ANNALES DE L'I. H. P., SECTION A

EDWARD P. OSIPOV

# The Yukawa quantum field theory : linear $N_\tau$ bound, locally Fock property

*Annales de l'I. H. P., section A*, tome 30, nº 3 (1979), p. 159-192
<http://www.numdam.org/item?id=AIHPA_1979__30_3_159_0>

*Ann. Inst. Henri Poincaré*, Section A :
Vol. XXX, nº 3, 1979, p. 159-192. *Physique théorique.*

# The Yukawa$_2$ quantum field theory : linear $N_\tau$ bound, locally Fock property

by

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ABSTRACT. — The Yukawa quantum field theory in two-dimensional space-time is considered. Using the Matthews-Salam integral representation a linear $N_\tau$ bound is proved. As a consequence, this bound implies that the Yukawa$_2$ theory in the infinite volume is locally Fock.

RÉSUMÉ. — On considère la théorie quantique des champs de Yukawa dans un espace-temps de dimension 2. Une majoration linéaire pour $N_\tau$ est démontrée en utilisant la représentation intégrale de Matthews et Salam. En conséquence, cette majoration implique que la théorie de Yukawa$_2$ dans le volume infini est localement Fock.

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## 1. INTRODUCTION

In the present paper we prove the locally Fock property (the definition of the locally Fock property see [*1*]) for the (renormalized) Yukawa$_2$ interaction ($= Y_2$) in the two-dimensional space-time with the free or periodic (in the spatial direction) boundary conditions. For the main results and references on the $Y_2$ theory, see, for instance [*2-6*, *10*, *11*, *13*, *22*, *23*, *26*, *24*, *27*].

The locally Fock property for $P(\varphi)_2$ models has first been proved by Glimm and Jaffe [*1*, *7*].

For the $Y_2$ theory the locally Fock property has been proved by Schrader [*3*] (for the free boundary conditions and under some restrictions on the smoothness of the space cut-off).

In the present paper we get rid of these restrictions and also give a new (semi-) Euclidean proof which is valid both for the free and for the periodic boundary conditions. Namely, the following theorem is valid:

THEOREM 1.1. — *The* $Y_2$ *theory with the free or periodic boundary conditions is locally Fock.*

Theorem 1.1 is used in the proof of the Lorentz invariance of the $Y_2$ theory with the periodic boundary conditions [*24*].

Since the locally Fock property is, in fact, a consequence of the linear $N_\tau$ bound, see [*1*, *7*, *2*, *3*], so we shall not give the proof of this theorem and fix our attention on the proof of the linear $N_\tau$ bound.

Let an ultraviolet cut-off in the spatial direction be given by a function $\eta(p_2/\sigma)$, where $\eta(.) \in \mathcal{S}(\mathbb{R})$ and $\eta(0) = 1$.

Let $\chi_\alpha(x)$ be the interval $[\alpha - 1/2, \alpha + 1/2]$. Let

$$g' := \{\alpha \in \mathbb{Z}^1 \mid \operatorname{supp} \chi_\alpha \cap \operatorname{supp} g \neq \varnothing\}$$

and $|g| :=$ cardinality of $g'$. We suppose that our space cut-offs satisfy the following conditions: *i)* $g(x) \in L_\infty(dx)$, *ii)* uniformly in $|g|$

$$\sup_\alpha \| [\chi_\alpha g]^\sim(k) \ln(2 + |k|) \|_{4/3} < \text{const},$$

where $[\chi_\alpha g]^\sim(k)$ is the Fourier transform of $\chi_\alpha g$.

THEOREM 1.2. — *Let* $H(g, \sigma)$ *be the (renormalized) cut-off Yukawa*$_2$ *Hamiltonian with the free or periodic boundary conditions. Let* $\tau < 1$. *Then, for some* $a > 0$, *depending on* $\tau$ *only,*

$$H(g, \sigma) - aN_\tau \geq -c_1 |g|$$

*uniformly in* $g$, $\sigma$, $1 \leq \sigma \leq \infty$.

*Remark*. — We note that Theorems 1.1 and 1.2 are also valid for $P(\varphi)_2 + Y_2$ models. The proofs are in this case the same as for the case of pure $Y_2$ theory.

To obtain a linear $N_\tau$ bound we use the (semi-) Euclidean formalism of Matthews-Salam, Seiler [*8-10*, *5*, *26*, *28*] in which the fermions have been integrated out. The idea of the proof consists in the following. We rewrite expressions like $(F, \exp(-t(H(g, \sigma) - aN_\tau))F)$ with the help of the Matthews-Salam integral representation with the Gaussian measure corresponding to the perturbed two-point function

$$G_\tau(t_1 - t_2, x_1 - x_2) = (\Omega_0, \phi(x_1) \exp(-|t_1 - t_2|(H_0 - aN_\tau))\phi(x_2)\Omega_0)$$

and then we use the technique of Refs. [*4*, *6*, *10*, *11*] to obtain the bounds on $\det_{\mathrm{ren}}(1 + K)$. Here the complications appear, which are connected with the fact that the Gaussian measure has no Markov property in the spatial direction. Nevertheless, the perturbed two-point function decreases exponentially on large (Euclidean) distances. This decrease allows us, similarly to Refs. [*6*, *12*], to obtain linear bounds.

The proof of the Matthews-Salam formulas, which we need, has been considered in Ref. [*28*].

The exposition is made in the following way. In Sec. 2 we formulate the Matthews-Salam integral representation for the theory with the Hamiltonian $H(g, \sigma) - aN_\tau$. In Sec. 3,4 we obtain the estimates which we need for the (perturbed) functions $G_\tau$, $S_\tau$ and for the integral operators connected with them. In Sec. 5 we obtain some necessary estimates of the functional integral over the (perturbed) measure $\mu_\tau$ and in Sec. 6,7 we prove the linear $N_\tau$ bound, i. e. Theorem 1.2.

Since the proofs of the linear $N_\tau$ bound for the free and periodic boundary conditions coincide essentially, so we give the detailed version for the free boundary conditions and make some remarks for the changes needed for the case of the periodic boundary conditions.

In the following $f^\sim$, $f^\wedge$ denote the direct and inverse Fourier transform of the function $f$. We define $\det_n$ as

$$\det_n(1 + A) := \det\left[(1 + A)\exp\left[\sum_{k=1}^{n-1}(-A)^k/k\right]\right].$$

By $\mathscr{C}_p$ we denote the set of compact operators with the norm

$$\|A\|_p = [\mathrm{Tr}(A^*A)^{p/2}]^{1/p}.$$

$P := -i$ (the gradient operator), $c_1, c_2, \ldots$ denote strictly positive constants possibly depending on unessential variables.

## 2. INTEGRAL REPRESENTATION OF MATTHEWS-SALAM

We want to obtain the integral representation of the Matthews-Salam type for the Hamiltonian expressions of the form

$$(\Omega_0, \exp(-t_1 H')F_1 \exp((t_1 - t_2)H')F_2 \ldots \exp((t_{n-1} - t_n)H')\Omega_0) \quad (2.1)$$

where F is either a fermion field $\psi$ or its Dirac conjugate $\overline{\psi} := \psi^+\gamma_0$, or a function of the boson field $\phi(0, x)$ at time zero,

$$H' = H_0 - aN_\tau + H_I \quad (2.2)$$

is the (renormalized, cut-off) Hamiltonian of the $Y_2$ theory perturbed by the « number » operator with $\tau < 1$ and the constant $a > 0$ is chosen such that

$$\mu' := \mu - a\mu^\tau > 0, \qquad \omega' := \omega - a\omega^\tau > 0,$$

where $\mu(k) := (k^2 + m_b^2)^{1/2}$, $\omega(k) := (k^2 + m_f^2)^{1/2}$.

We want to consider the (cut-off) interactions of the form

$$H_I = \int dx[:\overline{\psi}_\sigma(x)\Gamma\psi_\sigma(x): W(\phi_k(0, x))g(x) + W_1(\phi_k(0, x))g_1(x)] \quad (2.3)$$

$\Gamma = \alpha + i\beta\gamma_5$ with real $\alpha$, $\beta$ and $\gamma_5 = \gamma_5^+$ and where $\psi_\sigma(x) = \int dy\sigma(x-y)\psi(y)$ and $\sigma(x) = \sigma(-x)$ is a function from $\mathscr{S}(\mathbb{R})$. Let, for simplicity, W be either a bounded analytic real-valued function on $\mathbb{R}$ or $W(\phi) = \phi$, $W_1 \in \mathscr{S}(\mathbb{R})$ and an ultraviolet cut-off $k$ be made with the help of a function from $\mathscr{S}(\mathbb{R})$.

Let $\mu_\tau$ be the Gaussian mean zero measure on $\mathscr{S}'(\mathbb{R}^2)$ with the two-point function

$$G_\tau(x) = \int dk\mu(k)^{-1}e^{-|x_1|\mu'(k) - ikx} = \int d^2k \frac{\mu'(k_2)e^{-ikx}}{\mu(k_2)(k_1^2 + \mu'(k_2)^2)} \quad (2.4)$$

We remark that the Gaussian measure $\mu_\tau$ is hypercontractive and has the Markov property in the temporal direction [*14*], but it has no Markov property in the spatial direction [*15*].

We also introduce the (perturbed, Euclidean) fermion two-point function

$$S_\tau(t_1 - t_2, x_1 - x_2)_{\alpha\beta} = \begin{cases} (\Omega_0, \psi_\alpha(x_1)\exp((t_1 - t_2)(H_0 - aN_\tau))\overline{\psi}_\beta(x_2)\Omega_0) & \text{for } t_1 \leq t_2 \\ -(\Omega_0, \overline{\psi}_\beta(x_2)\exp((t_2 - t_1)(H_0 - aN_\tau))\psi_\alpha(x_1)\Omega_0) & \text{for } t_1 > t_2 \end{cases} \quad (2.5)$$

The following theorem is valid.

THEOREM 2.1. — *(Matthews-Salam formulas). Let* H′ *be given by* (2.2)-(2.3). *Then the Hamiltonian objects* (2.1) *are given by Matthews-Salam formulas with the Gaussian measure* $\mu_\tau$ *and the fermion two-point function* (2.5).

This theorem is proved in [*28*].

## 3. ESTIMATES ON FUNCTIONS $S_\tau$, $G_\tau$

In this section we consider properties of the functions $S_\tau$ and $G_\tau$ and of some integral operator connected with $S_\tau$ and $G_\tau$.

It follows from the definition (2.5) that

$$S_\tau(x) = S_1(x) + S_2(x) \tag{3.1}$$

where

$$S_1(x) = (2\pi)^{-2}\int d^2k e^{-ikx}\frac{(m - i\gamma_1^E k_1 - i\gamma_2^E k_2)\omega'(k_2)}{(k_1^2 + \omega'(k_2)^2)\omega(k_2)},$$
$$S_2(x) = \varepsilon(x_1)S_3(x),$$
$$S_3(x) = -\gamma_1^E a(2\pi)^{-2}\int d^2k e^{-ikx}\frac{\omega'(k_2)\omega(k_2)^{-1}}{k_1^2 + \omega'(k_2)^2}.$$

Here $\gamma_1^E = \gamma_0$, $\gamma_2^E = i\gamma_1$, $\{\gamma_i^E, \gamma_j^E\} = 2\delta_{ij}$ and $\varepsilon(x_1) = -1$ for $x_1 \leq 0$ and $\varepsilon(x_1) = 1$ for $x_1 > 0$.

The following lemmas contain the main properties of the functions $S_\tau$ and $G_\tau$.

LEMMA 3.1.

$$|S_2^\sim(k)| \leq c(\varepsilon)(k^2+1)^{-1/2+\varepsilon}(k_2^2+1)^{(\tau-1)/2}, \quad \varepsilon > 0.$$

*Proof of Lemma 3.1.* — In the sense of distributions

$$\varepsilon^\sim(k_1) = (\pi i)^{-1}\ \text{P. V.}\ k_1^{-1}$$

and, thus, as it can be easily seen

$$S_2^\sim(k) = (\pi i)^{-1}\int_{-1}^{1} dk_1' k_1'^{-1}(S_3^\sim(k_1 - k_1', k_2) - S_3^\sim(k_1, k_2)) + (\pi i)^{-1}\int_{\mathbb{R}\setminus[-1,1]} dk_1' S_3^\sim(k_1 - k_1', k_2)k_1'^{-1} \tag{3.2}$$

in the sense of distributions. Since the integrals in (3.2) are convergent, eq. (3.2) is valid in the sense of usual functions. To estimate the first integral we write

$$\int_{-1}^{1} dk_1' k_1'^{-1}(S_3^\sim(k_1 - k_1', k_2) - S_3^\sim(k_1, k_2)) = -\int_{-1}^{1} dk_1' \int_0^1 ds \frac{\partial S_3^\sim}{\partial k_1}(k_1 - sk_1', k_2) \tag{3.3}$$

and, thus, this integral is estimated by

$$\leq 2 \sup_{\substack{k_1' \in [-1,1] \\ s \in [0,1]}} \left| \frac{\partial S_3}{\partial k_1}(k_1 - sk_1', k_2) \right| \leq c(k^2 + 1)^{-3/2+\tau/2}$$

To estimate the second integral we divide the domain of the integration into two subdomains

$$\mathrm{I} = (\mathbb{R}\backslash[-1, 1]) \cap \{ k_1' \in \mathbb{R} \mid | k' | \leq | k |/2 \}$$
$$\mathrm{II} = (\mathbb{R}\backslash[-1, 1]) \cap \{ k_1' \in \mathbb{R} \mid | k' | > | k |/2 \}$$

where $k' = (k_1', 0)$. The contribution of domain I is less than

$$\sup_{k_1' \in \mathrm{I}} ((| k_1 - k_1' | + 1)^{2\varepsilon} S_3^{\sim}(k_1 - k_1', k_2)) \int_{\mathbb{R}\backslash[-1,1]} dk_1' \, | k_1' |^{-1} (| k_1 - k_1' | + 1)^{-2\varepsilon}.$$

Hölder's inequality gives $\int_{\mathbb{R}\backslash[-1,1]} dk_1'(.) \leq \text{const}.$

In domain I $| k - k' | \geq |k| - | k' | \geq | k | / 2$, thus,

$$\sup_{k_1' \in \mathrm{I}} [(| k_1 - k_1' | + 1)^{2\varepsilon} S_3^{\sim}(k_1 - k_1', k_2)] \leq \text{const}\, (k^2 + 1)^{-1+\tau/2+\varepsilon}$$

and so the integral over domain I is estimated by

$$\leq c(\varepsilon)(k^2 + 1)^{-1+\tau/2+\varepsilon}.$$

The integral over domain II is less than

$$\sup_{k_1' \in \mathrm{II}} | k_1' |^{-1} \int_{\mathrm{II}} dk_1' S_3^{\sim}(k_1 - k_1', k_2).$$

But $\sup_{k_1' \in \mathrm{II}} | k_1' |^{-1} \leq c(k^2 + 1)^{-1/2}$, and the integral is less than

$$\int_{\mathrm{II}} dk_1' S_3^{\sim}(k_1 - k_1', k_2) \leq c \int dk_1' (k_1'^2 + k_2^2 + 1)^{-1+\tau/2} \leq c_1 (k_2^2 + 1)^{(\tau-1)/2}$$

and so the integral over domain II is less than

$$\leq c(\varepsilon)(k^2 + 1)^{-1/2}(k_2^2 + 1)^{(\tau-1)/2}.$$

To prove the lemma we join together the obtained estimates. Lemma 3.1 is proved.

We need the estimates of the functions $S_\tau^{\sim}$ and its hermitian adjoint with some localization conditions in $x$ – space. Moreover, since we want to obtain the estimates uniform in cut-offs, we need the uniform estimates of the fermion propagator with cut-offs.

Let $\eta(.) \in \mathscr{S}(\mathbb{R})$, $\eta(0) = 1$ and $\eta_\sigma(p) = \eta(p_1/\sigma_1)\eta(p_2/\sigma_2)$ be an ultraviolet cut-off in the temporal $\sigma_1$ and spatial $\sigma_2$ directions. Let $1 - \zeta(p)$, $\zeta(p) \in \mathscr{S}(\mathbb{R}^2)$, where $\zeta(p) = 1$ for $|p| \leq \zeta$ and $\zeta(p) = 0$ for $|p| \geq \zeta + 1$, be an infrared (low momentum) cut-off.

The estimates we need are formulated in the following lemma:

LEMMA 3.2. — *Let* $\mathrm{I}(k)$ *be one of the functions* $\mathrm{S}_\tau^\sim(k)$, $\mathrm{S}_\tau^{+\sim}$ *or*

$$(k^2 + m^2)^{1/2}\mathrm{S}_\tau^{+\sim}(k)\mathrm{S}_\tau^\sim(k).$$

*Then*

*a)* $\| k_1^{n_1}k_2^{n_2}[h(x)\mathrm{I}^\wedge(x)]^\sim(k) \|_{\mathrm{L}_\infty} \leq c_1 \exp(-c_2 d)$ for $h(x) \in \mathrm{C}_0^\infty((\mathbb{R}\backslash\{0\}) \times \mathbb{R})$, $d = \mathrm{dist}(\{0\}, \mathrm{supp}\, h)$ ;

*b)* $\| (k_1^2 + 1)^{r/2}k_2^n[h(x_2)\mathrm{I}^\wedge(x)]^\sim(k) \|_{\mathrm{L}_\infty} \leq c_1 \exp(-c_2 d)$ for $h(x_2) \in \mathrm{C}_0^\infty(\mathbb{R}\backslash\{0\})$, $d = \mathrm{dist}(\{0\}, \mathrm{supp}\, h)$, $0 \leq r < 1$ ;

*c)* $\| h(x)\mathrm{I}^\wedge(x) \|_{\mathrm{L}_p} \leq c_1 \exp(-c_2 d)$ for $h(x) \in \mathrm{C}_0^\infty(\mathbb{R}^2\backslash\{0\})$, $d = \mathrm{dist}(\{0\}, \mathrm{supp}\, h)$, $2 \leq p < \infty$.

*If the function* $\mathrm{S}_\tau^\sim(k)$ *is replased by the function* $\mathrm{S}_\tau^\sim(k)\eta_\sigma(k)(1 - \zeta(k))$, *then the estimates a), b), c) are valid with the exponential decrease being replaced by the decrease faster than any power and the estimates are uniform in ultraviolet cut-off* $\sigma$ *for* $1 \leq \sigma_{1,2} \leq \infty$.

*Remark.* — The replacement of the exponential decrease by the decrease faster than any power in the case with cut-offs is connected with the decrease faster than any power of the functions $\eta^\wedge(x)$, $\zeta^\wedge(x)$.

*Proof of Lemma 3.2.* — Let us first consider the case *b*) of the lemma fo$_\tau$ the function $\mathrm{S}_\tau^\sim$. We shall obtain the representation for the function Sr in the form of a contour integral. The statements will follow from this representation.

In the sense of distributions

$$\int d^2k\mathrm{S}_\tau^\sim(k)e^{-ikx} = \lim_{\mathrm{R}_1\to\infty}\int_{-\mathrm{R}_1}^{\mathrm{R}_1} dk_1 \lim_{\mathrm{R}_2\to\infty}\int_{-\mathrm{R}_2}^{\mathrm{R}_2} dk_2\mathrm{S}_\tau^\sim(k)e^{-ikx} \tag{3.4}$$

Let us denote the complex variable $k_2$ by $\zeta$. We cut the complex plane $\zeta$ by $(-i\infty, -im]$ and $[im, i\infty)$ and choose the branches of the analytic functions $(\zeta^2 + m^2)^{1/2}$, $(\zeta^2 + m^2)^{\tau/2}$ such that they would be positive on the real semiaxis. Let, for definiteness, supp $h$ lie in the half-plane $x_2 < 0$. Let $\mathrm{C}(\delta, \varphi)$ be the contour chosen in Fig. 1, where $\delta$ and $\varphi$ are sufficiently small positive numbers which depend only on $m$ and $\tau$ and which will be chosen later (for $x_2 > 0$ we should choose the contour in the lower half-plane $\zeta$).

We assert that for sufficiently small $\delta$ and $\varphi$

$$\mathrm{Re}\,[k_1^2 + ((\zeta^2 + m^2)^{1/2} - a(\zeta^2 + m^2)^{\tau/2})^2] \geq c(k_1^2 + 1) \tag{3.5}$$

in and on the contour $\mathrm{C}(\delta, \varphi)$ for some constant $c > 0$.

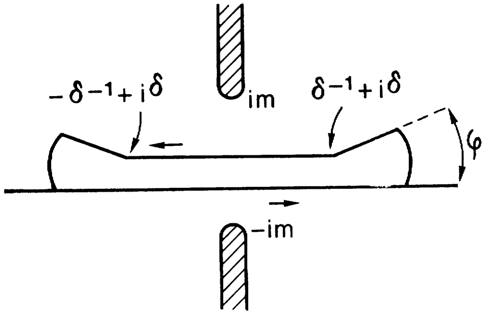


FIG. 1. — Contour $C(\delta, \varphi)$.

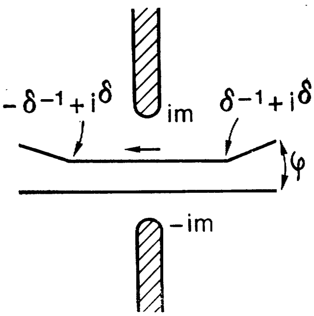


FIG. 2. — Contour $C'(\delta, \varphi)$.

To prove this inequality it is sufficient to show that

$$(\operatorname{Re} b(\zeta))^2 - (\operatorname{Im} b(\zeta))^2 \geq c_1 > 0. \tag{3.6}$$

in and on the contour $C(\delta, \varphi)$, where

$$b(\zeta) = (\zeta^2 + m^2)^{1/2} - a(\zeta^2 + m^2)^{\tau/2}.$$

We write

$$b(\zeta) = b(\operatorname{Re} \zeta) + i \operatorname{Im} \zeta \int_0^1 ds \frac{db}{d\zeta}(\operatorname{Re} \zeta + is \operatorname{Im} \zeta).$$

Since $b(\operatorname{Re} \zeta) > 0$, so $b(\operatorname{Re} \zeta) \geq c_3 > 0$. In the domain bounded by the contour $C(\delta, \varphi)$ $| db/d\zeta | \leq c_2$, where the constant $c_2$ is independent of $\delta$, $\varphi$ for sufficiently small $\delta$ and $\varphi$. Thus, in the domain

$$\{ \zeta \in \mathbb{C} \mid | \operatorname{Re} \zeta | \leq \delta^{-1}, | \operatorname{Im} \zeta | \leq \delta \}$$
$$\operatorname{Re} b(\zeta) \geq c_3 - \delta c_2, \ | \operatorname{Im} b(\zeta) | \leq \delta c_2$$

and thus in this domain inequality (3.6) is fulfilled for sufficiently small $\delta$. Since

$$b(\operatorname{Re} \zeta) = \operatorname{Re} b(\operatorname{Re} \zeta) \geq (1 - \nu) | \operatorname{Re} \zeta | - c_4(\nu), \nu > 0$$

and since in the domain bounded by the contour $C(\delta, \varphi)$

$$| \operatorname{Im} \zeta | \leq | \operatorname{Re} \zeta | \sin \varphi$$

so in this domain

$$\operatorname{Re} b(\zeta) \geq (1 - \nu - c_2 \sin \varphi) | \operatorname{Re} \zeta | - c_4(\nu),$$
$$| \operatorname{Im} b(\zeta) | \leq c_2 | \operatorname{Re} \zeta | \sin \varphi.$$

Thus, in the domain bounded by the contour $C(\delta, \varphi)$ for $| \operatorname{Re} \zeta | > \delta^{-1}$

$$(\operatorname{Re} b(\zeta))^2 - (\operatorname{Im} b(\zeta))^2 \geq ( | \operatorname{Re} b(\zeta) | - | \operatorname{Im} b(\zeta) | ) | \operatorname{Re} b(\zeta) |$$
$$\geq \delta^{-1}[(1 - \nu - 2c_2 \sin \varphi) | \operatorname{Re} \zeta | - c_4(\nu)]$$

But this is greater than a positive constant for sufficiently small $\delta$, $\varphi$.

These arguments and eqs. (3.2), (3.3) imply that the function $S_\tau^{\sim}(k_1, \zeta)$

is analytic in and on the contour $C(\delta, \varphi)$ for sufficiently small $\delta$, $\varphi$. It is easy to see, making estimates of the term $S_2^{\sim}(k_1, \zeta)$ for complex $\zeta \in C(\delta, \varphi)$ in the same way in the proof of Lemma 3.1 for real $k_2$, that

$$| S_\tau^{\sim}(k_1, \zeta) | \leq o(| \zeta |^{-1})$$

for $| \zeta | \to \infty$, $\zeta \in C(\delta, \varphi)$, may be nonuniformly in $k_1$. But then the integral of $S_\tau^{\sim}(k_1, \zeta)$ over the arcs of the circle entering the contour $C(\delta, \varphi)$ tends to zero when the radius of the circle increases. Let $C'(\delta, \varphi)$ be the contour shown in Fig. 2 (or, in the case $x_2 > 0$, the analogous contour in the lower half-plane). Then, by Cauchy's theorem

$$h(x_2)\int dk_2 S_\tau^{\sim}(k_1, k_2)e^{-ikx} = h(x_2)\int_{C'(\delta,\varphi)} d\zeta S_\tau^{\sim}(k_1, \zeta)e^{-ik_1x_1 - i\zeta x_2} \quad (3.7)$$

Using estimate (3.5) it is easy to see that (the term $S_2^{\sim}(k_1, \zeta)$ is estimated in the same way, as in the proof of Lemma 3.1)

$$| S_\tau^{\sim}(k_1, \zeta) | \leq c(| \zeta |)(k_1^2 + 1)^{-1/2+\varepsilon} \quad , \quad \varepsilon > 0,$$

where $c(| \zeta |)$ is a function increasing slower than a polynomial. Thus, in the sense of distributions

$$(k_1^2 + 1)^{r/2}k_2^n[h(x_2)S_\tau(x)]^{\sim}(k) = (k_1^2 + 1)^{r/2}\left[i^n \frac{\partial^n}{\partial x_2^n}(h(x_2)S_\tau(x))\right]^{\sim}(k)$$

and

$$\left| \left[i^n \frac{\partial^n}{\partial x_2^n}(h(x_2)S_\tau(x))\right]^{\sim}(k) \right| = \left| \int dx_2 e^{ik_2x_2} i^n \frac{\partial^n}{\partial x_2^n} h(x_2)\int_{C'(\delta,\varphi)} d\zeta S_\tau^{\sim}(k_1, \zeta)e^{-i\zeta x_2} \right|$$
$$\leq c_1(k_1^2 + 1)^{-1/2+\varepsilon} \exp(-c_2 d)$$

which proves the part *b*) of the lemma for the function $S_\tau$.

In the same way we consider the functions $S_\tau^+$ and $(k^2 + m^2)^{1/2}S_\tau^{+\sim}S_\tau^{\sim}$.

Now, we consider the case *a*) of the lemma for the function $S_\tau$. For definiteness, we suppose that supp $h$ lies in the half-plane $x_1 < 0$. Then, in the sense of distributions

$$h(x)\int d^2k e^{-ikx}S_\tau^{\sim}(k) = \lim_{\varepsilon\to+0} h(x) \lim_{R_2\to\infty}\int_{-R_2}^{R_2} dk_2 \lim_{R_1\to\infty}\int_{-R_1}^{R_1} dk_1 e^{-ikx}S_\tau(k, \varepsilon) \quad (3.8)$$

where

$$S_\tau(k, \varepsilon) = S_1^{\sim}(k) + \frac{1}{\pi i}\int dk_1' S_3^{\sim}(k_1', k_2)(k_1 - k_1' + i\varepsilon)^{-1} + S_3^{\sim}(k).$$

We use the relation $\lim_{\varepsilon\to+0}(k \pm i\varepsilon)^{-1} = \text{P. V. } k^{-1} \mp i\pi\delta(x)$ which is valid in the sense of distributions, see [*16*].

We want to write the integral over $k_1$ and $k_2$ as the integrals over some contours in the complex planes $\zeta_1$ and $\zeta_2$.

First we consider the integral over $k_1$. We cut the complex plane $\zeta_1$ by $[i\omega'(k_2), i\infty)$. Then the integrand is an analytic function in the upper half-plane with the cut. The integrand in (3.8) decreases in $\zeta_1$ in the upper half-plane as $|\zeta_1| \to \infty$. This is evident for the functions $S_1^{\sim}(\zeta_1, k_2)$, $S_3^{\sim}(\zeta_1, k_2)$ and for the integral over $k_1'$ it follows from the estimates

$$\left| \int dk_1' \frac{S_3^{\sim}(k_1', k_2)}{\zeta_1 - k_1' + i\varepsilon} \right| \leq \left| \int_{|k_1'| \leq |\zeta_1|/2} dk_1'(.) \right| + \left| \int_{|k_1'| \geq |\zeta_1|/2} dk_1'(.) \right|$$

$$\leq 2 |\zeta_1|^{-1} \int dk_1' S_3^{\sim}(k_1', k_2) + \sup_{|k_1'| \geq |\zeta_1|/2} [(1 + |k_1'|^{\delta}) S_3^{\sim}(k_1', k_2)]$$

$$\int dk_1'(|\operatorname{Re}\zeta_1 - k_1'| + \varepsilon)^{-1}(1 + |k_1'|^{\delta})^{-1} \leq c |\zeta_1|^{-1}.$$

Now, choosing the contour, as it is shown in Fig. 3, it is easy to see that the integral over the arcs of the circle tends to zero and, by Cauchy's theorem, the integral of $k_1$ over the real axis is equal to the integral over the contour $C'(k_2)$. As a result

$$h(x) \int dk_1 e^{-ik_1x_1} S_\tau(k_1, k_2, \varepsilon) = h(x) \int_{C'(k_2)} d\zeta_1 e^{-i\zeta_1 x_1} S_\tau(\zeta_1, k_2, \varepsilon) \quad (3.9)$$

It is easy to see that the right side of (3.9) is analytic in $\zeta_2$ for sufficiently small $|\operatorname{Im} \zeta_2|$ (in particular, for the integral over $k_1'$, the analyticity in this domain follows from the fact that $|\operatorname{Im}\zeta_2|$ is small). Besides, the integral of (3.9) over $\zeta_2$ over the intervals

$$\{ \zeta_2 \in \mathbb{C} \mid \operatorname{Re}\zeta_2 = R, 0 \leq \pm \operatorname{Im}\zeta_2 \leq \delta \}$$

for sufficiently small positive $\delta$ tends, as it is easy to see, to zero when $R \to \infty$. So, the contour of the integration over $\zeta_2$ may be shifted up or down, depending on the sign of $x_2$. As a result,

$$h(x) \int dk_2 \int dk_1 e^{-ikx} S_\tau(k, \varepsilon)$$
$$= h(x) \int_{\pm \operatorname{Im}\zeta_2 = \delta} d\zeta_2 \int_{C'(\zeta_2)} d\zeta_1 e^{-i\zeta_1 x_1 - i\zeta_2 x_2} S_\tau(\zeta_1, \zeta_2, \varepsilon)$$

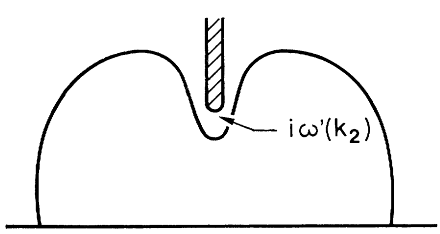


FIG. 3. — Contour $C(k_2)$.

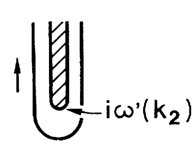


FIG. 4. — Contour $C'(k_2)$.

the integral over $\zeta_2$ converges due to the properties of the contour $C'(\zeta_2)$. This expression defines a function smooth in $x$ (this can be easily proved by interchanging the order of the differentiation and the integration), increasing exponentially in $x_1$ and $x_2$, and, thus, in $x$ for $x_1 \neq 0$. Thus, $L_1(d^2x)$ norms of derivatives of this function satisfy the bound $c_1 \exp(-c_2 d)$ and by the Hausdorff-Young inequality $L_\infty(d^2p)$ norms satisfy the same inequality.

Thus, the case *a*) of the lemma is proved for the function $S_\tau$.

In the same way we consider the other functions.

Consider the case *c*). By *a*) and *b*)

$$\begin{aligned} \left| \int d^2x f(x) h(x) I^\wedge(x) \right| &= \left| \int d^2k f^\sim(k) [h I^\wedge]^\sim(k) \right| \\ &\leq \int d^2k \, | f^\sim(k) | \, ((k_1^2 + 1)^{r/2} + k_2^{2n})^{-1} c_1 \exp(-c_2 d) \\ &\leq \| f^\sim \|_p c_1 \exp(-c_2 d) \| ((k_1^2 + 1)^{r/2} + k_2^{2n})^{-1} \|_{p'} \\ &\leq \| f \|_{p'} c_3 \exp(-c_2 d). \end{aligned}$$

We have used the Hölder and Hausdorff-Young inequalities. The obtained inequality proves the case *c*) of the lemma.

In the case with cut-offs it is necessary to estimate, for example, the case *b*) with an ultraviolet cut-off, the expression of the form

$$h\eta_\sigma^1 * S_\tau = h(x_2) \int d^2y \sigma_1 \eta(\sigma_1(x_1 - y_1)) \sigma_2 \eta(\sigma_2(x_2 - y_2)) S_\tau(y_1, y_2)$$

Let $\{ e_i \}_{i \in \mathbb{Z}^1}$ be a partition of unity in the variable $y_2$ obtained by the translations of a fixed $C_0^\infty$ function with the support in $[-3/4d, 3/4d]$ so $\operatorname{supp} e_i \subset [(i - 3/4)d, (i + 3/4)d]$. We insert this partition of unity between $\eta_\sigma^\wedge$ and $S_\tau$. After that the term

$$[h\eta_\sigma^\wedge * (e_0 S_\tau)]^\sim(k) = \eta(k_1/\sigma_1) \int dx_2 e^{ik_2x_2} \int dp_2 \\ [h(x_2)\sigma_2\eta^\wedge(\sigma_2(x_2 - .))e_0(.)]^\sim(p_2) S_\tau^\sim(k_1, p_2)$$

is estimated by

$$\sup_{p_2} | S_\tau^\sim(k_1, p_2) | \, | h(x_2)\sigma_2\eta^\wedge(\sigma_2(x_2 - y_2)) e_0(y_2) |_{\mathscr{S}}$$

where $| \, . \, |_{\mathscr{S}}$ is some norm on the Schwartz space in variables $x_2$, $y_2$. Taking the location of the supports into account, it is easy to see that the above expression is less than

$$c_1 (k_1^2 + 1)^{-1/2+\varepsilon} (1 + d^n)^{-1}$$

uniformly in $\sigma$ for $\sigma \geq 1$.

The terms

$$[h\eta_\sigma^\wedge * (e_i S_\tau)]^\sim(k)$$

are estimated similarly to the previous one by the

$$c_1(k_1^2 + 1)^{-1/2+\varepsilon}(1 + d^n)^{-1}(1 + i^2)^{-1}.$$

Taking the sum over $i$ we obtain power estimate for the cut-off propagator.

The estimate for the case *b*) with a lower momentum cut-off is obtained with the use of a two-dimensional partition of unity and the bound *a*) of the lemma.

In the same way we consider the case *a*) with cut-offs. Lemma 3.2 is proved.

LEMMA 3.3. — *The function* $G_\tau(x) \in L_p(d^2x)$, $1 \leq p < \infty$. *The following estimates are valid*

$$|G_\tau(x)| \leq G_0(x) + c_1 \exp(-c_2 |x|),$$

*where*

$$G_0(x) = [(k^2 + m_b^2)^{-1}]^\wedge(x).$$

*Proof of Lemma 3.3.* — The proof is analogous to the proof of Lemma 3.2. The integral over $k_1$ is taken explicitly, and the integral over $k_2$ may be rewritten as the integral over the contour of the form $C'(\delta, \varphi)$, Fig. 2. Lemma 3.3 is proved.

LEMMAS $3.1_V$-$3.3_V$. — *The bounds corresponding to the bounds of Lemmas 3.1-3.3 are also valid in the case of the periodic boundary conditions, these estimates are uniform in the box cut-off* $V \geq 1$ (*naturally*, $|x_2| < V/2$).

*Proof of Lemmas* $3.1_V$-$3.3_V$. — Lemma $3.1_V$ may be proved in the same way as Lemma 3.1.

The statements of Lemma $3.2_V$ and $3.3_V$ follow from Lemmas 3.2 and 3.3, from the inequality for a norm $\| \Sigma . \| \leq \Sigma \| . \|$ and from the relations

$$f_V(x_1, x_2) = \Sigma_j f(x_1, x_2 + jV)$$

where the series converges in the sense of distributions and where $f$ is one of the functions $G_\tau$, $S_\tau$, $S_\tau^+$, $[(k^2 + m^2)^{1/2} S_\tau^{+\sim} S_\tau^\sim]^\wedge$ or these functions with cut-offs and $f_V$ is the corresponding function in the periodic boundary conditions case.

Let us consider, for instance, the case *b*) of Lemma 3.2. In this case $h(x_2) \in C_0^\infty([-V/2, V/2] \backslash \{0\})$ and

$$h(x) I_V^\wedge(x_2) = \Sigma_j h(x_2) I^\wedge(x + jV).$$

Then

$$\left\| (k_1^2 + 1)^{r/2} k_{2V}^2 [h(x_2) I_V^\wedge(x)]^\sim \right\|_{L_\infty} \leq \Sigma_j \left\| (k_1^2 + 1)^{r/2} k_2^n [h(x_2) I^\wedge(x + jV)]^\sim \right\|_{L_\infty}$$
$$= \Sigma_j \left\| (k_1^2 + 1)^{r/2} k_2^n \exp(-ikjV)[h(x_2 - jV) I^\wedge(x)]^\sim \right\|_{L_\infty}$$
$$\leq (\text{by Lemma } 3.2)\ \Sigma_j c_1 \exp(-c_2(d + |j|)) = c_3 \exp(-c_2 d).$$

In the same way we consider the other cases. Lemmas $3.1_V$-$3.3_V$ are proved.

## 4. ESTIMATES FOR SOME INTEGRAL OPERATORS

In this section we consider trace-norms of some integral operators. With the help of Lemma 3.2 we prove Theorem 4.1 (an analogue of Theorem 2.2 of Ref. [*4*]) which allows us to estimate contributions of the separate squares.

THEOREM 4.1. — *Let $\chi_\alpha$, $\alpha \in \mathbb{Z}^2$, be the characteristic function of the square with center at $\alpha$ and side 1. Fix $q \geq 1$. Suppose operators* A, B *are given with* $A\chi_\alpha(P^2 + m^2)^{-1/4+\varepsilon} \in \mathscr{C}_{q'}$ *and* $(P^2 + m^2)^{-1/4+\varepsilon}\chi_\beta B \in \mathscr{C}_{q''}$ *for some* $\varepsilon > 0$ *and where* $q'^{-1} + q''^{-1} = q^{-1}$.

*Then, for all $\alpha$, $\beta$* $A\chi_\alpha I(P)\chi_\beta B \in \mathscr{C}_q$, *where* I(P) *is an operator of multiplication on one of the functions* $S_\tau^\sim$, $S_\tau^{+\sim}$, $(p^2 + m^2)^{1/2} S_\tau^{+\sim} S_\tau^\sim$, *and*

$$\| A\chi_\alpha I(P)\chi_\beta B \|_q$$
$$\leq c_1 \exp(-c_2 |\alpha - \beta|) \| A\chi_\alpha(P^2 + m^2)^{-1/4+\varepsilon} \|_{q'} \| (P^2 + m^2)^{-1/4+\varepsilon}\chi_\beta B \|_{q''}$$

*If the function $S_\tau^\sim(p)$ is replaced by the function with cut-offs $\eta_\sigma(p)(1 - \zeta(p))S_\tau^\sim(p)$, then the assertion of the theorem is valid with the replacement of the exponential decrease by the power decrease with arbitrarily large degree, and this degree of the decrease is uniform in the ultraviolet cut-off $\sigma$ for $\sigma_{1,2} \geq 1$.*

*Proof of Theorem 4.1.* — The proof uses Lemmas 3.1, 3.2 and is an analogue of that of Theorem 2.2 Ref. [*4*].

Let $|\alpha - \beta| \leq \sqrt{2}$ (i. e. touching squares), then

$$A\chi_\alpha I(P)\chi_\beta B = A\chi_\alpha(P^2 + m^2)^{-1/4+\varepsilon}[(P^2 + m^2)^{1/2 - 2\varepsilon} I(P)](P^2 + m^2)^{-1/4+\varepsilon}\chi_\beta B.$$

Now, the assertion follows from Hölder's inequality for operators and from the boundedness of the operator in the square brakets (Lemma 3.1).

Now, let $|\alpha - \beta| > \sqrt{2}$ (i. e. non-touching squares), then either $|\alpha_1 - \beta_1| \geq 2$ or $|\alpha_1 - \beta_1| \leq 1$, $|\alpha_2 - \beta_2| \geq 2$. Let $h$ be a $C_0^\infty(\mathbb{R})$ function with support in $(-\sqrt{2} - \delta, \sqrt{2} + \delta)$ identically one in $(-\sqrt{2}, \sqrt{2})$, where $\delta$ is chosen with $\sqrt{2} + \delta < 2$.

Now, if $|\alpha_1 - \beta_1| \geq 2$, then

$$\chi_\alpha(x) I^\wedge(x - y)\chi_\beta(y) = \chi_\alpha(x) h(|x - y| - |\alpha - \beta|) I^\wedge(x - y)\chi_\beta(y)$$

and

$$A\chi_\alpha I(P)\chi_\beta B = A\chi_\alpha(P^2+m^2)^{-1/4+\varepsilon}[(P^2+m^2)^{1/2-2\varepsilon}(h(|\,.\,|-|\alpha-\beta|)I^\wedge)^\sim(P)] (P^2+m^2)^{-1/4+\varepsilon}\chi_\beta B.$$

The assertion follows from Hölder's inequality for trace-norms of operators and from the boundedness of the operator in the square brackets (Lemma 3.2*a*).

Now, if $|\alpha_1-\beta_1|\leq 1$, $|\alpha_2-\beta_2|\geq 2$, then

$$\chi_\alpha(x)I^\wedge(x-y)\chi_\beta(y) = \chi_\alpha(x)h(|x_2-y_2|-|\alpha_2-\beta_2|)I^\wedge(x-y)\chi_\beta(y)$$

and

$$A\chi_\alpha I(P)\chi_\beta B = A\chi_\alpha(P^2+m^2)^{-1/4+\varepsilon}[(P^2+m^2)^{1/2+2\varepsilon}(h(|\,.\,|-|\alpha_2-\beta_2|)I^\wedge)^\sim(P)] (P^2+m^2)^{-1/4+\varepsilon}\chi_\beta B.$$

Again the assertion follows from Hölder's inequality for operators, from the boundedness of the operator

$$(P^2+m^2)^{1/2-\varepsilon}.((P_1^2+1)^{1/2-\varepsilon}+P_2^{2k}+1)^{-1},$$

from Lemma 3.2 and from the inequality $|\alpha-\beta|\leq 1+|\alpha_2-\beta_2|$ for $|\alpha_1-\beta_1|\leq 1$.

Theorem 4.1 is proved.

## 5. $\mu_\tau$-INTEGRABILITY OF SOME FUNCTIONS

In this section we prove some integrability results for the Gaussian measure $\mu_\tau$, which is given by the two-point function $G_\tau(x-y)$ (formula (2.4)), and obtain some bounds.

THEOREM 5.1. — *Let $q\geq 2$ and $\nu$, $\lambda>0$ with $(\nu+\lambda)q>1$. Let $\mu_\tau$ be the Gaussian mean zero measure on $\mathscr{S}'(\mathbb{R}^2)$ which is given by the covariance $G_\tau(x-y)$ (eq. (2.4)). Let $A(\phi)=(P^2+m^2)^{-\nu}\phi\Lambda(P^2+m^2)^{-\lambda}$ be a linear map from some subspace of $\mathscr{S}'(\mathbb{R}^2)$ to operators. Here $\Lambda\in L_\infty(d^2x)$ and has a bounded support. Then $A(\phi)$ is $\mu_\tau$ almost everywhere in $\mathscr{C}_q$.*

*Proof of Theorem 5.1.* — Consider first the case $q=2k$ with $k$ an integer. Acting as in the proof of Theorem 3.4 Ref. [*4*], (see also Ref. [*17*], Theorem 9.1) and using Lemma 3.3 it is easy to see that

$$\|A(\phi)\|_{2k}^{2k}\in L_2(\mathscr{S}'(\mathbb{R}^2),d\mu_\tau).$$

When $q$ is not an even integer, we can, following Seiler and Simon [*4*], use the interpolation theorem. For the proof of the interpolation theorem see Gohberg, Krein [*18*, p. 137-139). Theorem 5.1 is proved.

THEOREM 5.2. — *Let $\zeta(.) \in L_\infty(d^2p)$ be a function with a bounded support. Then*

$$L(\phi) = (P^2 + m^2)^{1/4} S_\tau^\sim(P)\zeta(P)\phi\Lambda(P^2 + m^2)^{-1/4}$$

*belongs to $\mathscr{C}_1$ $\mu_\tau$ almost everywhere and*

$$\| L(\phi) \|_1 \leq 4^{-1}\delta^{-1} \| M \|_2^2 + \delta(\phi, C\phi)_{G_\tau^{\sim -1}}$$

*where*

$$M = (P^2 + m^2)^{1/4} S_\tau^\sim(P)\zeta(P)\chi(P^2 + m^2),$$
$$N = (P^2 + m^2)^{-1}\phi\Lambda(P^2 + m^2)^{-1/4},$$

*$\chi \in C_0^\infty(\mathbb{R}^2)$ and $\chi(x)$ is equal to one on* supp $\Lambda$

$$\operatorname{Tr} N * N = (\phi, C\phi)_{G_\tau^{\sim -1}}$$

*and C is trace class in the Hilbert space $L_2(G_\tau^\sim(p)^{-1}d^2p)$*

*Proof of Theorem 5.2.* — Let $\chi$ be a $C_0^\infty(\mathbb{R}^2)$ function which satisfies $0 \leq \chi \leq 1$ and is identically one on supp $\Lambda$ and zero on

$$\{ x \in \mathbb{R}^2 \mid \operatorname{dist}(x, \operatorname{supp} \Lambda) \geq 1 \}.$$

Then $L = M \cdot N$.

Since

$$\chi(P^2 + m^2) = (P^2 + m^2)\chi - 2iP\nabla\chi + \Delta\chi \qquad (5.1)$$

so, by Lemma 2.1, Ref. [*4*], $M \in \mathscr{C}_2$. Then, by Theorem 5.1, $N \in \mathscr{C}_2$ $\mu_\tau$ almost everywhere and

$$\operatorname{Tr} N * N = (\phi, C\phi)_{G_\tau^{\sim -1}},$$

where C is trace class in $L_2(G_\tau^\sim(p)^{-1}d^2p)$, namely,

$$C = G_\tau^\sim \Lambda E \Lambda \qquad (5.2)$$

where $G_\tau^\sim$ and E are the operators of multiplication in the momentum space by the functions, respectively, $G_\tau^\sim(k)$ and

$$E(k) = \int d^2p(p_+^2 + m^2)^{-1/2}(p_-^2 + m^2)^{-2} \qquad (5.3)$$

where $p_\pm = p \pm k/2$.

Using the number inequality

$$xy \leq y^2/4\delta + x^2, \qquad x, y \in \mathbb{R}, \delta > 0$$

we obtain

$$\| L(\phi) \|_1 \leq \| M \|_2 \| N \|_2 \leq 4^{-1}\delta^{-1} \| M \|_2^2 + \delta(\phi, C\phi)_{G_\tau^{\sim -1}}.$$

Theorem 5.2 is proved.

We also need some estimates for integrals of Wick monoms which are similar to the estimates of Dimock, Glimm [*12*, Lemma 2.4] and Glimm, Jaffe and Spencer [*17*, Theorem 9.4]. In contrast to the bound of Lemma 2.4

by Dimock and Glimm [12] we need an estimate in which the kernel is $L_p(d^2x)$ function with $p > 1$ and $p - 1$ is small (see also Ref. [17]).

We want to consider functions on $\mathscr{S}'(\mathbb{R}^2)$ of the form

$$R(w, n, \xi) = \int dy_1 \dots dy_N \Pi_{v=1}^N : \Pi_{k \in n_v} \phi(\xi_{v,k}(. - y_v)) : w(y_1, \dots, y_N) \quad (5.4)$$

where the normal ordering is made with respect to the measure $\mu_\tau$ and, for simplicity, all $\xi_{v,k}$ have the form $\xi_{v,k} = \rho(v, k)\eta(\rho(v, k)x)$, $\eta \in \mathscr{S}(\mathbb{R}^2)$, $\int d^2x\eta(x) = 1$, or all $\xi_{v,k}$ are $\xi_{v,k} = \delta(x_1)\rho(v, k)\eta(\rho(v, k)x_2)$, $\eta \in \mathscr{S}(\mathbb{R})$, $\int dx\eta(x) = 1$ (a more general case of different cut-offs in different directions can also be considered). We remark that for ultraviolet cut-offs of the form $\delta(x_1)\eta(x_2)$ the normal ordering with respect to the measure $\mu_\tau$ and with respect to the free boson measure with the mass $m_b$ coincide.

We suppose that the support $w$ lies in a set of the form

$$\Delta_{\omega(\cdot)} = \Delta_{\omega(1)} \times \dots \times \Delta_{\omega(N)}$$

where $\omega(v) \in \mathbb{Z}^2$ and where $\Delta_j \subset \mathbb{R}^2$ is the square with side 1 and center at $j \in \mathbb{Z}^2$. Let $\chi_j$ be the characteristic function of $\Delta_j$ and let $n_* = \sup_v n_v$. Let, also $\mathscr{N} = \{ 1, \dots, N \}$, $I_v = \{ (v, 1), \dots, (v, n_v) \}$, $I = \{ I_v \}_{v \in \mathscr{N}}$. The set of all possible graphs on I is denoted $\Gamma(I)$, the set of all vacuum graphs is denoted $\Gamma_0(I)$ and let $[I] = \bigcup_{v \in \mathscr{N}} I_v$ be the set of all legs [12].

LEMMA 5.3. — *Let $w \in L_p(\Delta_{\omega(.)})$, $p > 1$ and $p^{-1} + p'^{-1} = 1$, then*

$$\left| \int d\mu_\tau R(w, n, \xi) \right| \le \| w \|_p \sum_{\mathscr{G} \in \Gamma_0(I)} \prod_{l \in \mathscr{G}} \| G_l \|_{p'n*}$$

*where*

$$G_l(y, y') = \chi_{l_-}(y) \langle \xi_{l_-}(. - y), G_\tau \xi_{l_+}(. - y') \rangle \chi_{l_+}(y').$$

*Proof of Lemma 5.3.* — The proof is the same as that of Proposition 9.3 [17] or of Lemma 2.1 [12] with the replacement of Schwarz's inequality by Holder's inequality. Lemma 5.3 is proved.

We define a function $\widehat{\omega}(.)$ on [I] by $\widehat{\omega}((v, k)) = \omega(v)$ and for $j \in \mathbb{Z}^2$ define

$$N_j = \sum_{v \in \omega^{-1}(j)} n_v = | \widehat{\omega}^{-1}(j) |.$$

THEOREM 5.4. — *Let $w \in L_p(\Delta_{\omega(.)})$, $p > 1$. Let $\xi_{v,k}$ have the aforementioned form and* $\inf_{(v,k) \in I} \rho(v, k) = \rho > 0$, *then, as above for $\delta \in [0, 1)$*

$$\left| \int d\mu_\tau R(w, n, \xi) \right| \le \prod_{j \in \mathbb{Z}^2} N_j ! \, M_\delta^{N_j} \| w \|_p \prod_{(v,k) \in [I]} c(\eta, \rho, \delta, r) \left\| \tilde{\xi}_{v,k}(1 + q^2)^{-r} \right\|_\infty^\delta$$

*where $r < (2p'n_*)^{-1}$.*

*Proof of Theorem 5.4.* — The proof of this theorem is the same as that of Lemma 2.4 of Ref. [*12*]. Some slight modifications appear in the proof of estimates for the kernel $G_l(y, y')$ only, to which the kernel $C_l(y, y')$ corresponds in notations of Dimock and Glimm [*12*]. But these estimates can easily be obtained.

Since the function $G_\tau(x)$ has an exponential falloff for large distances (Lemma 3.3) and since for each $n$ the uniform in $\rho(v, k)$ for $\inf_{(v,k)\in[l]} \rho(v, k) = \rho > 0$ estimate

$$| \rho(v, k)\eta(\rho(v, k))x | \leq c(\eta, \rho, n)(| x | + 1)^{-n} \quad \text{for} \quad | x | \geq 1$$

is valid. So

$$\| G_l \|_{p'n_*} \leq c_1(\eta, \rho, n)(| \widehat{\omega}(l_-) - \widehat{\omega}(l_+) | + 1)^{-n}.$$

On the other hand, for $p'n_* \geq 2$ the following estimate is valid

$$\| G_l \|_{p'n_*} \leq \| \xi_{l_-} * G_\tau * \xi_{l_+} \|_{p'n_*} \leq \text{(Hausdorff-Young inequality)}$$

$$\begin{aligned} &\| \xi_{l_-}^\sim(q)\xi_{l_+}^\sim(q)G_\tau^\sim(q) \|_{(p'n_*)'} \\ &\qquad \leq \| \xi_{l_-}^\sim(q)(q^2 + 1)^{-1/2} \|_{2(p'n_*)'} \| \xi_{l_+}^\sim(q)(q^2 + 1)^{-1/2} \|_{2(p'n_*)'} \\ &\qquad \leq c(r) \| \xi_{l_-}^\sim(q)(q^2 + 1)^{-r} \|_\infty \| \xi_{l_+}^\sim(q)(q^2 + 1)^{-r} \|_\infty \end{aligned}$$

for $r < (2p'n_*)^{-1}$.

These estimates and the arguments of Ref. [*12*] imply Theorem 5.4. Theorem 5.4 is proved.

## 6. UPPER BOUND ON $\int d\mu_\tau \, | \det_{\text{ren}} (1 + K(\Lambda)) |$

In this section we prove the main technical result of this paper.

The operator $S_\tau\Gamma\phi\Lambda$ enters the Matthews-Salam formulas. Here, $\Lambda(x)$ is a space-time cut-off. The idea to consider this operator in the Hilbert space $L_2((p^2 + m^2)^{1/2}d^2p) \otimes \mathbb{C}^2$ belongs to Seiler [*10*]. In the Hilbert space $L_2(d^2p) \otimes \mathbb{C}^2$ this operator has the form

$$K(\Lambda) = (P^2 + m^2)^{1/4}S_\tau^\sim(P)\Gamma\phi(P^2 + m^2)^{-1/4}.$$

The operator with cut-offs $K(\Lambda, \sigma)$ is given by the analogous expression.

We define

$$\begin{aligned} \det_{\text{ren}}(1 + K(\Lambda, \sigma)) := \det_2(1 + K(\Lambda, \sigma)) \\ \exp(\mathscr{E}_{2,\tau}(\Lambda, \sigma) + (s - \delta\mu^2(\Lambda, \sigma))Q(\Lambda, \sigma)) \end{aligned}$$

where $Q(\Lambda, \sigma) = \int d^2x\Lambda(x)^2 : \phi_\sigma^2(x) :$, $s$ and $\delta\mu^2(\Lambda, \sigma)$ are a finite and the infinite boson mass renormalizations and $\mathscr{E}_{2,\tau}(\Lambda, \sigma) = \frac{1}{2}\int d\mu_\tau K(\Lambda, \sigma)^2$ is the second order Euclidean renormalization.

Let $\Lambda(x)$ be a space-time cut-off. Let $\chi_\alpha$ be the partition of $\mathbb{R}^2$ on squares with centres in $\alpha \in \mathbb{Z}^2$ and side 1. We define

$$\Lambda' := \{ \alpha \in \mathbb{Z}^2 \mid \operatorname{supp} \chi_\alpha \cap \operatorname{supp} \Lambda \neq \emptyset \}$$

and we shall identify $\Lambda'$ and $\bigcup_{\alpha \in \Lambda'} \chi_\alpha$. Let $|\Lambda| :=$ cardinality of $\Lambda'$.

We suppose that space-time cut-offs satisfy the following conditions: *i)* $\Lambda(x) \in L_\infty(d^2x)$, *ii)* uniformly in $|\Lambda|$

$$\sup_\alpha \| [\chi_\alpha \Lambda]^\sim(k) \ln (2 + |k|) \|_{4/3} < \text{const.}$$

We remark that, in particular, the indicator of a rectangle satisfies the condition *i)*, *ii)* and if $g(x_2)$ satisfies the conditions *i)*, *ii)* of Sec. 1, then $\chi_{a,b}(x_1)g(x_2)$, where $\chi_{a,b}$ is the indicator of $[a, b]$, satisfies the conditions *i)*, *ii)*.

THEOREM 6.1. — *Let $\Lambda(x)$ satisfy the above conditions i), ii). Then, for any p there is a constant c so that uniformly in ultraviolet cut-offs*

$$\int d\mu_\tau \,| \det_{\text{ren}}(1 + K(\Lambda, \sigma))^p \leq c^{|\Lambda|}$$

*The similar assertion is also valid in the case of the periodic boundary conditions.*

*Proof of Theorem 6.1.* — To prove the theorem we, as Seiler and Simon [*11*, *4*] split the operator K into two parts with large and small momenta. Let $\zeta(p)$ be a positive $C^\infty$ function, equal to 1 on the set $\{ p \in \mathbb{R}^2 \mid |p| \leq \zeta \}$, and equal to 0 on the set $\{ p \in \mathbb{R}^2 \mid |p| \geq \zeta + 1 \}$. The function $\zeta(p)$ gives a low momentum cut-off. The positive constant $\zeta$ will be determined later.

Let

$$K = L + H$$

where

$$L = \zeta(P)(P^2 + m^2)^{1/4} S_\tau^\sim(P) \Gamma \phi \Lambda (P^2 + m^2)^{-1/4}.$$

Taking in the inequality of Theorem 4.1 [*4*] $A = H$, $B = 0$, $C = L$ we obtain the estimate

$$| \det_{\text{ren}}(1 + K(\Lambda, \sigma)) | \leq \Big[ \det (1 + A_+(\sigma))]^{1/2} \exp [b \| L(\sigma) \|_1 - \frac{1}{2} \operatorname{Tr} A_-(\sigma) - \frac{1}{4} \operatorname{Tr} (A_-(\sigma)^2) - \operatorname{Re} \operatorname{Tr} H(\sigma) + \mathscr{E}_{2,\tau}(\Lambda, \sigma) + (s - \delta\mu^2(\Lambda, \sigma)) Q(\Lambda, \sigma) \Big] \tag{6.1}$$

where $b = 1 + e^{5/4} = 4{,}4\ldots$, $A = H + H^* + H^*H$, $A_+$ is its non-negative part and $A_-$ – its negative part. Similarly we define $A(\sigma)$, etc.

It is easy to see that the right side of inequality (6.1) may be rewritten in the form

$$v_1 v_2 v_3$$

where

$$v_1 = \exp(b \| L(\sigma) \|_1),$$

$$v_2 = \exp\left[\frac{1}{4}\mathrm{Tr} : (H(\sigma)^* + H(\sigma))^2 : - \frac{1}{2}\mathrm{Re\,Tr}\,(H(\sigma))^2 + (s - \delta\mu^2(\Lambda, \sigma))Q(\Lambda, \sigma) + \mathscr{E}_{2,\tau}(\Lambda, \sigma)\right],$$

$$v_3 = [\det{}_3 (1 + A_+(\sigma))]^{1/2} \exp\left[-\frac{1}{4}\mathrm{Tr}\,((H(\sigma)^* H(\sigma))^2) - \mathrm{Re\,Tr}\,(H(\sigma)^2 H(\sigma)^*) - \frac{1}{4}\mathrm{Tr} : (H(\sigma)^* + H(\sigma))^2 :\right].$$

To prove the theorem it is sufficient to show that for each $p \in [1, \infty)$, $i = 1, 2, 3$

$$\| v_i \|_p \leq \mathrm{const}^{|\Lambda|}$$

uniformly in $\sigma$.

First, we consider the factor $v_1$. By Theorem 5.2

$$\| L(\sigma) \|_1 \leq 4^{-1}\delta^{-1} \| M \|_2^2 + \delta(\phi, C\phi)_{G_\tau^{\sim -1}}.$$

Since $\| 2p\delta C \| \leq 1/2$ for sufficiently small $\delta$, depending only on $\| \Lambda \|_\infty$ and $p$, so for such $\delta$ $1 - 2p\delta C$ is a positive operator in $L_2(G_\tau^{\sim}(p)^{-1}d^2p)$ and by a direct calculation of the Gaussian integrals (cf. [*10*, Lemma 3.3], [*13*, Lemma 2.1]) we obtain the bound

$$\| v_1 \|_p^p = \int d\mu_\tau \exp(4^{-1}\delta^{-1}p \| M \|_2^2 + p\delta(\phi, C\phi)_{G_\tau^{\sim -1}})$$
$$= \exp(4^{-1}\delta^{-1}p \| M \|_2^2) \det(1 - 2p\delta C)^{-1/2}$$
$$\leq \exp(4^{-1}\delta^{-1}p \| M \|_2^2 + 2p\delta \| C \|_1)$$

Estimating the determinant we have used the inequality

$$-\ln(1 - x) \leq \ln(1 + 2x) \leq 2x \quad \text{for} \quad 0 \leq x \leq 1/2.$$

Lemma 2.1 [*4*] and eq. (5.1) imply that

$$\| M \|_2 \leq c(\| \chi \|_2 + \| \nabla\chi \|_2 + \| \Delta\chi \|_2) \leq c_1 | \Lambda |^{1/2}.$$

The operator C in $L_2(G_\tau^{\sim}(p)^{-1}d^2p)$ is given by eq. (5.2). In correspondence with the unitary equivalence in the Hilbert space $L_2(d^2p)$ the operator C has the form

$$C = G_\tau^{\sim 1/2}\Lambda E \Lambda G_\tau^{\sim 1/2}.$$

We note that the operator E is the operator of convolution (see (5.3)) by the function $F_{1/2}(x)F_2(x)$, where

$$F_\nu(x) = [(p^2 + m^2)^{-\nu}]^\wedge(x).$$

To estimate $\mathscr{C}_1$ norm of the operator C we write

$$\| C \|_1 \leq \| G_\tau^{\sim 1/2} \Lambda E^{1/2} \|_2 \| E^{1/2} \Lambda G_\tau^{\sim 1/2} \|_2 = \| G_\tau^{\sim 1/2} \Lambda E^{1/2} \|_2^2$$

and

$$\| G_\tau^{\sim 1/2} \Lambda E^{1/2} \|_2^2 = \int d^2x d^2y G_\tau(x-y) \Lambda(y) E(y-x) \Lambda(x)$$
$$\leq \| \Lambda \|_\infty \| \Lambda \|_1 \| G_\tau(x) E(x) \|_1 \leq c \mid \Lambda \mid$$

So the linear bound for the factor $v_1$ is obtained.

Now we proceed to the Gaussian factor $v_2$.

To estimate this factor we cancel the divergences in the explicit form. For this purpose we rewrite this factor in the following form

$$v_2 = \exp \left[ \frac{1}{2} \operatorname{Tr}_{\text{reg}} : H(\sigma)^* H(\sigma) : + \mathscr{E}_{2,\tau}(\Lambda, \sigma) - \mathscr{E}_{2,\tau}(\Lambda, \sigma, \zeta) \right]$$

Here

$$\operatorname{Tr}_{\text{reg}} : H(\sigma)^* H(\sigma) : = - : (\phi, B(\sigma)\phi)_{G_\tau^{\sim -1}} :$$

where B is a positive (for large $\zeta$) Hilbert-Schmidt operator in the Hilbert space $L_2(G_\tau^\sim(p)^{-1} d^2p)$ (see below) and $B(\sigma)$—the corresponding operator with ultraviolet cut-offs. The operator B is equal to

$$B = G_\tau^\sim (\Lambda G_{\text{reg}} \Lambda - \Lambda^2 s)$$

where $G_{\text{reg}}$ is the operator of multiplication in the momentum space by the function

$$G_{\text{reg}}(k) = - \int d^2p [\operatorname{Sp} \Gamma^+ S_{\tau,\zeta}^{+\sim}(p_+) S_{\tau,\zeta}^\sim(p_+) \Gamma (p_+^2 + m^2)^{1/2} (p_-^2 + m^2)^{-1/2}$$
$$- 2(\alpha^2 + \beta^2)(p^2 + m^2)^{-1}] \quad (6.2)$$

Here, $p_\pm = p \pm k/2$, $S_{\tau,\zeta}^\sim(p) = (1 - \zeta(p)) S_\tau^\sim(p)$ is the perturbed two-point fermion function with a low momentum cut-off, Sp is the operation of taking the trace over spinor indices and we have written the counterterm of the boson mass renormalization for the Yukawa$_2$ interaction with $\Gamma = \alpha + i\beta\gamma_5$.

Furthermore,

$$\mathscr{E}_{2,\tau}(\Lambda, \sigma, \zeta) = \frac{1}{2} \int d\mu_\tau \operatorname{Tr} (H(\Lambda, \sigma)^2),$$
$$\mathscr{E}_{2,\tau}(\Lambda, \sigma) = \frac{1}{2} \int d\mu_\tau \operatorname{Tr} (K(\Lambda, \sigma)^2)$$

are the second order Euclidean renormalization with and without a low momentum cut-off.

It is easy to see that

$$| \mathscr{E}_{2,\tau}(\Lambda, \sigma) - \mathscr{E}_{2,\tau}(\Lambda, \sigma, \zeta) | \leq c \mid \Lambda \mid$$

uniformly in $\sigma$ and $\Lambda$.

In addition, if $B \geq 0$, then

$$\| v_2 \|_p^p = \| \exp(- :(\phi, B\phi)_{G_\tau^{\sim -1}}:) \|_p^p = [\det_2(1 + 2pB)]^{-1/2} \leq \exp\left(2p^2 \| B \|_2^2\right).$$

Thus, to obtain a linear estimate for the Gaussian factors it is sufficient to prove that $B \geq 0$ for sufficiently large $\zeta$ and to prove a linear bound for the square of the Hilbert-Schmidt norm of B.

Let us show that for sufficiently large $\zeta$ B is a positive Hilbert-Schmidt operator. Writing $S_\tau = S_0 + S_4$, where $S_0$ is the usual Euclidean two-point fermion function, and using eqs. (3.1) and the estimate of Lemma 3.1, we see that the function

$$G_{reg}(k) + \int d^2p[|1 - \zeta(p_+)|^2 \,\mathrm{Sp}\, S_0^{+\sim}(p_+)S_0^\sim(p_+)(p_+^2 + m^2)^{1/2} (p_-^2 + m^2)^{-1/2} - 2(\alpha^2 + \beta^2)(p^2 + m^2)^{-1}] \quad (6.3)$$

is bounded by a constant uniformly in $\zeta$. Estimating the second item in (6.3) in the same way as in [11] we obtain that

$$B \geq [\pi(\alpha^2 + \beta^2) \ln(1 + \zeta^2/m^2) + \mathrm{const} - s]G_\tau^\sim \Lambda^2.$$

So the operator B is positive for sufficiently large $\zeta$.

Now we estimate the Hilbert-Schmidt norm of the operator B in the space $L_2(G_\tau^\sim(p)^{-1}d^2p)$. We replace the space $L_2(G_\tau^\sim(p)^{-1}d^2p)$ by $L_2(d^2p)$ and write $\Lambda_\alpha := \Lambda\chi_\alpha$, where $\chi_\alpha$ is the indicator of the squares with center in $\alpha \in \mathbb{Z}^2$ and side 1. Then

$$B = B' + B''$$

where

$$B' = -sG_\tau^{\sim 1/2}\Lambda^2 G_\tau^{\sim 1/2},$$
$$B'' = \Sigma_{\alpha,\beta \in \Lambda'} B_{\alpha\beta},$$
$$B_{\alpha\beta} = G_\tau^{\sim 1/2}\Lambda_\alpha G_{reg}\Lambda_\beta G_\tau^{\sim 1/2}.$$

It is easy to see that

$$\| B' \|_2^2 = s^2 \int dxdy G_\tau(x - y)^2 \Lambda(x)^2 \Lambda(y)^2 \leq s^2 \| \Lambda(x) \|_4^4 \| G_\tau(x) \|_2^2 \leq s^2 |\Lambda| \| G_\tau(x) \|_2^2 \sup_\alpha \| [\chi_\alpha \Lambda]^\sim \|_{4/3}^4 \leq c |\Lambda|.$$

Let us consider operators $B_{\alpha\beta}$. First we show that $B_{\alpha\beta} \in \mathscr{C}_2$. It is easy to see that

$$\| B_{\alpha\beta} \|_2^2 = \int d^2kd^2rd^2sd^2u G_\tau^\sim(k)\Lambda_\beta^\sim(k - r)G_{reg}(r)\Lambda_\alpha^\sim(r - s)G_\tau^\sim(s) \Lambda_\alpha^\sim(s - u)G_{reg}(u)\Lambda_\beta^\sim(u - k).$$

Since (6.3) is bounded, so the direct calculation (see Seiler [3, A 12]) and the triangle inequality give

$$\begin{aligned} | G_{reg}(r) | &\leq c_1 \ln (2 + | r |) \leq c_1 \ln (2 + | k - r |) \ln (2 + | k |) \\ | G_{reg}(u) | &\leq c_1 \ln (2 + | u |) \leqslant c_1 \ln (2 + | u - k |) \ln (2 + | k |) \end{aligned} \tag{6.4}$$

Then

$$\| B_{\alpha\beta} \|_2^2 \leq c \int d^2k d^2r d^2s d^2u G_\tau^\sim (k) \ln^2 (2 + | k |) | \Lambda_\beta^\sim (k - r) |$$
$$\ln (2 + | k - r |) | \Lambda_\alpha^\sim (r - s) | G_\tau^\sim (s) | \Lambda_\alpha^\sim (s - u) | | \Lambda_\beta^\sim (u - k) | \ln (2 + | u - k |)$$
$$\leq c \| \Lambda_\alpha^\sim (p) \ln (2 + | p |) \|_{4/3}^2 \| \Lambda_\beta^\sim (p) \|_{4/3}^2 \| G_\tau^\sim (p) \|_2 \| G_\tau^\sim (p) \ln^2 (2 + | p |) \|_2$$

where we use Schwarz's and Young's inequalities to obtain the last expression. Now the condition *ii*) for $\Lambda$ implies that $B_{\alpha\beta} \in \mathscr{C}_2$.

Now

$$\| B'' \|_2^2 = \Sigma_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Lambda'} \operatorname{Tr} B_{\alpha_1\alpha_2} B_{\alpha_3\alpha_4}.$$

To prove a linear estimate it is sufficient to prove that

$$| \operatorname{Tr} B_{\alpha_1\alpha_2} B_{\alpha_3\alpha_4} | \leq c \prod_{i=1}^{4} (1 + | \alpha_i - \alpha_{i+1} |)^{-3}$$

where $\alpha_5 = \alpha_1$. This estimate follows from the equality

$$\operatorname{Tr} AB = \operatorname{Tr} BA$$

and from the estimate

$$| \operatorname{Tr} B_{\alpha_1\alpha_2} B_{\alpha_3\alpha_4} | \leq c \prod_{i=1}^{2} (1 + | \alpha_i - \alpha_{i+1} |)^{-6}.$$

$B_{\alpha_1\alpha_2} \in \mathscr{C}_2$ and Ch. XI.8, exercise D.49*c* [21] imply that

$$\operatorname{Tr} B_{\alpha_1\alpha_2} B_{\alpha_3\alpha_4} = \int d^2k d^2r d^2s d^2u G_\tau^\sim (k) \Lambda_{\alpha_1}^\sim (k - r) G_{reg}(r)$$
$$\Lambda_{\alpha_2}^\sim (r - s) G_\tau^\sim (s) \Lambda_{\alpha_3}^\sim (s - u) G_{reg}(u) \Lambda_{\alpha_4}^\sim (u - k) \tag{6.5}$$

Let $h$ be a $C_0^\infty(\mathbb{R})$ function with support in $(- \sqrt{2} - \delta, \sqrt{2} + \delta)$ identically one in $(- \sqrt{2}, \sqrt{2})$ where $\delta > 0$ is chosen with $\sqrt{2} + \delta < 2$.

Now, if $| \alpha_1 - \alpha_2 | > \sqrt{2}$, $| \alpha_2 - \alpha_3 | > \sqrt{2}$ (non-touching squares), then it is easy to see that (6.5) is the limit of the analogous expression in which $G_{reg}(r)$ is replaced by $G_{reg}(r, \varkappa)$, where $G_{reg}(r, \varkappa)$ is a function obtained by performing an ultraviolet cut-off in the fermion propagators (i. e. to bound the integration domain in (6.2)). Thus, we have

$$\int d^2r \Lambda_{\alpha_1}^\sim (k - r) G_{reg}(r) \Lambda_{\alpha_2}^\sim (r - s) = A^\sim (k, s)$$

where

$$\begin{aligned} A(x, y) &= \Lambda_{\alpha_1}(x)h(|x-y| - |\alpha_1 - \alpha_2|)F_1(x-y)F_2(x-y)\Lambda_{\alpha_2}(y), \\ F_1(x) &= [|1-\zeta(p)|^2 \operatorname{Sp} S_\tau^{+\sim}(p)S_\tau^\sim(p)(p^2+m^2)^{1/2}]^\wedge(x), \\ F_2(x) &= [(p^2+m^2)^{-1/2}]^\wedge(x). \end{aligned}$$

In the same way we can replace the function $G_\tau^\sim(s)$ by $A^\sim(s)$, where $A(x) = h(|x| - |\alpha_2 - \alpha_3|)G_\tau(x)$. Thus in the case of non-touching squares

$$\operatorname{Tr} B_{\alpha_1\alpha_2}B_{\alpha_3\alpha_4} = \int d^2k d^2s d^2u G_\tau^\sim(k)A^\sim(k, s)A(s)\Lambda_{\alpha_3}^\sim(s-u)G_{\text{reg}}(u)\Lambda_{\alpha_4}^\sim(u-k)$$

Using inequality (6.4) we see that

$$\begin{aligned} |\operatorname{Tr} B_{\alpha_1\alpha_2}B_{\alpha_3\alpha_4}| &\le \int d^2k d^2s d^2u \, | G_\tau^\sim(k) \ln(2+|k|)A^\sim(k, s)A^\sim(s) \\ &\qquad \Lambda_{\alpha_3}^\sim(s-u)\Lambda_{\alpha_4}^\sim(u-k)\ln(2+|u-k|)| \\ &\le (\text{Hölder's inequality}) \, \| G_\tau^\sim(k)\ln(2+|k|)\|_\infty \| A^\sim(k,s)\|_2 \\ &\quad \left\| A^\sim(s)\int d^2u \Lambda_{\alpha_3}^\sim(u)\Lambda_{\alpha_4}^\sim(u-k)\ln(2+|u-k|)\right\|_2 \\ &\le (\text{Young's inequality}) \, c\|A(x,y)\|_2 \|A(x)\|_2 \|\Lambda_{\alpha_3}^\sim\|_{4/3}\|\Lambda_{\alpha_3}^\sim(k)\ln(2+|k|)\|_{4/3} \\ &\le (\text{Lemmas 3.2, 3.3}) c\prod_{i=1}^{2}(1+|\alpha_i - \alpha_{i+1}|)^{-6}. \end{aligned}$$

In the same way we consider the other cases of mutual locations of $\alpha_1$, $\alpha_2$, $\alpha_3$.

Thus,

$$\begin{aligned} \|B\|_2^2 &= \Sigma_{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \Lambda'} \operatorname{Tr} B_{\alpha_1\alpha_2}B_{\alpha_3\alpha_4} \\ &\le \Sigma_{\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\Lambda'} c\prod_{i=1}^{4}(1+|\alpha_i - \alpha_{i+1}|)^{-3} \le c'|\Lambda| \end{aligned}$$

uniformly in $|\Lambda|$.

The estimation of the Gaussian factors is finished.

Now we proceed to the factor $v_3$. Estimating this factor we shall follow in the main the proof of Proposition II.1 given by Magnen and Seneor [*6*] (remark that we follow the proof of Proposition II.1 which was given in the preprint only).

Let $v_3(\sigma)$ be the factor $v_3$ with an ultraviolet cut-off $\sigma$. First, we show that $v_3(\sigma)$ defines $\mu_\tau$ measurable function for $\sigma \to \infty$. Theorem 5.1, the expansion $S_\tau = S_0 + S_4$, eqs. (3.1) and the estimates of Lemma 3.1 imply that $H \in \mathscr{C}_p$, $p > 2$, $\mu_\tau$ almost everywhere. Thus, to show that $v_3(\sigma)$ defines $\mu_\tau$ measurable function for $\sigma \to \infty$ it is sufficient to show that

$$\operatorname{Tr} : (H(\sigma)^* + H(\sigma))^2 : \to :(\phi, B_1\phi):_{G_\tau^{\sim -1}} \quad \text{in} \quad L_2(\mathscr{S}'(\mathbb{R}^2), d\mu_\tau)$$

for $\sigma \to \infty$. For this it is sufficient to show that the operator $B_1$ is Hilbert-Schmidt in $L_2(G_\tau^\sim(p)^{-1}d^2p)$. But $B_1 = G_\tau\Lambda(G_{reg} + F_{reg})\Lambda$, where $G_{reg}$ and $F_{reg}$ are the operators of multiplication by functions, respectively, (6.2) and

$$F_{reg}(k) = \int d^2p[\mathrm{Sp}\,\Gamma S^\sim_{\tau,\zeta}(p_+)\Gamma S^\sim_{\tau,\zeta}(p_-) + 2(\alpha^2 + \beta^2)(p^2 + m^2)^{-1}].$$

In the Hilbert space $L_2(d^2p)$ $B_1 = G_\tau^{1/2}\Lambda(G_{reg} + F_{reg})\Lambda G_\tau^{1/2}$. Since eqs. (3.1), the estimates of Lemma 3.1 and Seiler's calculations [*10*, Appendix] imply that $G_{reg} + F_{reg}$ is a bounded operator and Lemma 2.1 [*4*] implies that $\Lambda G_\tau^{1/2} \in \mathscr{C}_4$, so the operator $B_1$ is Hilbert-Schmidt.

Moreover, it can be shown, as above for the kernel $G_{reg}$, that for $|\alpha - \beta| > \sqrt{2}$ the operator $\Lambda_\alpha F_{reg}\Lambda_\beta$ has the kernel

$$\Lambda_\alpha(x)\,\mathrm{Sp}\,\Gamma S_{\tau,\zeta}(x-y)\Gamma S_{\tau,\zeta}(x-y)\Lambda_\beta(y),$$

which satisfies the estimate $c_1 \mid \Lambda_\alpha(x)\Lambda_\beta(y) \mid (1 + |\alpha - \beta|)^{-4}$. The proof of this estimate is similar to that of Lemma 3.2.

In the following we shall not write down the ultraviolet cut-off $\sigma$, keeping in mind that our bounds are uniform in $\sigma \le \infty$.

We introduce an increasing sequence of cut-offs $\rho_0, \rho_1, \ldots, \rho_n, \ldots$ with $\rho_0 = 0$ and $\rho_n = be^n$, $b \ge 1$. Let the values of ultraviolet cut-offs $k_\Delta$ belong to this sequence. We also define cut-off fields by

$$\chi_\Delta(x)\phi_{k_\Delta}(x) := \chi_\Delta(x)\int d^2k\phi^\sim(k)\eta_{k_\Delta}(k)\exp(ikx),$$

where $\eta_{k_\Delta}(k) = \eta(k/k_\Delta)$ and $\eta$ is a positive $C^\infty$ function, $\eta(k) = 1$ for $|k| \le 1/2$, $0 \le \eta(k) \le 1$ for $1/2 \le |k| \le 2$, $\eta(k) = 0$ for $2 \le k$. Then, for a set $\{k_{\Delta_\alpha}\}_{\alpha\in\mathbb{Z}^2}$ of localized cut-offs we define

$$H_k = \sum_\Delta (1 - \zeta(P))(P^2 + m^2)^{1/4}S^\sim_\tau(P)\Gamma\Lambda\chi_\Delta\phi_{k_\Delta}(P^2 + m^2)^{-1/4},$$

$$A_k = H_k + H_k^* + H_k^*H_k.$$

We write

$$\ln v_3 \le \frac{1}{2}\ln\det_4(1 + A_{k+}) + \frac{1}{2}|\ln\det_4(1 + A_{k+}) - \ln\det_4(1 + A_+)|$$
$$+ \frac{1}{6}\mathrm{Tr}\,A_+^3 - \frac{1}{4}\mathrm{Tr}\,((H^*H)^2) - \mathrm{Re}\,\mathrm{Tr}\,(H^2H^*) - \frac{1}{4}\mathrm{Tr} : (H^* + H)^2 :$$

We use the inequality

$$\ln\det_4(1 + A_{k+}) \le \frac{1}{2}\mathrm{Tr}\,A_{k+}^2 - \frac{1}{3}\mathrm{Tr}\,A_{k+}^3 \le \frac{1}{2}\mathrm{Tr}\,A_k^2 - \frac{1}{3}\mathrm{Tr}\,A_{k+}^3$$

for $A_k \geq -1$, and Lemma 6.2 to obtain the bound

$$\ln v_3 \leq \frac{1}{6}\operatorname{Tr}(A_+^3 - A_{k+}^3) - \frac{1}{4}\operatorname{Tr}((H^*H)^2 - (H_k^*H_k)^2)$$
$$- \operatorname{Re}\operatorname{Tr}(H^2H^* - H_k^2H_k^*) + \frac{1}{8}\| A^4 - A_k^4 \|_1$$
$$+ \frac{1}{4}\operatorname{Tr}(H_k^* + H_k)^2 - \frac{1}{4}\operatorname{Tr} : (H^* + H)^2 :$$

Applying Seiler's arguments [*10*, Lemma 3.2] we obtain the bound

$$\ln v_3 \leq \frac{1}{6}\| A^3 - A_k^3 \|_1 + \frac{1}{4}\| (H^*H)^2 - H_k^*H_k)^2 \|_1$$
$$+ \| H^2H^* - H_k^2H_k^* \|_1 + \frac{1}{8}\| A^4 - A_k^4 \|_1$$
$$+ \frac{1}{4}| \operatorname{Tr} : (H_k^* + H_k)^2 : - \operatorname{Tr} : (H^* + H)^2 : |$$
$$+ \frac{1}{4}\int d\mu_\tau \operatorname{Tr}(H_k^* + H_k)^2 \qquad (6.6)$$

LEMMA 6.2. — *(Seiler [10], Magnen, Seneor [6]). Let* A, $B \in \mathscr{C}_4$ *be two self-adjoint operators, then*

$$d = | \ln \det_4 (1 + A_+) - \ln \det_4 (1 + B_+) | \leq \frac{1}{4}\| A^4 - B^4 \|_1$$

Since the proof of Lemma 6.2 given by Magnen and Seneor [*6*, Lemma III.1] is contained only in the preprint, we give here its proof.

*Proof of Lemma 6.2* [*6*, preprint, Lemma III.1]. — It is sufficient to prove the inequality for operators of finite rank, then the proof follows by approximations. Let $\lambda_i^+$ and $\mu_i^+$ be the eigenvalues of $A_+$, and $B_+$ form a decreasing sequence, then with

$$v_i(t) = t\lambda_i^+ + (1 - t)\mu_i^+$$

we have

$$d = \left| \sum_i \left( \ln(1 + \lambda_i^+) - \lambda_i^+ + \frac{1}{2}\lambda_i^{+2} - \frac{1}{3}\lambda_i^{+3} - \ln(1 + \mu_i^+) + \mu_i^+ \right.\right.$$
$$\left.\left. - \frac{1}{2}\mu_i^{+2} + \frac{1}{3}\mu_i^{+3} \right) \right| = \left| \sum_i \int_0^1 dt \frac{d}{dt}\left[ \ln(1 + v_i(t)) - v_i(t) \right.\right.$$
$$\left.\left. + \frac{1}{2}v_i(t)^2 - \frac{1}{3}v_i(t)^3 \right] \right|$$
$$\leq \sum_i \int_0^1 dt(1 + v_i(t))^{-1} | \lambda_i^+ - \mu_i^+ | v_i(t)^3$$

Since $1 + v_i(t) \geq 1$, so

$$d \leq \sum_i |\lambda_i^+ - \mu_i^+| \int_0^1 dt v_i(t)^3 = 4^{-1} \sum_i |\lambda_i^+ - \mu_i^+| (\lambda_i^{+3} + \lambda_i^{+2}\mu_i^+ + \lambda_i^+ \mu_i^{+2} + \mu_i^{+3}) \leq \frac{1}{4} \sum_i |\lambda_i^{+4} - \mu_i^{+4}|$$

Since $\lambda_i^{+4}$ and $\mu_i^{+4}$ are subsets of the sets of the eigen-values of $A^4$ and $B^4$, thus, by applying the generalization of Lidskii's theorem (Kato [*19*, II Theorem 6.10 and 6.11] and Seiler [*10*, Lemma 3.2]) we have

$$d \leq \frac{1}{4} \| A^4 - B^4 \|_1$$

Lemma 6.2 is proved.

Eqs. (3.1), Lemma 3.1 and Seiler's calculations [*10*, Appendix A] imply that

$$\int d\mu_\tau \operatorname{Tr} (H_k^* + H_k)^2 \leq c \sum_\Delta \ln k_\Delta$$

uniformly in a space-time cut-off $\Lambda$.

We transform the expression (6.6) in the following way. Any difference term between quantities with and without ultraviolet cut-offs is developed as a sum of terms containing the difference $H - H_k$, or $H^* - H_k^*$, or $\phi - \phi_k$. Using linearity in the fields, we replace each field $\phi$ by $\sum_\Delta \phi\chi_\Delta$ and use the triangle inequality

$$\left\| \sum_\Delta \mathscr{A}(\Delta) \sum_{\Delta'} \mathscr{A}(\Delta') \dots \right\| \leq \sum_\Delta \sum_{\Delta'} \dots \| \mathscr{A}(\Delta)\mathscr{A}(\Delta') \dots \|.$$

Now each $\phi$ localized in $\Delta$ with an ultraviolet cut-off $k_\Delta = \rho_{n(\Delta)}$, for some $n(\Delta) \in \mathbb{Z}^+$, will be decomposed in (this is not done for the fields $\phi - \phi_k$)

$$\phi_{k_\Delta} = \sum_{i=1}^{n(\Delta)} \phi_i$$

with

$$\phi_i(x) = \int d^2k \phi^\sim(k)(\eta_{\rho_i}(k) - \eta_{\rho_{i-1}}(k)) \exp(ikx)$$

and $\eta_\rho(k) = \eta(k_1/\rho)\eta(k_2/\rho)$ (or $\eta_\rho(k) = \eta(k_2/\rho)$) and where $\eta$ is defined above.

After this we also use linearity in the fields or the triangle inequality

$$\left\| \sum_{i_1} \mathscr{A}_{i_1} \sum_{i_2} \mathscr{A}_{i_2} \dots \right\|_p \leq \sum_{i_1} \sum_{i_2} \dots \| \mathscr{A}_{i_1}\mathscr{A}_{i_2} \dots \|_p$$

to rewrite each localized expression as a sum over $i_1, i_2, \dots$

After these transformations expression (6.6) is bounded by a sum of expressions with each field being localized. Each of these expressions contains a difference $\phi - \phi_k$ localized in some square $\Delta_0$, which we call a reference square. Moreover each of the rest of the fields is localized and contains an upper $\rho_i$ and a lower $\rho_{i-1}$ ultraviolet cut-off.

We estimate with the help of Lemma 4.1 each of the expressions $\| \mathscr{A}(\Delta_1) \dots \mathscr{A}(\Delta_p) \|_1$, where $\mathscr{A}(\Delta)$ is $H(\Delta)$, or $H_k(\Delta)$, or $H(\Delta) - H_k(\Delta)$, or their hermitian conjugates. Taking into account that H has a low momentum cut-off which is performed with the help of a smooth function $\zeta(k)$, we obtain a bound of the form $c\prod_{r=1}^{p} d(\Delta_{r-1}, \Delta_r)^{-b} \| \mathscr{A}'(\Delta_r) \|_p$ for any given $b$. Here $d(\Delta, \Delta') = \max(1, \operatorname{dist}(\Delta, \Delta'))$ and $\mathscr{A}'(\Delta)$ is the operator with the kernel which is obtained from the kernel of the operator $\mathscr{A}(\Delta)$ by replacing the fermion propagator $S_\tau^\sim$ with cut-offs by $(p^2 + m^2)^{-1/4+\varepsilon}$. With the help of the triangle inequality we estimate the product

$$\prod_{r=1}^{p} d(\Delta_{r-1}, \Delta_r)^{-b} \quad \text{by} \quad \prod_{r=1}^{p} d(\Delta_0, \Delta_r)^{-b}$$

(probably, with another $b$). As a result, we obtain the bound

$$\| \mathscr{A}(\Delta_1) \dots \mathscr{A}(\Delta_p) \|_1 \leq c\prod_{r=1}^{p} d(\Delta_0, \Delta_r)^{-b} \| \mathscr{A}'(\Delta_r) \|_p .$$

And finally, for the terms appearing in the expansion of

$$| \operatorname{Tr} : (H_k^* + H_k)^2 : - \operatorname{Tr} : (H^* + H)^2 : |$$

we replace the kernels by the same ones multiplied by $d(\Delta_0, \Delta)^{-3}$.

We want to estimate the expression generated from (6.6) by a sum over the reference squares $\Delta_0$ of expressions depending only on the ultraviolet cut-off in $\Delta_0$ and independent of the values $k_\Delta$ in other squares. The dependence on $k_\Delta$ in our bound is through the sum over $\{ i \}$. Since each item of the sum is positive, we extend the sum over $\{ i \}$ up to infinity, and in this way it does not depend on $k_\Delta$, $\Delta \neq \Delta_0$.

We, thus, get the bound

$$\ln v_3 \leq O(1) \sum_{\Delta_0} \{ \ln \rho_{n(\Delta_0)} + O(1) + V(\Delta_0, \rho_{n(\Delta_0)}) \}$$

where

$$V(\Delta_0, \rho_{n(\Delta_0)}) = \sum_{s \in S} Q(\Delta_0, s) \tag{6.7}$$

and

$$Q(\Delta_0, s) = O(1)\left(\prod_{\substack{\text{over fields in } j\text{th term,}\\ \text{except the field } \phi-\phi_k}} O(1)d(\Delta_0, \Delta)^{-3}\right)R_{j,\Delta_0}(i, \Delta)$$

Here, $s = (j, i, \Delta) \in S$ is a vector, the components of which belong to the following sets. $j \in J$, where J labels the terms appearing after pointing out the difference $\phi - \phi_k$. $i \in \{ i, j \}$, $\Delta \in \{ \Delta, j \}$, where the sets $\{ i, j \}$ and $\{ \Delta, j \}$ are defined in the following way. If the $j$th term of the expansion contains the difference $\phi - \phi_k$ and $s - 1$ fields $\phi$, then $\{ \Delta, j \}$ is the set of sequences $(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \ldots, \Delta_{\alpha_{s-1}})$, where $\Delta_{\alpha_r} \in \Lambda'$, and $\{ i, j \}$ is the set of sequences $(i_1, i_2, \ldots, i_{s-1})$, where $i_r \in \mathbb{Z}^+$, $r = 1, \ldots, s - 1$.

We are now in a position to apply the modified argument of Dimock and Glimm (see Magnen, Seneor [*6*], Dimock, Glimm [*12*]).

We define a partition $\mathscr{D}_n(\Delta)$ of the space $\mathscr{S}'(\mathbb{R}^2)$. We define it by induction. Let $\mathscr{D}_1(\Delta)$ be the subset $\mathscr{S}'$ where $V(\Delta, \rho_1) = 1$. Let $\mathscr{D}_n(\Delta)$ be the subset of $\mathscr{S}'$ where $V(\Delta, \rho_{n-1}) > 1$ and $V(\Delta, \rho_n) \leq 1$. Since, as we will see below (Lemma 6.3), $\int d\mu_\tau \mid V(\Delta, \rho_n) \mid^2 \to 0$ as $n \to \infty$, so $\{ \mathscr{D}_n \}_{n \in \mathbb{Z}^+}$ defines a partition of the space $\mathscr{S}'(\mathbb{R}^2)$.

Further,

$$\bigcup_{\{n(\Delta)\}} \bigcap_{\Delta} \mathscr{D}_{n(\Delta)}(\Delta) = \bigcap_{\Delta} \bigcup_{n} \mathscr{D}_n(\Delta) = \mathscr{S}' \quad \mu_\tau \text{ a. e.}$$

where $\{ n(\Delta) \}$ is the set of all the functions from $\bigcup_{\Delta \in \Lambda'} \{ \Delta \}$ into $\mathbb{Z}^+$.

Thus,

$$\int d\mu_\tau(.) = \sum_{\{n(\Delta)\}} \int_{\bigcap_{\Delta} \mathscr{D}_{n(\Delta)}(\Delta)} d\mu_\tau(.)$$

and we obtain the bound

$$\int d\mu_\tau v_3^p \leq \sum_{\{n(\Delta)\}} \int_{\bigcap_{\Delta} \mathscr{D}_{n(\Delta)}(\Delta)} d\mu_\tau v_3^p$$

$$\leq \sum_{\{n(\Delta)\}} \int d\mu_\tau \prod_{\Delta} \{ V(\Delta, \rho_{n(\Delta)-1})^{p(n(\Delta))} \exp [pO(1)(O(1) + \ln \rho_{n(\Delta)})] \}$$

for any sequence of positive integers $p(n(\Delta))$.

To obtain a linear bound it is sufficient to prove the following lemma.

LEMMA 6.3. — *Whatever be the sequence* $\{ p(n(\Delta)) \}$, *there exists* $\varepsilon > 0$ *such that*

$$\int d\mu_\tau [V(\Delta, \rho_{n(\Delta)-1})]^{p(n(\Delta))} \leq \prod_{\Delta} O(1)^{p(n(\Delta))} (p(n(\Delta))\,!)^{O(1)} \rho_{n(\Delta)}^{-\varepsilon p(n(\Delta))}$$

In fact, assume that Lemma 6.3 is proved. Then

$$\int d\mu_\tau v_3^p \leq \sum_{\{n(\Delta)\}}\prod_{\Delta} O(1)O(1)^{p(n(\Delta))}(p(n(\Delta))\,!)^{O(1)} \exp(O(1)\ln \rho_{n(\Delta)})\rho_{n(\Delta)}^{-\varepsilon p(n(\Delta))}$$

$$\leq \prod_{\Delta}\left\{\sum_{n}(O(1)O(1)^{p(n)}(p(n)\,!)^{O(1)} \exp(O(1)\ln \rho_n)\rho_n^{-\varepsilon p(n)} \leq O(1)^{|\Lambda|}\right\}$$

The last inequality is obtained by chosing an appropriate sequence $\{ p(n) \}$ (for example, as in [12], $p(n) = \exp(\delta n)$ with sufficiently small $\delta > 0$).
It remains to prove Lemma 6.3.

*Proof of Lemma 6.3.* — Change the order of summation and multiplication in

$$\prod_{\Delta_0}[V(\Delta_0, \rho_{n(\Delta_0)-1})]^{p(n(\Delta_0))}$$

$$\prod_{\Delta_0}[V(\Delta_0, \rho_{n(\Delta_0)-1})]^{p(n(\Delta_0))} = \sum_{r(.,.)\in\mathcal{R}}\prod_{\Delta_0}\prod_{q_{\Delta_0}=1}^{p(n(\Delta_0))} Q(r(\Delta_0, q_{\Delta_0})) \qquad (6.8)$$

where $\mathcal{R}$ is the set of all the vector-functions depending on two arguments $\Delta_0 \in \bigcup_{\Delta_0\in\Lambda'}\{ \Delta_0 \}$ and $q_{\Delta_0} \in \{ 1, \ldots, p(n(\Delta_0)) \}$ and taking their values in $\left(\bigcup_{\Delta_0\in\Lambda'}\{ \Delta_0 \}\right) \times S$.

To estimate the integral of (6.8) over $d\mu_\tau$ we use the method of combinatoric factors [20]. It is sufficient to choose the following combinatoric factors: to each non reference localization square we assign the factor $O(1)d(\Delta_0, \Delta)^3$, to each boson field entering $Q(r(\Delta_0, q_{\Delta_0}))$ (except for the difference $\phi - \phi_k$) with a low cut-off $\rho_i$ we assign the factor $O(1)\rho_i^\varepsilon$ for some $\varepsilon > 0$. Then, to prove Lemma 6.3 it is sufficient to show that

$$\sup_{r(.,.)\in\mathcal{R}}\int d\mu_\tau\prod_{\Delta_0}\prod_{q_{\Delta_0}=1}^{p(n(\Delta_0))} Q_1(r(\Delta_0, q_{\Delta_0})) \leq \prod_{\Delta_0} O(1)^{p(n(\Delta_0))}(p(n(\Delta_0))\,!)^{O(1)}\rho_{n(\Delta_0)}^{-\varepsilon p(n(\Delta_0))}$$

where $Q_1$ is Q with the combinatoric factors, i. e. (cf. (6.7))

$$Q_1(\Delta_0, s) = O(1)\left(\prod_{\substack{\text{over fields in } j\text{th term,}\\ \text{except the field } \phi-\phi_k}} O(1)\rho_i^\varepsilon\right)R_{j,\Delta_0}(i, \Delta)$$

Let us consider the expression

$$\int d\mu_\tau\prod_{\Delta_0}\prod_{q_{\Delta_0}=1}^{p(n(\Delta_0))} Q_1(r(\Delta_0, q_{\Delta_0}))$$

It contains trace-norms $\| \cdot \|_a$ of the operators $\mathscr{A}'(\Delta)$ with $a \geq 3$. We estimate the integral by replacing all trace-norms $\| \cdot \|_a$ by $\| \cdot \|_3$. We use the interpolation theorem [*19*, p. 137-139, 11, p. 2290] to estimate

$$\| \mathscr{A}' \|_3 \leq \| \mathscr{B}' \|_2^{1/3} \| \mathscr{B}'' \|_4^{2/3}$$

where $\mathscr{B}'$, $\mathscr{B}''$ are appropriately chosen expressions (if

$$\mathscr{A}_z = (\mathrm{P}^2 + m^2)^{-z}\phi\Lambda(\mathrm{P}^2 + m^2)^{-1/4},$$

then $\mathscr{A}' = \mathscr{A}_{z=1/4-\varepsilon}$ and one may put $\mathscr{B}' = \mathscr{A}_{z=1-\varepsilon}$, $\mathscr{B}'' = \mathscr{A}_{z=1/8-\varepsilon}$).

Now we apply to the integral over $d\mu_\tau$ Holder's inequality in order to replace $| \cdot |$, $\| \cdot \|_2^{1/3}$, etc., by $| \cdot |^2$, $\| \cdot \|_2^2$, etc.

As a result we obtain the integral

$$\left[ \int d\mu_\tau \prod_{\Delta_0 \in \Lambda'} \prod_{q(\Delta_0)=1}^{p(n(\Delta_0))} \mathrm{Q}_2(r(\Delta_0, q_{\Delta_0}))^m \right]^{1/m} \tag{6.10}$$

where $\mathrm{Q}_2$ is obtained from $\mathrm{Q}_1$ by replacing some H by the appropriately chosen H′ and H″ (see above). The integrand in (6.10) has the form of (5.4) with localized $w \in \mathrm{L}_{1+\varepsilon_1}$, for some $\varepsilon_1 > 0$. For the kernels of H, H′, H″ this follows immediately and for the kernels appearing from

$$| \operatorname{Tr} : (\mathrm{H}_k^* + \mathrm{H}_k)^2 : - \operatorname{Tr} : (\mathrm{H}^* + \mathrm{H})^2 : |$$

this follows from Lemma 3.2*c* and Hölder's inequality.

We apply to (6.10) the bound of Theorem 5.4. It is evident that the localization numbers $\mathrm{N}_{\Delta_0} \leq \mathrm{O}(1)p(n(\Delta_0))$, $\| w \|_{\mathrm{L}_1+\varepsilon_1} \leq \prod_{\Delta_0} \mathrm{O}(1)^{p(n(\Delta_0))}$, $n_* \leq 2$.

Each combinatoric factor $0(1)\rho_i^\varepsilon$ is compensated by the same factor presented in the bound of Theorem 5.4. In addition, each factor in the product over $q_{\Delta_0}$ gives the factor $\rho_{n(\Delta_0)}^{-\varepsilon}$. As a result we obtain the bound (6.9).

Thus, Lemma 6.2 and so a linear bound for $v_3$ is proved.

Theorem 6.1 is proved.

## 7. BOUNDS ON SCHWINGER FUNCTIONS, LINEAR $\mathrm{N}_\tau$ BOUND

Let us write the Matthews-Salam formulas for the unnormalized Schwinger functions

$$\mathfrak{S}(h_1, \ldots, h_n ; f_1, \ldots, f_k; g_1, \ldots, g_k)$$

$$:= \int d\mu_\tau \prod_{r=1}^{n} \phi(h_r)\mathscr{D}_k(f, g, \phi) \det{}_{\text{ren}} (1 + \mathrm{K}(\Lambda))$$

where

$$\mathcal{D}_k(f, g, \phi) = (-1)^{k(k-1)/2} \Big\langle \bigwedge_{i=1}^{k} (\mathrm{P}^2 + m^2)^{-1/4} f_i,$$
$$\bigwedge_{i=1}^{k} \{ (1 + \mathrm{K}(\Lambda))^{-1} (\mathrm{P}^2 + m^2)^{1/4} \mathrm{S}_\tau^{\sim}(\mathrm{P}) g_i \} \Big\rangle$$

THEOREM 7.1. — *Let each $h_i$ be localized in some square $\chi_\alpha$ with exactly $n_\alpha$ localized in square $\alpha$. Then, for suitable constants $c_1$, $c_2$*

$$| \mathfrak{S}(h_1, \ldots, h_n; f_1, \ldots, f_k; g_1, \ldots, g_k) |$$
$$\leq c_1^{|\Lambda|} c_2^{n+k} \prod_\alpha (n_\alpha !)^{1/2} \prod_{r+1}^{n} \| h_r \| \prod_{i=1}^{k} \| f_i \|_{-1/2} \| g_i \|_{-1/2+\varepsilon}$$

*Analogous statements are valid also in the case of the periodic boundary conditions.*

*Proof of Theorem 7.1.* — We use the method of Frohlich, Seiler-Simon [*4*], namely, we obtain bounds on the Schwinger generating function and use Cauchy estimates to bound Schwinger functions. Let $a_i = (\mathrm{P}^2 + m^2)^{-1/4} f_i$, $b_i = (\mathrm{P}^2 + m^2)^{1/4} \mathrm{S}_\tau^{\sim}(\mathrm{P}) g_i$. By homogeneity, we can suppose that

$$\| a_i \| = \| b_i \| = 1$$

in $\mathrm{L}_2(d^2p) \otimes \mathbb{C}^2$. Let $\mathrm{C}_i$ be the rank one operator $\mathrm{C}_i u = a_i \langle b_i, u \rangle$. Then

$$\mathfrak{S}(h\,; f\,; g) = \partial^{n+k} / \partial\mu_1 \ldots \partial\mu_n \partial\lambda_1 \ldots \partial\lambda_k \int d\mu_\tau$$
$$\exp\left( \sum_{i=1}^{n} \mu_i \phi(h_i) \right) \det_{\mathrm{ren}} \left( 1 + \mathrm{K}(\Lambda) + \sum_{j=1}^{k} \lambda_j \mathrm{C}_j \right) \Bigg|_{\mu=\lambda=0} \qquad (7.1)$$

where the definition of $\det_{\mathrm{ren}}$ $(1 + \mathrm{K}(\Lambda) + \mathrm{C})$ uses the same counterterms as before (cf. Sec. 6). One can demonstrate the validity of this equality using the formulas of [*21*, ch. XI, 9.23].

Denote the function whose derivatives occur on the right of (7.1), $\mathrm{G}(\mu, \lambda)$ with $\mu = (\mu_1, \ldots, \mu_n)$, $\lambda = (\lambda_1, \ldots, \lambda_k)$. Then it is easy to see that G is an entire function on $\mathbb{C}^{n+k}$.

By using, as in Sec. 6, the inequality 4.1 of Seiler, Simon [*4*] (including $\sum_{j=1}^{k} \lambda_j \mathrm{C}_j$ into the term C), the inequality

$$\int d\mu_\tau \, | \exp(\mu\phi(h)) |^r \leq \exp\left(c \, | \mu |^2 \, \| h \|_{-1}^2\right)$$

and acting as in Sec. 6, we obtain that if $\| h_i \|_{-1} = 1$, $\| a_i \| = \| b_i \| = 1$, then

$$| G(\mu, \lambda) | \leq c_1^{|\Lambda|} \exp \left[ c \sum | \lambda_i | + c \sum_{\alpha \in \mathbb{Z}^2} \Big( \sum_{i \in S_\alpha} | \mu_i |^2 \Big) \right]$$

where $S_\alpha$ is the set of $i$ with supp $h_i \subset$ supp $\chi_\alpha$.

Thus, by Cauchy estimates

$$| \partial^{n+k} G / \partial \mu_1 \ldots \partial \lambda_k | \leq c_3^{n+k} c_1^{|\Lambda|} \prod_{\alpha \in \mathbb{Z}^2} R_\alpha^{-n_\alpha} \exp [c(n_\alpha R_\alpha)^2]$$

Taking $R_\alpha = n_\alpha^{-1/2}$, we get

$$| \mathfrak{S}(h ; f ; g) | \leq c_1^{|\Lambda|} c_2^{n+k} \prod_\alpha (n_\alpha !)^{1/2}$$

Since $\| a_i \| = \| f_i \|_{-1/2}$, $\| b_i \| = \| (P^2 + m^2)^{1/4} S_\tau^{\sim} g_i \|_{L_2}$, so homogeneity, eqs. (3.1) and Lemma 3.1 yield the bounds of the theorem.

Theorem 7.1 is proved.

*Proof of Theorem 1.2.* — The proof is similar to the arguments at the end of Sec. VI [*4*] and to the proof of Theorem 3.4 [*5*]. The density of Euclidean Jost states for the Hamiltonian $H_0 - aN_\tau$ is obvious (for the definition of Euclidean Jost states see [*5*]).

Moreover, the additional factor $\exp \left[ \int dt \chi(t) E_2(g, \sigma) - \mathscr{E}_{2,\tau}(\chi g, \sigma) \right]$ appears, where $E_2$ is the Hamiltonian energy renormalization in the second order and $\chi(t)$ is a time cut-off. If $\chi(t)$ is the indicator of $[a, b]$, then the direct calculation [*13*, § 3] implies the uniform bound

$$| (b - a) E_2(g, \sigma) - \mathscr{E}_{2,\tau}(\chi g, \sigma) | \leq c_1 (b - a) | g | .$$

Theorem 1.2 is proved.

ACKNOWLEDGMENTS

The author is grateful to M. I. Shirokov and A. M. Vasilev for their friendly encouragement. He is also indebted to J. Magnen and R. Seneor for having sent him the preprint of [*6*] prior to publication.

## REFERENCES

[*1*] J. Glimm and A. Jaffe, The $\lambda\varphi_2^4$ Quantum Field Theory without cut-offs. III. The Physical Vacuum, *Acta Math.*, t. **125**, 1970, p. 203-260.

[*2*] J. Glimm and A. Jaffe, *Quantum Field Models, in Statistical Mechanics and Quantum Field Theory*, Les Houches, 1970, ed. C. De Witt and R. Stora, Gordon and Breach, New York, 1971, p. 1-108.

[3] R. SCHRADER, Yukawa quantum field theory in two space-time dimensions without cut-offs, *Ann. Phys.* (N. Y.), t. **70**, 1972, p. 412-457.

[4] E. SEILER and B. SIMON, Bounds in the Yukawa$_2$ quantum field theory: upper bound on the pressure, hamiltonian bound and linear lower bound, *Commun. Math. Phys.*, t. **45**, 1975, p. 99-114.

[5] E. SEILER and B. SIMON, Nelson's symmetry and all that in the Yukawa$_2$ and $\phi_3^4$ field theories, *Ann. Phys.* (N. Y.), t. **97**, 1976, p. 470-518.

[6] J. MAGNEN and R. SENEOR, The Wightman axioms for the weakly coupled Yukawa model in two dimensions, *Commun. Math. Phys.*, t. **51**, 1976, p. 297-313; Preprint de l'École Polytechnique, octobre 1975.

[7] J. GLIMM and A. JAFFE, The energy-momentum spectrum and vacuum expectation values in quantum field theory, II, *Commun. Math. Phys.*, t. **22**, 1971, p. 1-22.

[8] P. T. MATTHEWS and A. SALAM, The Green's functions of quantized fields, *Nuovo Cimento*, t. **12**, 1954, p. 563-565.

[9] P. T. MATTHEWS and A. SALAM, Propagators of quantized fields, *Nuovo Cimento*, t. **2**, 1955, p. 120-134.

[10] E. SEILER, Schwinger functions for the Yukawa model in two dimensions with space-time cutoff, *Commun. Math. Phys.*, t. **42**, 1975, p. 163-182.

[11] E. SEILER and B. SIMON, On finite mass renormalizations in the two-dimensional Yukawa model, *J. Math. Phys.*, t. **16**, 1975, p. 2289-2293.

[12] J. DIMOCK and J. GLIMM, Measures on the Schwartz distribution space and application to $P(\varphi)_2$ field theories, *Adv. Math.*, t. **12**, 1974, p. 58-83.

[13] O. A. MCBRYAN, Volume dependence of Schwinger functions in the Yukawa$_2$ quantum field theory, *Commun. Math. Phys.*, t. **45**, 1975, p. 279-294.

[14] B. SIMON, *The $P(\varphi)_2$ euclidean (quantum) field theory*, Princeton University Press, 1974.

[15] G. M. MOLCHAN, The characterization of gaussian fields with the Markov property (in Russian), *Dokladi AN SSSR*, t. **197**, 1971, p. 784-787.

[16] I. M. GELFAND and G. E. SHILOV, *Generalized functions and their applications*. vol. I (In Russian), Fizmatgiz, Moscow, 1959.

[17] J. GLIMM, A. JAFFE and T. SPENCER, The particle structure of the weakly coupled $P(\varphi)_2$ model and other applications of high temperature expansions. Part II. In Constructive quantum field theory, ed. G. Velo and A. S. Wightman, *Lecture Notes in Physics*, vol. 25, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[18] I. C. GOHBERG and M. G. KREIN, Introduction to the theory of linear nonselfadjoint operators (In Russian), Nauka, Moscow, 1965 and Translations of mathematical monographs, vol. 18, *Amer. Math. Soc.*, Providence, R. I., 1969.

[19] T. KATO, *Perturbation theory for linear operators*, Springer, Berlin-Heidelberg-New York, 1966.

[20] J. GLIMM and A. JAFFE, The $\lambda(\varphi)^4)_2$ quantum field theory without cutoffs. IV. Perturbation of the hamiltonian, *J. Math. Phys.*, t. **13**, 1972, p. 1568-1584.

[21] N. DUNFORD and J. SCHWARTZ, *Linear Operators*, vol. II, Interscience, New York, 1963.

[22] A. COOPER and L. ROSEN, The weakly coupled Yukawa$_2$ field theory: cluster expansion and Wightman axioms, *Trans. Amer. Math. Soc.*, t. **234**, 1977, p. 1-88.

[23] L. ROSEN, Construction of the Yukawa$_2$ field theory with a large external field, *J. Math. Phys.*, t. **18**, 1977, p. 894-897.

[24] E. P. OSIPOV, Connection between the spectrum condition and the Lorentz invariance of the Yukawa$_2$ quantum field theory, *Commun. Math. Phys.*, t. **57**, 1977, p. 111-116.

[25] B. SIMON, Notes on infinite determinants of Hilbert space operators, *Adv. Math.*, t. **24**, 1977, p. 244-273.

[26] L. GROSS, On the formula of Matthews and Salam, *J. Funct. Anal.*, t. **25**, 1977, p. 162-209.

[27] E. P. HEIFETS and E. P. OSIPOV, The energy-momentum spectrum in the $Yukawa_2$ quantum field theory, *Commun. Math. Phys.*, t. **57**, 1977, p. 31-50.
[28] E. P. OSIPOV, *The $Yukawa_2$ quantum field theory: linear* $N_\tau$ *bound, locally Fock property,* Preprint TPh-95, Institute for mathematics, Novosibirsk, 1977 and *Ann. Inst. H. Poincaré*, t. **30**A, 1979, p. 193-206.

*(Manuscrit reçu le 6 novembre 1978)*