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Section A :
Physique théorique.

Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem (*)

by

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ABSTRACT. — The global presymplectic geometry of degenerate Lagrangian systems is investigated. A geometric constraint algorithm proposed earlier by us is used, in conjunction with techniques developed by Klein, to define and solve « consistent Lagrangian equations of motion ». This algorithm enables us to prove an equivalence theorem for the Lagrangian and Hamiltonian formulations of dynamical systems which are described by degenerate Lagrangians.

RÉSUMÉ. — On examine, dans ce travail, la géométrie présymplectique globale des systèmes Lagrangiens irréguliers. L'emploi d'un « algorithme des contraintes » précédemment proposé par les auteurs [*5*], et des techniques développées par J. Klein [*2*], permet d'établir, sous certaines hypothèses précises, un « théorème d'équivalence » pour les formulations Lagrangienne et Hamiltonienne de systèmes dynamiques décrits par des Lagrangiens irréguliers.

I. INTRODUCTION

The Lagrangian formulation of classical mechanics, from both philosophical and calculational standpoints, is the most fundamental approach

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to dynamics. Nevertheless, it is usually found necessary (and/or convenient) to cast the theory into Hamiltonian form, especially for the purpose of quantization. One underlying mathematical reason for this is that the Hamiltonian phasespace T^*Q (where Q is configuration space) is *canonically* a symplectic manifold whereas velocity phasespace TQ is not. It is the symplectic structure on phasespace which gives rise to the elegant simplicity of the Hamiltonian formalism [*1*].

In the so-called regular case, this « defect » of the Lagrangian formulation can easily be remedied. It has long been established, given a regular Lagrangian L on TQ, that it is possible to induce a symplectic structure Ω on TQ by either of two equivalent methods: via the « almost tangent » structure naturally associated to TQ (studied extensively by Klein [*2*]), or by using the Legendre transformation FL to pullback to TQ the canonical symplectic structure on T^*Q (the method of Abraham [*1*]). When the Lagrangian is regular, then, a canonical formalism can be established *on* TQ *itself* by means of this induced symplectic structure; it is apparent that one need not resort to the Hamiltonian formulation on T^*Q in order to « cast the theory into canonical form », or even to quantize the theory [*3*].

Thus, in the regular case, the basic mathematical object of the Lagrangian formulation of mechanics is the symplectic velocity phasespace (TQ, L, Ω). The dynamics of a given system is determined by solving the Lagrange equations

$$i(X)\Omega = dE\,, \tag{1.1}$$

where the « energy » E plays the role of the Hamiltonian on TQ. As Ω is nondegenerate, these equations possess unique solutions.

The assumption of regularity is, however, too restrictive (e. g., the Maxwell and Einstein systems). The major implication of the degeneracy of the Lagrangian is that Ω will now be merely *presymplectic* (i. e., Ω is no longer of maximal rank). Consequently, the equations of motion (1.1) as they stand need not possess solutions at all (and in general, even if solutions exist they will not be unique).

The study of degenerate or irregular Lagrangians was initiated by Dirac and Bergmann [*4*]. For many years it was thought that the degeneracy of Ω precluded the possibility of establishing a canonical formalism on TQ itself. Consequently, in order to « canonicalize » such degenerate systems it was believed necessary to change the theory into Hamiltonian form. Dirac and Bergmann developed an algorithm for doing this, and, at the same time, for solving the resulting Hamilton equations. No method was known for dealing *directly* (i. e., within the Lagrangian framework itself) with the equations (1.1).

The most important result of this paper is that the *presymplectic* structure on TQ is sufficient *in and by itself* to establish a canonical formalism on TQ. Put another way, the presymplectic geometry itself *is* the canonical formalism.

In a companion paper [*5*] we have developed a constraint algorithm which generalizes and improves upon the Dirac-Bergmann technique. Our algorithm is phrased in the context of global presymplectic geometry and consequently can be directly applied to the Lagrangian case. This generality is not illusory, as there exist well-behaved Lagrangian systems whose Hamiltonian counterparts are highly singular or even nonexistent (Section IV). This algorithm has several major advantages over the Dirac-Bergmann procedure. Specifically, the Dirac-Bergmann algorithm can only be applied locally. Insofar as the quantization of a theory is thought to depend significantly upon global considerations [*3*], this drawback becomes a major shortcoming. More importantly for our considerations, the application of the Dirac-Bergmann algorithm depends crucially upon the fact that the presymplectic manifold of interest (viz. the primary constraint submanifold FL(TQ)) is imbedded in a symplectic manifold (viz. T*Q). Even though the Lagrangian description can be cast into canonical form on TQ itself, one cannot transfer the Dirac-Bergmann results to velocity phasespace as there does not exist a Lagrangian counterpart to the symplectic imbedding space T*Q. The algorithm we have proposed does not suffer from this defect.

With regard to the Lagrange equations (1.1), our algorithm finds whether or not there exists a submanifold P of TQ along which these equations hold; if such a submanifold exists, then the algorithm gives a *constructive* method for finding it. In other words, the algorithm can be used to define and solve « consistent Lagrangian equations of motion ».

To the authors' knowledge, such an algorithm has never before been successfully attempted in the Lagrangian formulation. A superficially similar algorithm was proposed by Künzle [*6*] who was concerned only with homogeneous Lagrangians. Hence, he did not consider the « Dirac-Bergmann » problem per se [*7*] but rather a related question concerning the nature of solutions to the Lagrange equations, which we call the « second-order equation » problem. This is not directly relevant to our work here and will be discussed in depth in a companion paper [*8*] (see also Sections II and III).

The algorithm can be used to compare the Lagrangian and Hamiltonian formalisms. In the hyperregular case [*1*], it is well-known that the two formulations are completely equivalent. If L is regular but not hyperregular, then generally there will *not* exist a global Hamiltonian formulation of the dynamics of the system, as we show later (Section IV). In the homogeneous case, Sniatycki [*9*] distinguishes a class of « almost regular » Lagrangian systems and shows that they possess Hamiltonian counterparts. We generalize Sniatycki's definition of almost regular system to the heretofore untreated degenerate inhomogeneous case, and apply the algorithm in order to prove an *equivalence theorem* which states that to every (consistent) almost regular Lagrangian system there exists a corresponding (consistent) Hamiltonian formulation.

Section II provides a very brief introduction to the almost tangent structure canonically associated to TQ and its exterior calculus; its application to Lagrangian systems [*10*] is briefly outlined. A much more comprehensive treatment of these topics is found in the text by Godbillon [*11*]. In the third section, we review the constraint algorithm alluded to earlier [*5*] and apply it to the Lagrangian formalism, while in Section IV we state and prove the equivalence theorem for degenerate Hamiltonian and Lagrangian systems. In general, we try to keep our notation and terminology [*12*] consistent with that of reference [*11*].

II. ALMOST TANGENT STRUCTURE AND THE PRESYMPLECTIC GEOMETRY OF LAGRANGIAN SYSTEMS [*2*, *11*]

A manifold is said to be *symplectic* if it carries a distinguished closed nondegenerate 2-form. If we drop the requirement that this form have maximal rank the manifold is *presymplectic*. In this section, we develop just enough of the theory of vector-valued differential forms to enable us to put a presymplectic structure on velocity phasespace. We first establish some notation.

Let Q be a manifold with tangent bundle TQ and second tangent bundle T(TQ). The bundle projections are τ_Q: TQ $\rightarrow$ Q and τ_{TQ}: T(TQ) $\rightarrow$ TQ. The prolongation of τ_Q to T(TQ) is denoted $T\tau_Q$, and is such that the following diagram commutes:

$$\begin{array}{ccc} T(TQ) & \xrightarrow{T\tau_Q} & TQ \\ {\scriptstyle \tau_{TQ}}\downarrow & & \downarrow{\scriptstyle \tau_Q} \\ TQ & \xrightarrow[\tau_Q]{} & Q \end{array}$$

The *vertical bundle* V(TQ) is the subbundle of T(TQ) defined by [*12*] $V(TQ) := \ker T\tau_Q$.

Let ξ_y denote the vertical lift $T_xQ \rightarrow V_y(TQ)$, that is,

$$\xi_y(w) := \frac{d}{dT}(y + Tw)\,|_{T=0}$$

where $x = \tau_Q(y) = \tau_Q(w)$.

Using this we can define a map $J_y : T_y(TQ) \rightarrow T_y(TQ)$ by

$$J_y(Z) := \xi_y \circ T\tau_Q(Z) \tag{2.1}$$

for all $Z \in T_y(TQ)$. We thus obtain a linear endomorphism

$$J : T(TQ) \rightarrow T(TQ)$$

such that $J^2 = 0$, $\ker J = \operatorname{Im} J = V(TQ)$. The vector-valued 1-form J is called the *almost tangent structure* naturally associated to TQ. In a natural bundle chart on T(TQ), the action of J is $J(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, v^i, 0, \dot{q}^i)$.

We define the *adjoint* J^* of J to be the linear endomorphism of the exterior algebra Λ (TQ) given by

$$\begin{aligned} J^* f &:= f \\ \langle X \mid J^*\alpha \rangle &:= \langle JX \mid \alpha \rangle , \end{aligned} \tag{2.2}$$

where $f \in C^\infty(TQ)$, $\alpha \in T^*(TQ)$ and $X \in T(TQ)$. J^* is then defined on $\wedge$ (TQ) by homomorphic extension. We define the *interior product* of J with a p-form β by

$$i_J\beta(X_1, \ldots, X_p) := \sum_{i=1}^{p} \beta(X_1, \ldots, JX_i, \ldots, X_p) \tag{2.3}$$

where $X_1, \ldots, X_p \in T(TQ)$, and set $i_J f := 0$ for any function f. Finally, the *vertical derivative* d_J is

$$d_J := [i_J, d] . \tag{2.4}$$

It is apparent that d and d_J anticommute, and furthermore that $d_J^2 = 0$. Also, from (2.2) and the definition of i_J it follows that $d_J f = J^* df$.

In order to apply this machinery to physics, take Q to be the configuration space of some physical system; TQ is the velocity phasespace. The almost tangent geometry of TQ, in and by itself, is not enough to define a presymplectic structure. However, if we are given a distinguished function $L : TQ \rightarrow R$ (the Lagrangian), then J determines a preferred presymplectic form

$$\Omega := - dd_J L . \tag{2.5}$$

By construction and (2.3), J is Hamiltonian for Ω, i. e.,

$$i_J\Omega = 0 . \tag{2.6}$$

In a natural bundle chart (2.5) becomes simply

$$\Omega = \frac{\partial^2 L}{\partial v^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dq^i \wedge dv^j . \tag{2.7}$$

The Lagrangian L is said to be *regular* iff Ω is nondegenerate; otherwise L is *degenerate* or *irregular*. In more familiar terms, (2.7) shows that Ω is nondegenerate iff the velocity Hessian $\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right)$ of L is invertible. The triple (TQ, L, Ω) is said to be a *Lagrangian system*.

In order to geometrize the equations of motion we need to define yet one other object, the « fiber derivative ». Let $f \in C^\infty(TQ)$ so that $df \in T^*(TQ)$, and suppose $y \in TQ$. Consider the map $Ff(y)$ defined by

$$Ff(y) := df(y) \circ \xi_y : T_xQ \rightarrow R \tag{2.8}$$

where $x = \tau_Q(y)$. $Ff(y)$ is clearly linear on T_xQ, and consequently is a 1-form on Q. The map $Ff : TQ \rightarrow T^*Q$ is called the *fiber derivative* of f. It is fiber-preserving, but not necessarily a bundle map as it may not be linear on the fibers [*13*].

We call the function $A_f : TQ \rightarrow R$ defined by $A_f(w) := \langle w \mid Ff(w) \rangle$ for $w \in TQ$ the *action* of f. The *Liouville vectorfield* V is then defined by its action on $C^\infty(TQ)$ as follows:

$$V[f] := A_f . \tag{2.9}$$

Alternately, it is possible to characterize V by

$$V(w) = \xi_w(w) .$$

The operations d_J, i_J and i_V are related by

$$i_V d_J + d_J i_V = i_J . \tag{2.10}$$

Note that V is vertical, that is, $JV = 0$.

Return now to the Lagrangian system (TQ, L, Ω). The fiber derivative FL naturally associated to this system is said to be a *Legendre transformation.* When L is regular, FL can be shown [*1*] to be a local diffeomorphism. It may happen that FL is in fact a diffeomorphism, in which case L is said to be *hyperregular.* If A_L is the action of L, then the *energy* E of L is simply $A_L - L$. In these terms, the Lagrangian equations of motion are

$$i(X)\Omega = dE \tag{2.11}$$

as can easily be checked in a natural bundle chart using (2.7) and (2.8).

Variational as well as physical considerations [*14*] require that one should append to equations (2.11) the *second-order equation* condition

$$JX = V , \tag{2.12}$$

or in more familiar terms,

$$T\tau_Q(X) = \tau_{TQ}(X) .$$

This effectively demands that the Lagrange equations (2.11) be second-order differential equations, a property that is not usually possessed by these equations in the degenerate case [*6*, *8*].

It is clear that the presymplectic structure (2.5) is sufficient to establish a canonical formalism for Lagrangian dynamics. We now turn to a discussion of how to solve the above equations in the degenerate case.

III. THE CONSTRAINT ALGORITHM AND DEGENERATE LAGRANGIAN SYSTEMS [*5*, *18*]

Let (TQ, L, Ω) be a Lagrangian system. We search for necessary and sufficient conditions which will enable us to solve the Lagrangian equations of motion

$$i(X)\Omega = dE. \tag{3.1}$$

As was mentioned in the introduction, if Ω is symplectic, then (3.1) has a unique solution X.

When L is degenerate, however, this will not be so. Nonetheless, it may be possible to solve (3.1) in the following restricted sense: there exists a submanifold [*15*] P_2 of TQ and a vector field X on TQ such that equation (3.1) holds when restricted to P_2. Suppose this is the case and let $Z \in T(TQ)^{\perp}$ (note that $T(TQ)^{\perp}$ is just ker Ω) [*12*]. Then $i(Z)i(X)\Omega \mid P_2 = 0$, so that (3.1) requires that $i(Z)dE \mid P_2 = 0$. Conversely [*5*], it is possible to show that if P_2 is a submanifold of TQ such that $Z \in T(TQ)^{\perp}$ implies that $i(Z)dE \mid P_2 = 0$, then there exists a vectorfield X on TQ | P_2 which satisfies (3.1). Thus, the points of TQ where the equation (3.1) is inconsistent are exactly those for which $i(Z)dE \neq 0$ for any $Z \in T(TQ)^{\perp}$. Consequently we can characterize P_2 as follows

$$P_2 = \{ y \in TQ \text{ such that } \langle T(TQ)^{\perp} \mid dE \rangle (y) = 0 \}$$

with obvious shorthand notation.

We now try to solve the equation (3.1) restricted to P_2, viz.

$$(i(X)\Omega - dE) \mid P_2 = 0. \tag{3.2}$$

Equation (3.2) evidently can be solved algebraically for X, but this is not enough. *Physically*, we must demand that the motion of the system be constrained to lie in P_2. Thus the vectorfield X appearing in (3.2) must be *tangent* to P_2. This additional requirement will not necessarily be satisfied, leading to further conditions as follows: since X must be tangent to P_2, $TP_2^{\perp}$ annihilates the first term in (3.2). Consistency then demands that we *again* restrict to the set $P_3 \subseteq P_2$ of points where $TP_2^{\perp}$ annihilates the second term.

It is clear now how the algorithm must proceed. We generate a string of imbedded submanifolds

$$\cdots \to P_3 \xrightarrow{j_3} P_2 \xrightarrow{j_2} P_1 := TQ$$

defined as follows [*15*]: for $l \geqq 1$,

$$P_{l+1} = \{ y \in P_l \text{ such that } \langle TP_l^{\perp} \mid dE \rangle (y) = 0 \}.$$

The functions on P_{l-1} which define the *l-ary constraint submanifold* P_l are called *l-ary Lagrangian constraints* and are of the form $\langle TP_{l-1}^{\perp} \mid dE \rangle = 0$.

Once the constraint algorithm so defined is « set into motion », it is not difficult to show [5] that the equations (3.1) will be solvable iff the algorithm terminates with some nonempty *final constraint submanifold* P_K. On P_K we have completely consistent equations of motion of the form

$$(i(X)\Omega - dE) \mid P_K = 0 . \tag{3.3}$$

However, the solutions of (3.3) tangent to P_K are not necessarily unique, being determined only up to vectorfields in $\ker \Omega \cap TP_K$.

This algorithm completely solves the problem of degenerate Lagrangian systems in the following sense; it tells us whether or not the Lagrange equations (3.1) have solutions: if they are solvable, the algorithm provides a *constructive* method for finding the submanifold P_K along which tangential solutions exist. Moreover, it is possible to show that P_K is maximal in that if N is any other submanifold along which the equations (3.1) are satisfied, then $N \subseteq P_K$.

The constraint algorithm does not, however, assure us that the solutions of (3.3) will satisfy the second-order equation condition (2.12). Künzle [6] has addressed this issue (for homogeneous Lagrangians) and has developed a technique for finding simultaneous solutions of (3.3) and (2.12). We have developed an alternative approach to this problem [8] and have found that it can be dealt with separately (without invalidating any of the conclusions of this work) and so we shall not consider it further here.

The algorithm as detailed above in the Lagrangian case is but an example of a much more general technique [5] applicable to any presymplectic manifold (M, ω) and equations on M of the form $i(X)\omega = \alpha$ where α is a closed 1-form. The algorithm is global, geometrically natural, and well adapted to calculation (for an example, see ref. [5]). In addition, the algorithm requires very few assumptions for its applicability (besides the usual manifold-theoretic considerations) and thus furnishes a powerful physical tool.

IV. LEGENDRE TRANSFORMATIONS AND THE EQUIVALENCE THEOREM [1, 9]

Given a Lagrangian system (TQ, L, Ω), we consider the following two questions:

A) Does there exist a Hamiltonian formulation of the dynamics of the system?

B) If so, then when and in what sense are the Hamiltonian and Lagrangian descriptions equivalent?

As noted in the introduction, a number of results have already been achieved along this line. We extend these to the irregular inhomogeneous case.

A *special Hamiltonian system* consists of a submanifold M_1 of T^*Q, a function H_1 on M_1 (the Hamiltonian) and a presymplectic form ω_1 on M_1. The presymplectic structure on M_1 is induced by the canonical symplectic form ω on T^*Q; this comes about as follows: on T^*Q, there exists a canonical 1-form Θ given by

$$\langle W \mid \Theta \rangle := \langle T\pi_Q(W) \mid \tau_{T^*Q}(W) \rangle \tag{4.1}$$

where $W \in T(T^*Q)$ and $\pi_Q : T^*Q \to Q$, $\tau_{T^*Q} : T(T^*Q) \to T^*Q$ are the bundle projections. The exact symplectic structure on T^*Q is then $\omega = -d\Theta$; the inherited presymplectic structure on M_1 being given by $\omega_1 = g_1{}^*\omega$, where g_1 is the inclusion $M_1 \to T^*Q$.

Generically, the Hamilton equations of motion

$$i(X)\omega_1 = dH_1 \tag{4.2}$$

will not be consistent as they stand, and to solve them one must apply the constraint algorithm encountered in Section III (as elaborated in ref. [5]) with (M_1, H_1, ω_1) replacing (TQ, L, Ω).

In this paper, we are only interested in those special Hamiltonian systems which arise in a natural way from Lagrangian systems. The transition from Lagrangian to Hamiltonian form is accomplished via the Legendre transformation $FL : TQ \to T^*Q$ defined by (2.8). The image of TQ under FL defines a submanifold [*15*] M_1 of T^*Q, the *primary constraint submanifold* [*4*, *5*].

As was mentioned earlier, the symplectic structure ω on T^*Q is intimately related to the presymplectic structure Ω on TQ. In fact, we shall now show that $FL^*\Theta = J^*dL$. Let $Z \in T_y(TQ)$; then by (4.1),

$$\begin{aligned} \langle Z \mid FL^*\Theta \rangle &= \langle FL_*Z \mid \Theta \rangle \\ &= \langle T\pi_Q(FL_*Z) \mid \tau_{T^*Q}(FL_*Z) \rangle . \end{aligned}$$

Since the diagram

$$\begin{tikzcd} & T(TQ) \arrow[dl, "FL_*"'] \arrow[d, "\tau_{TQ}"] \\ T(T^*Q) \arrow[d, "\tau_{T^*Q}"'] & TQ \arrow[dl, "FL"'] \arrow[d, "\tau_Q"] \\ T^*Q \arrow[r, "\pi_Q"'] & Q \end{tikzcd}$$

commutes, we have by (2.8) and (2.1), where $y = \tau_{TQ}(Z)$,

$$\begin{aligned} \langle Z \mid FL^*\Theta \rangle &= \langle T\tau_Q(Z) \mid FL(y) \rangle \\ &= \langle \xi_y(T\tau_Q(Z)) \mid dL(y) \rangle \\ &= \langle J_y(Z) \mid dL(y) \rangle \\ &= \langle Z \mid J_y{}^*dL(y) \rangle . \end{aligned}$$

Thus $J^*dL = FL^*\Theta$, so that

$$\Omega = -dd_J L = -dJ^*dL = -dFL^*\Theta = FL^*\omega . \tag{4.3}$$

With regard to question (A), we first consider the hyperregular case. Then FL is a diffeomorphism, and $(T^*Q, E \circ FL^{-1}, \omega)$ is the required special Hamiltonian system. In the general case, FL is no longer a diffeomorphism of TQ onto T*Q. Consequently, we cannot define the Hamiltonian to be $E \circ FL^{-1}$; however, it may be possible to define H_1 implicitly by

$$H_1 \circ FL = E \tag{4.4}$$

This will give a well-defined function H_1 on FL(TQ) iff for any two points $w, z \in TQ$ such that $FL(w) = FL(z)$, we have $E(w) = E(z)$. There is of course no particular reason why this should be true, and thus we see that in the general case, there will *not* exist a special Hamiltonian formalism corresponding to the Lagrangian system (TQ, L, Ω) [*16*].

This motivates the following definition. A Lagrangian system satisfying the following two physically reasonable assumptions:

AR1) FL is a submersion onto its image, and

AR2) for $v \in TQ$, the fibers $FL^{-1}\{FL(v)\}$ of FL are connected submanifolds of TQ,

will be called an *almost regular* system.

For such a system, FL(TQ) can be canonically identified with the leaf space of the foliation of TQ generated by the involutive distribution ker FL_*. In particular, a regular Lagrangian is almost regular iff FL is injective.

We now prove that every almost regular system has a special Hamiltonian counterpart. Indeed, we need only show that (4.4) defines a single-valued Hamiltonian. Since each fiber of the submersion FL is connected, it is sufficient to show that $L_Z E = 0$ for $Z \in \ker FL_*$. From (2.9),

$$L_Z E = L_Z(A_L - L) = L_Z \langle V \mid dL \rangle - Z[L].$$

Now Z is vertical and so there (locally) exists a vectorfield W such that $Z = JW$. Thus,

$$\begin{aligned} L_Z E &= L_{JW} \langle V \mid dL \rangle - JW[L] \\ &= i(W)[J^* d \langle V \mid dL \rangle - J^* dL] \\ &= i(W)[d_J \langle V \mid dL \rangle - J^* dL] \\ &= i(W)[- i(V) d_J dL + i_J dL - J^* dL] \\ &= \Omega(W, V) \end{aligned}$$

by (2.10). But V is vertical so locally $V = JY$ for some vectorfield Y. Then by (2.6) and (4.3),

$$\begin{aligned} \Omega(W, JY) &= -\Omega(JW, Y) = -\Omega(Z, Y) \\ &= -FL^*\omega(Z, Y) = i(Y)[FL^*\omega](Z) \\ &= i(Y)FL^*[\omega(FL_* Z)] \\ &= 0 \end{aligned}$$

as $Z \in \ker FL_*$.

Consider a Lagrangian system (TQ, L, Ω) and its special Hamiltonian counterpart (M_1, H_1, ω_1). We say that the Hamiltonian and Lagrangian descriptions of the system are *equivalent* provided:

1) for every solution X_L of the Lagrange equations, $FL_*(X_L)$ (*if it exists*) satisfies the Hamilton equations, and

2) if X_H satisfies the Hamilton equations, then every $X_L \in FL_*^{-1}\{X_H\}$ solves the Lagrange equations.

When L is hyperregular, FL is a diffeomorphism of symplectic manifolds. Consequently, if X_L satisfies (3.1) we have, using (4.4):

$$\begin{aligned} 0 &= i(X_L)\Omega - dE \\ &= i(X_L)FL^*\omega - dFL^*H_1 \\ &= FL^*[i(FL_*X_L)\omega - dH_1] \end{aligned}$$

which implies that FL_*X_L solves (4.2). That requirement (2) is satisfied follows simply by reversing the above calculation.

The almost regular case is complicated by the fact that the Lagrange equations will in general be inconsistent, forcing us to apply the algorithm. However, we now show that this will also necessitate the application of the algorithm to the special Hamiltonian system (M_1, H_1, ω_1) as well, and moreover that the two algorithms differ in no essential respect. More precisely, we will define submersions FL_l such that the following diagram will commute (g_l is the inclusion $M_l \to M_{l-1}$):

$$\begin{tikzcd}
TQ \arrow[r, "FL"] \arrow[dr, "FL_1"] & T^*Q \\
P_2 \arrow[u, "j_2"] \arrow[dr, "FL_2"] & M_1 \arrow[u, "g_1"'] \\
P_3 \arrow[u, "j_3"] & M_2 \arrow[u, "g_2"'] \\
\vdots & \vdots \\
P_K \arrow[dr, "FL_K"] & M_{K-1} \\
 & M_K . \arrow[u, "g_K"']
\end{tikzcd}$$

The strategy will be to show that $FL_l(P_l) = M_l$. We do this step-by-step, beginning with $l = 2$. The core of the proof is contained in the following result [*17*]: in a point-wise sense, $FL_{1*}(T(TQ)^\perp) = TM_1^\perp$, where FL_1 is defined implicitly via $g_1 \circ FL_1 = FL$. To see this, let $Z \in T(TQ)^\perp$. Then by (4.3),

$$\begin{aligned} 0 &= i(Z)\Omega \\ &= i(Z)(g_1 \circ FL_1)^*\omega \\ &= FL_1^*[i(FL_{1*}Z)\omega_1] \end{aligned}$$

which implies that $i(\mathrm{FL}_{1*}\mathrm{Z})\omega_1 = 0$ as FL and consequently FL_1 are submersions. Hence, $\mathrm{FL}_{1*}\mathrm{Z} \in \mathrm{TM}_1^{\perp}$. The converse is similar. In fact, this suffices to show that

$$\mathrm{T(TQ)}^{\perp} = \mathrm{TM}_1^{\perp} \oplus \ker \mathrm{FL}_{1*} . \tag{4.5}$$

Now, if $\mathrm{Z} \in \mathrm{T(TQ)}^{\perp}$ is such that $\langle \mathrm{Z} \mid d\mathrm{E} \rangle = 0$, then (4.4), (4.5) and a pointwise calculation show that $\langle \mathrm{FL}_{1*}(\mathrm{Z}) \mid d\mathrm{H}_1 \rangle = 0$. Again, the converse is obtained by reversing the calculation.

The above results only hold in a pointwise sense, but it is possible to find a local basis of vectorfields in T(TQ) which locally span $\mathrm{T(TQ)}^{\perp}$ such that their prolongations by FL_1 exist and locally span $\mathrm{TM}_1^{\perp}$. Consequently, the above calculations imply that if ϕ_{L} is a secondary Lagrangian constraint of the form $\langle \mathrm{Z} \mid d\mathrm{E} \rangle$, then there exists a secondary Hamiltonian constraint $\phi_{\mathrm{H}} = \langle \mathrm{FL}_{1*}\mathrm{Z} \mid d\mathrm{H}_1 \rangle$ such that $\phi_{\mathrm{L}} = \phi_{\mathrm{H}} \circ \mathrm{FL}_1$ and vice-versa. In other words, we have shown that $\mathrm{FL}_1 \circ j_2(\mathrm{P}_2) = g_2(\mathrm{M}_2)$. If we define the map FL_2 by $g_2 \circ \mathrm{FL}_2 = \mathrm{FL}_1 \circ j_2$, it follows that $\mathrm{FL}_2(\mathrm{P}_2) = \mathrm{M}_2$. Furthermore, as FL_1 is a submersion, FL_2 is also.

The proof for cases $l \geqq 3$ consists of similar and only slightly more complicated calculations.

Iterating this procedure, we have that $\mathrm{FL}_1 \circ k_l(\mathrm{P}_l) = h_l(\mathrm{M}_l)$, where $k_l = j_2 \circ \ldots \circ j_l$ and h_l is defined similarly in terms of the g_l. Consequently the maps FL_l given by $h_l \circ \mathrm{FL}_l = \mathrm{FL}_1 \circ k_l$ are submersions of P_l onto M_l, thereby proving the commutativity of the diagram.

Turning now to requirement (1), suppose that X_{L} satisfies the Lagrange equations (3.3). Then, *if* $\mathrm{FL}_{1*}(\mathrm{X}_{\mathrm{L}})$ *exists*,

$$\begin{aligned} 0 &= (i(\mathrm{X}_{\mathrm{L}})\Omega - d\mathrm{E}) \mid \mathrm{P}_{\mathrm{K}} \\ &= (i(\mathrm{X}_{\mathrm{L}})\mathrm{FL}_{1*}\omega_1 - d\mathrm{FL}_1^*\mathrm{H}_1) \mid \mathrm{P}_{\mathrm{K}} \\ &= [\mathrm{FL}_1^*(i(\mathrm{FL}_{1*}\mathrm{X}_{\mathrm{L}})\omega_1 - d\mathrm{H}_1)] \mid \mathrm{P}_{\mathrm{K}} \\ &= \mathrm{FL}_1^*[(i(\mathrm{FL}_{1*}\mathrm{X}_{\mathrm{L}})\omega_1 - d\mathrm{H}_1) \mid \mathrm{M}_{\mathrm{K}}] \end{aligned}$$

by the commutativity of the diagram. Thus, as FL_1 is a submersion, $\mathrm{FL}_{1*}\mathrm{X}_{\mathrm{L}}$ solves the Hamilton equations

$$(i(\mathrm{FL}_{1*}\mathrm{X}_{\mathrm{L}})\omega_1 - d\mathrm{H}_1) \mid \mathrm{M}_{\mathrm{K}} = 0 .$$

Conversely, we can show that requirement (2) is satisfied by running the above calculation backwards.

Thus, we have proved the following (no period).

Equivalence Theorem. — Let (TQ, L, Ω) be an almost regular Lagrangian system. Then

1) There exists a special Hamiltonian formulation (FL(TQ), H_1, ω_1) of the dynamics of the system, and

2) The Lagrangian and Hamiltonian formulations are equivalent.

This theorem generalizes the work of Sniatycki [9], who proved a similar result for homogeneous systems. Sniatycki, in fact, also proved a converse to the above theorem in the homogeneous case stating that to every reasonable Hamiltonian system $(M_1, 0, \omega_1)$ there corresponds an almost regular homogeneous Lagrangian system. Such a converse does not appear to exist when $H_1 \neq 0$.

We briefly discuss the assumptions (AR1) and (AR2) which define an almost regular system. Actually, it takes little consideration to see that they are seldom true as global statements; however, this need not result in sharply restricting either the algorithm or the theory we have developed as a useful tool. In practice, it is almost always possible to find reasonable physical systems which will violate any sort of regularity assumption. Such violations are dealt with routinely; for example, one may work locally where everything is manageable, or one may use rather more sophisticated techniques of a global nature.

As an amusing aside, we note from the above commutative diagram that one need not transform to the Hamiltonian formulation directly from TQ. One could, if so desired, and work through the algorithm to the lth step in the Lagrangian formalism, then transform to the Hamiltonian description via FL_l. In this case, M_l would be the « primary constraint submanifold ».

V. DISCUSSION

We have emphasized the fact that a Lagrangian system in and by itself defines a canonical formalism, as embodied in its natural presymplectic structure. From a geometric stance, the Lagrangian formulation of dynamics is therefore essentially no different than the Hamiltonian description. Furthermore, we have developed a constraint algorithm which allows us to treat degenerate Lagrangian systems (i. e., to define and solve consistent Lagrangian equations of motion) directly on velocity phasespace itself. This, combined with the fact that our techniques are global, enables us to completely cope even with those Lagrangian systems which have no Hamiltonian counterparts. Even though this extreme case may be rare, in practice the methods we have developed are still useful for it may be much easier to work on velocity phasespace rather than on the Hamiltonian phasespace depending on how pathological the Legendre transformation is. In particular, the system can be quantized directly in the Lagrangian description [3], the transition to the Hamiltonian formalism being either unnecessary or perhaps impossible.

In addition, we have distinguished a class of « almost regular » Lagrangian systems and have shown that every such system possesses an equivalent Hamiltonian formulation.

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[3] D. J. SIMMS and N. M. J. WOODHOUSE, *Lectures on Geometric Quantization*, Lecture Notes in Physics, t. **53**, Springer-Verlag, Berlin, 1976.

[4] A nice summary is given in P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monograph Series, t. **2**, 1964. Several examples are presented in Hanson, Regge and Teitelboim, Accademia Nazionale dei Lincei (Rome), t. **22**, 1976.

[5] M. J. GOTAY, J. M. NESTER and G. HINDS, Presymplectic Manifolds and the Dirac-Bergmann Theory of Constraints, *J. Math. Phys.*, t. **19**, 1978, p. 2388.

[6] H. P. KÜNZLE, *Ann. Inst. H. Poincaré*, t. **A 11**, 1969, p. 393.

[7] From the point of view of the constraint algorithm, the homogeneous case is trivial because $E \equiv 0$ (see section III).

[8] J. M. NESTER and M. J. Gotay, *Presymplectic Lagrangian Systems II: The Second-Order Equation Problem* (in preparation).

[9] J. SNIATYCKI, Proc. 13th Biennial Seminar of the *Canadian Math. Cong.*, t. **2**, 1972, p. 125.

[10] Throughout this paper, we assume for simplicity that all physical systems under consideration are time-independent and that all relevant phasespaces are finite-dimensional; however, all of the theory developed in this paper can be applied when these restrictions are removed with little or no modification. For details concerning the infinite-dimensional case, see refs. [5] and [18].

[11] C. GODBILLON, *Géométrie Différentielle et Mécanique Analytique*, Hermann, Paris, 1969.

[12] We herein establish some notation and terminology. All manifolds and maps appearing in this paper are assumed to be C^∞. We designate the natural pairing $TM \times T^*M \to R$ by $\langle \mid \rangle$. The symbol i denotes the interior product. Note that if γ is a p-form, and $X_1, \ldots, X_p$ are vectorfields, then

$$i(X_1) \ldots i(X_p)\gamma = \gamma(X_p, \ldots, X_1).$$

The symbol « $\mid N$ » means « restriction to the submanifold N ». If $j : N \to M$ is the inclusion, then we denote by $\gamma \mid N$ the restriction of γ to N. Given a 2-form Ω on M, we define the « Ω-orthogonal complement » of TN in TM to be $TN^\perp = \{ Z \varepsilon TM$ such that $\Omega(Z, Y) = 0$ for all $Y \varepsilon TN \}$. Furthermore, we define $ker\ \Omega = \{ Y \varepsilon TM$ such that $i(Y)\Omega = 0 \}$. If $f : M \to P$ is smooth, then we denote by Tf or f_* the derived mapping $TM \to TP$. We have $ker\ Tf = \{ Y \varepsilon TM$ such that $Tf(Y) = 0 \}$.

[13] For another definition of FL (which is logically independent of the almost tangent structure J), see ref. [1].

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[15] We assume that all of the P_l appearing in the algorithm are in fact imbedded submanifolds. Otherwise, one must resort to standard tricks, e. g., work locally where everything is manageable (see Section IV).

[16] In fact, there does not even exist a unique *local* Hamiltonian formalism corresponding to such a Lagrangian system, as, e. g., with $L = 1/4v^4 - 1/2v^2$.

[17] In the following, $TM_1^\perp$ denotes the ω_1-orthogonal complement (see [12]).

[18] M. J. GOTAY, *Presymplectic Manifolds, Geometric Constraint Theory and the Dirac-Bergmann Theory of Constraints*, Ph. D. Thesis, University of Maryland, 1979.

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