m-h}}, \frac{\partial}{\partial y^{m-h+1}} \\ + i \frac{\partial}{\partial y^{m+1}}, \dots, \frac{\partial}{\partial y^m} + i \frac{\partial}{\partial y^{m+h}} \quad (i = \sqrt{-1})$$

for some fixed integer $h \geq 0$.

The atlas defined by these local coordinates will be called *adapted to S* and, henceforth, the integrable distributions S will always be considered via their adapted atlases.

We shall make the following index conventions:

$$(1.2) \quad a, b, \dots = 1, \dots, m-h; \quad \alpha, \beta, \dots = m-h+1, \dots, m; \\ \bar{\alpha}, \bar{\beta}, \dots = m+1, \dots, m+h; \quad u, v, \dots = m+h+1, \dots, d; \\ j, k, \dots = 1, \dots, d.$$

Formulas (1.1) suggest introducing

$$(1.3) \quad z^\alpha = y^\alpha + iy^{\alpha+h}, \quad \bar{z}^{\bar{\alpha}} = \bar{z}^\alpha$$

after what S has the local bases

$$(1.4) \quad \left\{ \frac{\partial}{\partial y^a}, \frac{\partial}{\partial \bar{z}^\alpha} \right\}.$$

Now we have in the adapted atlas mixed (real-complex) local coordinates $(y^a, z^\alpha, \bar{z}^\alpha, y^\mu)$ and it is easy to see that every point of the intersection of two such coordinate neighbourhoods has an open neighbourhood where the coordinate transformation takes the form

$$(1.5) \quad \begin{aligned} \tilde{y}^a &= \tilde{y}^a(y^b, z^\alpha, \bar{z}^\alpha, y^\mu), \\ \tilde{z}^\alpha &= \tilde{z}^\alpha(z^\beta, y^\mu), \\ \tilde{y}^\mu &= \tilde{y}^\mu(y^\nu), \end{aligned}$$

and, particularly, \tilde{z}^α depend analytically on z^β .

Clearly, Formulas (1.5) are characteristic for an adapted atlas. Except for the analyticity of \tilde{z}^α , these formulas also show that there are two real foliations related to S. One is defined by the local equations $y^\mu = \text{const.}$; we denote it by Δ and call it the *large foliation*. The other one D, the *small foliation* is a subfoliation of Δ and is defined locally by $y^\mu = \text{const.}$, $z^\alpha = \text{const.}$, $\bar{z}^\alpha = \text{const.}$

The obtained structure allows considering various important classes of functions and other geometric objects on M by the following rules: i) everything which depends locally on the coordinates $z^\alpha, \bar{z}^\alpha, y^\mu$ only is called *D-foliate*; ii) everything which depends locally on y^μ only is called *Δ -foliate*; iii) everything which depends locally on y^μ and, analytically, on z^α is called *adapted*. Particularly, we shall make an essential use of *adapted functions* and *adapted bundles*, where the last are characterized by the existence of a local trivialization with adapted transition functions.

The main aim of this Section is to show that one can associate Hilbert spaces to some integrable distributions S. We do this by a usual pattern in geometric quantization [1].

We begin by imposing to S a supplementary condition which is equivalent to the existence of a so-called *metilinear structure*. Consider an intersection of two coordinate neighbourhoods U, \tilde{U} of the adapted atlas of S with the transition functions (1.5). Put

$$A_{U\tilde{U}} = \det \left(\frac{\partial z^\alpha}{\partial \tilde{z}^\beta} \right) \det \left(\frac{\partial y^\mu}{\partial \tilde{y}^\nu} \right).$$

Then the condition which we request is: (C) *there is an adapted subatlas for which one of the values of $(A_{U\tilde{U}})^{\frac{1}{2}}$ can be fixed continuously and such that*

$$(1.6) \quad (A_{U\tilde{U}})^{\frac{1}{2}} (A_{\tilde{U}\tilde{U}})^{\frac{1}{2}} (A_{\tilde{U}U})^{\frac{1}{2}} = 1.$$

(This is the so-called *cocycle condition* and a change of the sub-atlas leads to a *cohomologous cocycle*).

An S which satisfies (C) together with a fixed $(A_{U\bar{U}})^{\frac{1}{2}}$ will be called *meta-linear*. The interested reader is referred to Appendix B of [6] for the discussion of the existence of a metalinear structure on S . The existence condition is the vanishing of some characteristic class in $H^2(M, Z_2)$. We take this opportunity to note that the existence of an integrable S imposes by itself some rather restrictive topological conditions, e. g. the vanishing of the high enough Pontryagin classes of the transverse bundles of the foliations D and Δ [2].

Now, let S be an integrable metalinear distribution on M . Then, an *adapted half-form* of S is a geometric object (quantity [8]) having a single component ρ with respect to every local chart of the adapted atlas such that $\rho : U \rightarrow C$ is an adapted function (locally $\rho = \rho(z^\alpha, y^\mu)$ analytic in z^α) and the components for two charts are related by

$$(1.7) \quad \tilde{\rho} = (A_{U\bar{U}})^{\frac{1}{2}} \rho.$$

Obviously, the adapted half-forms are adapted sections of an adapted complex line bundle on M with transition functions $(A_{U\bar{U}})^{\frac{1}{2}}$, and we shall denote this line bundle by $L(S)$.

It is interesting to note that these objects are acted on by diffeomorphisms $\Phi : M \rightarrow M$ which preserve S . In fact, take two corresponding points x_0 and $y_0 = \Phi(x_0)$ in M and let be $(U; y^a, z^\alpha, y^\mu)$, $(U'; y'^a, z'^\alpha, y'^\mu)$ local adapted charts at x_0, y_0 . Then Φ is given locally by

$$(1.8) \quad y'^a = y'^a(y^b, z^\alpha, \bar{z}^\alpha, y^\mu), \quad z'^\alpha = z'^\alpha(z^\beta, y^\mu), \quad y'^\mu = y'^\mu(y^\nu)$$

and a continuous determination of

$$B_{U'U} = [\det(\partial z^\alpha / \partial z'^\beta) \det(\partial y^\mu \partial y'^\nu)]^{\frac{1}{2}}$$

can be fixed by choosing it arbitrarily at y_0 . Furthermore, if compatibility conditions are required, the similar quantity is fixed for any two charts at x_0, y_0 and, if M is connected, it is also fixed for charts at arbitrary points $x, y = \Phi(x)$ by going from y_0 to y through a chain of consecutively intersecting coordinate neighbourhoods. (If M is not connected, we must fix arbitrarily B at a point of every connected component of M .)

Now, a pull-back of half-forms by Φ is defined by

$$(1.9) \quad (\Phi_* \rho)_U(x) = B_{U'U} \rho_{U'}(y).$$

It is an important fact that Formulas (1.7) and (1.9) do not involve derivatives of ρ . Hence, we may also consider non-differentiable half-forms.

Before proceeding, let us impose one more condition for S . Namely, we shall ask that its small foliation D be *strongly regular* in the sense that the coset space $N = M/D$ of the leaves of D is a Hausdorff manifold. In

this case, N has an induced atlas with the local coordinates $(z^\alpha, \bar{z}^\alpha, y^\mu)$.

Then, the adapted half-forms of M are pull-backs of *half-forms of N* . Moreover, if ρ and ρ' are two adapted half-forms, then the components $\rho \bar{\rho}'$ define a *density* [8] on N , i. e. an object which can be integrated over N .

In this context, a (non-necessarily differentiable) adapted half-form ρ on M , whose components depend analytically on the z^α and for which $\rho \bar{\rho}$ is (Lebesgues) integrable over N $\left(\int_N \rho \bar{\rho} < +\infty \right)$ is called a *square integrable adapted half form* on M .

For two such half-forms ρ, ρ' , a *scalar product* can be defined by

$$(1.10) \quad \langle \rho, \rho' \rangle = \int_N \rho \bar{\rho}'$$

and (since a corresponding version of the Schwartz inequality is obviously available) we see that the square integrable adapted half-forms of M generate a Hilbert space $\mathcal{H}(M, S)$ which we shall call the *adapted Hilbert space* of (M, S) .

If $\Phi : M \rightarrow M$ is a diffeomorphism which preserves S , Φ_* of (1.9) acts as a unitary operator on the adapted Hilbert space.

It is also interesting to consider the pre-Hilbert subspace $\mathcal{H}'(M, S)$ of $\mathcal{H}(M, S)$ which consists of the differentiable adapted half-forms of M whose support projects to a compact subset of N . Namely, there is an interesting action of some tangent vector fields of M on $\mathcal{H}'(M, S)$.

Indeed, a real tangent vector field ξ on M is called *adapted* if $\exp(t\xi)$ preserve S and, in this case, the radicals needed for (1.9) can be fixed by requiring continuity with respect to t and fixing them to be 1 for $t = 0$. Then, a *Lie derivative* of half-forms ρ can be defined by the usual formula

$$(1.11) \quad L_\xi \rho = \frac{d}{dt} [\exp(t\xi)]_* \rho|_{t=0}$$

and it can be calculated by the general method in [10].

Namely, put

$$(1.12) \quad \xi = \xi^a \frac{\partial}{\partial y^a} + \lambda^\alpha \frac{\partial}{\partial z^\alpha} + \bar{\mu}^{\bar{\alpha}} \frac{\partial}{\partial \bar{z}^{\bar{\alpha}}} + \eta^\mu \frac{\partial}{\partial y^\mu}$$

This is real iff $\bar{\mu}^{\bar{\alpha}} = \bar{\lambda}^{\bar{\alpha}}$ and adapted iff $[\xi, \partial/\partial y^a]$ and $[\xi, \partial/\partial \bar{z}^{\bar{\alpha}}]$ belong to S , i. e. iff λ^α are adapted and η^μ are Δ -foliate functions. Then one gets [10]

$$(1.13) \quad L_\xi \rho = \lambda^\alpha \frac{\partial \rho}{\partial z^\alpha} + \eta^\mu \frac{\partial \rho}{\partial y^\mu} + \frac{\rho}{2} \left(\frac{\partial \lambda^\alpha}{\partial z^\alpha} + \frac{\partial \eta^\mu}{\partial y^\mu} \right),$$

which is again an (adapted) half-form.

Moreover, formally, the operator L_ξ can be defined by (1.13) for complex vector fields ξ given by (1.12) with adapted λ^α and Δ -foliate η^μ , i. e. fields which satisfy the bracket conditions stated above. These will be called

almost adapted fields or, if both their real and imaginary parts are adapted, *adapted fields*. The last means η^u - Δ -foliate, λ^α -adapted and $\mu^{\bar{\alpha}}$ -adapted.

It is by (1.13) that such fields act on $\mathcal{H}'(M, S)$ and this action is lineary. Moreover, the general formula [10]

$$L_{[\xi, \eta]} = L_\xi L_\eta - L_\eta L_\xi$$

shows that we have actually a representation of the Lie algebra of the almost adapted vector fields of M on $\mathcal{H}'(M, S)$.

Furthermore, an adapted field ξ projects to a well defined field $\tilde{\xi}$ on N . The last defines similarly a Lie derivative of densities φ on N which is given by [10]

$$(1.14) \quad L_{\tilde{\xi}}\varphi = \lambda^\alpha \frac{\partial \varphi}{\partial z^\alpha} + \mu^{\bar{\alpha}} \frac{\partial \varphi}{\partial \bar{z}^\alpha} + \eta^u \frac{\partial \varphi}{\partial y^u} + \left(\frac{1}{2} \frac{\partial \lambda^\alpha}{\partial z^\alpha} + \frac{1}{2} \frac{\partial \mu^{\bar{\alpha}}}{\partial \bar{z}^\alpha} + \hat{c}\eta^u \right) \varphi$$

and, when applied to a product $\rho \bar{\rho}'$ of half-forms, yields

$$(1.15) \quad L_{\tilde{\xi}}(\rho \bar{\rho}') = (L_{\tilde{\xi}}\rho) \bar{\rho}' + \rho \overline{(L_{\tilde{\xi}}\rho')},$$

where $\bar{\xi}$ is the complex conjugate field of ξ .

Now, by using the same proof like the one given in [8] for exact forms (see [9]), a variant of the Stokes formula can be obtained to the effect that

$$(1.16) \quad \int_N L_{\tilde{\xi}}\varphi = 0$$

for every tangent field $\tilde{\xi}$ of N and every density φ .

Hence (1.15) yields

$$(1.17) \quad \langle L_{\tilde{\xi}}\rho, \rho' \rangle + \langle \rho, L_{\bar{\xi}}\rho' \rangle = 0,$$

i. e., $-L_{\bar{\xi}}$ is the adjoint of $L_{\tilde{\xi}}$. If ξ is real $L_{\tilde{\xi}}$ is skew-Hermitian and if ξ is imaginary $L_{\tilde{\xi}}$ is a Hermitian operator on $\mathcal{H}'(M, S)$.

2. QUANTIZATION OF POLARIZED SYMPLECTIC MANIFOLDS

In this Section, the manifold M of Section 1 will be a symplectic manifold with $d = 2n$ and with a fundamental 2-form Ω satisfying $d\Omega = 0$. We shall also assume that it satisfies the so-called *integrality-condition* [6], i. e. that Ω represents via de Rham's theorem a real image of an integral cohomology class.

Furthermore, S will be a complex integrable n -dimensional distribution on M endowed with a metalinear structure, D -strongly-regular and such that $\Omega(\xi, \eta) = 0$ for every pair $\xi, \eta \in S$ (i. e. S is *Lagrangian*). Such an S will

be called a *nice polarization* of M and the triple (M, Ω, S) is a *polarized symplectic manifold*.

The quantization problem is that of representing functions on M by linear (Hermitian) operators on an associated Hilbert space, compatibly with the Poisson bracket.

We shall represent S , like in Section 1, by an adapted atlas and we shall use the same notation and the same index conventions, taking of course $d = 2n$ and $m = n$.

The fact that S is Lagrangian means

$$(2.1) \quad \Omega\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) = 0, \quad \Omega\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial \bar{z}^\alpha}\right) = 0, \quad \Omega\left(\frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}\right) = 0,$$

and we deduce the basic fact that, with respect to adapted coordinates one has

$$(2.2) \quad \Omega = A_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \theta_u \wedge dy^u,$$

where $A_{\alpha\bar{\beta}}$ is a skew-Hermitian matrix and $\theta_u = B_i dy^i$ are some Pfaff forms on the respective coordinate neighbourhood.

It is known from the Kostant-Souriau prequantization theory that there is a complex line bundle K on M endowed with a Hermitian metric h and with a Hermitian connection ∇ such that $2\pi i\Omega$ be the curvature form of ∇ . We shall fix such a K and add it to the configuration (M, Ω, S) . The following main result can now be proven:

THEOREM. — *With the notation above, K is an adapted line bundle on M and its metric h is D-foliate.*

Proof. — Take an adapted coordinate neighbourhood U which is trivializing for K and a basic cross-section σ of K over U . On U , the connection ∇ is defined by its local connection form

$$(2.3) \quad \alpha = C_a dy^a + C_\alpha dz^\alpha + C_{\bar{\alpha}} d\bar{z}^\alpha + C_u dy^u.$$

For this we have $d\alpha = 2\pi i\Omega$, which implies, in view of (2.2):

$$(2.4) \quad \frac{\partial C_a}{\partial y^b} = \frac{\partial C_b}{\partial y^a}, \quad \frac{\partial C_{\bar{\alpha}}}{\partial \bar{z}^\beta} = \frac{\partial C_{\bar{\beta}}}{\partial \bar{z}^\alpha}, \quad \frac{\partial C_a}{\partial \bar{z}^\alpha} = \frac{\partial C_{\bar{\alpha}}}{\partial y^a}.$$

Now, let us go over to the basic section $\sigma' = b'\sigma$. Then α is replaced by $\alpha + d \ln b'$ and we want to chose b' such that

$$\frac{\partial \ln b'}{\partial \bar{z}^\alpha} = - C_\alpha.$$

If U is taken sufficiently small, then, because of the second relation (2.4), the existence of such a b' follows from the classical Grothendick-Dolbeault lemma [3].

With respect to σ' , the connection ∇ is given by a form (2.3) where $C_{\bar{z}} = 0$ and hence, by the last relation (2.4), we also have that the new C_a are analytic in z^α .

Furthermore, we change again the basic section by $\sigma'' = b''\sigma'$, where b'' will be required to satisfy, with respect to the new coefficients C , the condition

$$\frac{\partial \ln b''}{\partial y^a} = -C_a.$$

The first relation (2.4) assures, by a well known lemma of Poincaré that such a b'' exists and, in view of the last relation (2.4), it is analytic in z^α .

Now, with respect to σ'' the local connection form becomes

$$(2.5) \quad \beta = \beta_\alpha dz^\alpha + \beta_u dy^u$$

and $d\beta = 2\pi i\Omega$ implies

$$(2.6) \quad \frac{\partial \beta_\alpha}{\partial y^u} = 0$$

i. e. β_α are D-foliate functions.

A basic local cross-section of K which satisfies (2.5) will be called *distinguished* and we just proved that K admits a local trivialization endowed with distinguished bases.

One can prove that the distinguished bases are given by the following geometric construction. Take a sufficiently small cubical adapted coordinate neighbourhood U and fix in it arbitrarily (by some equations $y^a = \text{const.}$, $z^\alpha = \text{const.}$) a slice Σ_0 transversal to the large foliation Δ and a differentiable basis σ_0 of K/Σ_0 . Consider next the transversal slice Σ'_0 of the small foliation D which contains Σ_0 and extend σ_0 to a basis σ'_0 of Σ'_0 which is parallel with respect to the fields $\partial/\partial \bar{z}^\alpha$. (The existence of σ'_0 is deducible from the Grothendieck-Dolbeault lemma.) Finally, translate σ'_0 parallelly along the slices of the small foliation to get the desired distinguished basis σ'' . The last is a correct operation since ∇ clearly induces a flat connection on the restriction of K to the leaves of the small foliation.

Now, if σ_1 and σ_2 are distinguished bases of K/U_1 and K/U_2 and if $\sigma_2 = f\sigma_1$ over $U_1 \cap U_2$, we must have for the corresponding connection forms $\beta_2 = \beta_1 + d \ln f$. Since $\beta_{1,2}$ are both of the form (2.5), we get $\partial f/\partial y^a = 0$, $\partial f/\partial \bar{z}^\alpha = 0$, i. e. f is an adapted function. This ends the proof of the fact that K is an adapted line bundle.

As for its metric h we have

$$\frac{\partial}{\partial y^a} [h(\sigma, \sigma)] = h(\nabla_{\partial/\partial y^a} \sigma, \sigma) + h(\sigma, \nabla_{\partial/\partial y^a} \sigma) = 0,$$

if σ is distinguished, since in this case $\nabla_{\partial/\partial y^a} \sigma = 0$ by (2.5). This is just the meaning of the fact that h is a D-foliate metric.

The stated Theorem is thereby proven.

In view of this Theorem it is now meaningful to speak of *adapted cross-sections* of the line bundle $K \otimes L(S)$ ($L(S)$ defined by the half forms of (M, S)). These will be sections which have local expressions of the form $\varphi(\sigma \otimes \delta)$ where σ and δ are respectively distinguished local bases of K and $L(S)$ and where φ is an adapted function. Moreover, we may admit non-differentiable functions φ which, however, are analytic with respect to the variables z^α .

Considering again the manifold $N = M/D$ we can try to define a scalar product for sections of $K \otimes L(S)$ by a natural extension of the formula

$$(2.7) \quad \langle s \otimes \rho, s' \otimes \rho' \rangle = \int_N h(s, s') \rho \bar{\rho}' ,$$

where s, s' are adapted sections of K and ρ, ρ' adapted half-forms. The integrand of (2.7) is clearly a density on N since the metric h is D -foliate.

The scalar product (2.7) makes sense for what we shall call *square integrable adapted sections* of $K \otimes L(S)$, i. e. sections for which the scalar product of the section with itself exists and which, also, are analytic in the z^α .

Then, these sections generate a Hilbert space which we denote by $\mathcal{H}(M, \Omega, S)$ and call it the *adapted Hilbert space of the triple* (M, Ω, S) . The elements of \mathcal{H} will be called *wave functions*.

Particularly, we get an interesting pre-Hilbert subspace of the adapted space if we take the wave functions which are also differentiable with respect to all the variables and whose support projects to a compact subset of N . This subspace will be denoted by $\mathcal{H}'(M, \Omega, S)$.

And now about operators.

Let $f : M \rightarrow \mathbb{R}$ be a differentiable function, i. e. on *observable* on M . Then, we shall denote by $\text{sg } f$ its *symplectic gradient* defined by

$$(2.8) \quad (\text{sg } f) \lrcorner \Omega = df .$$

This allows defining the Kostant-Souriau prequantization operator \hat{f} associated to f which, following [6], we shall take as

$$(2.9) \quad \hat{f}(s) = \frac{1}{2\pi i} \nabla_{\text{sg } f} s + fs ,$$

where s is a differentiable section of K .

If we put locally $s = t\sigma$, where σ is a distinguished basis of K , (2.9) becomes

$$(2.10) \quad \hat{f}(s) = \frac{1}{2\pi i} \{ [\text{sg } f](t) + \beta(\text{sg } f)t + 2\pi i f t \} \sigma ,$$

where β is the connection form (2.5).

Now it is natural to try extending \hat{f} to $\mathcal{H}'(\mathbf{M}, \Omega, \mathbf{S})$ by a definition of the type

$$(2.11) \quad \hat{f}(s \otimes \rho) = \hat{f}(s) \otimes \rho + \frac{1}{2\pi i} s \otimes L_{\text{sg } f} \rho.$$

This is not well defined for every f , however. First, for $L_{\text{sg } f} \rho$ to be adapted we have to ask $\text{sg } f$ to be an adapted field. Then, the first term of (2.10) is also adapted and we still have to ask that

$$\Xi = \beta(\text{sg } f) + 2\pi i f$$

be an adapted function.

Nevertheless, we can see that this last condition is implied by the first one. Indeed, if

$$\text{sg } f = \zeta^a \frac{\partial}{\partial y^a} + \lambda^\alpha \frac{\partial}{\partial z^\alpha} + \bar{\lambda}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} + \eta^\mu \frac{\partial}{\partial y^\mu}$$

we have by (2.8) and by $\Omega = (1/2\pi i)d\beta$ together with (2.5), (2.6):

$$\begin{aligned} \frac{\partial \beta_u}{\partial y^a} \eta^\mu + 2\pi i \frac{\partial f}{\partial y^a} &= 0, \\ \frac{\partial \beta_\alpha}{\partial \bar{z}^\beta} \lambda^\alpha + \frac{\partial \beta_u}{\partial \bar{z}^\beta} \eta^\mu + 2\pi i \frac{\partial f}{\partial \bar{z}^\beta} &= 0. \end{aligned}$$

Under the hypotheses that the functions λ^α are adapted and y^μ Δ -foliate these conditions are just

$$\frac{\partial \Xi}{\partial y^a} = 0, \quad \frac{\partial \Xi}{\partial \bar{z}^\beta} = 0,$$

i. e. Ξ is an adapted function.

Hence, (2.11) yields a well defined operator

$$\hat{f} : \mathcal{H}'(\mathbf{M}, \Omega, \mathbf{S}) \rightarrow \mathcal{H}'(\mathbf{M}, \Omega, \mathbf{S})$$

for every observable f for which $\text{sg } f$ is an adapted vector field on \mathbf{M} . \hat{f} will be called the *quantization* of f , and it follows easily from (1.15) and (1.16) that this \hat{f} is a Hermitian operator.

A particularly important case is obtained by asking f itself to be an adapted function, i. e., actually, a Δ -foliate function (since it is real). In this case, (2.10) and (2.11) yield

$$(2.12) \quad \hat{f}(s \otimes \rho) = f s \otimes \rho,$$

i. e. the quantization is simply multiplication by f . Clearly, such a quantization can be extended to arbitrary wave functions of the whole adapted Hilbert space $\mathcal{H}(\mathbf{M}, \Omega, \mathbf{S})$.

Note also the following commutation formula which follows from (2.9) and (2.11)

$$\widehat{f}\widehat{g} - \widehat{g}\widehat{f} = \frac{1}{2\pi i} \widehat{\{f, g\}},$$

where $\{f, g\}$ is the Poisson bracket of the two functions.

Next since we should be interested in quantizing more general observables, new instruments must be considered. The main idea now used in geometric quantization is based on *pairing* the Hilbert spaces of wave functions of two different polarizations (see [I], [6]). We shall present here a simple case when pairing is possible and which generalizes the case of two transverse real polarizations studied in [I] (*).

Let (M, Ω) be a symplectic manifold and S_1, S_2 two nice polarizations on M . We call them complementary polarizations if every point $x \in M$ has coordinate neighbourhoods $\{U_i; y_i^{a_i}, z_i^{\alpha_i}, y_i^{u_i}\}$ respectively adapted to S_i ($i = 1, 2$) such that the following transition relations hold

$$(2.13) \quad y_2^{a_2} = y_1^{u_1}, \quad z_2^{\alpha_2} = z_1^{\alpha_1}, \quad y_2^{u_2} = y_1^{a_1}$$

for some convenient ordering of the indices. Two such charts at x will also be called *complementary*.

It is clear now that those charts of the adapted atlas of S_1 which admit a complementary chart define an atlas of M whose coordinate transformations are locally of the form

$$(2.14) \quad \tilde{y}^a = \tilde{y}^a(y^b), \quad \tilde{z}^\alpha = \tilde{z}^\alpha(z^\beta), \quad \tilde{y}^u = \tilde{y}^u(y^v),$$

and which is (up to a permutation of the coordinates) a common adapted atlas of S_1 and S_2 .

With respect to this atlas, the form Ω becomes in view of (2.2) and of the analogon of (2.1) for S_2 :

$$(2.15) \quad \Omega = A_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta + B_{au} dy^a \wedge dy^u.$$

In this formula, every term is a well defined 2-form on M . We call $B = B_{au} dy^a \wedge dy^u$ the *kernel form* of the pair (S_1, S_2) and note that (2.14) imply the transformation law

$$(2.16) \quad B_{au} = \tilde{B}_{bv} \frac{\partial \tilde{y}^b}{\partial y^a} \frac{\partial \tilde{y}^v}{\partial y^u}.$$

Furthermore, suppose that S_1 and S_2 are transversally orientable with

(*) A more general pairing is discussed in R. J. Blattner, *The metilinear geometry of non-real polarizations*, Conference on « Differential Geometrical Methods in Mathematical Physics », Bonn, 1975, Lect. Notes in Math. 570, Springer-Verlag, Berlin, 1977.

respect to their large foliations, i. e. that the atlas we are working with can be assumed to satisfy the conditions

$$\det \left(\frac{\partial \tilde{y}^a}{\partial y^b} \right) > 0, \quad \det \left(\frac{\partial \tilde{y}^u}{\partial y^v} \right) > 0.$$

Then, if we denote by $\mathcal{H}_i = \mathcal{H}(M, \Omega, S_i)$ ($i = 1, 2$), a Hermitian pairing $(,) : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{C}$ can be defined by

$$(2.17) \quad (s_1 \otimes \rho_1, s_2 \otimes \rho_2) = c \int_M |\det(B_{au})|^{\frac{1}{2}} \hbar(s_1, s_2) \rho_1 \bar{\rho}_2.$$

This is well defined since by (1.7) and (2.16) the integrand of (2.17) is a density on M provided that the metalinear structures of S_1 and S_2 be conveniently choosen. (Again, we do not discuss the convergence of the considered integral.) As for c in (2.17), it is a constant number which may be conveniently chosen in every concrete case.

The relation between pairing and quantization is based on the following idea [1, 6]. If f is a real observable on M then $[\exp(t \operatorname{sg} f)]_* S = S_t$ should be again a nice polarization. If $\mathcal{H}(M, \Omega, S)$ and $\mathcal{H}(M, \Omega, S_t)$ can be paired in such a manner that the pairing define a unitary *interwinning operator* $U_t : \mathcal{H}(M, \Omega, S_t) \rightarrow \mathcal{H}(M, \Omega, S)$ then a one-parameter group of unitary transformations representing something of the kind $U_t \circ [\exp(t \operatorname{sg} f)]_*$ should be expected on $\mathcal{H}(M, \Omega, S)$. Its « generator » will be the quantization \hat{f} of f (See examples in [1, 6]).

A similar idea should be considered in discussing the relation between the quantizations defined by two different polarizations.

Of course, there are many other problems to be discussed such as the so-called Bohr-Sommerfeld conditions, etc. [7].

We shall end this section by a supplementary remark concerning the integrality condition. Namely, suppose it is not satisfied for (M, Ω) but it is satisfied for a lift (M', Ω') to some covering manifold M' of M . Then everything can be lifted to M' and we can quantize the observables on M by operators on the wave functions on M' . E. g., if the second homotopy group $\pi_2(M) = 0$, the integrality condition will always be satisfied on the universal covering space M' of M which can lead in this case to a quantization.

3. CLASSICAL EXAMPLES. THE HARMONIC OSCILLATOR

In the present Section we shall see how the formulation of the geometric quantization given in Section 2 works in some classical cases. We shall thereby compare this formulation with those already used in the literature (c. g. [6]).

The basic case is

$$(3.1) \quad M = \mathbb{R}^{2n} = \{ (p_a, q^a) \mid a = 1, \dots, n \}, \quad \Omega = dp_a \wedge dq^a$$

(with the summation convention, of course).

Various polarizations are then available, two of them being used most often. Namely, the real polarization S_1 defined by the Δ -transverse coordinates q^a (i. e. the complexification of the tangent distribution of the leaves of the foliation $q^a = \text{const.}$), and the real polarization S_2 with the Δ -transverse coordinates p_a . These are clearly complementary polarizations.

S_1 -quantization is quite simple. Namely, Ω is integral and exact, hence we can take a trivial line bundle K with the basic section 1 , with respect to which the Hermitian metric is given by $h(1, 1) = 1$ and the connection is given by the connection form $\beta = 2\pi i p_a dq^a$. (2.5) shows that 1 is actually a distinguished basis. Since, on the other side, there is an adapted atlas consisting of a single chart, we see that the wave functions are just square integrable complex valued functions $\varphi(q^a)$. It is also easy to see that (2.11) yields quantizations for those observables which are at most linear with respect to the variables p_a .

As for S_2 , we have the same connection, but 1 is no more a distinguished basis. Using the method of the proof of the Theorem given in Section 2, we see that

$$\sigma = e^{-2\pi i p_a q^a}$$

yields a distinguished basis of K , since the connection form is now $\beta + d \ln \sigma = -2\pi i q^a dp_a$. Hence the S_2 -wave functions are of the form

$$(3.2) \quad \chi = e^{-2\pi i p_a q^a} \psi(p_a),$$

where $\psi(p_a)$ is a square integrable function.

The coefficients of the kernel form B (which, in this case, equals Ω) form the unit matrix, and by (2.17) we obtain the pairing formula

$$(3.3) \quad (\varphi, \chi) = c \int_M e^{2\pi i p_a q^a} \varphi(q^a) \bar{\psi}(p_a),$$

which can be arranged to lead to the Fourier transform as intertwining operator [6].

A third interesting polarization can be obtained on M in the following manner. Put

$$(3.4) \quad z^a = p_a - i q^a \quad (a = 1, \dots, n).$$

Then we get

$$(3.5) \quad \Omega = -\frac{i}{2} \sum_{a=1}^n dz^a \wedge d\bar{z}^a$$

and we see that $\{ \partial/\partial \bar{z}_a \}$ span a nice polarization S_3 for which the leaves of the small foliation are the points of M .

We take again as K the trivial bundle with the basis 1 and note that

$$(3.6) \quad \omega = \frac{\pi}{2} (z^a d \bar{z}^a - \bar{z}^a dz^a)$$

is a Hermitian connection for the metric $h(1, 1) = 1$, whose curvature is $2\pi i \Omega$.

Then

$$(3.7) \quad \tau = e^{-\frac{\pi}{2} \sum_{a=1}^n z^a \bar{z}^a}$$

defines a distinguished basis with respect to which the connection form

$$\text{becomes } \varpi = -\pi \sum_{a=1}^n \bar{z}^a dz^a.$$

It follows that the wave functions are functions of the form

$$(3.8) \quad \Psi = e^{-\frac{\pi}{2} \sum_{a=1}^n z^a \bar{z}^a} \psi(z^a),$$

where ψ are complex analytic functions in z^a , and their scalar product is

$$(3.9) \quad \langle \Psi_1, \Psi_2 \rangle = \int_M e^{-\pi \sum_{a=1}^n z^a \bar{z}^a} \psi_1 \bar{\psi}_2$$

(to be compared with a formula of Bargman mentioned in [6, p. 109]).

Finally, we want to make a more complete discussion of an important physical example, which is that of a *harmonic oscillator*. This could be done as in [6] by means of the above polarization S_3 , but we shall prefer to proceed like in Simms [5] since this provides a fuller illustration of the general schema of Section 2.

Following [5], the harmonic oscillator is defined by the symplectic manifold

$$(3.10) \quad M = \mathbb{R}^{2n} - \{0\}, \quad \Omega = h^{-1} \sum_{a=1}^n dp_a \wedge dq^a,$$

where $\mathbb{R}^{2n} = \{(p_a, q^a)\}$ and h is the Planck constant.

Its Hamiltonian is the function

$$(3.11) \quad H = \frac{1}{2m} \sum_{a=1}^n [(p_a)^2 + k^2(q^a)^2],$$

where $k = m\chi$, m is the mass and χ the frequency of the oscillator, and the problem is to get a representation of H by a quantum operator.

Consider the complex coordinates [5]

$$(3.12) \quad z^a = \frac{1}{(2m)^{\frac{1}{2}}} (p_a - ikq^a).$$

Then we have

$$(3.13) \quad H = \sum_{a=1}^n z^a \bar{z}^a = r^2$$

and we should expect a simple quantization of H by the help of a polarization for which r is a Δ -transverse coordinate.

Now, since we want r to be a coordinate it is natural to try some kind of polar coordinates. Following [5], we put

$$(3.14) \quad M = \bigcup_{j=1}^n U_j, \quad U_j = \{ z \in M / z^j \neq 0 \}$$

and define on U_j the local coordinates (t_j, u_j^k, r) , where

$$(3.15) \quad z^j = |z^j| e^{it_j}, \quad u_j^k = z^k / z^j \quad (k \neq j),$$

$$z^j = r e^{it_j} \left(1 + \sum_{h \neq j} u_j^h \bar{u}_j^h \right)^{-\frac{1}{2}},$$

$$(3.16) \quad z^k = r e^{it_j} u_j^k \left(1 + \sum_{h \neq j} u_j^h \bar{u}_j^h \right)^{-\frac{1}{2}}, \quad (i = \sqrt{-1}).$$

Hereafter we change our index convention and agree that $h, j, k = 1, \dots, n$. Then, we get on $U_j \cap U_h$

$$(3.17) \quad u_j^h = \frac{1}{u_h^j}, \quad u_j^k = \frac{u_h^k}{u_h^j} \quad (k \neq j, h), \quad r = r,$$

and it follows that we have here an adapted atlas, while the corresponding polarization S is generated over U_j by $\{ \partial / \partial t_j, \partial / \partial \bar{u}_j^k \}$ ($k \neq j$). (The ambiguity in the definition of t_j in (3.15) results in translations of this coordinate by multiples of 2π , which does not influence S.)

This S actually is a polarization since it follows from (3.10) and (3.12) that

$$(3.18) \quad \Omega = -i\kappa \sum_{a=1}^n dz^a \wedge d\bar{z}^a = -i\kappa \left\{ \sum_{k \neq j} \Phi^2 du_j^k \wedge d\bar{u}_j^k + \frac{\Phi^4}{r^2} \left(\sum_{h \neq j} u_j^h d\bar{u}_j^h \right) \wedge \left(\sum_{h \neq j} \bar{u}_j^h du_j^h \right) + \left[2irdt_j + \frac{\Phi^2}{r^2} \sum_{k \neq j} (\bar{u}_j^k du_j^k - u_j^k d\bar{u}_j^k) \right] \wedge dr \right\},$$

where $\varkappa = 1/(h\chi)$, $\Phi = r \left(1 + \sum_{k \neq j} u_j^k \bar{u}_j^k \right)^{-\frac{1}{2}}$, and this shows that the conditions corresponding to (2.1) are satisfied.

Moreover, S is a nice polarization. Indeed, the definition of the local coordinates (t_j, u_j^k, r) yields a diffeomorphism

$$\mathbf{R}^{2n} - \{0\} \approx \mathbf{CP}^{n-1} \times \mathbf{R}_+ \times \mathbf{S}^1,$$

where \mathbf{S}^1 is the unit circle, \mathbf{R}_+ is the set of the positive real numbers and \mathbf{CP}^{n-1} is the $(n-1)$ -dimensional complex projective space. Now, the leaves of the small foliation \mathbf{D} of the polarization S correspond to \mathbf{S}^1 and it follows that S is \mathbf{D} -strongly regular with the quotient manifold $\mathbf{M}/\mathbf{D} \approx \mathbf{CP}^{n-1} \times \mathbf{R}_+$.

Finally, S admits metalinear structures. This follows by cohomology arguments which yield the precise result that there are essentially two such structures for $n = 1$ and one for $n > 1$ [5].

This ends the proof of the fact that S is nice.

We shall give in the sequel a straightforward elementary construction of the metalinear structures of S .

For $n = 1$, only the transverse coordinate r is to be considered in the plane $\mathbf{M} = \mathbf{U}_1$ of the complex coordinate z^1 with deleted origin. In order to get both structures, let us put $\mathbf{M} = \mathbf{U}_1^+ \cup \mathbf{U}_1^-$, where

$$(3.19) \quad \begin{aligned} \mathbf{U}_1^+ &= \{ z^1 = re^{it_1} \mid 0 < t_1 < 2\pi \}, \\ \mathbf{U}_1^- &= \{ z^1 = re^{it_1} \mid -\pi < t_1 < \pi \}, \end{aligned}$$

and $\mathbf{U}_1^+ \cap \mathbf{U}_1^-$ has two connected components: $0 < t_1 < \pi$ and $\pi < t_1 < 2\pi$.

Now, the transition function for the coordinate r is always equal to 1, which is also its determinant, and we can take

$$(3.20) \quad \begin{aligned} a) \quad \sqrt{1} &= 1 \text{ on the whole of } \mathbf{U}_1; \\ b) \quad \sqrt{1} &= \begin{cases} 1 & \text{on } 0 < t_1 < \pi, \\ -1 & \text{on } \pi < t_1 < 2\pi. \end{cases} \end{aligned}$$

This clearly defines the two metalinear structures of S in the case $n = 1$.

For $n > 1$, a simple calculation based on (3.17) yields

$$(3.21) \quad \det \left(\frac{\partial u_j^k}{\partial u_h^l} \right) \det \left(\frac{dr}{dr} \right) = \frac{(-1)^h (z^h)^n}{(-1)^j (z^j)^n}$$

and it is obviously possible to fix the radical if n is even.

For an arbitrary n , we put $\mathbf{U}_n = \mathbf{U}_n^+ \cup \mathbf{U}_n^-$, where the two subsets are defined by (3.20) with the index 1 replaced by h , and fix arbitrarily a continuous value of $(z^h)^{\frac{1}{2}}$ on \mathbf{U}_n^+ and on \mathbf{U}_n^- . But then we must also consider the intersections $\mathbf{U}_n^+ \cap \mathbf{U}_n^-$ with the identity as a coordinate transformation

and 1 as its determinant. For them $\sqrt{1}$ will be fixed on every connected component of $U_h^+ \cap U_h^-$ in such a manner that $\sqrt{1}(z^h)_{\pm}^{\frac{1}{2}} = \sqrt{1}(z^h)_{\pm}^{\frac{1}{2}}$ on the respective component. It is now simple to see that this provides a metalinear structure on S . (Of course, we shall also have to choose $\sqrt{(-1)^h}$, but this can be done arbitrarily for every index h .)

We shall discuss, next, adapted half-forms and wave functions.

We begin by noticing that there is a global basic half-form β in each one of the cases above. Namely, for $n = 1$ we take $\beta = 1$ on U_1 in the case (3.20) a), and $\beta = 1$ on U_1^+ , $\beta = -1$ on U_1^- in the case (3.20) b). Then, every adapted half-form is of the type $\rho = F(r)\beta$.

For $n > 1$, the basic half-form β is defined by taking $\beta_h = (-1)^{h/2}(z^h)^{n/2}$ on U_h ($h = 1, \dots, n$), where the radicals are the fixed ones. Hence, again every half-form is of the type $\rho = F\beta$ where F is a globally defined function on M . But β is not adapted and, therefore, the condition that ρ be adapted must be added separately.

Let us go over now to the Kostant-Souriau line bundle K . Since Ω is exact, K is to be taken trivial, i. e. $K = M \times C$ with the initial basis 1 and the Hermitian metric $h(1, 1) = 1$.

In view of (3.18), it follows that the corresponding Hermitian connection is given by the connection form

$$(3.22) \quad \omega = \pi\kappa \sum_{a=1}^n (z^a d \bar{z}^a - \bar{z}^a dz^a),$$

which using (3.16) is represented in U_j by

$$(3.23) \quad \omega_j = -\pi\kappa \left\{ \Phi^2 \sum_{h \neq j} (\bar{u}_j^h du_j^h - u_j^h d\bar{u}_j^h) + 2ir^2 dt_j \right\},$$

with the same Φ like in (3.18).

Next, to go over to a distinguished basis of K we put first ω_j under the equivalent form

$$(3.24) \quad \omega_j = -2\pi\kappa r^2 \left\{ d \left(\ln \frac{\Phi}{r} \right) + idt_j + \frac{\Phi^2}{r^2} \sum_{h \neq j} \bar{u}_j^h du_j^h \right\}.$$

Then, if we define over U_j the local section

$$(3.25) \quad b_j = e^{2\pi\kappa r^2 \ln z^j}$$

(where determination is obtained by putting $U_j = U_j^+ \cup U_j^-$ as before) we get new connection forms $\varpi_j = \omega_j + d \ln b_j$, and these are

$$(3.26) \quad \varpi_j = -2\pi\kappa \Phi^2 \sum_{h \neq j} \bar{u}_j^h du_j^h + 2\pi\kappa r(1 + 2 \ln z^j) dr.$$

Hence the b_j yield a distinguished basis of K and an adapted section of this line bundle is a complex valued function s on M such that $s_j = sb_j^{-1}$ are adapted functions for every $j = 1, \dots, n$.

It follows that a generating element of the adapted Hilbert space $\mathcal{H}(M, \Omega, S)$, which is of the form $s \otimes \rho$ is given by

$$(3.27) \quad s \otimes \rho / U_j = s_j \rho_j \beta_j^{-1} b_j (1 \otimes \beta),$$

where s_j and ρ_j are adapted functions. Hence a general adapted section γ of $K \otimes L(S)$ is defined by its restrictions

$$(3.28) \quad \gamma / U_j = \gamma_j \beta_j^{-1} b_j (1 \otimes \beta),$$

where γ_j is adapted on U_j .

More exactly, for the case $n = 1, a)$ we have

$$(3.29) \quad \gamma / U_1 = v(r) e^{2\pi\kappa r^2 i t_1} \cdot 1,$$

where 1 is the basic section and we fix the argument t_1 on U_1^+ and U_1^- .

For the case $n = 1, b)$, we have

$$(3.30) \quad \begin{aligned} \gamma / U_1^+ &= v(r) e^{2\pi\kappa r^2 i t_1} \cdot 1, \\ \gamma / U_1^- &= -v(r) e^{2\pi\kappa r^2 i t_1} \cdot 1. \end{aligned}$$

Both in (3.29) and in (3.30) $v(r)$ has to be a square integrable function.

For the case $n \geq 2$, we have

$$(3.31) \quad \begin{aligned} \gamma / U_j &= (-1)^{-j/2} (z^j)^{-n/2} \gamma_j(u_j^k, r) e^{2\pi\kappa r^2 \ln z^j} (1 \otimes \beta) \\ &= v_j(u_j^k, r) \Phi^{-n/2} e^{2\pi\kappa r^2 \ln \Phi} e^{\frac{4\pi\kappa r^2 - n}{2} i t} (1 \otimes \beta), \end{aligned}$$

where we must take separately the determination of t_j on U_j^+ and U_j^- , and v_j are square integrable functions which are analytic with respect to u_j^k .

Moreover, because of the ambiguity in the determination of t_j , the fact that γ is a global section of $K \otimes L(S)$ implies:

- i) in the case $n=1, a)$, $v(r) \neq 0$ only where $2\pi\kappa r^2$ is an integer;
- ii) in the case $n=1, b)$, $v(r) \neq 0$ only where $2\pi\kappa r^2$ is the half of an odd integer;

- iii) in the case $n \geq 2$, $v_j \neq 0$ only where $\frac{4\pi\kappa r^2 - n}{2}$ is an integer.

Consider the function

$$\delta(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \neq 0, \end{cases}$$

Then, it follows from the discussion above that we can get wave functions if we take respectively, in (3.29), (3.30), (3.31) the following:

- i) in (3.29), $v(r) = v'(r) \delta(2\pi\kappa r^2 - K)$,

ii) in (3.30), $v(r) = v'(r)\delta\left(2\pi\kappa r^2 - K - \frac{1}{2}\right),$

iii) in (3.31), $v_j = v'_j(u_j^h, r)\delta\left(\frac{4\pi\kappa r^2 - n}{2} - K\right),$

where the v' are square integrable and analytic in u_j^h , and $K = 0, 1, 2, \dots$

In agreement with Section 2, the quantization of the energy H given by (3.13) consist in multiplication of the wave functions by r^2 , and it is easy to see that we get, correspondingly, the following eigenvalues of this operator:

i) $n = 1, a): \quad \lambda = \frac{K}{2\pi\kappa} = Kh\zeta,$

ii) $n = 1, b): \quad \lambda = \left(K + \frac{1}{2}\right)h\zeta,$

iii) $n \geq 2: \quad \lambda = \left(K + \frac{n}{2}\right)h\zeta.$

where $h = \hbar/2\pi$.

These are the classical energy levels for the harmonic oscillator [5].

Finally, in the case $n \geq 2$, we have one more condition for γ to be a global section. Namely, the form of the functions v'_j at points where $4\pi\kappa r^2 - n = 2K$ is determined by asking that $\gamma/U_j = \gamma/U_k$ for every pair (j, k) . It follows from (3.31) that this condition means

$$(3.32) \quad v'_j\left(\frac{z^h}{z^j}\right)(z^j)^K = v'_k\left(\frac{z^h}{z^k}\right)(z^k)^K,$$

where the functions v' are analytic in their $n - 1$ arguments.

By replacing in (3.32) the functions v' by corresponding Taylor developments we see that the equality cannot hold unless v' are polynomials of total degree $\leq K$. (Particularly, this shows why we must take in this case also $K \geq 0$.)

Clearly, the number of linearly independent such polynomials gives us the multiplicity of the corresponding eigenvalue and this number is $\binom{K+n-1}{K}$, which is again in agreement with the classical results [5].

REFERENCES

[1] R. J. BLATTNER, Quantization and representation theory, in: Harmonic Analysis on Homogeneous Spaces, A. M. S., *Proc. Symp. Pure Math.*, **XXVI**, 1974, p. 147-165.
 [2] R. BOTT, Lectures on characteristic classes and foliations, in: *Lecture Notes in Math.*, t. **279**, Springer-Verlag, Berlin, 1972, p. 1-94.
 [3] S. S. CHERN, *Complex manifolds without potential theory*. D. Van Nostrand, Comp., Princeton, New Jersey, 1967.

- [4] L. NIRENBERG, A complex Frobenius theorem, in: *Sem. on Analytic Functions, Inst. for Adv. Study*, Princeton, vol. 1, 1957, p. 172-179.
- [5] D. J. SIMMS, Metalinear structures and a geometric quantization of the harmonic oscillator, in: Colloque du CNRS, n° 237, *Géométrie symplectique et physique mathématique*, Aix-en-Provence, 1974, p. 163-173.
- [6] D. J. SIMMS and N. M. J. WOODHOUSE, Lectures on geometric quantization. *Lecture Notes in Physics*, 53, Springer-Verlag, Berlin, 1976.
- [7] J. Sniatycki and S. TOPOROWSKI, On representation spaces in geometric quantization, *Int. J. of Theoretical Physics*, t. 16, 1977, p. 615-633.
- [8] S. ŠTERNBERG, *Lectures on differential geometry*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1976.
- [9] I. VAISMAN, *Basic ideas of geometric quantization*, *Rend. Sem. Mat.*, Torino (to appear).
- [10] K. YANO, *The theory of Lie derivatives and its applications*, North Holland Publ., Amsterdam, 1957.

(Manuscrit reçu le 4 mai 1979)