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On the role of certain symmetry properties in relativistic kinetic theory and thermodynamics

by

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ABSTRACT. — This paper focuses attention on the important role played by certain symmetry properties (when they are admitted) in examining thermodynamic or kinetic theory models for matter field space-times. Symmetry properties belonging to the family of contracted Ricci collineations are of particular interest. The members of this family of symmetry mappings are rather diverse in nature but all satisfy the relation $g^{ij}\mathscr{L}R_{ij} = 0$. Among other topics, in the area of thermodynamics the connection of these symmetry properties with certain equations of state is investigated. Also the relationship between FCRC symmetry properties and « matter symmetries » (Berezdivin and Sachs, 1973) of kinetic theory is considered.

1. INTRODUCTION

In this paper we wish to draw further attention to the important role played by certain symmetry mappings and the related local conservation laws when they are admitted by matter field space-times (1) (MFS). Recent investigations [*1-5*] have examined some of the consequences of conformal

(1) A matter field space-time is defined to be a space-time with a matter tensor that has a unique timelike eigenvector $u^i(u^iu_i = 1$ with signature of metric $-2)$ with positive eigenvalue ρ.

motions and symmetry mappings which satisfy (2) $g^{ij}\mathscr{L}R_{ij} = 0$ [*i. e.*, members of the family of contracted Ricci collineations (3) (FCRC)] in terms of the hydrodynamic description and the evolution of a MFS. In conjunction with the hydrodynamic description of a MFS, one often uses thermodynamic considerations and sometimes it is appropriate to consider an underlying kinetic theory model of the MFS. Here we will be concerned with the consequences of the above mentioned symmetry mappings (when admitted) in relation to the thermodynamic and kinetic theory models of the MFS which admit these symmetries.

In Section 2 we will briefly review the thermodynamic and kinetic theory descriptions of a MFS. Also, we will review some results of earlier symmetry investigations that relate to the topic at hand. In Section 3 we investigate the thermodynamic properties of perfect fluid MFS admitting degenerate and nondegenerate Ricci collineations. The results of this investigation are stated in the form of several theorems which, among other things, serve to partially determine the explicit form of the equations of state. The role of symmetries in kinetic theory is investigated in Section 4. In particular, the possible connections between the symmetry mappings constituting members of the FCRC (see Symmetry Property Inclusion Diagram (4) and the so-called matter symmetries introduced by Berezdivin and Sachs [*8*] are considered together with suggested generalizations of the properties defining matter symmetries. Section 5 provides a summary of the results obtained in this paper and comments relating to some areas for further work.

2. REVIEW OF CERTAIN RESULTS FROM KINETIC THEORY, THERMODYNAMICS, AND EARLIER SYMMETRY INVESTIGATIONS

The natural structure for relativistic kinetic theory (5) is the tangent bundle (TM) over the space-time manifold (M). The tangent bundle [*11*]

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(2) In accord with the notations and definitions used by J. A. Shouten [*6*], here and throughout the paper we use *i*) ∇_k for the operation of covariant differentiation, *ii*) $\mathscr{L}$ for the operation of Lie differentiation with respect to the vector ξ^i (unless otherwise noted) and *iii*) round and square brackets on indices for the operations of symmetrization and antisymmetrization, respectively.

(3) The FCRC family of symmetries was first introduced by Davis, Green and Norris [*8*]. A more thorough discussion of this family of symmetries and the related conservation expression generator $[\nabla_j(\sqrt{-g}\,R^j_k\xi^k) = 0]$ can be found in Davis and Oliver [*5*]. Here we simply note that nondegerate FCRC members can be expressed as conditional relations which take the form $\mathscr{L}R_{ij} = H_{ij}$, $g^{ij}H_{ij} = 0$.

(4) A more complete version of this diagram and other references relating to these symmetries can be found in Davis, Green and Norris [*7*].

(5) For a more complete description of relativistic kinetic theory see J. Ehlers [*9*], J. M. Stewart [*10*] and the references in these papers.

is an eight-dimensional manifold with a point of TM determined by a point x of M and a tangent vector at x. The set of all points in TM determined by a point x of M is denoted the fibre over x and consists of all tangent vectors at x. An important set of local coordinates for TM is given by (x^a, p^a) where x^a are local coordinates for the point x of M and p^a are the contravariant components of the vector with respect to the coordinate basis $\{ \partial/\partial x^a \}$. The above coordinates and a connection on M lead to the connection basis for vectors on TM, $\{ D_a, \partial/\partial p^a \}$ where $D_a = (\partial/\partial x^a - \Gamma^c_{ab}p^b(\partial/\partial p^c))$. A vector on TM will be called horizontal (vertical) if its last (first) four components are zero with respect to the connection basis. In the absence of a macroscopic electromagnetic field, an important horizontal vector field in relativistic kinetic theory is $L = p^a D_a$, the geodesic spray or Liouville operator.

Relativistic kinetic theory ([5]) is concerned with the determination of the distribution of particles on TM as represented by the distribution function f which satisfies the relativistic Boltzmann equation $L(f) = C(f)$ where $C(f)$ is the collision term. If the distribution function is given, then the energy-momentum tensor, particle four-current density and entropy flux density of the matter can be determined as follows:

$$T^{ab}(x) = \int f p^a p^b \sqrt{-g}\, d^4p\,, \qquad (2.1)$$

$$N^a(x) = \int f p^a \sqrt{-g}\, d^4p \qquad (2.2)$$

and

$$S^a(x) = \int (f \ln f) p^a \sqrt{-g}\, d^4p\,. \qquad (2.3)$$

Using Boltzmann's equation one can show that these quantities obey the following relations:

$$\nabla_a T^{ab} = 0\,, \qquad (2.4)$$

$$\nabla_a N^a = 0 \qquad (2.5)$$

and

$$\nabla_a S^a \geqq 0\,. \qquad (2.6)$$

Solutions to the relativistic Boltzmann can be divided into two classes. First, there are solutions where the matter can be viewed as test particles moving in a fixed background geometry. Second, there are solutions where the matter described by the distribution function generates the geometry via Einstein's equation

$$\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = \varkappa T_{ab} \qquad (2.7)$$

where T_{ab} is given by equations (2.1).

([5]) Voir note précédente.

Equilibrium solutions are a simple but important set of solutions with members in both previously mentioned classes. Equilibrium solutions are defined by $\nabla_a S^a = 0$ whereas general solutions satisfy $\nabla_a S^a \geqq 0$ (the relativistic version of the H-theorem). The equilibrium condition can be shown to be equivalent to $L(f) = 0$ (Liouville's equation). Solutions to Liouville's equation can be further classified according to the reason for the vanishing of $C(f)$. For the first class of solutions $C(f)$ vanishes because there are no collisions while for the second class $C(f)$ vanishes because detailed balancing occurs. The second class of equilibrium solutions and solutions « near » to them are particularly important because from them one can derive the following thermodynamical results. For solutions to Liouville's equation of the second class one can show that (6)

$$T^{ab} = \rho u^a u^b - p\gamma^{ab},$$
$$N^a = nu^a,$$
$$S^a = su^a,$$

and

$$d\rho = T ds + n^{-1}(\rho + p - Ts)dn. \tag{2.8}$$

Here, ρ, n, s, p and T are respectively the energy density, mean particle density, mean entropy density, isotropic pressure and temperature as observed in the rest frame of u^a. For solutions « near » to detailed balancing equilibrium solutions, one can demonstrate the following:

$$\begin{aligned} T^{ab} &= \rho u^a u^b + 2u^{(a}q^{b)} - p\gamma^{ab} + \pi^{ab}, \\ N^a &= nu^a, \\ S^a &= su^a + s^a \quad \text{with} \quad s^a = q^a/T, \\ d\rho &= T ds + n^{-1}(\rho + p - Ts)dn, \\ q^a &= \lambda\gamma^{ab}(\partial_b T - T a_b), \end{aligned} \tag{2.9}$$

and

$$\pi_{ab} = \xi\theta\gamma_{ab} + 2\eta\sigma_{ab}.$$

Here q^a and s^a are respectively the heat flow and entropy diffusion current as measured in the rest frome of u^a and λ, ξ and η are non-negative constants.

Some results have already been obtained in the area of relating symmetry properties to relativistic kinetic theory and thermodynamics. For example, detailed balancing equilibrium solutions imply the existence of a timelike motion or conformal motion (7). Also, the relation between Killing vectors and solutions of Liouville's equation has been investigated (8). In addition,

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(6) The quantities which describe a timelike curve congruence with unit tangent vector u^i are defined as follows: *i*) the expansion $\theta = \nabla_j u^j$, *ii*) the acceleration $a_i = u^j \nabla_j u_i$, *iii*) the rotation $\omega_{ij} = \nabla_{[j}u_{i]} - a_{[i}u_{j]}$ and *iv*) the shear $\sigma_{ij} = \nabla_{(j}u_{i)} - a_{(i}u_{j)} - \frac{1}{3}\theta\gamma_{ij}$ where $\gamma_{ij} = g_{ij} - u_i u_j$ is the projection tensor.

(7) See, for example, Ehlers [*12*].

(8) See, for example, the article by Ehlers [*12*] and more recently the article by Ray and Zimmerman [*13*].

the properties of locally dynamic symmetric solutions [*10*, *14-16*] have been examined. Berezdivin and Sachs [*8*] have considered matter symmetries and their relation to motions. In the area of thermodynamics, Horwitz and Katz [*17*] have shown how Killing vectors can lead from local to global thermodynamics. The main restriction of the results just mentioned in this section is that they usually involve the existence of a Killing vector. In Sections 3 and 4 we will investigate some of the implications of certain FCRC symmetry properties more general than motions on a thermodynamic or a kinetic theory model of a MFS.

3. THE ROLE OF SYMMETRIES IN RELATIVISTIC THERMODYNAMICS

Horwitz and Katz [*17*] have shown that the existence of certain Killing vectors can lead from local definitions of thermodynamic quantities to global definitions. However, in general, a space-time will not admit these Killing vectors. In this section we will show how symmetry properties less restrictive than motions can partially determine the local thermodynamics of a MFS. In particular we will examine the effect of a Ricci collineation with symmetry vector proportional to the timelike eigenvector of the Ricci tensor on the thermodynamics of a perfect fluid MFS.

The matter tensor and the thermodynamics of a perfect fluid MFS have been previously given in Section 2 [equations (2.8)]. In addition, the familiar « dynamical » and « conservation » equations for the fluid follow from $\nabla_j T^{ij} = 0$ and take the form

$$(\rho + p)a_i = \gamma_i^k \partial_k p \tag{3.1}$$

and

$$u^i \partial_i \rho + (\rho + p)\theta = 0 . \tag{3.2}$$

Also Einstein's equations can be used to show that the Ricci tensor is given by

$$\varkappa^{-1} R_{ij} = \frac{1}{2}(\rho + 3p)u_i u_j - \frac{1}{2}(\rho - p)\gamma_{ij} . \tag{3.3}$$

Using the above one can state the conditions for the existence of a Ricci collineation with symmetry vector $\xi^i = \lambda u^i$ as in the following theorem.

THEOREM (9) 3.1. — *A perfect fluid MFS admits a Ricci collineation with symmetry vector $\xi^i = \lambda u^i$ if and only if (i) either $\rho = p$ or $\sigma_{ij} = 0$ and (ii) either $\rho + 3p = 0$ or $(\partial_i \lambda)/\lambda = - a_i + \theta u_i$ and $\nabla_k \{ (\rho + 3p)\lambda u^k \} = 0$.*

(9) See Corollary (2.1) in the paper by Davis and Oliver [*5*]. This theorem shows that vanishing shear, in general, partially underlies this particular perfect fluid symmetry.

As implied in the above theorem and as will be illustrated in the following theorems and corollaries, there exists a close relationship between certain equations of state and the above defined FCRC member.

THEOREM 3.2. — *If a perfect fluid MFS satisfies* (10) $\rho = p$, *then* $\xi^i = \rho^{-1/2}u^i$ *is a Ricci collineation vector.*

Proof. — The proof of this theorem follows from Theorem (3.1). Condition *i*) of Theorem (3.1) is satisfied since $\rho = p$. Condition *ii*) follows with $\lambda = \rho^{-1/2}$ because of equations (3.1) and (3.2).

In this case the temperature satisfies an interesting relationship as shown in the following corollary.

COROLLARY 3.1. — *If a perfect fluid* MFS *satisfies* $\rho = p$, *then the temperature is homogeneous of degree one in* n *and* s.

Proof. — From $d\rho = \mathrm{T}ds + n^{-1}(\rho + p - \mathrm{T}s)dn$, we find that $(\partial\rho/\partial s) = \mathrm{T}$ and $(\partial\rho/\partial n) = n^{-1}(\rho + p - \mathrm{T}s)$. By equating $(\partial^2\rho/\partial n\partial s)$ with $(\partial^2\rho/\partial s\partial n)$, we find that $n(\partial\mathrm{T}/\partial n) + s(\partial\mathrm{T}/\partial s) = (\partial p/\partial s)$. Since $\rho = p$ and $(\partial\rho/\partial s) = \mathrm{T}$, it follows that $n(\partial\mathrm{T}/\partial n) + s(\partial\mathrm{T}/\partial s) = \mathrm{T}$ which by Euler's Theorem implies that T is homogeneous of degree one in n and s.

As shown above, the equations of state (10) $\rho = p$ or $\rho + 3p = 0$ have special significance when considering Ricci collineations with symmetry vector $\xi^i = \lambda u^i$. In fact, if $\rho = p$, then $\xi^i = \rho^{-1/2}u^i$ is such a symmetry vector. These equations of state are special cases of equations of state of the form $\rho = \rho(p)$. In the following theorem it is shown that only certain equations of state of the form $\rho = \rho(p)$ are allowable if the space-time admits a Ricci collineation with symmetry vector $\xi^i = \lambda u^i$.

THEOREM 3.3. — *If a perfect fluid MFS with equation of state* $\rho = \rho(p)$ *admits a Ricci collineation with symmetry vector* $\xi^i = \lambda u^i$, *then either* ξ^i *degenerates to a Killing vector or* (11) $(\partial\rho/\partial p) = 3(\rho + p)/(\rho + 5p)$.

Proof. — We assume $(\partial\rho/\partial p) \neq 3(\rho + p)/(\rho + 5p)$ and show $\mathscr{L}g_{ij} = 0$. If $(\partial\rho/\partial p) \neq 3(\rho + p)/(\rho + 5p)$, then $\rho \neq p$ and $\rho + 3p \neq 0$. Thus by Theorem (3.1) $\sigma_{ij} = 0$, $(\partial_i\lambda)/\lambda = -a_i + \theta u_i$ and $\nabla_k[(\rho + 3p)\lambda u^k] = 0$. Using these results we find that $\mathscr{L}g_{ij} = \left(\frac{8}{3}\right)\lambda\theta\gamma_{ij}$. Since $\rho = \rho(p)$, $u^i\partial_i\rho = (\partial\rho/\partial p)u^i\partial_i p$. Combining this result with (3.2), $(\partial_i\lambda)/\lambda = -a_i + \theta u_i$ and $\nabla_k[(\rho + 3p)\lambda u^k] = 0$ gives $(\rho + p)\theta = (\partial\rho/\partial p)(\rho + 5p)(\theta/3)$. Since $(\partial\rho/\partial p) \neq 3(\rho + p)/(\rho + 5p)$, this implies $\theta = 0$ and thus $\mathscr{L}g_{ij} = 0$.

Finally, in Table 1 we provide a classification of Ricci collineations with symmetry vector $\xi^i = \lambda u^i$ in terms of the kinematical quantities associated

(10) See, for example, Wesson [*18*] for more on this special equation of state, $\rho = p$.

(11) The solutions to $(\partial\rho/\partial p) = 3(\rho + p)/(\rho + 5p)$ are $\rho = p$, $\rho + 3p = 0$, and $p = v\rho$ with $\rho = \varkappa(3v + 1)^{1/2}(v - 1)^{3/2}$ and $v > 1$.

TABLE 1. — *Ricci Collineations in perfect fluid MFS with $\xi^i = \lambda u^i(\rho \neq p, p + 3p \neq 0)$. The symbol* N *in the table means not zero.*

Case	σ_{ij}	a_i	w_{ij}	θ	Comments
a	0	N	N	N	Possible if $\rho \neq \rho(p)$
b	0	N	0	N	Not possible
c	0	N	N	0	Motion
d	0	N	0	0	Motion
e	0	0	N	N	Not possible
f	0	0	0	N	Possible (Either $\rho = \rho(p)$ or $\gamma_j^i\partial_i s = \gamma_j^i\partial_i n = 0$)
g	0	0	N	0	Motion
h	0	0	0	0	Motion

with u^i. We see from the table that cases *a* and *f* are the only possible cases which do not degenerate to motions and that certain restrictions on the thermodynamic variables are imposed in each of these cases. Thus we can see through this special example how the local thermodynamics of a MFS is affected by the existence of a symmetry more general than a motion.

In principle, it would not be difficult to extend the above type of classification of perfect fluid MFS to a specific but more general MFS admitting other given members of the FCRC. In this connection we now briefly consider a theorem and corollary relating to a member of the FCRC defined by the demand ([12]) $\mathscr{L}R_{ij} = \Lambda\left[R_{ij} - \frac{1}{4}g_{ij}R\right]$ for the special case of a perfect fluid MFS. It will be seen that the essential results of this theorem and corollary may be regarded as generalizations of some of the results given in connection with Theorems (3.1) and (3.3) above.

THEOREM 3.4. — *A perfect fluid MFS $[\rho \neq \pm p$ and $\rho + 3p \neq 0]$ admits a nondegenerate FCRC defined by $\mathscr{L}R_{ij} = \Lambda\left[R_{ij} - \frac{1}{4}g_{ij}R\right]$ with symmetry vector $\xi^i = \lambda u^i$ if and only if $\sigma_{ij} = 0$, $(\partial_i\lambda)/\lambda = -a_i + fu_i$ $(f \neq \theta)$ and $\nabla_k[(\rho + 3p)\lambda u^k] = 0$.*

Proof. — The proof of this theorem follows from the following two results. First, we note that $\mathscr{L}R_{ij}$ with $\xi^i = \lambda u^i$ can be expressed as ([13])

$$\mathscr{L}R_{ij} = \psi\left[R_{ij} - \frac{1}{4}g_{ij}R\right] + \varkappa(\rho + 3p)u_{(i}\gamma^k_{j)}(a_k\lambda + \partial_k\lambda) + \varkappa\lambda(p - \rho)\sigma_{ij} + (\varkappa/4)g_{ij}\nabla_k[(\rho + 3p)\lambda u^k]$$

([12]) This particular FCRC symmetry property was considered earlier by Green, Norris, Oliver and Davis [*2*]. Norris, Green and Davis [*20*] have also examined other FCRC symmetry properties which are more general than Ricci collineations.

([13]) This decomposition was discussed earlier by Oliver and Davis [*1*].

with

$$\psi = (\rho + p)^{-1} \left\{ \nabla_k[(\rho + 3p)\lambda u^k] - \frac{4}{3}(\rho + 3p)^{1/2}\lambda\nabla_k[(\rho + 3p)^{1/2}u^k] \right\}.$$

Second, it can also be shown that no sum of terms in the above decomposition of $\mathscr{L}R_{ij}$ can vanish without each individual term of that sum vanishing.

COROLLARY 3.2. — *A perfect fluid MFS with equation of state of the form* $\rho = \rho(p)$ *that admits an FCRC defined by Theorem* (3.4) *necessarily satisfies* $(\partial\rho/\partial p) = 3(\rho + p)\theta/[f\rho + (2\theta + 3f)p]$.

Proof. — The proof of this corollary follows along the same lines as the proof to Theorem (3.3).

Explicit results for the Robertson-Walker metric relating to the content of Theorem (3.4) and Corollary (3.2) were given earlier [*2*] in a somewhat different context.

4. THE ROLE OF MATTER AND FCRC SYMMETRIES IN RELATIVISTIC KINETIC THEORY

Berezdivin and Sachs [*8*] have previously examined the question of the connection between matter symmetries in general relativistic kinetic theory and isometries of the space-time. Here we will first review the concept of a matter symmetry. Next we will examine some properties of general as well as special matter symmetries. Finally, in order to relate « matter symmetries » and certain FCRC symmetry properties more general than isometries, we will be led to new possible definitions for « matter symmetries ».

Berezdivin and Sachs [*8*] have given the following definition for a matter symmetry and shown that it has the properties which follow the definition. A matter symmetry exists if there are two locally Lorentz frames at the same point or at different points of space-time such that the distribution function is the same with respect to each frame. A one parameter group of matter symmetries (by convention also called a matter symmetry) exists if and only if there is a vector field W on TM satisfying $Wf = 0$ and $W = H^aD_a + A^a{}_bp^b(\partial/\partial p^a)$ with H^a a vector field and A_{ab} an antisymmetric tensor field on M. Several important properties of matter symmetries are as follows:

1) They carry fibres into fibres linearly and isometrically.
2) If $A_{ab} = \nabla_bH_a$, then W is the natural lift of H^a and H^a is a Killing vector.
3) $$\int (Wf)p^ap^b\sqrt{-g}\,d^4p = \mathscr{L}_H T^{ab} - (A^a{}_d - \nabla_dH^a)T^{bd} - (A^b{}_d - \nabla_dH^b)T^{ad} = 0.$$

One important property of matter symmetries not discussed in the above work is given in the following theorem.

THEOREM 4.1. — *If* $W = H^a D_a + A^a{}_b p^b(\partial/\partial p^a)$ *is a matter symmetry vector, then* $\mathscr{L}_H T = 0$.

Proof. — From property three above, we see that

$$0 = g_{ab}\mathscr{L}_H T^{ab} + 2\nabla_{(a}H_{d)}T^{ad} = g_{ab}\mathscr{L}_H T^{ab} + (\mathscr{L}_H g_{ab})T^{ab} = \mathscr{L}_H T .$$

This property will be important later when we discuss generalizations of matter symmetries, but first it will be helpful to examine special cases of matter symmetries. One way to analyze matter symmetries is in terms of properties of H^a and $A^a{}_b$. We will be particularly interested when H^a is a symmetry vector and $A^a{}_b$ is closely related to H^a. One interesting example is provided by the case where $A^a{}_b$ is given as in the next theorem.

THEOREM 4.2. — *Let* $W = H^a D_a + A^a{}_b p^b(\partial/\partial p^a)$ *represent a matter symmetry with* $A_{ab} = \nabla_{[b}H_{a]}$. *If* $2\sigma g_{ab}$ *and* Λ_{ab} *are the trace and trace free parts of* $\mathscr{L}_H g_{ab}$, *then* $A_{ab} = \nabla_b H_a - \sigma g_{ab} - \frac{1}{2}\Lambda_{ab}$, $\mathscr{L}_H T_{ef} = 2\sigma T_{ef} + T^a{}_{(f}\Lambda_{e)a}$ *and* $\mathscr{L}_H R_{ef} = 2\sigma R_{ef} + R^a{}_{(f}\Lambda_{e)a}$.

Proof. — The form given for A_{ab} follows from $A_{ab} = \nabla_{[b}H_{a]}$ and $\mathscr{L}_H g_{ab} \equiv 2\sigma g_{ab} + \Lambda_{ab}$. The rest of the theorem follows from this, property three of matter symmetries and Einstein's equations.

Some motivation for considering symmetries like those described in Theorem (4.2) is provided in the following corollary.

COROLLARY 4.1. — *Let* W *describe a matter symmetry as in Theorem* (4.2). *If* H^a *is a Killing vector, then* W *is the natural lift of* H^a.

Proof. — For W to be natural lift of H^a, $A^a{}_b$ must equat $\nabla_b H^a$. This is true in this case since for a motion $\sigma = \Lambda_{ab} = 0$.

Thus we see that the matter symmetries in Theorem (4.2) are generalizations of the particular matter symmetries described previously in property two. Another important subcase of Theorem (4.2) matter symmetries is when H^a is a conformal Killing vector.

COROLLARY 4.2. — *Let* W *generate a matter symmetry as in Theorem* (4.2). *If* H^a *is a conformal Killing vector, then* $A_{ab} = \nabla_b H_a - \sigma g_{ab}$, $\mathscr{L}_H T_{ef} = 2\sigma T_{ef}$ *and* $\mathscr{L}_H R_{ef} = 2\sigma R_{ef}$.

Proof. — Since for a conformal Killing vector $\Lambda_{ab} = 0$, the proof follows from Theorem (4.2).

As can be seen from Corollary (4.2), not only does the metric exhibit conformal properties but so do the Ricci and matter tensors. Another reason this example is important is that W is the particular lift of H^a which Iwai [*19*] has shown to be important when discussing conformal motions in terms of the tangent bundle.

As a final special matter symmetry, we consider the following subcase of Theorem (4.2).

COROLLARY 4.3. — *Let* W *represent a matter symmetry as in Theorem* (4.2) *with* f *the distribution function for a perfect fluid. If* $\mathscr{L}_{\mathrm{H}} \mathrm{R}_{ab} = 0$ *with* $\rho \neq p, p \neq 0$ *and* $\rho + 3p \neq 0$, *then* $\mathscr{L}_{\mathrm{H}} g_{ab} = 0$.

Proof. — By Theorem (4.2), $\mathscr{L}_{\mathrm{H}} \mathrm{R}_{ab} = 0$ implies

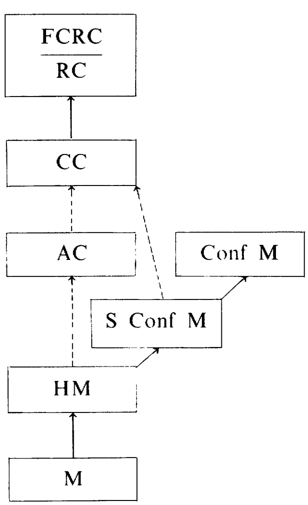
$$4\sigma \mathrm{R}_{ef} + \mathrm{R}^{a}{}_{f} \Lambda_{ea} + \mathrm{R}^{a}{}_{e} \Lambda_{fa} = 0 \tag{4.1}$$

with $\varkappa^{-1} \mathrm{R}_{ij} = \frac{1}{2}(\rho + 3p)u_i u_j - \frac{1}{2}(\rho - p)\gamma_{ij}$. Combining the contractions of $u^e u^f$ and g^{ef} on equation (4.1) gives $(\rho + 3p)(2\sigma + u^e u^f \Lambda_{ef}) = 0$ and $(\rho - p)\sigma = 0$. Since $\rho + 3p \neq 0$ and $\rho = p$, $\sigma = 0$ and $u^e u^f \Lambda_{ef} = 0$. Contracting equation (4.1) with $u^e \gamma^f{}_h$ and using $\sigma = 0$ and $p \neq 0$ gives $u^a \gamma^f{}_h \Lambda_{fa} = 0$. Summing $\gamma^e_i \gamma^f_j$ on equation (4.1) and using $\sigma = 0$ and $\rho \neq p$ gives $\gamma^e_i \gamma^f_j \Lambda_{ef} = 0$. Λ_{ab} can be written as

$$\Lambda_{ab} = (\Lambda_{ef} u^e u^f) u_a u_b + 2u_{(a} \gamma^e_{b)} \Lambda_{ef} u^f + \gamma^e_a \gamma^f_b \Lambda_{ef} .$$

We have shown each term of the sum vanishes and thus $\Lambda_{ab} = 0$. Since $\Lambda_{ab} = 0$ and $\sigma = 0$, $\mathscr{L}_{\mathrm{H}} g_{ab} = 0$.

This theorem serves to illustrate the kind of problem which arises in trying to relate FCRC symmetry properties more general than motions to matter symmetries. This fact can also be seen from the result of Theorem (4.1) as shown in the following theorem.



FCRC - Members of Family of Contracted Ricci Collineations: $g^{ij} \mathscr{L} \mathrm{R}_{ij} = 0$
RC - Ricci Collineation: $\mathscr{L} \mathrm{R}_{ij} = 0$
CC - Curvature Collineation: $\mathscr{L} \mathrm{R}_{jkm}{}^{i} = 0$
AC - Affine Collineation: $\mathscr{L} \mathrm{R}^{i}_{jk} = 0$
Conf M - Conformal Motion: $\mathscr{L} g_{ij} = 2\sigma g_{ij}$
S Conf M - Special Conformal Motions:
$\mathscr{L} g_{ij} = 2\sigma_{ij}, \nabla_j \nabla_k \sigma = 0$
HM - Homothetic Motion:
$\mathscr{L} g_{ij} = 2\sigma g_{ij}, \sigma = \text{constant}$
M - Motions: $\mathscr{L} g_{ij} = 0$

FIG 1. — *Symmetry Property Inclusion Diagram.*

THEOREM 4.3. — *If* $W = H^a D_a + A^a{}_b p^b (\partial/\partial p^a)$ *generates a matter symmetry and* H^a *generates an FCRC symmetry property, then* $R^{ij}\Lambda_{ij} + 2\sigma R = 0$.

Proof. — Theorem (4.1) shows that for a matter symmetry $\mathscr{L}_H T = 0$ and hence $\mathscr{L}_H R = 0$. A FCRC symmetry property satisfies $g^{ij}\mathscr{L}_H R_{ij} = 0$. Therefore it follows that

$$0 = \mathscr{L}_H R = g^{ij}\mathscr{L}_H R_{ij} + R_{ij}\mathscr{L}_H g^{ij} = -2\sigma R - \Lambda^{ij}R_{ij}.$$

As illustrated below, FCRC symmetry properties more general than motions do not, in general, satisfy the relation in Theorem (4.3),

$$R^{ij}\Lambda_{ij} + 2\sigma R = 0.$$

COROLLARY 4.4. — *Suppose* H^a *in Theorem* (4.3) *generates a special conformal motion. Then if* $R \neq 0$, H^a *is a Killing vector.*

Proof. — The proof follows from Theorem (4.3) and the fact that a special conformal motion is a FCRC symmetry property with $\Lambda_{ij} = 0$.

Thus in order to relate FCRC symmetry properties to « matter symmetries », we consider two generalizations of the definition of matter symmetries. First, we look at the condition $Wf = 0$. This condition is reasonable for an isometry but seems too restrictive for a FCRC symmetry property. Thus one possible extension of the definition of matter symmetry would be to allow Wf to equal some specified non-zero function. The other generalization we consider is relaxing the condition that $A_{(ab)} = 0$. There are two reasons to consider this generalization. First, this restriction is imposed to make W map fibres into fibres isometrically. Thus when we look at symmetries more general than a motion, it makes sense to remove this restriction. Secondly, if H^a is a projective collineation vector, then $M = H^a D_a + A^a{}_b p^b(\partial/\partial p^a)$ with $A_{ab} = \nabla_b H_a - \sigma g_{ab}$ is the particular lift of H^a which Iwai [*19*] has shown to be important when discussing projective collineations in terms of the tangent bundle. We note that $A_{(ab)}$ is non-zero in this case. We conclude this section by showing how a generalization of matter symmetry can remove the problem indicated in Corollary (4.4).

COROLLARY 4.5. — *Let* $W = H^a D_a + A^a{}_b p^b(\partial/\partial p^a)$ *satisfy* $Wf = -2\sigma f$ *with* $4\sigma = \nabla_a H^a$ *and* $A_{ab} = \nabla_b H_a - \sigma g_{ab}$. *Further suppose that* H^a *is a special conformal Killing vector. Then* $\mathscr{L}_H T^{ef}$ *as calculated from property three of matter symmetries agrees with* $\mathscr{L}_H T^{ef}$ *as calculated from its space-time symmetry with no further restriction on* σ.

Proof. — Using

$$\int (Wf) p^a p^b \sqrt{-g}\, d^4p = \mathscr{L}_H T^{ab} - (A^a{}_d - \nabla_d H^a)T^{bd} - (A^b{}_d - \nabla_d H^b)T^{ad}$$

with A_{ab} and Wf given as above yields

$$\mathscr{L}_H T^{ab} = -2\sigma T^{ab} - 2\sigma \int p^a p^b \sqrt{-g}\, d^4 p = -4\sigma T^{ab}.$$

A special conformal motion satisfies $\mathscr{L}_H g_{ab} = 2\sigma g_{ab}$ and $\mathscr{L}_H R_{ab} = 0$. Using these properties of a special conformal motion with Einstein's equations gives $\mathscr{L}_H T^{ab} = -4\sigma T^{ab}$.

5. SUMMARY AND COMMENTS RELATING TO FURTHER WORK

In this paper we have examined the important role played by FCRC symmetry properties (which include motions as a special case) when a thermodynamic or a kinetic theory model is used for a MFS. In Section 3 the connection between certain equations of state and FCRC symmetry properties was considered. In Section 4 the concept of matter symmetry was further examined. In particular several special examples of matter symmetries were considered and it was shown how problems arose when trying to relate matter symmetries and general FCRC symmetry properties. For this reason, several generalizations of matter symmetries were proposed which could be used in relating general FCRC symmetry properties to « matter symmetries ».

The above discussion suggests several topics for future investigation. First, in this paper we have only investigated the relations between certain special FCRC symmetry properties and equations of state for perfect fluid MFS. One could extend these investigations to include other symmetry properties and/or matter tensors more general than that of a perfect fluid. Second, the relation of the « lift » vectors suggested by Iwai [*19*] and matter symmetries or generalizations of matter symmetries should be further examined. Finally, one could further consider the relation between FCRC symmetry properties and the proposed generalizations of matter symmetries. Specifically, results which give conditions under which these generalizations imply FCRC symmetry properties would be important.

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